A NON-GORENSTEIN POINT ON A SMOOTHABLE HILBERT SCHEME

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ABSTRACT. The square of the maximal ideal in nine variables defines a non-Gorenstein point of the reduced smoothable component of the Hilbert scheme of ten points. We prove this by comparing tangent determinant characters at two smooth points in an affine chart contracted to the square-zero point. The same comparison shows that the normalization is not $\mathbb{Q}$-Gorenstein above this point. The proof uses a smooth homogeneous algebra of Hilbert function $(1, 5, 4)$; an explicit example and exact verification are included. A boundary dimension count proves regularity in codimension one. Normality, Cohen–Macaulayness, and rational singularities of the original germ are not determined here.

1. INTRODUCTION

Work over $\mathbb{C}$. Let

$$\mathcal{H} = \mathrm{Hilb}^{10}_{\mathrm{sm}}(\mathbb{A}^9)_{\mathrm{red}}$$

be the reduced closure of the locus of ten distinct points, and put

$$Z_0 = \mathrm{Spec}\big(\mathbb{C}[z_1, \dots, z_9]/(z_1, \dots, z_9)^2\big).$$

Thus $\mathcal{H}$ is an integral variety of dimension 90. All assertions about Gorensteinness below concern the local ring of $\mathcal{H}$, rather than the coordinate algebra of the finite scheme Z_0.

Theorem 1.1. *The local ring $\mathcal{O}_{\mathcal{H},[Z_0]}$ is not Gorenstein. Moreover, the normalization $\widetilde{\mathcal{H}} \to \mathcal{H}$ has a unique point above $[Z_0]$, and its canonical divisor is not $\mathbb{Q}$-Cartier at that point.*

The argument is a character obstruction. On a positively graded affine variety, an equivariant rank-one module which is invertible at the vertex is generated by one homogeneous element. For a GL_9-action, its linearization is therefore a power of the determinant. A curvilinear point forces the anticanonical character to be $\det^2$. A second smooth point, obtained from a homogeneous algebra with Hilbert function $(1, 5, 4)$, gives an incompatible character.

The existence of the latter point is proved by Gałązka, Keneshlou and Šivic in the proof of [GKS25, Proposition 4.2]. Only its smoothness and Hilbert function are needed. In particular, the main argument does not depend on a numerical approximation or on identifying a component from equations.

We also prove that $\mathcal{H}$ is regular in codimension one (Proposition 5.1). Thus normality of the local ring in Theorem 1.1 is equivalent to Serre's condition S_2; Cohen–Macaulayness would imply normality. These two properties are predicted for smoothable components by a consequence of Haiman's conjecture on isospectral Hilbert schemes [Hai01, Conjecture 5.2.1]; see also [JRS26, §1]. Section 6 records the exact scope of the obstruction and a further invariant-ring calculation.

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2. The centered chart and equivariant line bundles

Let $U \subset \mathcal{H}$ be the affine open subset on which

$$1, z_1, \dots, z_9$$

is a basis of the universal quotient. The chart is invariant under linear coordinate changes and translations. It contains $[Z_0]$: indeed, the ideals

$$\big(z_i z_j \ (i<j),\ z_i^2 - s z_i \ (1 \le i \le 9)\big) \subset \mathbb{C}[s, z_1, \dots, z_9] \tag{2.1}$$

define a free family of rank 10, whose fiber for $s \neq 0$ consists of $0, se_1, \dots, se_9$.

For a quotient in U, write M_i for multiplication by z_i. Its trace barycenter is

$$c = (c_1, \dots, c_9), \qquad c_i = \frac{1}{10}\operatorname{tr}(M_i).$$

Translation by $-c$ gives a GL_9-equivariant isomorphism

$$U \simeq \mathbb{A}^9 \times X, \qquad X = \{\, \operatorname{tr}(M_1) = \cdots = \operatorname{tr}(M_9) = 0 \,\} \subset U. \tag{2.2}$$

Here GL_9 acts in the standard way on the first factor. The centered variety X is integral of dimension 81; denote by $o \in X$ the point $[Z_0]$.

Write the products in the chosen basis as

$$z_i z_j = b_{ij} + \sum_{k=1}^{9} p_{ij}^k z_k.$$

Under dilation of $\mathbb{A}^9$ by t, the coefficients p_{ij}^k and b_{ij} have geometric weights 1 and 2, respectively. Thus scalar dilation extends to a contraction $\mathbb{A}^1 \times X \to X$ with value o at $t = 0$. In particular,

$$R = \mathbb{C}[X] = \bigoplus_{d \ge 0} R_d, \qquad R_0 = \mathbb{C}, \qquad \mathfrak{m}_o = \bigoplus_{d>0} R_d. \tag{2.3}$$

Lemma 2.1. *Let R be a positively graded integral $\mathbb{C}$-algebra with $R_0 = \mathbb{C}$, and let $G = \mathrm{GL}(V)$ act on* Spec R *so that its scalar subgroup induces the grading. Let M be a finite graded, G-equivariant, torsion-free R-module of rank one. If $M_{\mathfrak{m}_o}$ is free of rank one, then M is free of rank one. Its equivariant structure is a character* $\det^k$ *for some $k \in \mathbb{Z}$.*

Proof. The vector space $M/\mathfrak{m}_o M$ is one-dimensional. Choose a homogeneous lift s of a nonzero element. Graded Nakayama gives $M = Rs$, and torsion-freeness makes $R \to M$, $1 \mapsto s$, injective.

Every unit of R is constant: this follows by comparing highest degrees in a product equal to 1. Hence the G-action on the nowhere-vanishing generator is given by a character $G \to \mathbb{G}_m$. Every character of $\mathrm{GL}(V)$ is an integral power of the determinant. □

Corollary 2.2. *If X is Gorenstein at o, there exists $k \in \mathbb{Z}$ such that, for every smooth point $x \in X$ and every one-parameter subgroup $\lambda : \mathbb{G}_m \to \mathrm{GL}_9$ fixing x, the character on* $\det T_x X$ *is* $(\det \lambda)^k$.

Proof. The non-Gorenstein locus of X is closed and invariant under scalar dilation. Any nonempty such locus contains the contracting vertex. Thus the hypothesis implies that X is Gorenstein everywhere. Apply Lemma 2.1 to the naturally linearized anticanonical module $\omega_X^\vee$. At a smooth point its fiber is $\det T_x X$. □

We use geometric tangent weights throughout. If coordinates have specified positive weights and a tangent map sends an ideal generator of weight d to a quotient monomial of weight e, then its tangent weight is $d - e$. Equivalently, the coefficient of that monomial in the deformed equation is multiplied by t^{d-e}. This convention fixes the sign of the anticanonical character.

3. Two smooth fixed points

3.1. **The curvilinear point.** Let $p \in U$ be the quotient $\mathbb{C}[t]/(t^{10})$ embedded by $z_i = t^i$. Its ideal is

$$(z_2 - z_1^2, \dots, z_9 - z_1^9, z_1^{10}).$$

It is smoothable, for instance by replacing t^{10} with $t^{10} - s$, and it is a complete intersection. Consequently its Hilbert-scheme tangent space has dimension $9 \cdot 10 = 90$, so p is smooth on the full Hilbert scheme. All nonconstant multiplication operators are nilpotent, hence $p \in X$.

The subgroup

$$\lambda_0(u) = \operatorname{diag}(u, u^2, \dots, u^9)$$

fixes p. The nine ideal generators have weights $2, 3, \dots, 10$, and the quotient basis has weights $0, 1, \dots, 9$. The determinant weight of the full tangent space is therefore

$$10 \sum_{d=2}^{10} d - 9 \sum_{e=0}^{9} e = 10 \cdot 54 - 9 \cdot 45 = 135.$$

Removing the translation factor in (2.2) subtracts $1 + \cdots + 9 = 45$, giving

$$\operatorname{wt}_{\lambda_0}(\det T_p X) = 90, \qquad \det \lambda_0(u) = u^{45}. \tag{3.1}$$

Thus the character in Corollary 2.2, if it exists, must satisfy $45k = 90$, or $k = 2$.

3.2. **A point of Hilbert function $(1, 5, 4)$.**

Lemma 3.1. *There is a homogeneous ideal $J \subset S = \mathbb{C}[x_1, \dots, x_5]$ such that $A = S/J$ has Hilbert function $(1, 5, 4)$ and $[J]$ is a smooth point of $\operatorname{Hilb}^{10}(\mathbb{A}^5)$ on its smoothable component. For any such ideal, the scalar-dilation tangent representation is*

$$T_{[J]} \operatorname{Hilb}^{10}(\mathbb{A}^5) \simeq \mathbb{C}(0)^{44} \oplus \mathbb{C}(1)^5 \oplus \mathbb{C}(r) \quad \text{for some } r \in \{1, 2, 3\}, \tag{3.2}$$

where $\mathbb{C}(w)$ denotes the one-dimensional representation of weight w.

Proof. The existence assertion is the construction in the proof of [GKS25, Proposition 4.2]; an explicit verification is given in Appendix A.

For the weight statement, use $T_{[J]} \operatorname{Hilb}^{10}(\mathbb{A}^5) = \operatorname{Hom}_S(J, A)$. Since $J_1 = 0$, $\dim J_2 = 15 - 4 = 11$, and $A_{\geq 3} = 0$, every linear map $J_2 \to A_2$ extends uniquely to a degree-zero S-module map $J \to A$. Thus the weight-zero tangent space has dimension $11 \cdot 4 = 44$.

The ideal J is generated in degrees at most 3, so all remaining geometric weights lie in $\{1, 2, 3\}$. Translation gives five independent weight-one directions: the differential of the trace barycenter is a left inverse to the translation map. Smoothness gives total tangent dimension 50, leaving one further one-dimensional representation of weight $r \in \{1, 2, 3\}$. □

Choose homogeneous quadrics $q_1, \dots, q_4$ whose classes form a basis of A_2. Embed $\operatorname{Spec} A$ in $\mathbb{A}^9 = \mathbb{A}^5_x \times \mathbb{A}^4_y$ by the graph

$$y_j = q_j(x) \quad (1 \leq j \leq 4).$$

Denote the resulting point by q. The nine nonconstant basis elements have nilpotent multiplication operators, so $q \in X$. It is fixed by

$$\lambda_1(u) = \operatorname{diag}(uI_5, u^2 I_4).$$

The open locus in $\operatorname{Hilb}^{10}(\mathbb{A}^9)$ on which projection to $\mathbb{A}^5$ is a closed immersion is the total space of four copies of the rank-ten tautological bundle on $\operatorname{Hilb}^{10}(\mathbb{A}^5)$. Indeed, a graph is specified by four elements of the universal quotient algebra. It follows that q is smooth on the full Hilbert scheme. Graphs over the smoothable component are smoothable, so this is also a smooth point of $\mathcal{H}$.

For each extra coordinate, a coefficient along an element of A_d has weight $2-d$. The vertical tangent representation is therefore

$$\mathbb{C}(0)^{16} \oplus \mathbb{C}(1)^{20} \oplus \mathbb{C}(2)^4,$$

of determinant weight 28. The translation factor of $\mathbb{A}^9$ has weights $1^5, 2^4$, of sum 13. The equivariant tangent sequences, together with (3.2), now give

$$\mathrm{wt}_{\lambda_1}(\det T_q X) = (5+r) + 28 - 13 = 20 + r \leq 23, \qquad \det \lambda_1(u) = u^{13}. \tag{3.3}$$

Proof of the first assertion of Theorem 1.1. If $\mathcal{O}_{\mathcal{H},[Z_0]}$ were Gorenstein, the product (2.2) would imply that X is Gorenstein at o. Corollary 2.2 and (3.1) would then force the character $\det^2$. At q this has weight 26, contradicting (3.3). □

4. The normalization

Let $\nu : \overline{X} \to X$ be the normalization, with $\overline{R} = \mathbb{C}[\overline{X}]$. The GL_9-action lifts to $\overline{X}$. The grading lifts as well, and $\overline{R}$ is finite over R.

Lemma 4.1. *The algebra $\overline{R}$ is nonnegatively graded with $\overline{R}_0 = \mathbb{C}$. There is a unique point $\overline{o}$ above o, and scalar dilation contracts $\overline{X}$ to $\overline{o}$.*

Proof. As a finite graded R-module, $\overline{R}$ is bounded below. A nonzero homogeneous element of negative degree would have nonzero powers of arbitrarily negative degree, which is impossible. The degree-zero part is a finite-dimensional domain over $\mathbb{C}$, hence equals $\mathbb{C}$. The finite algebra $\overline{R}/\mathfrak{m}_o\overline{R}$ is nonnegatively graded with degree-zero part $\mathbb{C}$; its positive part is nilpotent. It therefore has a unique maximal ideal. The assertion about contraction follows from the grading. □

Proposition 4.2. *The canonical divisor of $\overline{X}$ is not $\mathbb{Q}$-Cartier at $\overline{o}$.*

Proof. Suppose otherwise. For some integer $m > 0$, the reflexive anticanonical power

$$\omega_{\overline{X}}^{[-m]} := \left((\omega_{\overline{X}}^{\vee})^{\otimes m}\right)^{\vee\vee}$$

would be invertible at $\overline{o}$. This is a finite graded, equivariant rank-one reflexive module. By Lemma 2.1, it is free with character $\det^k$ for some $k \in \mathbb{Z}$.

Normalization is an isomorphism over the smooth locus of X, so both p and q lift unchanged. At these two points, (3.1) and (3.3) yield

$$45k = 90m, \qquad 13k = (20+r)m.$$

The first equality gives $k = 2m$; the second would then give $r = 6$, whereas $r \leq 3$. This is a contradiction. □

Normalization commutes here with the product by affine space: the normalization of U is $\mathbb{A}^9 \times \overline{X}$. Lemma 4.1 gives the unique point above $[Z_0]$. The canonical divisor on this product is pulled back from $\overline{X}$, since affine space has trivial canonical bundle. Being $\mathbb{Q}$-Cartier is preserved and reflected by this smooth product. Proposition 4.2 therefore proves the second assertion of Theorem 1.1.

5. Regularity in codimension one

The normality question admits a useful reduction. Dimension statements in this section refer to the underlying reduced loci.

Proposition 5.1. *The divisors in the complement of the reduced-subscheme locus of $\mathcal{H}$ are exactly*

(i) *the closure of the locus consisting of one double point and eight reduced points with pairwise disjoint supports;*

(ii) *the closure of the punctual locus of Hilbert function $(1,5,4)$, with moving support.*

Both divisors are generically smooth on the full Hilbert scheme. In particular, $\mathcal{H}$ satisfies Serre's condition R_1.

Proof. We first record how punctual dimensions enter the calculation. Let a locus of length-ℓ algebras of embedding dimension r, embedded in $\mathbb{A}^r$ and supported at the origin, have dimension d. On an open set where projection from $\mathbb{A}^N$ to $\mathbb{A}^r$ restricts to a closed immersion, the remaining coordinates are arbitrary functions on the length-ℓ scheme. They contribute $\ell(N-r)$ parameters. Allowing the support in $\mathbb{A}^r$ to move, the dimension is therefore

$$r + d + \ell(N-r).$$

Consequently, the corresponding locus in the smoothable component has codimension at least

$$c = r(\ell-1) - d. \tag{5.1}$$

We may use the dimension of the entire punctual locus for an upper bound on its smoothable part. For disjoint unions the codimensions add, by the étale local product structure of the Hilbert scheme.

For $\ell \leq 8$, the punctual dimension is at most $(r-1)(\ell-1)$ by [JK24, Theorem 1.2]; thus $c \geq \ell - 1$. For $\ell = 9, 10$, the same bound applies when $r \leq 3$, by [JK24, Theorem 1.3] and the graph calculation above. We may hence assume $r \geq 4$. When $H(2)$ is 1 or 2, the weak bound $d \leq (r-1)(\ell-1)$ follows from [JK24, Theorem 5.7]. If $H(2) = 0$, the algebra is square-zero, $d = 0$, and $c = r^2$.

Suppose first that the Hilbert function is $(1, r, s)$, with $s \geq 3$. The punctual locus is the Grassmannian $\mathrm{Gr}(s, \binom{r+1}{2})$, so

$$c = r^2 + s^2 - \frac{sr(r-1)}{2}. \tag{5.2}$$

For length nine the possibilities with $r \geq 4$ are $(r,s) = (4,4), (5,3)$; the respective bounds are $8, 4$. For length ten they are $(4,5), (5,4), (6,3)$, with bounds $11, 1, 0$. The smoothable $(1,6,3)$ locus has codimension six in $\mathrm{Gr}(3,21)$ by [EV10, Theorem 1.4 and Corollary 5.6]. Its codimension in the smoothable component is therefore six. The $(1,5,4)$ locus, on the other hand, is irreducible, smoothable, and has codimension one. It is generically smooth on the full Hilbert scheme by [GKS25, Proposition 4.2 and its proof].

It remains to consider Hilbert functions with $r \geq 4$, $H(2) \geq 3$, and $H(3) > 0$. Since the length is at most ten, the following list is exhaustive. The last two columns give an upper bound on the fixed-support dimension in $\mathbb{A}^r$ and the resulting lower bound in (5.1).

ℓ	Hilbert function	$d \leq$	$c \geq$
9	$(1,4,3,1)$	24	8
10	$(1,5,3,1)$	40	5
10	$(1,4,4,1)$	27	9
10	$(1,4,3,2)$	27	9
10	$(1,4,3,1,1)$	34	2

We justify all entries. For a homogeneous ideal with Hilbert function $(1, r, s, 1)$, its inverse system is determined by a cubic F up to scalar, an s-dimensional space of quadrics containing the first partials of F, and all linear forms. Put $N_r = \binom{r+1}{2}$. If F has t essential variables, this parameter space has dimension

$$t(r-t) + \binom{t+2}{3} - 1 + (s-t)(N_r - s), \qquad 1 \le t \le \min(r, s).$$

The maxima for $(r, s) = (4, 3), (5, 3), (4, 4)$ are $17, 28, 21$, respectively. The fibers of the associated-graded map have dimension $N_r - s$ by [JK24, Proposition 5.11]. Adding these dimensions gives $24, 40, 27$. For $(1, 4, 3, 2)$, the homogeneous locus has components of dimensions 13 and 11 by [GKS25, Proposition 2.6]; the associated-graded fibers have dimension $2(10 - 3) = 14$, again by [JK24, Proposition 5.11]. This gives the fourth row.

For the final row, a homogeneous inverse system has degree-four and degree-three pieces $\langle L^4 \rangle$ and $\langle L^3 \rangle$ for a linear form L. Its quadratic piece is a three-dimensional space containing L^2. Hence the homogeneous locus has dimension

$$\dim \mathbb{P}^3 + \dim \mathrm{Gr}(2, 9) = 3 + 14 = 17.$$

Fix its homogeneous ideal J and write $A = \mathbb{C}[x_1, \ldots, x_4]/J$. There are seven quadratic minimal generators. The quotient by these quadrics has degree-two dimension three; Macaulay's growth bound gives degree-three dimension at most four. Thus J has at most three additional cubic minimal generators. A filtered lift of a quadratic generator is determined by a tail in $A_3 \oplus A_4$, and a cubic generator by a tail in A_4. Generators of degree at least four have no tails. Indeed, choose complements to J in each degree and reduce successive higher terms into these complements; filtered Nakayama then shows that the chosen lifts generate every ideal with associated graded ideal J. The fiber is therefore parametrized by at most

$$7 \dim(A_3 \oplus A_4) + 3 \dim A_4 = 14 + 3 = 17$$

coefficients, subject to further equations. Its dimension is at most seventeen, giving $d \le 34$.

We have proved that every punctual length-nine locus has codimension at least four, and that the only punctual length-ten divisor is the $(1, 5, 4)$ locus. By additivity, a nonpunctual boundary divisor must consist of a single length-two factor and eight simple factors. This locus is irreducible and lies in the smooth locus of the full Hilbert scheme: its local Hilbert scheme is a product of smooth Hilbert schemes of one or two points. The $(1, 5, 4)$ divisor is also irreducible and generically smooth, as noted above. Every codimension-one point of $\mathcal{H}$ is consequently regular. □

Corollary 5.2. *The centered basis chart X is normal if and only if it satisfies S_2. The same equivalence holds for $\mathcal{O}_{X,o}$. In particular, Cohen–Macaulayness of the local ring at its contracting vertex would imply normality there.*

Proof. The product decomposition of the basis chart with the translation space transfers R_1 from $\mathcal{H}$ to X. Apply Serre's criterion. □

6. Normality and rational singularities

Theorem 1.1 concerns Gorensteinness and the canonical divisor of the normalization; Proposition 5.1 proves R_1. Neither result decides S_2, normality, or rational singularities. We can, however, rule out a possible obstruction to normality coming from scalar determinant functions.

For the universal centered algebra on X, set

$$T_{ij} = \operatorname{tr}(M_i M_j), \qquad \Delta = \det(T_{ij})_{1\le i,j\le 9}.$$

The function Δ is homogeneous of degree 18. The full trace pairing in the basis $1, z_1, \dots, z_9$ has matrix $\operatorname{diag}(10, T)$, so $\Delta \neq 0$ precisely on the reduced locus.

Proposition 6.1. *The invariant rings of X and its normalization satisfy*

$$R^{\mathrm{SL}_9} = \overline{R}^{\mathrm{SL}_9} = \mathbb{C}[\Delta].$$

Proof. A centered configuration of ten points in the basis chart is an affine simplex with barycenter zero. Hence

$$X_\Delta \simeq \mathrm{GL}_9 / \mathfrak{S}_{10},$$

where $\mathfrak{S}_{10}$ acts in its nine-dimensional standard representation. Its determinant is the sign character. Write D for the determinant function on GL_9. Taking left SL_9-invariants and then right $\mathfrak{S}_{10}$-invariants gives

$$\mathbb{C}[X_\Delta]^{\mathrm{SL}_9} = \mathbb{C}[D, D^{-1}]^{\mathfrak{S}_{10}} = \mathbb{C}[D^2, D^{-2}] = \mathbb{C}[\Delta, \Delta^{-1}].$$

The last equality holds because $\Delta(g \cdot x)$ is $\det(g)^2 \Delta(x)$.

Both R and $\overline{R}$ embed equivariantly into $\mathbb{C}[X_\Delta]$: normalization changes nothing on this smooth open orbit. Their gradings have no negative degrees, whereas $\deg \Delta = 18$. Thus their invariant rings contain no negative power of Δ. Since $\Delta \in R$, both invariant rings equal $\mathbb{C}[\Delta]$. □

This calculation tests only the trivial SL_9-representation. It does not exclude integral elements in other representation types, so it is not a normality certificate.

Remark 6.2. Non-$\mathbb{Q}$-Gorensteinness does not obstruct rational singularities. For example, the affine cone over $\mathbb{P}^1 \times \mathbb{P}^2$ with polarization $\mathcal{O}(1,1)$ is normal and has rational singularities. Its coordinate semigroup is saturated, and the resolution given by the total space of $\mathcal{O}(-1,-1)$ has vanishing higher cohomology of its structure sheaf, since $H^i(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}(n,n)) = 0$ for $i > 0$ and $n \ge 0$. Nevertheless its canonical class is not $\mathbb{Q}$-Cartier at the vertex: $K_{\mathbb{P}^1\times\mathbb{P}^2} = \mathcal{O}(-2,-3)$ is not rationally proportional to $\mathcal{O}(1,1)$.

Question 6.3. Is $\mathcal{O}_{\mathcal{H},[Z_0]}$ normal? Is it Cohen–Macaulay? Does it have rational singularities?

These questions are left unresolved by the present argument. In particular, the non-Gorenstein conclusion must not be interpreted as a depth obstruction.

APPENDIX A. AN EXPLICIT SMOOTH AUXILIARY POINT

The following example is extracted from the matrix family in the proof of [GKS25, Proposition 4.2]. It makes the existence assertion in Lemma 3.1 independently reproducible. All the calculations below use rational arithmetic.

In $S = \mathbb{Q}[a,b,c,d,e]$, let J be generated by

$$\begin{aligned} &ae - a^2, \qquad 3b^2 - ab - 10ac, \qquad bc - 3ad, \\ &3bd - 8a^2 + ad, \qquad 3be - a^2 + 4ab, \qquad c^2 - 3a^2, \\ &cd - a^2 + 2ab, \qquad ce + ac, \qquad 3d^2 + ab + 4ac, \\ &3de - ac + 2ad, \qquad 3e^2 + a^2 - 2ad. \end{aligned} \tag{A.1}$$

The quotient has basis

$$1, a, b, c, d, e, a^2, ab, ac, ad. \tag{A.2}$$

Indeed, the eleven displayed quadrics have distinct pivot monomials outside $\{a^2, ab, ac, ad\}$, and the multiplication map $S_1 \otimes J_2 \to S_3$ has rank 35. Thus $S_3 \subset J$, and the Hilbert function is $(1,5,4)$.

A.1. **A flat smoothing.** Put

$$E = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

With block sizes $4, 5, 1$ and $e_1 \in \mathbb{Q}^5$, define

$$T = T(s) = \begin{pmatrix} sE & B & 0 \\ 0 & sF & e_1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The five multiplication operators are

$$\begin{aligned} A_1 &= T, \\ A_2 &= s^{-1}T^2 + \tfrac{1}{3}T - \tfrac{1}{3}s^{-5}T^6, & A_3 &= s^{-2}T^3, \\ A_4 &= s^{-3}T^4 - \tfrac{1}{3}s^{-2}T^3 + \tfrac{1}{3}s^{-7}T^8, \\ A_5 &= s^{-4}T^5 - \tfrac{2}{3}s^{-3}T^4 + \tfrac{2}{3}s^{-8}T^9. \end{aligned}$$

Direct expansion shows that all negative powers of s cancel: every A_i has entries in $\mathbb{Q}[s]$, of degree at most one. They commute, since over $\mathbb{Q}(s)$ they are polynomials in T.

Let v be the last standard basis vector of $\mathbb{Q}[s]^{10}$. The vectors $A_i v$ are the five middle-block standard vectors. The top 4×4 block of

$$T^2 v, \quad TA_2 v, \quad TA_3 v, \quad TA_4 v$$

is constant with determinant -1. Together with $v, A_1 v, \ldots, A_5 v$ these vectors form a $\mathbb{Q}[s]$-basis. The commuting operators thus define a free rank-ten algebra family with cyclic vector v. Its special fiber has exactly the multiplication table (A.1)–(A.2).

Finally,

$$\det(tI - T(s)) = t(t^4 + s^4)(t^5 - s^5)$$

has ten distinct roots over $\mathbb{C}$ for $s \neq 0$. Indeed, a common nonzero root of the last two factors would give $\zeta^4 = -1$ and $\zeta^5 = 1$, hence both $\zeta^{20} = -1$ and $\zeta^{20} = 1$. Thus the nonzero fibers are reduced. This proves smoothability of J.

A.2. **Tangent-space verification.** Normalize the generators in (A.1) so that the coefficient of each of the eleven nonbasis quadratic monomials is 1, and denote them by $f_1, \dots, f_{11}$. The kernel of

$$\mu : S_1 \otimes \langle f_1, \dots, f_{11} \rangle \longrightarrow S_3$$

has dimension $55 - 35 = 20$. Write its elements as linear syzygies $\sum_{j,i} s_{ji} x_i f_j = 0$, where $(x_1, \dots, x_5) = (a, b, c, d, e)$.

For a tangent map of weight one, write $f_j \mapsto \sum_\ell u_{j\ell} x_\ell$. Each linear syzygy then gives the equation

$$\sum_{j,i,\ell} s_{ji} u_{j\ell} x_i x_\ell = 0 \quad \text{in } (S/J)_2.$$

These equations form an 80×55 matrix of rank 49. For weight two, write $f_j \mapsto v_j \in \mathbb{Q}$. The equations $\sum_j s_{ji} v_j = 0$ form a 100×11 matrix of rank 11. Weight zero has dimension 44, as in Lemma 3.1, and there are no weights above two because J is generated by quadrics.

Matrix	Rows	Columns	Rank over $\mathbb{Q}$
Cubic multiplication μ	35	55	35
Weight-one syzygy constraints	80	55	49
Weight-two syzygy constraints	100	11	11

These ranks give $\dim \mathrm{Hom}_S(J, S/J) \le 44 + (55 - 49) + (11 - 11) = 50$. The smoothing places $[J]$ on a component of dimension 50, so equality holds and $[J]$ is smooth on the full Hilbert scheme. Its geometric tangent weights are exactly $0^{44}, 1^6$. For this explicit choice, the two determinant weights in the proof are therefore 21 and 26.

The accompanying script `verify_small_witness.py` constructs the matrices from the formulas above, checks the cancellations and cyclic basis, and computes all three ranks by exact rational row reduction. It also records nonzero minors modulo 32003, with determinants respectively 21492, 9763, and 4926; all denominators are checked to be invertible at that prime. The row and column indices are saved in `small_witness_certificate.json`. The independent script `check_small_witness.m2` computes $\mathrm{Hom}_S(J, S/J)$ directly over $\mathbb{Q}$ and obtains dimensions $0, 0, 6, 44$ in internal degrees $-3, -2, -1, 0$.

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