

NORMALITY OF SMOOTH LATTICE THREE-POLYTOPES

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ABSTRACT. We prove that every smooth lattice three-polytope is normal, settling Oda’s normality question in dimension three. By Ogata–Nakagawa stabilization it suffices to prove 2-normality. A hypothetical hole produces a minimal centrally symmetric strip body disjoint from the lattice, with three or four strip pairs and all nonzero lattice widths at least three. The four-strip case is excluded by Howe’s theorem, Pick’s formula and Ball’s plank theorem; the three-strip case reduces to a sharp 3×3 matrix inequality proved by exact case analysis in the appendix.

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Everything in this article, apart from this section and citation [14], is completely AI generated (and nothing in this section is). Thus the main aim of this section is to explain why I post such an article, why I think it is very useful. First, let me say what I believe are the facts:

- Oda’s conjecture is one of the central ones in toric geometry. It predicts that every smooth polytope is normal (definitions are in the article). While most experts believe it may be simply false in higher dimensions, the case of dimension three is particularly challenging - here I do not think most people had a clear opinion. But certainly many tried (including the human author of this article).
- There seemed to be a belief (hope?) among mathematicians that AI is very good at producing counterexamples, but is far below humans in reasoning. Maybe even that AI generated mathematics is low quality.

In my opinion the article below is very well written, contains a lot of very good ideas, is creative, novel and important for the community. Should I keep it to myself? Should I pretend I wrote it? (I know many people do!) I believe most mathematicians would answer no to two previous questions. There was also an option of completely rewriting it, revealing that it was in cooperation with AI — this seems to be what community is now most happy about. Still, the aim seems to be mostly psychological, and hides the true amount of work done by AI. Also, say a year ago, I was hoping I will be giving all ideas of statements and proofs to AI and it will be writing it in a good, nice form. I am somehow unwilling to reverse it and reduce me to the re-writer. Not to mention that while I would write many parts differently (mostly shorter), the style it uses, for me, is perfectly fine. I was asking myself: if I got this article half a year ago, as a reviewer say for Duke, would I be recommending accept? The answer is yes. Irrespectively of this article, I think we have to accept the fact that already now the strong AI models are capable of generating a top-quality article. So at the very least, I hope the article below could help the community understand where AI is. This is not the only article I generated with AI in last month, but the only one I am willing to share for now. But based on one other I shared with a colleague, I wanted to cite him: ”Are you saying this pdf is AI generated, or has there been some heavy editing on your part?”

I also just glanced at it, and the scariest part is not that it can do the math, but that I can’t tell who wrote the final version.”

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What was I using? The decisive part of the proof was achieved by OpenAI Sol Ultra in Codex. Checking parts in Lean, as well as writing was done in Anthropic Fable 5 and Opus 5. I did modify Codex setup to use more agents than it does by default. I also used the 'goal' option, that prevents from quitting too early. I used only publicly available models, tokens through abo (no extra tokens bought). I am sure people who have access to latest models, and/or many more tokens are playing in a different league.

How was I using it? After Matt Larson's counterexample to the strong version of White's conjecture [14], some students came to me and said that they found it particularly interesting, as he wrote how he did it. If there is one thing I learned about AI in last two months it would be: prompting matters, models matter, resources matter. (The first one will probably become obsolete in a year or two, but for now I believe it holds.) Probably for experts in AI, what I wrote is similar to: humans need food, water and air, but I think many mathematicians (including me two months ago) underestimate all three.

The main prompt I used was based on the solution to cycle double cover conjecture https://cdn.openai.com/pdf/04d1d1e4-bc75-476a-97cf-49055cd98d31/cdc_prompt.pdf. I hand modified it, so that it matches Oda's conjecture, but note that majority of the prompt is not about mathematics at all. Nowadays, I would not hand modify it, I would probably ask AI to do it. A thing I noticed immediately: there is no way a user like me could use a similar prompt. On 64 agents the system was dragon-like: heating everything and eating tokens at incredible rate — or actually protesting to even try to behave as asked, but after reducing considerably the resources one could still get the idea where this is going. I should also say, it did not work the first time I tried.

For writing the article: the first version it gave was not very good. The version you see I think is version ten. I never rewrote a part, but I did prompt it to make it shorter/longer, more readable, change notation — I do not think any mathematical knowledge was needed at that point. The versions differed in length (from about 20 pages to over 50) and style. I sometimes asked to add/remove part only to find out that was a bad idea.

For Lean: I should start by saying that I am as far from a lean expert as one can be. Thomas Kahle pointed out that I have to be very careful: the system may just want to give me the answer I would be happy with, which will have nothing to do with the lean statement I want to prove. Parallely, I was talking to Christian Ikenmeyer about lean formal proofs. He suggested the way that I proceeded. Use one AI system to just make a formal statement of the theorem you want to prove, with all definitions needed, but no proofs. Formally this is achieved by an axiom called sorry, which implies everything. This is usually an easy task, the AI has no incentive here to cheat. Then, you use a different AI system to make the formal proof of the given formalised statement (based on the informal proof it has), i.e. to not use the sorry axiom. This task takes a lot of time, but can be achieved without the use of top models, second-tier model seem to be doing just fine.

How long did it take? Just constructing the proof (not the formalisation) took about 2 days of AI work. However, as I was frequently running out of tokens, the task was paused several times, and the proof took about three weeks. Additional one-two weeks was used to produce lean formalisation and the written text.

How sure am I this is correct? I would say I am more sure this is correct than some of the articles that were published in the past, with no erratum, cited etc. But there are some (very) technical parts - no matter if such things are human or AI written, there is always positive probability of an error.

Where are we, as a community? I am not sure if people realize that in last two years the number of math articles produced essentially more than doubled. Sorry to bring this up, but I am afraid this means half of the mathematical output is currently AI produced. The problem is that most people do not admit it. Even though more than a year ago (which is like different era from my perspective in AI), the (mostly used) AI's were producing really bad math. The arXiv initiative, that blocks users who put false, AI generated articles would

have been perfect a year ago. I am afraid now, it will not be punishing people who do not reveal they use AI, it will be punishing people who only use old, free or cheap models. (I recently asked AI to go through one human made reference list - it found two mistakes.) Again, sorry — please do not kill the messenger.

Particularities:

- I did not ask AI to put me as an author. I just asked it to turn the proof into an article and there I was as an author anyway. It is impossible to say how much I influenced the model. I did have quite a few conversations with it about possible counterexamples to Oda's conjecture. Certainly, it was told that faces are no problem, and also vertices near to faces are not problematic, by induction on dimension. This is how it started the proof. But this was also common knowledge for experts. I am not sure if coincidentally, I am a coauthor of an article about centrally symmetric body case of Oda's conjecture. That result is not needed for the proof, but the proof in a central way focuses on symmetric bodies.
- I hand removed last remark of the paper. It was a page long explanation how the proof of the last lemma may be machine checked.
- It was pointed out to me that AI's have a tendency to give strange names to lemmas and propositions — indeed you can see it below. Many of those names actually make a lot of sense. I remember just a few weeks ago, my students were brain storming about a (completely different) proof. They were using all the time 'by the tricky-trick', by 'the 8-hyper-edge lemma' etc. Of course, it did not make any sense to anyone outside of the room, but to everyone in the room that was the way to think fast. In articles, at least in my community, we very rarely do this. Maybe in latex we have a fancy ref command for a lemma, but in the article it is Lemma X.Y. I used to do it sometimes with definitions (to put in brackets the name of object I define), but I remember at least once the reviewer criticized it. Now I think — isn't it better to do it with lemmas? Referring to Lemma X.Y in many cases means the reader needs to go through article, find it and recall what it was, while a name —even strange— could help us to note which lemma that was.
- From my experience, model is the human analogue of intelligence. And those newest, strongest models, well they are probably still behind the very best mathematicians, but I am not sure if they are behind, say me. Using multiple agents seems analogous to making a group of mathematicians work, instead of just one. And tokens plus computing power seems analogue of how long we work on the problem.
- I believe that for every provable statement, and every model, there exists a function, S shaped $f : \mathbb{R}_{\geq 0} \rightarrow [0, 1]$. This function is weakly increasing, with $f(0) = 0$, but $\lim_{t \rightarrow \infty} f(t) = 1$. It assigns to the number of tokens (possibly also disk space could be taken into account) the probability of finding the proof of the statement. (The function probably also depends on a few other factor, like how we present the problem, or how it is transformed before using the model etc.) It could very well be that one could change above 'model' to 'human' and 'tokens' to 'time'. What we called a stronger mathematician, was probably, higher values of f on different problems. It could be interesting to see how such a function looks for different models and problems.

1. INTRODUCTION

Let M be a lattice of rank d and let $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. A *lattice polytope* is the convex hull of finitely many points of M . For subsets $A, B \subset M_{\mathbb{R}}$, we write

$$A + B = \{a + b : a \in A, b \in B\}$$

for their Minkowski sum, and $kP = P + \dots + P$ for the k -fold Minkowski sum of a polytope P .

A lattice polytope P is k -normal if

$$(kP) \cap M = \underbrace{(P \cap M) + \cdots + (P \cap M)}_{k \text{ summands}}.$$

It is *normal*, or has the *integer decomposition property*, if it is k -normal for every $k \geq 1$. We always use normality with respect to the ambient lattice. A d -dimensional lattice polytope is *smooth* if it is simple and, at every vertex, the primitive lattice directions of the d incident edges form a basis of M .

For the affine-semigroup approach to normal lattice polytopes we refer to [4]; for the toric dictionary, including the correspondence between lattice-point addition and multiplication of sections, see [7, 17]. Oda's ample–nef multiplication question [18, Problem 3] asks, in polytope language, whether

$$(P \cap M) + (Q \cap M) = (P + Q) \cap M$$

whenever the normal fan of Q coarsens that of P ; Oda hopes for an affirmative answer at least when P is smooth and the two normal fans coincide, which covers the case considered here. The terminology of fans is recalled in Section 3. The specialization $Q = mP$ asks whether every smooth lattice polytope is normal. In dimension three, the next result shows that it is enough to consider the single case $Q = P$.

Proposition 1.1 (Ogata–Nakagawa stabilization). *Let $P \subset M_{\mathbb{R}}$ be a d -dimensional lattice polytope, where $d \geq 1$. Then*

$$(1.1) \quad (iP \cap M) + (P \cap M) = ((i+1)P) \cap M \quad \text{for every } i \geq d-1,$$

with the convention $0P = \{0\}$.

We use the precise stepwise form proved by Ogata and Nakagawa [20]; see also [19, (1.2) and Proposition 1.1] for the explicit lattice-point formulation. The underlying Hilbert-basis height estimate goes back to Ewald–Wessels [8]; Liu–Trotter–Ziegler [16] give an elementary proof. For the affine-semigroup formulation, see [4].

Corollary 1.2. *A three-dimensional lattice polytope is normal if and only if it is 2-normal. Every lattice polytope of dimension at most two is normal.*

Our main theorem is the following.

Theorem 1.3. *Let $M \simeq \mathbb{Z}^3$, and let $P \subset M_{\mathbb{R}}$ be a three-dimensional smooth lattice polytope. Then P is normal.*

In dimension two the question has a positive answer in the full ample–nef form: Fakhruddin [9] proved it for smooth P , and Haase, Nill, Paffenholz and Santos [11] gave a combinatorial proof without the smoothness hypothesis; the latter is recorded below as (S1). Previously known three-dimensional cases of normality include smooth centrally symmetric polytopes [2] and smooth polytopes whose adjoint bundle is not big, in particular those without interior lattice points [21]. Smoothness cannot be dropped: edge polytopes of graphs may fail to be normal [22], and very ample non-normal polytopes exist in every dimension at least three [12, 15]. Without smoothness, normality holds for polytopes with sufficiently long edges [10]; for the integer decomposition property of dilated polytopes see [6], and for the structure of the family of all normal polytopes [5]. For an overview of Oda's question and of the integer decomposition property, see [13].

Remark 1.4 (Scope). The theorem proves the normality specialization $Q = mP$ of Oda's question; this specialization is [18, Problem 2]. It does not prove the stronger ample–nef statement for an arbitrary lattice polytope Q whose normal fan coarsens that of P .

2. STRATEGY OF THE PROOF

By Corollary 1.2, it is enough to prove

$$(2.1) \quad (2P) \cap M = (P \cap M) + (P \cap M)$$

for every three-dimensional smooth lattice polytope $P \subset M_{\mathbb{R}}$. Suppose, for a contradiction, that $z \in (2P) \cap M$ is not a sum of two lattice points of P , and put $c = z/2$. The argument has the following six steps.

(1) Write

$$P = \{x \in M_{\mathbb{R}} : \langle u_{\rho}, x \rangle \leq a_{\rho} \text{ for every facet } \rho\}, \quad d_{\rho} = 2a_{\rho} - \langle u_{\rho}, z \rangle.$$

Smooth facet slices show that every d_{ρ} is at least three, that $c \in \text{int } P$, and that $z \notin 2M$ (Section 4).

(2) The centrally symmetric polytope

$$C = (2P - z) \cap (z - 2P) = \{t : |\langle u_{\rho}, t \rangle| \leq d_{\rho} \text{ for every } \rho\}$$

satisfies $h_C(b) \geq 3$ for every nonzero b in the dual lattice $N = \text{Hom}(M, \mathbb{Z})$ (Section 5). This is the second and last place where smoothness is used.

(3) The parity change of variables $t = 2x - z$ turns decompositions of z into points of the coset $z + 2M$ lying in C . Deleting strip pairs while preserving disjointness from M produces an inclusion-minimal strip body

$$B_I = \{x : |\langle u_i, x - c \rangle| \leq d_i/2 \ (i \in I)\}.$$

The projection defined by the selected normals shows that they span $N_{\mathbb{R}} = N \otimes \mathbb{R}$; so $|I| \geq 3$. Enlarging B_I to contact position and applying the contact-polytope lemma shows that its $2|I|$ contacts occupy distinct classes of $M/2M$; hence $|I| \leq 4$. Moreover,

$$\text{width}_b(B_I) \geq 3 \quad (b \in N \setminus \{0\}).$$

These facts are assembled in Section 6.

(4) If $|I| = 4$, the contact polytope has eight vertices and no other lattice points. Howe's theorem puts it in two consecutive lattice planes; Pick's formula identifies each layer as a fundamental parallelogram; and Ball's affine plank theorem yields a nonzero lattice direction in which the enlarged strip body has width at most three. The enlargement was strict, so B_I has width strictly less than three, a contradiction (Section 7). Thus $|I| = 3$.

(5) For three strips, contact coordinates identify the lattice with

$$D_3 = \{m \in \mathbb{Z}^3 : m_1 + m_2 + m_3 \equiv 0 \pmod{2}\}$$

and the enlarged body with $\{x : \|Ax\|_{\infty} \leq 1\}$, where A has diagonal entries one and off-diagonal entries of absolute value less than one. Lattice-freeness gives $\|A\varepsilon\|_{\infty} \geq 1$ for every sign vector ε (Section 8).

(6) The matrix lemma, proved by exact case analysis in Appendix A, supplies a sign vector s with $\|A^{-T}s\|_1 \leq 3$ (Section 9). Since $s/2$ pairs integrally with D_3 , it pulls back to a nonzero covector in N along which the enlarged body has width at most three. Strict enlargement again makes the corresponding width of B_I smaller than three, contradicting Step (3) (Section 10).

Therefore no such z exists, so (2.1) holds; the stabilization corollary then proves Theorem 1.3.

3. NOTATION AND STANDARD FACTS

3.1. Lattices, fans, support functions, widths. Throughout, $M \simeq \mathbb{Z}^3$ is the primal lattice and $N = M^* = \text{Hom}(M, \mathbb{Z})$ its dual. We write $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, and $\langle b, x \rangle$ for the pairing of $b \in N_{\mathbb{R}}$ with $x \in M_{\mathbb{R}}$; for $X \subseteq M_{\mathbb{R}}$ we abbreviate $\langle b, X \rangle = \{\langle b, x \rangle : x \in X\}$. A nonzero lattice vector or covector is *primitive* if it is not of the form kv with v in the lattice and $k \geq 2$ an integer. A lattice polytope is *empty* if its only lattice points are its vertices.

Let $P \subset M_{\mathbb{R}}$ be a full-dimensional polytope. The *normal cone* of a face F of P is

$$\sigma_F = \{u \in N_{\mathbb{R}} : \langle u, \cdot \rangle \text{ attains its maximum over } P \text{ at every point of } F\},$$

a closed convex cone, and the collection Σ of these cones is the *normal fan* of P , a complete fan in $N_{\mathbb{R}}$. Its maximal cones are the normal cones σ_v of the vertices, and its rays are the normal cones of the facets, generated by the primitive outward facet normals. A fan is *smooth* if every maximal cone is generated by a basis of the lattice; a fan Σ' is a *coarsening* of Σ if every cone of Σ lies in a cone of Σ' .

Remark 3.1. The normal fan of a smooth lattice polytope P of full dimension d is complete and smooth. Indeed, let v be a vertex of P , let $e_1, \dots, e_d$ be the primitive edge directions at v and $u_1, \dots, u_d$ the primitive outward normals of the facets through v , labelled so that the facet with normal u_i contains every edge except the i -th; the labelling is possible because P is simple. Then $\langle u_i, e_j \rangle = 0$ for $j \neq i$ and $\langle u_i, e_i \rangle < 0$, since moving from v along e_i leaves that facet while staying in P . By smoothness $(e_1, \dots, e_d)$ is a basis of M ; writing $(f_1, \dots, f_d)$ for the dual basis, the displayed relations give $u_i = \langle u_i, e_i \rangle f_i$, and primitivity forces $u_i = -f_i$. Hence σ_v is generated by the basis $-f_1, \dots, -f_d$ of N .

For a nonempty closed convex $K \subseteq M_{\mathbb{R}}$, the *support function* is $h_K(b) = \sup_{x \in K} \langle b, x \rangle \in \mathbb{R} \cup \{+\infty\}$, $b \in N_{\mathbb{R}}$. It is sublinear, that is $h_K(b + b') \leq h_K(b) + h_K(b')$ and $h_K(\lambda b) = \lambda h_K(b)$ for $\lambda \geq 0$; it is monotone in K ; and it is finite everywhere if and only if K is bounded, in which case the supremum is attained. The *width* of K in a direction $b \in N_{\mathbb{R}} \setminus \{0\}$ is

$$(3.1) \quad \text{width}_b(K) = h_K(b) + h_K(-b) = \sup_{x \in K} \langle b, x \rangle - \inf_{x \in K} \langle b, x \rangle;$$

for $b \in N \setminus \{0\}$ this is a *lattice width* of K , and the *lattice width* of K is the minimum of (3.1) over all primitive $b \in N$. For $c \in M_{\mathbb{R}}$ and $\lambda > 0$ the homothet $c + \lambda(K - c)$ has support function $b \mapsto \langle b, c \rangle + \lambda(h_K(b) - \langle b, c \rangle)$, whence

$$(3.2) \quad \text{width}_b(c + \lambda(K - c)) = \lambda \text{width}_b(K) \quad (b \in N_{\mathbb{R}} \setminus \{0\}),$$

an identity in $\mathbb{R} \cup \{+\infty\}$.

For a compact convex $K \subset M_{\mathbb{R}}$ with $0 \in \text{int } K$, the *polar* is $K^\circ = \{b \in N_{\mathbb{R}} : \langle b, x \rangle \leq 1 \text{ for all } x \in K\}$. Two facts about it are used, both immediate from the definition: polarity reverses inclusions, and, in coordinates given by a basis, $([-\frac{1}{2}, \frac{1}{2}]^3)^\circ = \{a : \|a\|_1 \leq 2\}$, since $h_{[-1/2, 1/2]^3}(a) = \frac{1}{2}\|a\|_1$.

Finally, once linear coordinates are fixed we identify $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ with $\mathbb{R}^3$, the pairing becoming the standard scalar product. For $A \in \text{GL}_3(\mathbb{R})$ put $K_A = \{x \in \mathbb{R}^3 : \|Ax\|_\infty \leq 1\} = A^{-1}[-1, 1]^3$. Using $\langle b, A^{-1}y \rangle = \langle A^{-T}b, y \rangle$, $h_{[-1, 1]^3}(v) = \|v\|_1$ and the central symmetry of K_A ,

$$(3.3) \quad h_{K_A}(b) = \|A^{-T}b\|_1, \quad \text{width}_b(K_A) = 2\|A^{-T}b\|_1.$$

3.2. Four geometric inputs. The degree-two contradiction uses the following four substantive geometric results. We also use standard facts from convexity, linear programming and finite group theory; the Ogata–Nakagawa stabilization used to pass from degree two to all degrees was stated in the introduction.

Definition 3.2 (Plank and relative width). A *plank* is a set $\{x \in M_{\mathbb{R}} : |\langle b, x - x_0 \rangle| \leq s\}$ with $b \in N_{\mathbb{R}} \setminus \{0\}$, $x_0 \in M_{\mathbb{R}}$ and $s \geq 0$; by (3.1) its width in the direction b is $2s$. Its *relative*

width with respect to a convex body E is $2s/\text{width}_b(E)$, the ratio of the widths of the plank and of E measured by one and the same functional; the ratio is unchanged when b is rescaled.

- (S1) (*Minkowski sums of lattice polygons.*) Let L be a lattice of rank two and let $F, G \subset L_{\mathbb{R}}$ be two-dimensional lattice polygons such that the normal fan of G coarsens that of F . Then

$$(F \cap L) + (G \cap L) = (F + G) \cap L.$$

This is Theorem 1.1 of Haase, Nill, Paffenholz and Santos [11]; no smoothness is assumed, and the fan hypothesis cannot be omitted.

- (S2) (*Howe's theorem.*) Every three-dimensional lattice polytope whose only lattice points are its vertices has lattice width one; that is, it lies between two consecutive parallel lattice planes. This is Howe's theorem, published by Scarf as [24, 1.3 and 1.5]; Treutlein [25, Theorem 1.2] records it in the equivalent form that such a polytope is a Cayley polytope, the convex hull of two lattice polygons lying in parallel lattice planes at lattice distance one — for a three-dimensional lattice polytope, whose vertices are lattice points, that property is exactly lattice width one. See also the introduction of [3]. It is used in Section 7.

- (S3) (*Pick's formula* [23].) The normalized area of a two-dimensional lattice polygon R in a rank-two lattice, that is, twice its Euclidean area computed in a basis, equals

$$2 \cdot \#\{\text{interior lattice points of } R\} + \#\{\text{boundary lattice points of } R\} - 2.$$

- (S4) (*Ball's affine plank theorem* [1].) Let $E \subset \mathbb{R}^n$ be a convex body symmetric about the origin and covered by finitely many planks, each symmetric about the origin. Then the sum of their relative widths is at least one. (Planks and relative widths are defined in Definition 3.2.)

3.3. Strip bodies, contacts, and two reusable tools.

Definition 3.3 (Symmetric strip body). Let I be a finite index set, $c \in M_{\mathbb{R}}$, and let $u_i \in N_{\mathbb{R}} \setminus \{0\}$ and $r_i > 0$ for $i \in I$. The associated *symmetric strip body* centred at c is

$$E = \{x \in M_{\mathbb{R}} : |\langle u_i, x - c \rangle| \leq r_i \ (i \in I)\}.$$

Each constraint is a *strip pair*; for $J \subseteq I$, retaining only those indexed by J gives the *subsystem indexed by J* . Two properties are used repeatedly. First, E is bounded if and only if the u_i span $N_{\mathbb{R}}$: if they do, they contain a basis and all coordinates of $x - c$ in the dual basis are bounded on E ; if they do not, any nonzero $v \in \bigcap_i \ker u_i$ gives a line $c + \mathbb{R}v \subseteq E$. Second, the interior of E is the set where all the inequalities are strict: that set is open and contained in E , while if $\langle u_i, x - c \rangle = \varepsilon r_i$ with $\varepsilon \in \{1, -1\}$, a v with $\langle u_i, v \rangle = \varepsilon$ gives $|\langle u_i, x + tv - c \rangle| = r_i + t > r_i$ for every $t > 0$, so $x \notin \text{int } E$.

Definition 3.4 (Witnesses and contacts). Let $\Lambda \subseteq M_{\mathbb{R}}$. The strip body E above, or its defining system, is *inclusion-minimal Λ -empty* if $E \cap \Lambda = \emptyset$ while for every $i \in I$ the body of the subsystem indexed by $I \setminus \{i\}$ meets Λ ; a point of Λ in that body is a *deletion witness* for i . A *contact* for one of the defining one-sided inequalities of E is a point of E satisfying that inequality with equality and all the others strictly.

The first tool enlarges all half-widths by a common factor greater than one, placing the body in contact position while its interior remains empty.

Lemma 3.5 (Enlargement to contact position). Let $c \in M_{\mathbb{R}}$ and let $\Lambda \subseteq M$ be nonempty and invariant under the point reflection $\iota(x) = 2c - x$. Let $(u_j)_{j \in J}$ be a nonempty finite family of nonzero covectors in N and let $(w_j)_{j \in J}$ be positive reals. For a vector $r = (r_j)_{j \in J}$ of positive reals put

$$(3.4) \quad S(r) = \{x \in M_{\mathbb{R}} : |\langle u_j, x - c \rangle| \leq r_j \ (j \in J)\};$$

its interior $\text{int } S(r)$ is the set where every inequality is strict (Definition 3.3). Assume:

- (i) $S(w) \cap \Lambda = \emptyset$;
- (ii) for every $j \in J$ there is $\lambda^{(j)} \in \Lambda$ with $|\langle u_k, \lambda^{(j)} - c \rangle| \leq w_k$ for every $k \neq j$.

Then there are positive reals $W = (W_j)_{j \in J}$ and a real $\delta > 1$ with $W_j \geq \delta w_j$ for every j , such that:

- (a) $\text{int } S(W) \cap \Lambda = \emptyset$ and $S(W) \supseteq c + \delta(S(w) - c)$;
- (b) for every $j \in J$ there is $\lambda_j \in \Lambda$ with $|\langle u_j, \lambda_j - c \rangle| = W_j$ and $|\langle u_k, \lambda_j - c \rangle| < W_k$ for every $k \neq j$; its reflection $\iota(\lambda_j) \in \Lambda$ satisfies $\langle u_k, \iota(\lambda_j) - c \rangle = -\langle u_k, \lambda_j - c \rangle$ for every $k \in J$.

Proof. Since $\Lambda \subseteq M$ and $u_j \in N$, the value set $\langle u_j, \Lambda - c \rangle$ is contained in the coset $\mathbb{Z} - \langle u_j, c \rangle$ of $\mathbb{Z}$ in $\mathbb{R}$; in particular it meets every bounded interval in a finite set.

Step 1: a uniform strict enlargement. For $j \in J$ put

$$\varepsilon_j = \min\{|v| - w_j : v \in \langle u_j, \Lambda - c \rangle, w_j < |v| \leq w_j + 1\},$$

and $\varepsilon_j = 1$ if this set is empty; by the previous paragraph the minimum is taken over a finite set of positive numbers, so $0 < \varepsilon_j \leq 1$. Put $\eta = \frac{1}{2} \min_j \varepsilon_j$ and $W_j^0 = w_j + \eta$. Let $\lambda \in \Lambda$. By 3.5(i) there is a j with $|v| > w_j$, where $v = \langle u_j, \lambda - c \rangle$; if $|v| \leq w_j + 1$ then $|v| \geq w_j + \varepsilon_j > W_j^0$, and otherwise $|v| > w_j + 1 > W_j^0$. Hence $S(W^0) \cap \Lambda = \emptyset$. Put $\delta = 1 + \eta / \max_k w_k > 1$; then $\delta w_j \leq w_j + \eta = W_j^0$ for every j .

Step 2: a compact set of feasible enlargements. For a vector r of positive reals, $\text{int } S(r) \cap \Lambda = \emptyset$ holds if and only if every $\lambda \in \Lambda$ admits a $j \in J$ with $r_j \leq |\langle u_j, \lambda - c \rangle|$. Hence

$$F = \prod_{j \in J} [W_j^0, \infty) \cap \bigcap_{\lambda \in \Lambda} \bigcup_{j \in J} \{r \in \mathbb{R}^J : r_j \leq |\langle u_j, \lambda - c \rangle|\}$$

is exactly the set of $r \geq W^0$ with $\text{int } S(r) \cap \Lambda = \emptyset$. It is closed, being an intersection of finite unions of closed half-spaces with a closed orthant, and it contains W^0 by Step 1. It is bounded: let $r \in F$ and fix $j \in J$; the witness $\lambda^{(j)}$ of 3.5(ii) satisfies $|\langle u_k, \lambda^{(j)} - c \rangle| \leq w_k < W_k^0 \leq r_k$ for every $k \neq j$, so the index i with $r_i \leq |\langle u_i, \lambda^{(j)} - c \rangle|$, which exists because $r \in F$, can only be $i = j$; that is, $r_j \leq |\langle u_j, \lambda^{(j)} - c \rangle|$. Thus F is a nonempty compact subset of $\prod_j [W_j^0, |\langle u_j, \lambda^{(j)} - c \rangle|]$, and the continuous function $r \mapsto \sum_j r_j$ attains its maximum on F , at some W .

Since $W \in F$ we have $\text{int } S(W) \cap \Lambda = \emptyset$ and $W_j \geq W_j^0 \geq \delta w_j$; and if $x = c + \delta(y - c)$ with $y \in S(w)$ then $|\langle u_j, x - c \rangle| = \delta |\langle u_j, y - c \rangle| \leq \delta w_j \leq W_j$ for every j , so $x \in S(W)$. This is 3.5(a).

Step 3: contacts. Fix $j \in J$. As in Step 1 there is a $\theta \in (0, 1]$ such that no $v \in \langle u_j, \Lambda - c \rangle$ satisfies $W_j < |v| < W_j + \theta$; namely the minimum of $|v| - W_j$ over the finitely many v with $W_j < |v| \leq W_j + 1$, and $\theta = 1$ if there are none. Let e_j be the j -th coordinate vector of $\mathbb{R}^J$. The vector $W + \theta e_j$ lies in $\prod_k [W_k^0, \infty)$ and has a strictly larger coordinate sum than W , so by maximality $W + \theta e_j \notin F$: there is a $\lambda \in \Lambda$ with $|\langle u_j, \lambda - c \rangle| < W_j + \theta$ and $|\langle u_k, \lambda - c \rangle| < W_k$ for every $k \neq j$. As $W \in F$, some index i satisfies $W_i \leq |\langle u_i, \lambda - c \rangle|$, and the strict inequalities just listed force $i = j$; the choice of θ then gives $|\langle u_j, \lambda - c \rangle| = W_j$. So $\lambda_j := \lambda$ is as required. Finally $\iota(\Lambda) = \Lambda$ and $\langle u_k, \iota(x) - c \rangle = \langle u_k, c - x \rangle = -\langle u_k, x - c \rangle$ for every k , which gives the assertions about $\iota(\lambda_j)$ and proves 3.5(b). $\square$

The second tool turns a contact configuration into a lattice polytope.

Lemma 3.6 (Contact polytope). *Let I be a finite index set with $|I| \geq 3$, let $z \in M \setminus 2M$ and $c = z/2$, let ℓ_i ($i \in I$) be nonzero linear forms on $M_{\mathbb{R}}$, let $r_i > 0$, and put*

$$(3.5) \quad E = \{x \in M_{\mathbb{R}} : |\ell_i(x - c)| \leq r_i \ (i \in I)\}, \quad \text{int } E = \{x \in M_{\mathbb{R}} : |\ell_i(x - c)| < r_i \ (i \in I)\}.$$

Assume $\text{int } E \cap M = \emptyset$ and that for every $i \in I$ there is a lattice point $p_i \in M$ with

$$(3.6) \quad \ell_i(p_i - c) = r_i, \quad |\ell_j(p_i - c)| < r_j \quad (j \neq i).$$

Put $p_{-i} = z - p_i$ and $Q = \text{conv}\{p_{\pm i} : i \in I\}$. Then:

- (C1) $Q \cap M = \{p_{\pm i} : i \in I\} = \text{Vert } Q$; in particular the $2|I|$ contacts $p_{\pm i}$ are pairwise distinct and Q is an empty lattice polytope;
- (C2) the $2|I|$ contacts have pairwise distinct classes modulo $2M$; consequently Q is three-dimensional.

Proof. Reflection in c sends x to $z - x$ and preserves M . Hence $p_{-i} = z - p_i$ is a lattice point satisfying

$$\ell_i(p_{-i} - c) = -r_i, \quad |\ell_j(p_{-i} - c)| < r_j \quad (j \neq i).$$

The contact equations show that the $2|I|$ contacts are pairwise distinct, and each one-sided inequality defining E is tight at exactly one of them. Every convex combination with positive weight on at least two contacts is therefore strict in every one-sided inequality, and hence lies in $\text{int } E$. Since $\text{int } E \cap M = \emptyset$, this shows

$$Q \cap M = \{p_{\pm i} : i \in I\}.$$

Moreover p_i is the unique contact maximizing ℓ_i , and p_{-i} the unique contact maximizing $-\ell_i$; hence all contacts are exposed vertices. This proves (C1).

For every i , $p_i - p_{-i} = 2p_i - z \equiv z \pmod{2M}$, which is nonzero because $z \notin 2M$; thus opposite contacts are incongruent modulo $2M$. If two nonopposite contacts $a \neq b$ were congruent, then $(a + b)/2 \in M$ would be a nontrivial convex combination of contacts and hence would lie in $\text{int } E$, a contradiction. The contact classes are therefore pairwise distinct.

Finally, if Q had affine dimension at most two, all differences of its vertices would lie in a subgroup $L \subseteq M$ of rank at most two. Their classes modulo $2M$ would then lie in one coset of the image of L , which has at most $|L/2L| \leq 4$ elements. This contradicts the existence of $2|I| \geq 6$ distinct contact classes. Hence Q is three-dimensional, proving (C2). $\square$

4. FACET DEPTH OF A HYPOTHETICAL HOLE

Throughout this section $P \subset M_{\mathbb{R}}$ is a three-dimensional smooth lattice polytope, as in Theorem 1.3, written in its irredundant facet presentation

$$(4.1) \quad P = \{x \in M_{\mathbb{R}} : \langle u_{\rho}, x \rangle \leq a_{\rho} \text{ for every ray } \rho\},$$

where ρ runs over the rays of the normal fan Σ of P , equivalently over its facets, $u_{\rho} \in N$ is the primitive outward normal of the corresponding facet, and $a_{\rho} = \max_{x \in P} \langle u_{\rho}, x \rangle \in \mathbb{Z}$. Being smooth, P is simple: exactly three facets and exactly three edges meet at each vertex, and each of the three pairs of facets at a vertex meets in one of those edges.

Fix a facet F , say the one given by ρ , and put $u = u_{\rho}$, $a = a_{\rho}$. The *slack* of F is the affine function $h = a - \langle u, \cdot \rangle$ on $M_{\mathbb{R}}$; it is nonnegative on P , integral on M , and vanishes on P exactly along F . We write $\{h = t\}$ for the affine plane $\{x : h(x) = t\}$ and put

$$F_1 = P \cap \{h = 1\}.$$

Of the three edges of P at a vertex v of F , exactly two lie in F , namely the two edges of the polygon F at v ; let w_v be the primitive direction of the third, pointing away from v .

Lemma 4.1 (Slices above a facet). *For every vertex v of F one has $h(v + w_v) = 1$, and for $0 \leq t \leq 1$*

$$(4.2) \quad P \cap \{h = t\} = \text{conv}\{v + tw_v : v \in \text{Vert } F\} = (1 - t)F + tF_1.$$

In particular F_1 is a nonempty lattice polygon, possibly a segment or a point.

Proof. Let v' be the other endpoint of the edge from v in direction w_v , so $v' = v + mw_v$ for an integer $m \geq 1$. Smoothness makes the three primitive edge directions at v a basis of M . The integral linear form $x \mapsto h(v + x) = -\langle u, x \rangle$ vanishes on the two directions tangent to F ; since u is primitive, its value on w_v is therefore 1 or -1 . It is positive: the edge $[v, v']$ is the one edge at v not contained in F , and $F = P \cap \{h = 0\}$ is a face, so $v' \notin F$ and $h(v') > 0$. Hence $h(v + w_v) = 1$ and $v + tw_v \in P$ for $0 \leq t \leq 1$.

Fix $0 < t < 1$. No vertex of P has height t , because all vertex heights are integers. Hence every vertex of the section $P \cap \{h = t\}$ is the intersection with an edge of P . The endpoints of that edge have integral heights on opposite sides of t ; the lower height is therefore 0. Thus the lower endpoint is a vertex v of F , the edge is the unique edge from v not contained in F , and the intersection point is $v + tw_v$. This proves the first equality in (4.2) for $0 < t < 1$; the case $t = 0$ is trivial. At $t = 1$ the inclusion from right to left follows from the first paragraph. For the reverse inclusion, fix $q \in F_1$ and $v_0 \in F$. The points $(1 - t)v_0 + tq$ lie in the height- t section; letting $t \uparrow 1$ and passing to a convergent subsequence of their convex-hull coefficients puts q in $\text{conv}\{v + w_v : v \in \text{Vert } F\}$. Hence $F_1 = \text{conv}\{v + w_v : v \in \text{Vert } F\}$, so F_1 is a nonempty lattice polytope.

For $0 < t < 1$, each generator satisfies $v + tw_v = (1 - t)v + t(v + w_v)$, giving $P \cap \{h = t\} \subseteq (1 - t)F + tF_1$. Conversely, convexity of P shows that $(1 - t)F + tF_1$ lies in P , and every one of its points has height t . The endpoint cases are immediate. This proves (4.2). $\square$

The slices $P \cap \{h = t\}$, and in particular F and F_1 , lie in translates of the plane $\ker u \subset M_{\mathbb{R}}$. For a polytope $K \subset x_0 + \ker u$ the *normal fan of K* means that of $K - x_0 \subset \ker u$, a fan in the plane $(\ker u)^* = N_{\mathbb{R}}/\mathbb{R}u$; it does not depend on the choice of x_0 .

Lemma 4.2 (Coarsening). *The normal fan of F_1 coarsens the normal fan of F .*

Proof. Fix $0 < t < 1$ and put $S = P \cap \{h = t\}$. By (4.2), S contains a translate of $(1 - t)F$, so S , like F , is two-dimensional, and the normal fans of S and of F are complete fans in the plane $(\ker u)^*$. Let e be an edge of S . A point of relint e cannot lie in $\text{int } P$, since then a neighbourhood in $\{h = t\}$ would lie in S . Choose a facet $G \neq F$ of P containing such a point. No facet can have the nonintegral constant height t , so $S \cap G$ is a proper face of S containing e , hence equals e . Some vertex of G has height 0: otherwise all vertices of G would have integral height at least 1, whereas $e \subset G$ has height $t < 1$. Thus G meets F at a vertex, and simplicity of P implies that $F \cap G$ is an edge of F . Modulo the normal u of F , the facet normal of G is therefore the outward normal both of e and of $F \cap G$. Every ray of the normal fan of S is consequently a ray of the normal fan of F .

Choose $v_0 \in F$ and $q_0 \in F_1$, and translate F, F_1, S by $-v_0, -q_0, -((1 - t)v_0 + tq_0)$, respectively. The translated polytopes lie in $\ker u$, and Lemma 4.1 becomes $S' = (1 - t)F' + tF'_1$. The normal fan of a Minkowski sum is the common refinement of the normal fans of its summands, because the face exposed by a covector is the sum of the exposed faces. Hence the normal fan of S' refines that of F' . In a refinement of complete planar fans, every ray of the coarser fan is also a ray of the finer one. The preceding paragraph gives the reverse inclusion, so the two fans coincide. Since the normal fan of S' also refines that of F'_1 , and translation does not change normal fans, the normal fan of F_1 coarsens that of F . $\square$

Corollary 4.3. *With F and F_1 as above,*

$$(4.3) \quad (F \cap M) + (F \cap M) = (2F) \cap M,$$

$$(4.4) \quad (F \cap M) + (F_1 \cap M) = (F + F_1) \cap M,$$

$$(4.5) \quad (F_1 \cap M) + (F_1 \cap M) = (2F_1) \cap M.$$

Proof. Let $L = M \cap \ker u$, and translate F and F_1 by lattice points of their respective affine planes to lattice polytopes $F', F'_1 \subset L_{\mathbb{R}}$. Translating the three sumsets by the corresponding sums of the two translation vectors identifies their lattice points with those of $2F', F' + F'_1$ and $2F'_1$. The first and third identities follow from the normality of every lattice polytope of dimension at most two (Corollary 1.2). If F'_1 is two-dimensional, the middle identity follows from (S1) and Lemma 4.2; if it has dimension at most one, it follows from Remark 4.4. $\square$

Remark 4.4 (Degenerate slices). Let $F \subset L_{\mathbb{R}}$ be a two-dimensional lattice polygon and let $F_1 \subset L_{\mathbb{R}}$ be a lattice polytope of dimension at most one whose normal fan coarsens that of F . If F_1 is a lattice point, the equality $(F \cap L) + (F_1 \cap L) = (F + F_1) \cap L$ is just translation.

Suppose $F_1 = c_0 + [0, n]e$ with $c_0 \in L$, $e \in L$ primitive and $n \geq 1$ an integer. Choose lattice coordinates with $e = (1, 0)$ and translate so that $c_0 = 0$. The normal fan of F_1 consists of the line $\mathbb{R}(0, 1)$ and the two closed half-planes it bounds, so coarsening forces every cone of the normal fan of F into one of those half-planes. Thus neither $(0, 1)$ nor $(0, -1)$ can lie in the interior of a cone; completeness makes both rays of that fan. Hence the top and bottom faces of F are horizontal lattice edges. Write $F \cap \{y = t\} = [g(t), G(t)] \times \{t\}$. The length $G - g$ is concave and is at least one at both endpoint heights; hence $G(t) - g(t) \geq 1$ throughout.

For $(x, m) \in (F + F_1) \cap L$ one has $g(m) \leq x \leq G(m) + n$. Put $j = \max\{0, \lceil x - G(m) \rceil\}$. Then $0 \leq j \leq n$ and $g(m) \leq x - j \leq G(m)$: the upper bound is immediate, while if $j > 0$ then $x - j > G(m) - 1 \geq g(m)$. Thus $(x, m) = (x - j, m) + (j, 0)$ is the required lattice decomposition.

Lemma 4.5 (Shallow points decompose). *Let $z \in (2P) \cap M$ and suppose that $d = 2a - \langle u, z \rangle$ satisfies $d \leq 2$. Then $z \in (P \cap M) + (P \cap M)$.*

Proof. Since $z \in 2P$ we have $\langle u, z \rangle \leq 2a$, so $d \in \{0, 1, 2\}$. Moreover $z/2 \in P$ and $h(z/2) = d/2 \in [0, 1]$, so (4.2) gives

$$z \in 2\left(P \cap \left\{h = \frac{d}{2}\right\}\right) = (2 - d)F + dF_1;$$

that is, z lies in $2F$, in $F + F_1$, or in $2F_1$, according as $d = 0, 1$, or 2 . Since $z \in M$, the corresponding identity of Corollary 4.3 writes z as a sum of two lattice points of $F \cup F_1 \subseteq P$. $\square$

Convention 4.6 (The hypothetical hole). From here to the end of the proof of Theorem 1.3 we argue by contradiction: we fix a lattice point

$$z \in (2P) \cap M \quad \text{with} \quad z \notin (P \cap M) + (P \cap M),$$

which we call the *hole*, and we write $c = z/2$. For every ray ρ of Σ we call

$$(4.6) \quad d_\rho = 2a_\rho - \langle u_\rho, z \rangle \in \mathbb{Z}$$

the *depth* of z at the corresponding facet; it is the slack of z in the presentation of $2P$ obtained by doubling (4.1).

Proposition 4.7 (Facet depth). *Under Convention 4.6:*

- (D1) $d_\rho \geq 3$ for every ray ρ ;
- (D2) c lies in the interior of P ;
- (D3) $z \notin 2M$.

Proof. (D1) If $d_\rho \leq 2$ for some ρ , then Lemma 4.5, applied to the facet given by ρ (the facet F above was an arbitrary one), writes z as a sum of two lattice points of P , contrary to Convention 4.6.

(D2) By (D1), $\langle u_\rho, c \rangle = a_\rho - d_\rho/2 < a_\rho$ for every ρ , so c satisfies all the inequalities of (4.1) strictly.

(D3) If $z \in 2M$, then c is a lattice point of P , and $z = c + c$ contradicts Convention 4.6. $\square$

5. THE WIDTH LEMMA

Throughout this section Convention 4.6 and the notation of Section 4 stay in force: P is presented as in (4.1), the slacks d_ρ of (4.6) satisfy $d_\rho \geq 3$ by Proposition 4.7(D1), and Σ is the normal fan of P , a complete smooth fan in $N_{\mathbb{R}}$.

Let h_P be the support function of P and put

$$(5.1) \quad \phi(v) = 2h_P(v) - \langle v, z \rangle = h_{2P-z}(v), \quad v \in N_{\mathbb{R}}.$$

Thus ϕ is the support function of the lattice polytope $2P - z$: it is *sublinear*, that is, $\phi(v + v') \leq \phi(v) + \phi(v')$ and $\phi(\lambda v) = \lambda\phi(v)$ for $\lambda \geq 0$; it is integral on N , because $2P - z$ is a lattice polytope; and it is linear on each cone of Σ , because $2P - z$ has the same normal fan as P , with

$$\phi(u_\rho) = 2a_\rho - \langle u_\rho, z \rangle = d_\rho \geq 3.$$

Moreover $2h_P(v)$ is an even integer for $v \in N$, so ϕ has a fixed parity along N :

$$(5.2) \quad \phi(v) \equiv \langle v, z \rangle \pmod{2} \quad (v \in N).$$

By (4.1) we have $2P - z = \{t : \langle u_\rho, t \rangle \leq d_\rho \text{ for every } \rho\}$ and $z - 2P = -(2P - z)$, so the intersection of these two bodies is

$$(5.3) \quad C := (2P - z) \cap (z - 2P) = \{t \in M_{\mathbb{R}} : |\langle u_\rho, t \rangle| \leq d_\rho \text{ for every } \rho\}.$$

The rays of the complete fan Σ span $N_{\mathbb{R}}$, so the inequalities in (5.3) bound t and C is compact; it is centrally symmetric, $C = -C$; and it contains the origin in its interior, because every $d_\rho \geq 3$. In particular h_C is finite and even, $h_C(-b) = h_C(b)$, so $\text{width}_b(C) = 2h_C(b)$ for $b \in N_{\mathbb{R}} \setminus \{0\}$ by (3.1).

Lemma 5.1 (Width lemma). *For every $b \in N \setminus \{0\}$ we have $h_C(b) \geq 3$.*

Equivalently, every nonzero lattice width of C is at least 6.

Proof. Step 0: $\phi(x) \geq 3$ for every nonzero $x \in N$. Since Σ is complete, x lies in some maximal cone σ ; since P is smooth, the primitive generators u_1, u_2, u_3 of the rays of σ form a basis of N . Hence $x = n_1u_1 + n_2u_2 + n_3u_3$ with integers $n_j \geq 0$, not all zero, and linearity of ϕ on σ gives $\phi(x) = \sum_j n_j d_j \geq 3$, where $d_j = \phi(u_j)$ is the depth (4.6) of the ray spanned by u_j .

Step 1: the run inequality. Let $b \in N \setminus \{0\}$ and let $x_i = x_0 + ib \in N$ ($0 \leq i \leq m$, $m \geq 1$) be nonzero lattice covectors whose values $\phi(x_i)$ form an arithmetic progression with common difference r . Then

$$(5.4) \quad |r| \leq 2 \implies \phi(x_0) \geq 3m \text{ and } \phi(x_m) \geq 3m.$$

For $m = 1$ this is Step 0. Let $m \geq 2$ and choose $t \in 2P - z$ with $\langle x_1, t \rangle = \phi(x_1)$. The numbers $\phi(x_i) - \langle x_i, t \rangle$ form an arithmetic progression in i , since both terms do; they are nonnegative because $\phi = h_{2P-z}$ and $t \in 2P - z$; and they vanish at the interior index $i = 1$. A nonnegative arithmetic progression with an interior zero vanishes identically, since a common difference $\delta \neq 0$ would make one of the two neighbouring values $\mp\delta$ negative. Hence $\phi(x_i) = \langle x_i, t \rangle$ for all i , and sublinearity together with $t \in 2P - z$ gives

$$\langle x_0 + x_m, t \rangle = \phi(x_0) + \phi(x_m) \geq \phi(x_0 + x_m) \geq \langle x_0 + x_m, t \rangle,$$

so both inequalities are equalities. Pick a vertex w of $2P - z$ with $\langle x_0 + x_m, w \rangle = \phi(x_0 + x_m)$. Then $\langle x_0, w \rangle + \langle x_m, w \rangle = \phi(x_0) + \phi(x_m)$, while $\langle x_0, w \rangle \leq \phi(x_0)$ and $\langle x_m, w \rangle \leq \phi(x_m)$; hence both are equalities, which says precisely that x_0 and x_m lie in the normal cone σ_w of w , a maximal cone of Σ . In the ray basis of σ_w write $x_0 = \sum_j X_j u_j$ and $x_m = \sum_j Y_j u_j$ with $X, Y \in \mathbb{Z}_{\geq 0}^3$, and let $B \in \mathbb{Z}^3$ be the coordinate vector of b , so that $Y = X + mB$ and $B \neq 0$. Linearity of ϕ on σ_w gives $\phi(x_0) = \sum_j X_j d_j$, $\phi(x_m) = \sum_j Y_j d_j$ and $r = \sum_j B_j d_j$. If $\phi(x_0) < 3m$, then $\sum_j X_j \leq \phi(x_0)/3 < m$ because every $d_j \geq 3$, so every $X_j \leq m - 1$; then $mB_j = Y_j - X_j \geq -(m - 1) > -m$ forces the integer B_j to be nonnegative, and $B \neq 0$ gives $r = \sum_j B_j d_j \geq 3$. Contrapositively, $r \leq 2$ implies $\phi(x_0) \geq 3m$; applied to the reversed progression $x_m, x_{m-1}, \dots, x_0$, with b and r replaced by $-b$ and $-r$, this shows that $r \geq -2$ implies $\phi(x_m) \geq 3m$.

Step 2: the progression bound. If $b \in N \setminus \{0\}$ and $x_i = x_0 + ib \in N$ ($0 \leq i \leq q$, $q \geq 1$) are all nonzero, then

$$(5.5) \quad \phi(x_0) + \phi(x_q) \geq 3q.$$

Put $\phi_i = \phi(x_i)$ and $\delta_i = \phi_i - \phi_{i-1}$ for $1 \leq i \leq q$; these are integers because ϕ is integral on N . Sublinearity gives $\phi_{i-1} + \phi_{i+1} \geq \phi(x_{i-1} + x_{i+1}) = \phi(2x_i) = 2\phi_i$, so the integers $\delta_1 \leq \dots \leq \delta_q$ are nondecreasing; by (5.2), $\delta_i \equiv \langle x_i, z \rangle - \langle x_{i-1}, z \rangle = \langle b, z \rangle \pmod{2}$, so all δ_i share one parity. By Step 0, $\phi_{\min} := \min_i \phi_i \geq 3$.

Split $\{1, \dots, q\}$ into maximal runs of constant δ . A run of slope r and length m_r consists of lattice covectors $x_s, \dots, x_{s+m_r}$ on which ϕ is an arithmetic progression with common difference r , so Step 1 applies to it; call the run *shallow* if $|r| \leq 2$ and *steep* if $|r| \geq 3$.

Maximal runs have pairwise distinct slopes, increasing from left to right; as all slopes share one parity, the shallow slopes that occur form a subset of $\{-1, +1\}$ or of $\{-2, 0, +2\}$, and therefore

$$(5.6) \quad \sum_{\text{shallow}} (3 - |r|) \leq \max\{2 + 2, 1 + 3 + 1\} = 5.$$

Next, every shallow run has length $m_r \leq \phi_{\min}/3$. Indeed, let p be the number of negative increments, so that $\delta_i < 0$ exactly for $i \leq p$; then $\phi_0 > \dots > \phi_p$ and $\phi_p \leq \dots \leq \phi_q$, whence $\phi_{\min} = \phi_p$. A shallow negative run has slope -1 or -2 , the largest negative slope of its parity, so it is the last negative run and its right endpoint carries the value $\phi_p = \phi_{\min}$; symmetrically, a shallow positive run has slope $+1$ or $+2$, the smallest positive slope of its parity, so it is the first positive run and its left endpoint carries the value $\phi_{\min}$; and ϕ is constant equal to ϕ_p along a zero run, so both endpoints of such a run carry the value $\phi_{\min}$. In all three cases (5.4) gives $\phi_{\min} \geq 3m_r$.

Finally, summing $|\delta|$ over the negative and over the positive runs gives the total drop and the total rise of the sequence ϕ_i : $\sum_{r < 0} |r|m_r = \phi_0 - \phi_{\min}$ and $\sum_{r > 0} rm_r = \phi_q - \phi_{\min}$. Therefore

$$(5.7) \quad \begin{aligned} 3q &= \sum_{\text{runs}} 3m_r \leq \sum_{\text{runs}} |r|m_r + \sum_{\text{shallow}} (3 - |r|)m_r \\ &= (\phi_0 - \phi_{\min}) + (\phi_q - \phi_{\min}) + \sum_{\text{shallow}} (3 - |r|)m_r \\ &\leq \phi_0 + \phi_q - 2\phi_{\min} + \frac{5}{3}\phi_{\min} = \phi_0 + \phi_q - \frac{1}{3}\phi_{\min} < \phi_0 + \phi_q, \end{aligned}$$

where the first inequality uses $3 \leq |r|$ on steep runs and $3 = |r| + (3 - |r|)$ on shallow ones, the middle equality uses that the zero run contributes nothing to $\sum_{\text{runs}} |r|m_r$, the second inequality uses $m_r \leq \phi_{\min}/3$ together with (5.6), and the last step uses $\phi_{\min} \geq 3 > 0$. This proves (5.5).

Step 3: duality and descent. Fix $b \in N \setminus \{0\}$. Since C is compact and contains the origin, $h_C(b)$ is the optimal value of the feasible and bounded linear program $\max\{\langle b, t \rangle : \pm \langle u_\rho, t \rangle \leq d_\rho \text{ for every } \rho\}$, whose dual is

$$\min\left\{\sum_{\rho} d_\rho(\lambda_\rho^+ + \lambda_\rho^-) : \sum_{\rho} (\lambda_\rho^+ - \lambda_\rho^-)u_\rho = b, \lambda \geq 0\right\}.$$

The dual is feasible because the rays of the complete fan span $N_{\mathbb{R}}$, so by strong duality both optima are attained and equal. At an optimum $\min(\lambda_\rho^+, \lambda_\rho^-) = 0$ for every ρ : otherwise subtracting that minimum from both entries preserves feasibility and, as $d_\rho > 0$, strictly decreases the objective. Writing $\alpha_\rho = \lambda_\rho^+ - \lambda_\rho^-$, so that $|\alpha_\rho| = \lambda_\rho^+ + \lambda_\rho^-$, we conclude that

$$(5.8) \quad h_C(b) = \min\left\{\sum_{\rho} d_\rho |\alpha_\rho| : b = \sum_{\rho} \alpha_\rho u_\rho\right\}.$$

Moreover the (λ^+, λ^-) program is a rational linear program over a pointed polyhedron, as it lies in the nonnegative orthant, so it has a rational optimal solution; the corresponding $\alpha_\rho = \lambda_\rho^+ - \lambda_\rho^-$ is a rational optimal solution of (5.8).

Fix such an α and an integer $q \geq 1$ with all $q\alpha_\rho \in \mathbb{Z}$, and put

$$x = \sum_{\alpha_\rho > 0} q\alpha_\rho u_\rho, \quad y = \sum_{\alpha_\rho < 0} q|\alpha_\rho| u_\rho,$$

so that $x, y \in N$ and $x - y = q \sum_{\rho} \alpha_\rho u_\rho = qb$. Sublinearity of ϕ and $\phi(u_\rho) = d_\rho$ give

$$(5.9) \quad \phi(x) + \phi(y) \leq \sum_{\rho} q|\alpha_\rho| d_\rho = q h_C(b).$$

Consider the progression $x_i = y + ib$ ($0 \leq i \leq q$) running from y to x . Since $b \neq 0$, at most one x_k vanishes. If none does, then (5.5) gives $\phi(y) + \phi(x) \geq 3q$. If $x_k = 0$, then $y = -kb$ and

$x = (q - k)b$, so positive homogeneity and Step 0 give $\phi(y) + \phi(x) = k\phi(-b) + (q - k)\phi(b) \geq 3k + 3(q - k) = 3q$. Either way, (5.9) yields $q h_C(b) \geq 3q$, that is, $h_C(b) \geq 3$. $\square$

6. THE PARITY LENS AND THE MINIMAL CERTIFICATE

We keep the standing assumption and the notation of the two preceding sections. Thus

$$P = \{x \in M_{\mathbb{R}} : \langle u_{\rho}, x \rangle \leq a_{\rho} \text{ for every ray } \rho\}$$

is the irredundant facet presentation of P , with primitive outward facet normals $u_{\rho} \in N$; the lattice point $z \in (2P) \cap M$ is *not* a sum of two lattice points of P ; and

$$d_{\rho} = 2a_{\rho} - \langle u_{\rho}, z \rangle, \quad c = \tfrac{1}{2}z, \quad \Lambda = z + 2M.$$

By Proposition 4.7 we have $d_{\rho} \geq 3$ for every ray ρ , the point c lies in the interior of P , and $z \in M \setminus 2M$. By Lemma 5.1 the compact centrally symmetric polytope

$$C = (2P - z) \cap (z - 2P) = \{t \in M_{\mathbb{R}} : |\langle u_{\rho}, t \rangle| \leq d_{\rho} \text{ for every } \rho\}$$

satisfies $h_C(b) \geq 3$ for every $b \in N \setminus \{0\}$.

Lemma 6.1 (Parity lens). *The map $x \mapsto 2x - z$ is a bijection of M onto Λ , with inverse $t \mapsto \frac{1}{2}(t + z)$, and $\Lambda = -\Lambda$. For every $x \in M_{\mathbb{R}}$,*

$$(6.1) \quad x \in P \quad \text{and} \quad z - x \in P \quad \Longleftrightarrow \quad 2x - z \in C,$$

and $C \cap \Lambda = \emptyset$.

Proof. If $x \in M$ then $2x - z = z + 2(x - z) \in \Lambda$; if $t = z + 2m$ with $m \in M$ then $\frac{1}{2}(t + z) = z + m \in M$ and $2(z + m) - z = t$. The two maps are therefore mutually inverse bijections between M and Λ . Since $2z \in 2M$ we have $-(z + 2m) = z + 2(-z - m) \in \Lambda$, whence $\Lambda = -\Lambda$.

Let ρ be a ray and $t = 2x - z$. Substituting $d_{\rho} = 2a_{\rho} - \langle u_{\rho}, z \rangle$,

$$\langle u_{\rho}, t \rangle \leq d_{\rho} \Longleftrightarrow \langle u_{\rho}, x \rangle \leq a_{\rho}, \quad -\langle u_{\rho}, t \rangle \leq d_{\rho} \Longleftrightarrow \langle u_{\rho}, z - x \rangle \leq a_{\rho},$$

so $|\langle u_{\rho}, t \rangle| \leq d_{\rho}$ holds for every ρ exactly when both x and $z - x$ satisfy every facet inequality of P ; this is (6.1). Finally, if $t \in C \cap \Lambda$ then $x = \frac{1}{2}(t + z)$ lies in M and $2x - z = t \in C$, so $x, z - x \in P$ and $z = x + (z - x)$ contradicts the choice of z . $\square$

Proposition 6.2 (Minimal certificate). *There is a nonempty set I of rays of the normal fan of P such that, with*

$$(6.2) \quad C_I = \{t \in M_{\mathbb{R}} : |\langle u_i, t \rangle| \leq d_i \ (i \in I)\}, \quad B_I = c + \tfrac{1}{2}C_I,$$

the following hold.

- (a) $B_I \cap M = \emptyset$; equivalently, $C_I \cap \Lambda = \emptyset$.
- (b) For every $i \in I$ there is $m^{(i)} \in M$ satisfying

$$|\langle u_k, m^{(i)} - c \rangle| \leq \tfrac{1}{2}d_k \quad (k \in I \setminus \{i\}).$$

We call such an I a minimal certificate.

Proof. With all facet normals retained, the body in (6.2) is $B = c + \frac{1}{2}C$, and $B \cap M = \emptyset$ by Lemma 6.1. Delete strip pairs while the remaining closed body still misses M , and stop when no further deletion is possible. This process terminates and leaves a nonempty index set, since the subsystem with no inequalities is all of $M_{\mathbb{R}}$ and therefore meets M .

The resulting body B_I is disjoint from M , and minimality gives, for every $i \in I$, a lattice point $m^{(i)}$ in the subsystem obtained by deleting the i th strip; this is (b). Finally, $x \in B_I$ is equivalent to $2x - z \in C_I$, while the parity lens identifies M with $\Lambda = z + 2M$. Hence $B_I \cap M = \emptyset$ is equivalent to $C_I \cap \Lambda = \emptyset$. $\square$

From here to the end of the proof, I denotes a fixed minimal certificate, and C_I, B_I are the bodies in (6.2).

Proposition 6.3 (Structure and contact enlargement). *The normals $(u_i)_{i \in I}$ span $N_{\mathbb{R}}$ and $|I| \in \{3, 4\}$. The body B_I is a compact convex body, symmetric about $c = z/2$, with $B_I \cap M = \emptyset$ and*

$$(6.3) \quad \text{width}_b(B_I) \geq 3 \quad (b \in N \setminus \{0\}).$$

Moreover, there are $\delta > 1$, numbers $W_i \geq \delta d_i/2$, and lattice points p_i ($i \in I$) such that, with

$$(6.4) \quad \widehat{B}_I = \{x \in M_{\mathbb{R}} : |\langle u_i, x - c \rangle| \leq W_i \ (i \in I)\},$$

one has

$$(6.5) \quad \text{int } \widehat{B}_I \cap M = \emptyset, \quad c + \delta(B_I - c) \subseteq \widehat{B}_I,$$

and

$$(6.6) \quad \langle u_i, p_i - c \rangle = W_i, \quad |\langle u_j, p_i - c \rangle| < W_j \quad (j \neq i).$$

Writing $p_{-i} = z - p_i$ and $Q_I = \text{conv}\{p_{\pm i} : i \in I\}$, the $2|I|$ contacts are pairwise incongruent modulo $2M$, and Q_I is a three-dimensional empty lattice polytope whose vertices are exactly those contacts.

Proof. We first prove that the selected normals span $N_{\mathbb{R}}$. Otherwise the map

$$U : M_{\mathbb{R}} \longrightarrow \mathbb{R}^I, \quad U(x) = (\langle u_i, x \rangle)_{i \in I},$$

has rank at most two. The image $U(M)$ is a lattice in its real span, and the lattice polytope $U(P)$ is full-dimensional there. It is therefore normal by Corollary 1.2. Since $U(z) \in 2U(P) \cap U(M)$, write $U(z) = y_1 + y_2$ with $y_r \in U(P) \cap U(M)$. Lift y_1 to $m \in M$; then $U(z - m) = y_2$. Because $y_1, y_2 \in U(P)$, for every $i \in I$ we have

$$\langle u_i, m \rangle \leq a_i, \quad \langle u_i, z - m \rangle \leq a_i,$$

which is equivalent to $|\langle u_i, m - c \rangle| \leq d_i/2$. Thus $m \in B_I \cap M$, contradicting Proposition 6.2(a).

The spanning property gives $|I| \geq 3$ and makes C_I and B_I bounded. They are closed, and $d_i > 0$ for every i , so 0 and c lie in their respective interiors. Since C_I is symmetric about the origin and $B_I = c + \frac{1}{2}C_I$,

$$\text{width}_b(B_I) = h_{C_I}(b) \geq h_C(b) \geq 3 \quad (b \in N \setminus \{0\});$$

the first equality follows from symmetry, the middle inequality from $C \subseteq C_I$, and the last from Lemma 5.1. This proves (6.3); compactness, symmetry and disjointness from M are immediate from the definition and Proposition 6.2(a).

Apply Lemma 3.5 to B_I , with M in the role of Λ and half-widths $d_i/2$. Its hypotheses are exactly the disjointness from M and the deletion witnesses in Proposition 6.2; reflection in c is $x \mapsto z - x$, which preserves M . The lemma gives δ, W_i and an enlargement satisfying (6.5). Reflecting each contact if necessary gives the points p_i in (6.6). Now Lemma 3.6 applies. It gives all assertions about Q_I and the contact classes. Since $M/2M$ has eight elements, $2|I| \leq 8$, hence $|I| \leq 4$. $\square$

We fix once and for all the enlargement $\widehat{B}_I$, the factor δ , the half-widths W_i , and the contacts $p_{\pm i}$ supplied by Proposition 6.3.

7. FOUR STRIP PAIRS ARE IMPOSSIBLE

We use the fixed data of Propositions 6.3 and 4.7. Thus $z \in M \setminus 2M$, $|I| \in \{3, 4\}$, the normals $(u_i)_{i \in I}$ span $N_{\mathbb{R}}$, and $\text{width}_b(B_I) \geq 3$ for every $b \in N \setminus \{0\}$.

Lemma 7.1 (No four strip pairs). $|I| \neq 4$.

Proof. Assume $|I| = 4$, put $K_4 = \widehat{B}_I$, and write $Q = Q_I$.

Step 1: Howe's two layers. By Proposition 6.3, Q is a three-dimensional empty lattice polytope with the eight vertices $p_{\pm i}$, and $Q = z - Q$. Howe's theorem (S2) gives a primitive $b \in N$ and an integer k such that

$$\langle b, Q \rangle = [k, k+1].$$

Central symmetry sends this interval to $\langle b, z \rangle - [k, k+1]$, so the two intervals are equal and

$$(7.1) \quad \langle b, z \rangle = 2k+1.$$

Every contact has integral b -value, and the two members of an opposite pair have values summing to $2k+1$; hence one lies in each of the planes $\{\langle b, \cdot \rangle = k\}$ and $\{\langle b, \cdot \rangle = k+1\}$. Each plane therefore contains four contacts.

Step 2: the layers are fundamental parallelograms. Let $H = \{\langle b, \cdot \rangle = k\}$ and $L = M \cap \ker b$; since b is primitive, $H \cap M$ is a translate of L . The face $Q \cap H$ has the four lower contacts as its vertices, each of them an exposed vertex of Q ; a face with four vertices is two-dimensional, so $Q \cap H$ is a convex lattice quadrilateral in the plane H , and by Lemma 3.6 its only lattice points are those vertices. Pick's formula (S3), applied in the lattice $H \cap M$, gives normalized area 2, so a diagonal splits it into two lattice triangles of positive normalized area summing to 2; both are therefore unimodular, that is, of normalized area one. Label the vertices cyclically as w_1, w_2, w_4, w_3 , so that $w_2 w_3$ is a diagonal; split along it, and send (w_1, w_2, w_3) to $((0, 0), (1, 0), (0, 1))$ in affine lattice coordinates on H . If $w_4 = (\alpha, \beta)$, the cyclic order gives $\alpha, \beta > 0$, hence $\alpha, \beta \geq 1$, while the second triangle has normalized area $\alpha + \beta - 1 = 1$. Thus $\alpha = \beta = 1$, and the lower layer is

$$q, \quad q + v_1, \quad q + v_2, \quad q + v_1 + v_2,$$

where v_1, v_2 form a basis of L .

Put $v_3 = z - 2q - v_1 - v_2$. By (7.1), $\langle b, v_3 \rangle = 1$, so $M = L \oplus \mathbb{Z}v_3$ and v_1, v_2, v_3 form a basis of M . Reflection in c carries the lower layer to the upper one; therefore the eight contacts are exactly $q + \theta_1 v_1 + \theta_2 v_2 + \theta_3 v_3$ with $\theta \in \{0, 1\}^3$. Since $2c = z = 2q + v_1 + v_2 + v_3$, translating by $-c$ and using this basis identifies the contacts with

$$(7.2) \quad \frac{s}{2}, \quad s \in \{\pm 1\}^3.$$

Let e_1, e_2, e_3 now denote the dual basis, so that a point x has coordinates $x_j = \langle e_j, x \rangle$, and put $K_4^0 = K_4 - c$. Since the four normals span $N_{\mathbb{R}}$, K_4 is bounded, and $0 \in \text{int } K_4^0$ because every $W_i > 0$; so K_4^0 is a convex body symmetric about the origin, and its polar $(K_4^0)^\circ$ is defined.

Step 3: three planks cover $2K_4^0$. The contact cube $[-\frac{1}{2}, \frac{1}{2}]^3$ lies in K_4^0 , so polarity gives

$$(K_4^0)^\circ \subseteq \{a : \|a\|_1 \leq 2\}.$$

For a sign vector s , the contact $s/2$ lies on one of the supporting planes of K_4^0 ; let $a_s = \pm u_i / W_i$ be the corresponding outward normal, chosen so that $\langle a_s, s/2 \rangle = 1$, and write $a_{s,j}$ for its coordinates in the basis e_1, e_2, e_3 , so that $\langle a_s, x \rangle = \sum_j a_{s,j} x_j$. Then $a_s \in (K_4^0)^\circ$, and

$$1 = \frac{1}{2} \sum_j s_j a_{s,j} \leq \frac{1}{2} \|a_s\|_1 \leq 1.$$

Thus $\|a_s\|_1 = 2$ and every nonzero coordinate of a_s has the sign prescribed by s .

It follows that the planks $\{|x_j| \leq 1\}$, $j = 1, 2, 3$, cover $2K_4^0$. Indeed, if $x \in 2K_4^0$ had $|x_j| > 1$ for all j , choose $s_j = \text{sgn}(x_j)$. The sign property gives $\langle a_s, x \rangle > \|a_s\|_1 = 2$, contradicting $\langle a_s, x/2 \rangle \leq 1$.

Step 4: Ball's theorem. The relative width of the j th plank with respect to $2K_4^0$ is $1/\text{width}_{e_j}(K_4)$. Applying Ball's theorem (S4) to this centrally symmetric cover yields

$$\sum_{j=1}^3 \frac{1}{\text{width}_{e_j}(K_4)} \geq 1,$$

so $\text{width}_{e_j}(K_4) \leq 3$ for some j . But $K_4 \supseteq c + \delta(B_I - c)$ with $\delta > 1$, and $e_j \in N \setminus \{0\}$; hence

$$\text{width}_{e_j}(K_4) \geq \delta \text{width}_{e_j}(B_I) \geq 3\delta > 3$$

by (6.3), a contradiction. $\square$

8. THE CONTACT LATTICE OF A MINIMAL THREE-STRIP BODY

By Lemma 7.1 the minimal certificate has $|I| = 3$; relabel $I = \{1, 2, 3\}$. Its three normals span $N_{\mathbb{R}}$ and are therefore linearly independent. Set $K_3 = \widehat{B}_I$, which is a parallelepiped, define $\ell_i(x) = \langle u_i, x \rangle / W_i$, and put $q_i = p_i - c$ and $p_{-i} = z - p_i$. The retained enlargement and contacts from Proposition 6.3 give

$$(8.1) \quad \begin{aligned} K_3 &= \{x \in M_{\mathbb{R}} : |\ell_i(x - c)| \leq 1 \ (i = 1, 2, 3)\}, \\ \text{int } K_3 \cap M &= \emptyset, \quad c + \delta(B_I - c) \subseteq K_3, \end{aligned}$$

and

$$(8.2) \quad \ell_i(q_i) = 1, \quad |\ell_j(q_i)| < 1 \quad (j \neq i).$$

The six centred contacts are $\pm q_1, \pm q_2, \pm q_3$.

Lemma 8.1 (Contact-index lemma). *The three lattice vectors*

$$(8.3) \quad 2q_1, \quad q_2 - q_1, \quad q_3 - q_1$$

form a basis of M .

They lie in M , since $2q_1 = 2p_1 - z$ and $q_i - q_1 = p_i - p_1$. We use two auxiliary lemmas, the second concerning the lattice

$$(8.4) \quad D_3 = \{m \in \mathbb{Z}^3 : m_1 + m_2 + m_3 \equiv 0 \pmod{2}\}$$

of integer vectors of even coordinate sum, of index two in $\mathbb{Z}^3$; its nontrivial coset is $e_1 + D_3$, the set of integer vectors of odd coordinate sum, where e_1, e_2, e_3 form the standard basis.

Lemma 8.2 (Interior criterion). *Let A be a real 3×3 matrix with $A_{ii} = 1$ for every i and $|A_{ij}| < 1$ for every $i \neq j$. If $x \in \mathbb{R}^3$ satisfies $\|x\|_1 \leq 1$ and $x \notin \{\pm e_1, \pm e_2, \pm e_3\}$, then $\|Ax\|_{\infty} < 1$.*

Proof. For every i ,

$$|(Ax)_i| \leq |x_i| + \sum_{j \neq i} |A_{ij}| |x_j| \leq \|x\|_1 \leq 1.$$

Equality can hold only if $x_j = 0$ for all $j \neq i$; it then forces $|x_i| = 1$. Thus equality would give $x = \pm e_i$, contrary to the hypothesis, and $\|Ax\|_{\infty} < 1$. $\square$

Lemma 8.3 (Odd-index lemma). *Let $\Gamma \subset \mathbb{R}^3$ be a subgroup with $\Gamma \supsetneq D_3$ and $[\Gamma : D_3]$ finite and odd. Then $e_1 + \Gamma$ contains a vector x with $\|x\|_1 \leq 1$ and $x \notin \{\pm e_1, \pm e_2, \pm e_3\}$.*

Proof. The notation here is local to this proof. The group Γ/D_3 is finite, nontrivial and of odd order; let $p \geq 3$ be a prime dividing that order, let $v \in \Gamma$ have order exactly p modulo D_3 (Cauchy's theorem), and put $a = pv \in D_3$.

Some coordinate of a is nonzero modulo p : otherwise $a = py$ with $y \in \mathbb{Z}^3$, and $\sum_j y_j = (\sum_j a_j)/p$ is even because $\sum_j a_j$ is even and p is odd, so $y \in D_3$; but $pv = a = py$ gives $v = y$, contradicting the order of v . Fix such a coordinate, say the i -th, choose $k \in \{1, \dots, p-1\}$ with $ka_i \equiv 1 \pmod{p}$, and let $r \in \mathbb{Z}^3$ be the vector of centred residues of ka modulo p : thus $r \equiv ka \pmod{p}$ coordinatewise and $|r_j| \leq (p-1)/2$ for every j . Since $p \geq 3$, the value 1 is

its own centred residue, so $r_i = 1$ and $\|r\|_1 \leq 1 + 2 \cdot \frac{p-1}{2} = p$. If $\sum_j r_j$ is odd we are done with r ; suppose it is even.

The two coordinates of r other than r_i have distinct absolute values: if both equalled m , then $\sum_j |r_j| = 1 + 2m$ would be odd, and $\sum_j r_j \equiv \sum_j |r_j| \pmod{2}$ would be odd as well. Let $l \neq i$ be the index of the coordinate with the strictly larger absolute value and l' the remaining index, so that $|r_{l'}| \leq |r_l| - 1$ and, being the larger of two distinct nonnegative integers, $|r_l| \geq 1$. Let $r' \in \mathbb{Z}^3$ agree with r in all coordinates except that $r'_l = r_l - p \operatorname{sgn}(r_l)$. Then $r' \equiv r \equiv ka \pmod{p}$; the coordinate sum $\sum_j r'_j = \sum_j r_j \mp p$ is odd, being an even number minus an odd one; and $0 < |r_l| \leq (p-1)/2 < p$ gives $|r'_l| = p - |r_l|$, so

$$\|r'\|_1 = 1 + |r_{l'}| + (p - |r_l|) \leq 1 + (|r_l| - 1) + (p - |r_l|) = p.$$

Replacing r by r' , we have in either case $r \equiv ka \pmod{p}$ with $\sum_j r_j$ odd and $\|r\|_1 \leq p$.

Put $x = r/p$ and $n = x - kv$. Since $a = pv$ we have $kv = ka/p$ and $n = (r - ka)/p \in \mathbb{Z}^3$, the latter because $r \equiv ka$ coordinatewise modulo p . From $p \sum_j n_j = \sum_j r_j - k \sum_j a_j$, with $\sum_j a_j$ even and p odd, the integer $\sum_j n_j$ is odd; hence $n \in e_1 + D_3$ and

$$x = kv + n \in \Gamma + (e_1 + D_3) = e_1 + \Gamma, \quad \|x\|_1 = \frac{\|r\|_1}{p} \leq 1.$$

Finally $x \neq \pm e_j$ for every j : otherwise x would be an integer vector of odd coordinate sum, so $kv = x - n$ would be an integer vector of even coordinate sum, i.e. $kv \in D_3$ with $1 \leq k \leq p-1$, contradicting the fact that v has order p modulo D_3 . $\square$

Proof of Lemma 8.1. Step 1: independence and odd index. By Proposition 6.3, $Q_I = c + \operatorname{conv}\{\pm q_1, \pm q_2, \pm q_3\}$ is three-dimensional. Hence q_1, q_2, q_3 are linearly independent.

The vectors $2q_1, q_2 - q_1, q_3 - q_1$ are linearly independent; let M_0 be the subgroup they generate. It has finite index in M . Let $\pi : M \rightarrow V = M/2M$ and put

$$\alpha = \pi(z), \quad \beta = \pi(p_2 - p_1), \quad \gamma = \pi(p_3 - p_1).$$

These are the images of the three generators of M_0 , because $2q_1 = 2p_1 - z$ and $q_i - q_1 = p_i - p_1$. The six pairwise distinct contact classes are $\pi(p_i)$ and $\pi(p_i) + \alpha$, with $\alpha \neq 0$. Their images under $\bar{\pi} : V \rightarrow V/\langle \alpha \rangle$ are therefore three distinct points. The differences from the first are $\bar{\pi}(\beta)$ and $\bar{\pi}(\gamma)$; they are nonzero and distinct, hence form a basis of this two-dimensional quotient. Thus α, β, γ span V , equivalently $M_0 + 2M = M$.

Multiplication by 2 is consequently surjective on the finite group M/M_0 , hence bijective. This group has no element of order two, so Cauchy's theorem implies that its order $[M : M_0]$ is odd.

Step 2: contact coordinates. There is a unique linear isomorphism $T : M_{\mathbb{R}} \rightarrow \mathbb{R}^3$ with $T(q_i) = e_i$ for $i = 1, 2, 3$. It carries M_0 onto the lattice generated by $2e_1, e_2 - e_1, e_3 - e_1$; these lie in D_3 , and the matrix having them as columns has determinant $2 = [\mathbb{Z}^3 : D_3]$, so the lattice they generate has index two in $\mathbb{Z}^3$ and hence equals D_3 . Putting $\Gamma = T(M)$ we thus have $D_3 = T(M_0) \subseteq \Gamma$ with $[\Gamma : D_3] = [M : M_0]$ finite and odd, by Step 1. Moreover $c = p_1 - q_1$ gives $T(c) = T(p_1) - e_1$ with $T(p_1) \in \Gamma$, whence

$$(8.5) \quad T(M - c) = \Gamma - T(c) = e_1 + \Gamma.$$

Let A be the 3×3 real matrix with entries $A_{ij} = \ell_i(q_j)$. A point $y = \sum_j x_j q_j$ of $M_{\mathbb{R}}$ has $T(y) = x$ and $\ell_i(y) = \sum_j A_{ij} x_j = (Ax)_i$, so $y \in K_3 - c$ if and only if $\|Ax\|_{\infty} \leq 1$. Together with (8.2), which gives $A_{ii} = \ell_i(q_i) = 1$ and $|A_{ij}| = |\ell_i(q_j)| < 1$ for $i \neq j$, this yields

$$(8.6) \quad T(K_3 - c) = \{x \in \mathbb{R}^3 : \|Ax\|_{\infty} \leq 1\}, \quad A_{ii} = 1, \quad |A_{ij}| < 1 \quad (i \neq j).$$

Finally A , being the matrix of the linearly independent functionals ℓ_i in the basis (q_1, q_2, q_3) of $M_{\mathbb{R}}$, is invertible; so $\{x : \|Ax\|_{\infty} \leq 1\}$ is the preimage of the unit cube under a linear isomorphism, and

$$(8.7) \quad T(\operatorname{int} K_3 - c) = \operatorname{int} T(K_3 - c) = \{x \in \mathbb{R}^3 : \|Ax\|_{\infty} < 1\}.$$

Step 3: conclusion. Suppose $\Gamma \neq D_3$. By Step 2 the subgroup $\Gamma \supsetneq D_3$ has finite odd index, so Lemma 8.3 produces $x \in e_1 + \Gamma$ with $\|x\|_1 \leq 1$ and $x \notin \{\pm e_1, \pm e_2, \pm e_3\}$. By (8.5) the point $m = c + T^{-1}(x)$ lies in M . By (8.6) the matrix A satisfies the hypotheses of Lemma 8.2, which therefore gives $\|Ax\|_\infty < 1$; by (8.7) this means $m \in \text{int } K_3$, contradicting $\text{int } K_3 \cap M = \emptyset$ in (8.1).

Hence $\Gamma = D_3$, that is, $T(M) = T(M_0)$; since T is injective, $M = M_0$. So the three linearly independent vectors (8.3) generate M , and they form a basis of M . $\square$

Proposition 8.4 (Normalized three-strip data). *There are a real number $\delta > 1$, a parallelepiped $K_3 \subset M_{\mathbb{R}}$ with*

$$K_3 \supseteq c + \delta(B_I - c), \quad \text{int } K_3 \cap M = \emptyset,$$

and a linear isomorphism $T : M_{\mathbb{R}} \rightarrow \mathbb{R}^3$ such that the following hold.

- (N1) $T(M) = D_3$; in particular $T(M - c) = e_1 + D_3$ is the set of integer vectors with odd coordinate sum.
- (N2) $T(K_3 - c) = \{x \in \mathbb{R}^3 : \|Ax\|_\infty \leq 1\}$ for an invertible matrix A with $A_{ii} = 1$ for every i and $|A_{ij}| < 1$ for $i \neq j$.
- (N3) $\|A\varepsilon\|_\infty \geq 1$ for every $\varepsilon \in \{\pm 1\}^3$.

Proof. Take δ , K_3 and the two displayed properties from (8.1), and T from Step 2 above. Then (N2) is (8.6), while Lemma 8.1 gives $M = M_0$ and hence $T(M) = T(M_0) = D_3$, which with (8.5) is (N1). It remains to prove (N3), that is,

$$(8.8) \quad \|A\varepsilon\|_\infty \geq 1 \quad \text{for every } \varepsilon \in \{\pm 1\}^3.$$

Every $\varepsilon \in \{\pm 1\}^3$ has odd coordinate sum, hence lies in $e_1 + D_3$, which is $T(M - c)$ by (N1); so $c + T^{-1}(\varepsilon)$ is a lattice point, and $\|A\varepsilon\|_\infty < 1$ would place it in $\text{int } K_3$ by (8.7), contradicting (8.1). $\square$

9. THE MATRIX LEMMA

This section and Appendix A speak only about real matrices, and the letters occurring in them are local to them; the only bridge to the geometry is that the matrix A of Lemma 9.1 will be the contact matrix (8.6) of Section 8.

Lemma 9.1 (Matrix lemma). *Let A be an invertible real 3×3 matrix with $A_{ii} = 1$, $|A_{ij}| < 1$ for $i \neq j$, and $\|A\varepsilon\|_\infty \geq 1$ for every $\varepsilon \in \{\pm 1\}^3$. Then*

$$(9.1) \quad \min_{s \in \{\pm 1\}^3} \|A^{-T}s\|_1 \leq 3.$$

Remark 9.2 (Sharpness). The identity matrix satisfies the hypotheses, with $\|\varepsilon\|_\infty = 1$ for each sign vector, and attains equality in (9.1), since $A^{-T}s = s$ and $\|s\|_1 = 3$. Thus the constant 3 is optimal, with equality already at $A = I_3$.

The proof is given in Appendix A; it is exact and includes all equality and zero cases. The hypothesis becomes a three-clause sign system inconsistent with $xyz = 1$. A direct classification gives four cases: a positive three-cycle, an exceptional tied pattern, a zero row, and an opposite-sign two-cycle. The first three are handled directly. In the last, a cofactor computation, an exact C_4 symmetry and a closed-sector sign analysis reduce the claim to six explicit algebraic certificates.

10. COMPLETION OF THE PROOF

Proof of Theorem 1.3. By Corollary 1.2, it is enough to prove 2-normality. Suppose that a hole $z \in (2P) \cap M$ exists. By Propositions 6.3 and 8.4 and Lemma 7.1 there are a three-strip body B_I , a number $\delta > 1$, a parallelepiped K_3 , a linear isomorphism $T : M_{\mathbb{R}} \rightarrow \mathbb{R}^3$, and an invertible matrix A such that

$$\text{width}_b(B_I) \geq 3 \quad (b \in N \setminus \{0\}), \quad K_3 \supseteq c + \delta(B_I - c),$$

$$T(M) = D_3, \quad T(K_3 - c) = \{x : \|Ax\|_\infty \leq 1\},$$

and A satisfies the hypotheses of Lemma 9.1.

Choose $s \in \{\pm 1\}^3$ with $\|A^{-T}s\|_1 \leq 3$ and put $b_0 = s/2$. For every $m \in D_3$,

$$\langle b_0, m \rangle = \frac{1}{2} \sum_{j=1}^3 s_j m_j \in \mathbb{Z},$$

because $s_j m_j \equiv m_j \pmod{2}$ and m has even coordinate sum. Since $T(M) = D_3$, the pullback $\tilde{b}_0 = b_0 \circ T \in N_{\mathbb{R}}$, which is nonzero because T is an isomorphism, takes integer values on M and so belongs to N . Translation invariance of width and (3.3) give

$$\text{width}_{\tilde{b}_0}(K_3) = \text{width}_{b_0}(T(K_3 - c)) = 2\|A^{-T}b_0\|_1 = \|A^{-T}s\|_1 \leq 3.$$

On the other hand, monotonicity and homogeneity of width give

$$\text{width}_{\tilde{b}_0}(K_3) \geq \delta \text{width}_{\tilde{b}_0}(B_I) \geq 3\delta > 3,$$

a contradiction. Hence P is 2-normal and therefore normal by Corollary 1.2. $\square$

Smoothness is used only in Sections 4 and 5: first to obtain depth at least three at every facet, and then to convert those ray values into the width bound through the smooth normal fan. The matrix constant is also three and is sharp by Remark 9.2; the strict inequality needed at the end comes entirely from the enlargement factor $\delta > 1$.

APPENDIX A. COMPLETE PROOF OF THE MATRIX LEMMA

This appendix gives a self-contained proof of Lemma 9.1, a statement about real matrices: apart from the matrix A , the letters occurring here bear no relation to those of the geometric part of the paper, and every symbol used below is defined inside this appendix. Symbols introduced in Sections A.1, A.3 and A.4 keep their meaning throughout; a letter introduced inside any other subsection is local to it.

The route is this. Section A.2 splits the hypothesis into four dominance patterns, three of which are settled at once. The fourth is reduced in Section A.3 to a planar inequality at the four corners of a square, and Section A.4 shows that only the ten pairs (word, sector) listed in (A.41) can occur; they fall into the six cells of (A.143), each settled by an exact certificate in Sections A.5 to A.10, and Section A.11 collects the pieces.

A.1. Statement. Let

$$(A.1) \quad A = \begin{pmatrix} 1 & a & b \\ c & 1 & d \\ e & f & 1 \end{pmatrix}$$

be invertible, with every off-diagonal entry of absolute value strictly less than one. Assume that no sign vector is mapped into the open cube:

$$(A.2) \quad \|A\varepsilon\|_\infty \geq 1 \quad (\varepsilon \in \{\pm 1\}^3).$$

Then

$$(A.3) \quad \min_{s \in \{\pm 1\}^3} \|A^{-T}s\|_1 \leq 3.$$

No genericity is assumed: the case analysis includes every boundary it introduces—the ties in the dominance classification of Section A.2, the zeros of the four ambiguous forms (A.34) whose signs label the cells, the diagonal $e = f$ between the two open sectors of (A.41), and the axes $e = 0$ and $f = 0$.

We shall repeatedly use signed-coordinate relabeling. If W is a signed permutation matrix and $A' = WAW^T$, then

$$(A.4) \quad \|A's\|_\infty = \|A(W^T s)\|_\infty, \quad A'^{-T}s = WA^{-T}W^T s.$$

Thus both (A.2) and the conclusion (A.3) are invariant under every permutation and diagonal sign conjugation used below.

A.2. The exhaustive dominance classification. For $\varepsilon \in \{\pm 1\}^3$, put

$$(A.5) \quad x = \varepsilon_1 \varepsilon_2, \quad y = \varepsilon_1 \varepsilon_3, \quad z = \varepsilon_2 \varepsilon_3 = xy.$$

Multiplication of the three rows of $A\varepsilon$ by $\varepsilon_1, \varepsilon_2, \varepsilon_3$, respectively, shows that $\|A\varepsilon\|_\infty < 1$ is equivalent to

$$(A.6) \quad ax + by < 0, \quad cx + dz < 0, \quad ey + fz < 0.$$

Indeed, for example, the first absolute value is $|1 + ax + by|$, and $-2 < ax + by < 2$, so it is less than one exactly when $ax + by < 0$.

Call x, y, z the *literals*, and call the set of values that one row of (A.6) forces on its two literals its *clause*. If the two coefficients in that row have unequal absolute values, strict negativity prescribes for the literal multiplying the larger one its *adverse* value, the one of sign opposite to that coefficient. If their common absolute value is nonzero and tied, it prescribes both adverse values at once; a tied clause is thus a conjunction of two prescriptions, not an either/or. The three rows give clauses on

$$(A.7) \quad \{x, y\}, \quad \{x, z\}, \quad \{y, z\}.$$

Condition (A.2) says that their full system is inconsistent with $xyz = 1$.

A *transversal* is a choice of one prescribed literal from every nonempty clause, and an off-diagonal entry of A is *weakly dominant* when its absolute value is at least that of the other off-diagonal entry in its row. An inconsistent transversal has one of two forms:

- (1) two rows prescribe opposite values of their common literal; the corresponding entries form an opposite-sign weakly dominant two-cycle;
- (2) the three chosen literals are x, y, z and their prescribed values have product -1 ; the corresponding entries form a weakly dominant directed three-cycle whose coefficient product is positive.

Suppose every such transversal is consistent although the full system is not. If some row has both off-diagonal entries zero, then that row already forces $\|A\varepsilon\|_\infty \geq 1$, and we are in the zero-row branch below; assume therefore that all three clauses are nonempty. No literal can then occur with opposite prescribed values in two clauses, since selecting those two values would give an inconsistent transversal of type (1). The prescriptions consequently agree wherever they overlap, so their union is a partial assignment; being inconsistent with $xyz = 1$, that assignment must prescribe all of x, y, z with product -1 . In particular no transversal meets all three literals, for such a transversal would carry exactly the values of that assignment and hence be inconsistent of type (2).

We claim that this forces two of the clauses to be singletons on the same literal. If no clause is tied, the three clauses are singletons whose union is $\{x, y, z\}$, hence singletons on three distinct literals, and the unique transversal meets all three. So some clause is tied; let $\{u, v\}$ be the pair it constrains and w the third literal, the two remaining clauses being those on $\{u, w\}$ and on $\{v, w\}$. If the clause on $\{u, w\}$ prescribes w , the clause on $\{v, w\}$ cannot prescribe v : u taken from the tied clause, w from the first and v from the second would form a transversal meeting all three literals. Being nonempty, the clause on $\{v, w\}$ is then the singleton $\{w\}$. Symmetrically, if the clause on $\{v, w\}$ prescribes w , then the clause on $\{u, w\}$ is the singleton $\{w\}$. Now w is prescribed by the union, and not by the tied clause, which is on $\{u, v\}$; so one of the two remaining clauses does prescribe w . That clause turns the other one into the singleton $\{w\}$, which in turn prescribes w and hence turns the first one into the singleton $\{w\}$ as well. Both are therefore the singleton $\{w\}$, with one and the same value, and the tied clause prescribes both u and v . Renaming w, u, v as x, y, z , which is legitimate because permuting the indices $1, 2, 3$ permutes the three literals (A.5) arbitrarily, the clauses

have the form

$$(A.8) \quad \{x = X\}, \quad \{x = X\}, \quad \{y = Y, z = Z\}, \quad XYZ = -1.$$

This is the exceptional tied-row branch. Therefore the following four cases are exhaustive:

- a positive dominant three-cycle;
- the exceptional tied pattern;
- a zero row;
- an opposite-sign dominant two-cycle.

A.2.1. *Positive dominant three-cycle.* After a permutation and a diagonal sign conjugation, write the cycle as

$$(A.9) \quad a, d, e \geq 0, \quad |b| \leq a, \quad |c| \leq d, \quad |f| \leq e.$$

Put $\lambda = A^{-T}\mathbf{1}$, where $\mathbf{1} = (1, 1, 1)^T$. The first Cramer numerator is

$$(A.10) \quad N_1 = 1 - df - c + cf + ed - e.$$

Writing $c = dC$, $f = eF$, where $|C|, |F| \leq 1$, gives

$$(A.11) \quad N_1 = (1 - e)(1 - dC) + de(1 - C)(1 - F) \geq 0.$$

The cyclic analogues give $N_2, N_3 \geq 0$. The row sums

$$(A.12) \quad 1 + a + b, \quad 1 + c + d, \quad 1 + e + f$$

are all at least one. If $\det A < 0$, then every coordinate of λ would be nonpositive, contradicting

$$(A.13) \quad 3 = \mathbf{1}^T A^T \lambda = (A\mathbf{1})^T \lambda.$$

Thus $\det A > 0$, $\lambda \geq 0$, and (A.12)–(A.13) yield

$$(A.14) \quad \|A^{-T}\mathbf{1}\|_1 \leq 3.$$

A.2.2. *Exceptional tied pattern.* After permuting indices, let the tied coefficients be $\alpha = A_{12}, \beta = A_{13}$, and let the two coefficients which dominate the repeated literal be $\delta = A_{23}, \varphi = A_{32}$. Consistency of the repeated prescription gives $\operatorname{sgn} \delta = \operatorname{sgn} \varphi$, while $XYZ = -1$ gives $\operatorname{sgn}(\alpha\beta\delta) = +1$. Choose diagonal signs which make α, δ, φ positive. The invariant cycle product then makes β positive as well. Since $|\alpha| = |\beta|$, and the other entries are weakly dominated, the normal form is

$$(A.15) \quad A = \begin{pmatrix} 1 & t & t \\ dU & 1 & d \\ fV & f & 1 \end{pmatrix}, \quad 0 \leq t, d, f < 1, \quad |U|, |V| \leq 1.$$

Here $t = |\alpha| = |\beta|$, $d = |\delta|$, $f = |\varphi|$, and U, V are the ratios of the dominated entries (these two letters are local to the present subsection). A zero value of t, d or f is included by closure: the certificates below are polynomial, and such a matrix, having a zero row, is in any case settled by Section A.2.3. For $\lambda = A^{-T}\mathbf{1}$, the Cramer numerators are

$$(A.16) \quad \begin{aligned} N_1 &= 1 - df + dU(f - 1) + fV(d - 1), \\ N_2 &= (1 - f)(1 - t), \\ N_3 &= (1 - d)(1 - t). \end{aligned}$$

The coefficients of U, V in N_1 are nonpositive, so its minimum on $[-1, 1]^2$ is attained at $U = V = 1$, where

$$(A.17) \quad N_1 = (1 - d)(1 - f) \geq 0.$$

Every row sum in (A.15) is at least one. The determinant-sign argument from (A.13) again gives $\det A > 0$, $\lambda \geq 0$, and

$$(A.18) \quad \|A^{-T}\mathbf{1}\|_1 \leq 3.$$

A.2.3. *A zero row.* Permute a zero row to the last position:

$$(A.19) \quad A = \begin{pmatrix} 1 & a & b \\ c & 1 & d \\ 0 & 0 & 1 \end{pmatrix}, \quad B_* = \begin{pmatrix} 1 & a \\ c & 1 \end{pmatrix}.$$

One of $u = (1, 1), (1, -1)$ satisfies

$$(A.20) \quad \|B_*^{-T}u\|_1 = \frac{2 - |a + c|}{1 - ac} \leq 2.$$

For example, when $a + c \geq 0$, the last inequality is $a + c \geq 2ac$, immediate if $ac \leq 0$, and otherwise equal to $a(1 - c) + c(1 - a) \geq 0$; the other sign follows after replacing (a, c) by $(-a, -c)$.

Let $(q_1, q_2) = B_*^{-T}u$. Choose $s_3 \in \{\pm 1\}$ nearest to $bq_1 + dq_2$. Since

$$|bq_1 + dq_2| < |q_1| + |q_2| \leq 2,$$

we have $|s_3 - bq_1 - dq_2| \leq 1$. Hence

$$(A.21) \quad \|A^{-T}(u_1, u_2, s_3)^T\|_1 \leq 3.$$

Only the opposite two-cycle remains.

A.3. Exact reduction of an opposite two-cycle to four corner costs. After a permutation and a diagonal sign conjugation the remaining matrix has the form

$$(A.22) \quad L = \begin{pmatrix} 1 & a & b \\ -p & 1 & d \\ e & f & 1 \end{pmatrix}, \quad 0 \leq a, p < 1, \quad |b| \leq a, \quad |d| \leq p, \quad |e|, |f| < 1.$$

We call $\{(a, b) : |b| \leq a \leq 1\}$ the *dominance triangle* of (a, b) , and similarly for (p, d) ; the product of the two is the *dominance box*. Put

$$(A.23) \quad A_0 = 1 - df, \quad B_0 = p + de, \quad P_0 = a - bf, \quad D_0 = 1 - be, \quad \Delta = A_0 D_0 + B_0 P_0,$$

and for $q \in \mathbb{R}^2$

$$(A.24) \quad X_q = \begin{pmatrix} A_0 & B_0 \\ -P_0 & D_0 \end{pmatrix} \left(q - \begin{pmatrix} e \\ f \end{pmatrix} \right).$$

The form

$$(A.25) \quad \Delta(x, y) = (1 - bx)(1 - dy) + (a - by)(p + dx) = 1 + ap + (ad - b)x - (bp + d)y$$

is affine, the two xy -terms cancelling; for $|x|, |y| \leq 1$ its four factors obey $1 - bx \geq 1 - a$, $1 - dy \geq 1 - p$, $a - by \geq 0$, $p + dx \geq 0$, so

$$(A.26) \quad \Delta(x, y) \geq (1 - a)(1 - p) > 0 \quad (|x|, |y| \leq 1).$$

Let $\text{cof } L$ be the cofactor matrix of L , so that $L^{-T} = \text{cof } L / \det L$. The nine cofactors give, for every $q \in \mathbb{R}^2$,

$$(A.27) \quad (\text{cof } L)(q_1, q_2, 1)^T = (X_{q,1}, X_{q,2}, \Delta(q))^T, \quad \det L = \Delta(e, f) = \Delta,$$

the second identity holding because the third row of L is $(e, f, 1)$, so that $(\text{cof } L)L^T = (\det L)I$ gives $(\text{cof } L)(e, f, 1)^T = (0, 0, \det L)^T$, which compared with the first identity at $q = (e, f)$, where $X_q = 0$, yields $\det L = \Delta(e, f)$. Expanding both sides,

$$(A.28) \quad \Delta(q) = \Delta - bX_{q,1} - dX_{q,2}.$$

By (A.26), $\Delta > 0$ and $\Delta(q) > 0$ for $q \in \{\pm 1\}^2$, so

$$(A.29) \quad \|L^{-T}(q_1, q_2, 1)^T\|_1 = \frac{|X_{q,1}| + |X_{q,2}| + \Delta(q)}{\Delta} = 1 + \frac{\Phi_q}{\Delta},$$

and therefore

$$(A.30) \quad \|L^{-T}(q_1, q_2, 1)^T\|_1 \leq 3 \iff \Phi_q \leq 2\Delta,$$

where

$$(A.31) \quad \Phi_q = |X_{q,1}| + |X_{q,2}| - bX_{q,1} - dX_{q,2}.$$

We call Φ_q the *corner cost* at q , and call a difference $2\Delta - \Phi_q$, or 2Δ minus a convex average of the four corner costs, a *gap*. All the work below goes into exhibiting, for each L , a corner whose gap is nonnegative.

Write

$$(A.32) \quad R = 1 - e, \quad S = 1 + e, \quad U = 1 - f, \quad V = 1 + f.$$

All four quantities are positive. Moreover

$$(A.32a) \quad A_0, D_0 > 0, \quad B_0 \geq p(1 - |e|) \geq 0, \quad P_0 \geq a(1 - |f|) \geq 0.$$

These inequalities justify the fixed coordinate signs in the following table. In cyclic order NE, NW, SW, SE , that is, for $q = (1, 1), (-1, 1), (-1, -1), (1, -1)$, the display (A.24) is

q	$X_{q,1}$	$X_{q,2}$
NE	$A_0R + B_0U$	$-P_0R + D_0U$
NW	$-A_0S + B_0U$	$P_0S + D_0U$
SW	$-A_0S - B_0V$	$P_0S - D_0V$
SE	$A_0R - B_0V$	$-P_0R - D_0V$

Thus one coordinate sign is fixed at every corner. Define the four ambiguous forms

$$(A.34) \quad J = -P_0R + D_0U, \quad K = -A_0S + B_0U, \quad J_2 = P_0S - D_0V, \quad K_2 = A_0R - B_0V,$$

and encode their signs in the *word* of L , the quadruple of bits $(J > 0, K > 0, J_2 > 0, K_2 > 0)$ written as a binary string such as 1001.

A.4. The exact C_4 symmetry and the complete sector table. Signed-coordinate relabeling acts by $L \mapsto WLW^T$; call it *universal* if the single matrix W maps the whole family (A.22) to itself, not merely one member of it. Modulo the central sign, the universal symmetries of that oriented family form the cyclic group generated by

$$(A.35) \quad \rho : (a, p, b, d, e, f) \mapsto (p, a, d, -b, f, -e).$$

Indeed, a universal relabeling must preserve the unordered distinguished pair $\{1, 2\}$; otherwise an unrestricted entry e or f would become a purportedly dominant entry. With the identity permutation, the first two coordinate signs must agree, giving I and $\text{diag}(1, 1, -1)$ modulo the central sign. With the swap (12), they must disagree, giving the generator (A.35) and its product with that half-turn. These four maps form C_4 ; none of them acts as a reflection of the (e, f) -plane. Individual degenerate matrices may have further accidental automorphisms; none is used. Explicitly, take

$$W = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L' = WLW^T.$$

For $q' = (q_2, -q_1)$, direct substitution gives

$$(A.35a) \quad \Delta' = \Delta, \quad X'_{q'} = (X_{q,2}, -X_{q,1}), \quad b'X'_{q',1} + d'X'_{q',2} = bX_{q,1} + dX_{q,2}.$$

Hence $\Phi'_{q'} = \Phi_q$, in agreement with the norm invariance (A.4). The map rotates both (b, d) and (e, f) by a quarter-turn. On words,

$$(A.36) \quad \rho(J, K, J_2, K_2) = (-K, J_2, -K_2, J),$$

so, as long as no ambiguous form vanishes, ρ sends the word (w_1, w_2, w_3, w_4) of L to the word $(\overline{w_2}, w_3, \overline{w_4}, w_1)$ of WLW^T , the bar denoting negation. This is a C_4 -action on the sixteen binary words, with six orbits. Rotating (e, f) into the closed first quadrant $e, f \geq 0$, we may assume from now on that L lies in one of the three sectors $e < f$, $f < e$ and $e = f$ of the table (A.41) below. The ten pairs (word, sector) listed in (A.41) are grouped into the six cells of (A.143): a cell is the set of matrices (A.22) whose word and sector lie in one of those groups, and each of the six subsections Sections A.5 to A.10 settles one cell.

Call a word *candidate* if the sign restrictions (A.38)–(A.40) proved below do not exclude it, and *feasible* if it is the word of some L in the normal form (A.22) with $e, f \geq 0$; feasible words are candidate. The sign restrictions cut the sixteen words down to six candidate ones, and these lie in exactly three of the six orbits, namely

$$(A.37) \quad \begin{aligned} &\{0000, 1010, 1111, 0101\}, \\ &\{0001, 1000, 1011, 1101\}, \\ &\{1001\}; \end{aligned}$$

the three remaining orbits $\{0010, 0100, 0111, 1110\}$, $\{0011, 1100\}$ and $\{0110\}$ contain none. The grouping is recorded only to show that the six words are not an arbitrary selection; what is used below is the sector table (A.41), read off from Corollary A.2.

All the sign information used below comes from four factored identities.

Lemma A.1. *For L in the normal form (A.22), the four ambiguous forms (A.34) satisfy, identically in (a, p, b, d, e, f) ,*

$$(A.37a) \quad \begin{aligned} J &= (1-a)U - (a-b)(f-e), & K &= -(1-p)S - (p-d)(e+f), \\ J_2 &= -(1-a)V + (a+b)(e-f), & K_2 &= (1-p)R - (p+d)(e+f), \end{aligned}$$

and the generator (A.35) carries each identity to the next:

$$J \circ \rho = -K, \quad K \circ \rho = J_2, \quad J_2 \circ \rho = -K_2, \quad K_2 \circ \rho = J.$$

Proof. By (A.23) and (A.34), $J = -P_0R + D_0U$ with $P_0 = a-bf$ and $D_0 = 1-be$. Expanding,

$$J = U - aR + b(fR - eU) = U - aR + b(f - e) = (1-a)U - (a-b)(f-e),$$

since $fR - eU = f(1-e) - e(1-f) = f - e$ and $-aR = -aU - a(f-e)$. The other three identities follow by the same two-line expansion. Substituting (A.35) into the four right-hand sides gives the four displayed ρ -relations, which are exactly (A.36). $\square$

Corollary A.2 (Signs). *On the domain (A.22),*

$$(A.37b) \quad \begin{aligned} e \geq f &\implies J \geq (1-a)U > 0, & e + f \geq 0 &\implies K \leq -(1-p)S < 0, \\ e \leq f &\implies J_2 \leq -(1-a)V < 0, & e + f \leq 0 &\implies K_2 \geq (1-p)R > 0. \end{aligned}$$

Proof. In the first implication $a-b \geq 0$ and $f-e \leq 0$, so the second summand of J in (A.37a) is nonnegative; the other three implications are proved identically, using $p-d \geq 0$, $a+b \geq 0$, $p+d \geq 0$ respectively. The leading terms are positive because $a, p < 1$ and $|e|, |f| < 1$. $\square$

Corollary A.3. *The first-quadrant inequalities (A.38), (A.39) and (A.40) below are the K -, J_2 - and J -identities of (A.37a), term for term, combined with (A.37b).*

In a given cell each of the four sign conditions either holds automatically, by Corollary A.2, or restricts one of the two dominance triangles by an affine inequality; we call the latter the *active cuts* of the cell.

Corollary A.4. *Each of the active-cut displays (A.45), the pair of cuts stated in Section A.7.1, (A.94), (A.111), (A.123) and (A.135) below is one of the four identities (A.37a) rearranged.*

By Corollary A.3, in the first quadrant one always has

$$(A.38) \quad K = -S + pU + d(e+f) \leq -(1-p)S \leq 0.$$

If $e \leq f$, then

$$(A.39) \quad J_2 = aS - V + b(e - f) \leq -(1 - a)V \leq 0,$$

whereas if $e \geq f$,

$$(A.40) \quad J = U - aR + b(f - e) \geq (1 - a)U \geq 0.$$

Therefore (on $e = f$ both (A.39) and (A.40) apply) the complete closed-sector list is

sector	candidate words
$0 \leq e < f < 1$	0000, 0001, 1000, 1001
$0 \leq f < e < 1$	1000, 1001, 1010, 1011
$e = f$	1000, 1001

No word outside (A.41) survives (A.38)–(A.40). When an ambiguous form vanishes, the word has that bit zero; for covering the parameter space, however, such a wall may be assigned to either adjacent closed sign domain, since all certificates below use weak inequalities and extend continuously to the wall. What remains to prove is therefore exactly the six cells of (A.143).

Remark A.5 (Attainment). Each of the six candidate words is in fact feasible, so (A.41) is not merely an upper bound for the list of cells. This is used nowhere: an unattained word would only make its cell vacuous. Witnesses in the normal form (A.22) with $e, f \geq 0$ are listed below, together with the values of the four ambiguous forms (A.34); none of them vanishes, and their signs give the word.

word	(a, p, b, d, e, f)	(J, K, J_2, K_2)
0000	$(\frac{3}{4}, \frac{3}{4}, -\frac{3}{4}, \frac{3}{8}, 0, \frac{1}{4})$	$(-\frac{3}{16}, -\frac{11}{32}, -\frac{5}{16}, -\frac{1}{32})$
0001	$(\frac{3}{4}, \frac{1}{4}, -\frac{3}{4}, -\frac{1}{4}, 0, \frac{1}{4})$	$(-\frac{3}{16}, -\frac{7}{8}, -\frac{5}{16}, \frac{3}{4})$
1000	$(\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, \frac{3}{8}, 0, \frac{1}{4})$	$(\frac{7}{16}, -\frac{11}{32}, -\frac{15}{16}, -\frac{1}{32})$
1001	$(\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, 0, 0)$	$(\frac{3}{4}, -\frac{3}{4}, -\frac{3}{4}, \frac{3}{4})$
1010	$(\frac{3}{4}, \frac{3}{4}, \frac{3}{8}, \frac{3}{8}, \frac{1}{4}, 0)$	$(\frac{11}{32}, -\frac{13}{32}, \frac{1}{32}, -\frac{3}{32})$
1011	$(\frac{3}{4}, \frac{1}{4}, \frac{3}{8}, -\frac{1}{4}, \frac{1}{4}, 0)$	$(\frac{11}{32}, -\frac{17}{16}, \frac{1}{32}, \frac{9}{16})$

Some vertex tables below use $a = 1$ or $p = 1$ to compactify an open dominance triangle. At such an auxiliary vertex only the polynomial gap formulas are evaluated; the lemma itself always has $a, p < 1$ and $\Delta > 0$, and the inequalities pass from the compactification to it by continuity, no inverse being invoked where Δ might vanish.

Most of the certificates below are read off from finitely many vertex values, by the following elementary device.

Remark A.6 (Vertex principle). Let $T = \text{conv}\{V_1, \dots, V_m\}$ and $T' = \text{conv}\{W_1, \dots, W_n\}$ be polytopes and let g be a real function on $T \times T'$ which is *bi-affine*: affine in each of its two arguments, called its *blocks*, for every fixed value of the other. Write $\xi = \sum_i \lambda_i V_i$ and $\eta = \sum_j \mu_j W_j$ with $\lambda_i, \mu_j \geq 0$ and $\sum_i \lambda_i = \sum_j \mu_j = 1$. Affinity in the second argument and then in the first gives

$$g(\xi, \eta) = \sum_j \mu_j g(\xi, W_j) = \sum_{i,j} \lambda_i \mu_j g(V_i, W_j).$$

The weights $\lambda_i \mu_j$ are nonnegative with sum one, so $\min_{T \times T'} g = \min_{i,j} g(V_i, W_j)$; in particular $g \geq 0$ on all of $T \times T'$ as soon as $g \geq 0$ at the mn pairs of vertices. A segment or a single point is allowed as a factor, and so is a function affine in one block only; a combination of such functions with fixed nonnegative coefficients is again bi-affine, so the conclusion covers the convex averages of gaps used below.

Unless other blocks are named, they are (a, b) and (p, d) , or linear substitutes for these, and the two polytopes are the closed dominance triangles cut down by the active cuts of the cell at hand. Each gap $2\Delta - \Phi_q$ is then bi-affine: by (A.24), $X_{q,1}$ depends only on (p, d) and $X_{q,2}$ only on (a, b) , while inside a cell the signs in (A.31) are fixed, so that Φ_q is a sum of products of a term affine in one block with a term affine in the other, and the same holds for $\Delta = A_0 D_0 + B_0 P_0$ by (A.23). Two arguments depart from this pattern and say so: in Section A.6 the blocks are the single variables B_0 and P_0 , and in Section A.9.2 the second block enters quadratically, a Bernstein expansion taking the place of the vertex values.

A.5. The 0000 cell, $0 \leq e < f < 1$. Here all four ambiguous forms are nonpositive, so by (A.33) the coordinate signs of X_q are $(+, -)$ at NE and $(-, +)$ at NW . The cell is settled by the nine vertex values (A.48) of the stronger two-corner certificate

$$(A.42) \quad \frac{S}{2} \Phi_{NE} + \frac{R}{2} \Phi_{NW} \leq 2\Delta.$$

Put

$$(A.43) \quad u = a - b, \quad v = a + b, \quad r = p - d, \quad t = p + d.$$

Then $u, v, r, t \geq 0$, $u + v \leq 2$, $r + t \leq 2$, and

$$(A.44) \quad P_0 = \frac{uV + vU}{2}, \quad D_0 = \frac{2 + e(u - v)}{2}, \quad A_0 = \frac{2 + f(r - t)}{2}, \quad B_0 = \frac{rR + tS}{2}.$$

By Corollary A.4, the two active cuts $J \leq 0$ and $K_2 \leq 0$ read, in the variables (A.43),

$$(A.45) \quad (V - 2e)u + Uv \geq 2U, \quad Rr + (S + 2f)t \geq 2R.$$

The other two are automatic by Corollary A.2: $J_2 \leq 0$ because $e < f$, and $K \leq 0$ because $e + f \geq 0$. Hence the closed domains are

$$(A.46) \quad \begin{aligned} T_1 &= \text{conv} \left\{ (2, 0), (0, 2), \left(\frac{2U}{1 + f - 2e}, 0 \right) \right\}, \\ T_2 &= \text{conv} \left\{ (2, 0), (0, 2), \left(0, \frac{2R}{1 + e + 2f} \right) \right\}. \end{aligned}$$

Let $G = 2\Delta - S\Phi_{NE}/2 - R\Phi_{NW}/2$. It is bi-affine in the blocks $(u, v) \in T_1$ and $(r, t) \in T_2$, so by Remark A.6 it is enough to evaluate G at the nine pairs of vertices of $T_1 \times T_2$. Set

$$(A.47) \quad \ell = 1 + f - 2e > 0, \quad k = 1 + e + 2f > 0.$$

In the vertex orders displayed in (A.46), the nine values of G are

	(2, 0)	(0, 2)	(0, 2R/k)
(2, 0)	$4(e^2 + f)$	$2S^2$	$\frac{2S(1 - f^2 + 2e(e + f))}{k}$
(0, 2)	$2R^2$	$4U$	$\frac{2R(U^2 + 1 - e^2)}{k}$
(2U/ℓ, 0)	$\frac{4R(e^2 + f)}{\ell}$	$\frac{2U(e^2 + 2f + 1)}{\ell}$	$\frac{2RN(e, f)}{\ell k}$

where

$$(A.49) \quad N(e, f) = 1 + 2e^2 + 2e^3 + 2ef + f(f^2 - f + 1) > 0.$$

Every entry is nonnegative, proving (A.42).

A.6. The 1001 cell throughout the first quadrant. Here $J, K_2 \geq 0 \geq K, J_2$, so by (A.33) every X_q lies in the closed quadrant of q , whatever the sector. We settle the cell by averaging the four corner costs over a random corner Q and proving that the resulting gap (A.53) is nonnegative: the boundary values (A.58) settle it off the diagonal $B_0 = P_0$, and (A.60)–(A.64) settle it there. Let Q be a random corner of the square $\{\pm 1\}^2$ with

$$(A.50) \quad \mathbb{E}Q = (e, f), \quad \mathbb{E}(Q_1 Q_2) = \tau.$$

Such a distribution exists exactly for

$$(A.51) \quad e + f - 1 \leq \tau \leq 1 - |e - f|, \quad \Pr(Q = (x, y)) = \frac{1 + xe + yf + xy\tau}{4}.$$

Because $\mathbb{E}X_Q = 0$ and $|X_{Q,1}| + |X_{Q,2}| = Q^T X_Q$,

$$(A.52) \quad \mathbb{E}\Phi_Q = A_0(1 - e^2) + D_0(1 - f^2) + (B_0 - P_0)(\tau - ef).$$

Choose the endpoint of (A.51) minimizing this expression. The gap becomes

$$(A.53) \quad \Gamma = 2A_0D_0 + 2B_0P_0 - A_0RS - D_0UV + h|B_0 - P_0|,$$

where

$$(A.54) \quad h = RU \ (B_0 \geq P_0), \quad h = RV \ (P_0 \geq B_0, e \geq f), \quad h = SU \ (P_0 \geq B_0, f \geq e).$$

We prove $\Gamma \geq 0$.

The dominance box gives $0 \leq B_0 \leq S$, $0 \leq P_0 \leq V$, and the sharp lower envelopes

$$(A.55) \quad A_0 \geq 1 - \frac{fB_0}{S} =: A_*(B_0),$$

$$(A.56) \quad D_0 \geq D_*(P_0) := \begin{cases} 1 - eP_0/U, & 0 \leq P_0 \leq U, \\ R + e(P_0 - U)/f, & U \leq P_0 \leq V. \end{cases}$$

For $f = 0$ the interval $U \leq P_0 \leq V$ collapses to the point $P_0 = 1$, and the second branch—like the second row of (A.58) below—is used only through its continuous limit as $f \downarrow 0$. Indeed, $B_0 - Sd = p - d \geq 0$, while $P_0 - bU = a - b \geq 0$ and $1 - P_0 - bf = 1 - a \geq 0$. Moreover

$$(A.57) \quad \frac{\partial \Gamma}{\partial A_0} = 2D_0 - RS \geq R^2, \quad \frac{\partial \Gamma}{\partial D_0} = 2A_0 - UV \geq U^2.$$

It is therefore enough to substitute (A.55)–(A.56); call the result $\gamma(B_0, P_0)$.

Here the blocks are the single variables B_0 and P_0 , and the domain is the rectangle $[0, S] \times [0, V]$, divided by the diagonal $B_0 = P_0$ and by the breakpoint $P_0 = U$ of (A.56) into four closed pieces. On each piece γ is bi-affine: the substituted A_* is affine in B_0 alone and the substituted D_* in P_0 alone, so the products $2A_*D_*$ and $2B_0P_0$ are bi-affine and the terms $-A_*RS$ and $-D_*UV$ are affine in a single variable, while $|B_0 - P_0|$ takes the sign fixed on the piece and the multiplier h of (A.54) is constant there.

Now fix P_0 and let B_0 traverse the fibre $[P_0, S]$ of the region $B_0 \geq P_0$. Along it γ is affine in the single variable B_0 , with no breakpoint, since the breakpoint of D_* concerns P_0 ; an affine function of one variable on a segment attains its minimum at an endpoint, so only $B_0 = P_0$ and $B_0 = S$ need to be examined. Symmetrically, fix B_0 and let P_0 traverse the fibre $[B_0, V]$ of the region $P_0 \geq B_0$. Along it γ is continuous and piecewise affine in P_0 with the single breakpoint $P_0 = U$, so its minimum is attained at $P_0 = B_0$, at $P_0 = U$ or at $P_0 = V$. This leaves the diagonal $B_0 = P_0$, treated below, and the three non-diagonal faces $B_0 = S$, $P_0 = U$ and $P_0 = V$. They are collected in the next table, split according to the branch of

(A.56) and to the sign of $e - f$, which fixes h ; the last column gives the value of γ .

	region	boundary	γ
	$B_0 \geq P_0$	$P_0 \leq U, B_0 = S$	$P_0(2e + f + 1) + U^2$
	$B_0 \geq P_0$	$P_0 \geq U, B_0 = S$	$\frac{P_0 V(e + f) + (f - e)U^2}{f}$
(A.58)	$P_0 \geq B_0, e \geq f$	$P_0 = U$	$\frac{B_0(S^2 - 4f) + R^2 S}{S}$
	$P_0 \geq B_0, e \geq f$	$P_0 = V$	$B_0 S + e^2 - 2ef + 2f^2 + 2f + 1$
	$P_0 \geq B_0, f \geq e$	$P_0 = U$	$\frac{B_0 R(1 + e - 2f) + S(U^2 + (f - e)^2)}{S}$
	$P_0 \geq B_0, f \geq e$	$P_0 = V$	$B_0(1 + 2f - e) + S^2$

All are nonnegative. In the second row the numerator is nondecreasing in P_0 , and $P_0 \geq U$ gives it the lower bound $2fUS$. In the third and fifth rows one has $0 \leq B_0 \leq U$; if the coefficient of B_0 in the displayed numerator is negative, substitute $B_0 = U$. The resulting numerator is

$$(A.59) \quad N_0 = \begin{cases} h_0^3 + 2fh_0^2 + U^2 h_0 + 2U^2, & e \geq f, h_0 = e - f, \\ U^2(2 - h_0) + h_0^2(e + f), & f \geq e, h_0 = f - e, \end{cases}$$

which is nonnegative.

It remains to check the diagonal $B_0 = P_0 = x$. For $0 \leq x \leq U$, write $x = Ut$, and put

$$c_* = S - f, \quad k_* = c_* + ef.$$

Then

$$(A.60) \quad \gamma(Ut, Ut) = e^2 + f^2 - \frac{(e + f)k_*}{S}t + \frac{2Uc_*}{S}t^2.$$

Its values at $t = 0, 1$ are $e^2 + f^2$ and N_0/S , respectively. If the vertex of this quadratic lies outside $[0, 1]$, the minimum of $\gamma(Ut, Ut)$ on $[0, 1]$ is one of those two endpoint values, both of which are nonnegative. Assume therefore that the vertex lies in $[0, 1]$, that is,

$$(e + f)k_* \leq 4Uc_*.$$

Under that condition nonnegativity in (A.60) amounts to the discriminant bound $(e + f)^2 k_*^2 \leq 8SUc_*(e^2 + f^2)$, which, by $(e + f)^2 \leq 2(e^2 + f^2)$, follows from

$$(A.61) \quad k_*^2 \leq 4SUc_*,$$

proved as follows. If $k_* \leq S(e + f)$, combine this inequality with $(e + f)k_* \leq 4Uc_*$. Otherwise $k_* > S(e + f)$, which is exactly $e^2 + 2f < 1$; then $U > 1/2$, $k_* \leq S$ and $2c_* \geq S$, so $k_*^2 \leq S^2 \leq 4SUc_*$, again (A.61).

If $f > 0$ and $U \leq x \leq V$, write $x = U + 2ft$. The exact Bernstein form is

$$(A.62) \quad \gamma(x, x) = (1 - t)^2 c_0 + 2t(1 - t)c_1 + t^2 c_2,$$

where

$$c_0 = \frac{N_0}{S}, \quad c_1 = \frac{e^3 - e^2 f + e^2 + ef^2 + 2e - f^2 - f + 2}{S},$$

$$(A.63) \quad c_2 = e^2 - ef + e + 2f^2 + 3f + 2.$$

When $f = 0$, the interval $[U, V]$ is the singleton $\{U\}$, already covered by (A.60) at $t = 1$. Here

$$(A.64) \quad Sc_1 = e(e^2 - ef + e + f^2 + 2) + (1 - f)(f + 2) \geq 0, \quad c_2 = e(1 + e - f) + 2f^2 + 3f + 2 \geq 0.$$

Thus $\Gamma \geq 0$. The probabilities in (A.51) form a convex combination of corner costs, so at least one $\Phi_q \leq 2\Delta$.

A.7. The 0001 cell, $0 \leq e < f < 1$. Here $K_2 \geq 0 \geq J, K, J_2$. Define the two vertical-edge averages

$$(A.65) \quad C_1 = \frac{V}{2}\Phi_{NE} + \frac{U}{2}\Phi_{SE} = R((1-b)A_0 + (1+d)P_0),$$

$$(A.66) \quad C_2 = \frac{V}{2}\Phi_{NW} + \frac{U}{2}\Phi_{SW}.$$

Both are convex averages of corner costs. We split only according to the signs of b and d ; the three pieces below cover the whole cell and are settled by the vertex tables (A.68) and (A.76) and by the identity (A.91).

A.7.1. The part $b \geq 0$. By Corollary A.4, the active cuts $J \leq 0$ and $K_2 \geq 0$ are $Ra - (f-e)b \geq U$ and $Vp + (e+f)d \leq R$; the closed domain of this part is therefore the product of the triangles

$$(a, b) \in \text{conv}\{(1, 1), (1, 0), (U/R, 0)\},$$

$$(A.67) \quad (p, d) \in \text{conv}\{(0, 0), (1, -1), (\sigma, \sigma)\}, \quad \sigma = \frac{R}{1+e+2f}.$$

The gap $G_1 = 2\Delta - C_1$ is bi-affine in the blocks (a, b) and (p, d) , which range over the two triangles just displayed, so Remark A.6 applies to their product. In the displayed vertex orders, its table is

	$(0, 0)$	$(1, -1)$	(σ, σ)
$(1, 1)$	RV	$4R$	$\frac{2R(2e+f^2+1)}{1+e+2f}$
$(1, 0)$	$2e$	$3+f-e+ef$	$\frac{(1-f)(1-e^2)+4e(1+f)}{1+e+2f}$
$(U/R, 0)$	$e+f$	$3+e-f+ef$	$\frac{1-f+e^2(1+f)+4e+2f^2}{1+e+2f}$

Every entry is nonnegative, so $G_1 \geq 0$.

A.7.2. The part $b \leq 0, d \leq 0$. Write

$$(A.69) \quad b = -c, \quad a = c + x, \quad d = -h, \quad p = h + q,$$

where all four new variables are nonnegative. The inequality $K_2 \geq 0$ is

$$(A.70) \quad Vq + Rh \leq R.$$

Put

$$(A.71) \quad k_0 = ef + 4e - f - 2, \quad F_1 = 2\Delta - C_1, \quad F_3 = 2\Delta - \Phi_{SW}.$$

Direct expansion gives

$$(A.72) \quad F_1 = S - Rx + ck_0 + h\{c(3+2f-3e) + 3Rx + fS\} + 2q(cV + x).$$

The second line is nonnegative. Thus $F_1 < 0$ implies

$$(A.73) \quad x > x_0 = \frac{S + ck_0}{R}.$$

The coefficient of x in F_3 is

$$(A.74) \quad S + (1-3e)h + 2q > 0,$$

whose lower bound is $2R$ when $1-3e < 0$, since $h \leq 1$ by (A.70). Hence $F_3 > F_3|_{x=x_0}$.

The restriction $F_3|_{x=x_0}$ is bi-affine in the blocks c and (h, q) , since F_3 is affine in (c, x) and in (h, q) separately and x_0 is affine in c . Its domain is contained in

$$(A.75) \quad [0, 1] \times \text{conv}\{(0, 0), (1, 0), (0, R/V)\},$$

because $c \in [0, 1]$, because $h, q \geq 0$, and because (A.70) is the inequality cutting the triangle off the nonnegative quadrant. By Remark A.6 it suffices to evaluate at $c = 0, 1$ and at the three vertices in (A.75); the six values are

	$(0, 0)$	$(1, 0)$	$(0, R/V)$
0	N_{00}/R	$2S$	$N_{02}/(RV)$
1	SN_{10}/R	$2(e f + 5e + f + 1)$	$N_{12}/(RV)$

where

$$(A.77) \quad \begin{aligned} N_{00} &= 1 - f + e(1 + f) + 2e^2, \\ N_{10} &= 1 - f + e(f + 3), \\ N_{02} &= (1 - f)(f + 2 - e^2) + e(1 + f)(f + 3), \\ N_{12} &= 1 - f^2 + 4e(f + 3) - e^2(1 - f)(f + 5). \end{aligned}$$

These are positive, the first three manifestly. For N_{12} , $e(1 - f)(f + 5) < 4(f + 3)$, so $4e(f + 3) - e^2(1 - f)(f + 5) \geq 0$ (strictly positive when $e > 0$). Therefore $F_1 < 0$ forces $F_3 > 0$, and some corner works.

A.7.3. *The part $b \leq 0 \leq d$.* Write

$$(A.78) \quad b = -c, \quad a = c + x, \quad d = h, \quad p = h + q, \quad c, x, h, q \geq 0.$$

The closed domain inequalities needed below are

$$(A.79) \quad c + x \leq 1, \quad Rx + (1 + f - 2e)c \geq U, \quad (S + 2f)h + Vq \leq R.$$

Set $P = cV + x$, and define

$$(A.80) \quad C_3 = \Phi_{SW}, \quad G_i = 2\Delta - C_i \quad (i = 1, 2, 3).$$

Put

$$(A.81) \quad g_{10} = S + c(ef + 4e - f - 2) - Rx,$$

$$(A.82) \quad \begin{aligned} g_{20} &= -cef + 2ce - cf^2 - cf + c - efx - e + f^2 - fx, \\ g_{30} &= c(3e + f + 2) + Sx - e - f, \\ c_1 &= c(1 + 3e + 2f) + x(1 + 3e) - fS, \\ c_2 &= c(3 + 3e + 2f) + 3Sx - fR, \\ c_3 &= c(4 + 3e + 3f) + 3Sx - e - 3f - 2, \\ c_y &= 3cV + 2x - V. \end{aligned}$$

The exact expansions are

$$(A.83) \quad \begin{aligned} G_1 &= g_{10} + c_1h + 2Pq, \\ G_2 &= g_{20} + c_2h + 2Pq, \\ G_3 &= g_{30} + c_3h + c_yq. \end{aligned}$$

Suppose all three gaps were negative. The following identities are exact:

$$\begin{aligned}
 Rc_1 + (1 + 3e)g_{10} &= \{S + c(3e - 1)\}\{U + e(f + 3)\}, \\
 c_2 &= c_1 + 2(c + ef + x), \\
 Rc_y &= -2g_{10} + (1 - c)U + e\{3 + f + c(5 - f)\}, \\
 g_{20} + fg_{30} &= V\{c(1 + 2e) - e\}.
 \end{aligned}
 \tag{A.84}$$

The right side of the first line of (A.84) is nonnegative: if $e < 1/3$, then $S + c(3e - 1) \geq 4e$, while for $e \geq 1/3$ both summands are nonnegative; the second factor is nonnegative since $f \leq 1$.

We first claim $g_{10} < 0$. Suppose instead $g_{10} \geq 0$. If $c_1 \geq 0$, the first line of (A.83) with $h, q, P \geq 0$ gives $G_1 \geq 0$, a contradiction. If $c_1 < 0$, (A.79), (A.84), and $(S + 2f) - (1 + 3e) = 2(f - e) > 0$ give

$$G_1 \geq g_{10} + \frac{R}{S + 2f}c_1 \geq 0, \tag{A.85}$$

again a contradiction. Hence $g_{10} < 0$, and (A.84) gives

$$c_1 > 0, \quad c_2 > 0, \quad c_y > 0. \tag{A.86}$$

The inequality $g_{10} < 0$ also forces

$$c(1 + 2e) - e > 0. \tag{A.87}$$

Indeed, since $x \leq 1 - c$,

$$g_{10} \geq 2e - cD_*, \quad D_* = 1 + f - 3e - ef. \tag{A.88}$$

If $c \leq e/(1 + 2e)$, the right side is nonnegative when $D_* \leq 0$, and when $D_* > 0$ it is at least

$$\frac{e(1 - f + 7e + ef)}{1 + 2e} \geq 0, \tag{A.89}$$

a contradiction.

Now $G_2 < 0$ and (A.86) imply $g_{20} < 0$. In the stated sector $f > e \geq 0$, so $f > 0$, and equations (A.84), (A.87) give $g_{30} > 0$. If one also takes the degenerate closure value $f = 0$, then necessarily $e = 0$, and $g_{20} = c > 0$ by (A.87), already contradicting $G_2 < 0$. Thus no division by a possibly zero f is hidden. Finally $G_3 < 0$ forces $c_3 < 0$. Dropping the nonnegative q -terms from the last two lines of (A.83) yields

$$h < -\frac{g_{20}}{c_2}, \quad h > -\frac{g_{30}}{c_3}. \tag{A.90}$$

But the exact identity is

$$\frac{3(-c_3g_{20} + c_2g_{30})}{V} = (c_3 + e + 2 - c)^2 + 3(e + 2 - c)\{(1 + 2e)c - e\} > 0. \tag{A.91}$$

Because $c_2c_3 < 0$, (A.91) says

$$-\frac{g_{20}}{c_2} < -\frac{g_{30}}{c_3},$$

contradicting (A.90). Thus at least one gap in (A.83) is nonnegative, and one of the four corner costs is at most 2Δ .

A.8. The 1000 cell for $0 \leq e \leq f < 1$. Here $J \geq 0 \geq K, J_2, K_2$. This sector is divided below into three regions, each settled by a nine-point vertex table for a convex average of gaps. Put

$$\ell = 1 + f - 2e, \quad k = 1 + e + 2f. \tag{A.92}$$

The two active cuts give exactly

$$\begin{aligned}
 T_{ab} &= \text{conv}\{O = (0, 0), E = (1, 1), F = (U/\ell, -U/\ell)\}, \\
 T_{pd} &= \text{conv}\{M = (1, -1), N = (1, 1), Z = (R/k, R/k)\}.
 \end{aligned}
 \tag{A.93}$$

Indeed, by Corollary A.4, the active cuts $J \geq 0$ and $K_2 \leq 0$ are

$$(A.94) \quad Ra - (f - e)b \leq U, \quad Vp + (e + f)d \geq R.$$

The other two sign conditions, $K \leq 0$ and $J_2 \leq 0$, are automatic: $-K = A_0S - B_0U$, at M, N, Z , has values

$$(A.95) \quad 2(e + f), \quad 0, \quad \frac{2S(e + f)}{k},$$

while $-J_2 = D_0V - P_0S$, at O, E, F , has values

$$(A.96) \quad V, \quad 2(f - e), \quad \frac{2V(f - e)}{\ell}.$$

Let $G_q = 2\Delta - \Phi_q$. Each gap, and hence every convex average of gaps with coefficients independent of (a, b) and (p, d) , is bi-affine in these two blocks on $T_{ab} \times T_{pd}$, so by Remark A.6 nonnegativity of such an average follows from nonnegativity at the nine pairs of vertices. In the three tables below the rows are the vertices O, E, F of T_{ab} and the columns are M, N, Z of T_{pd} .

If $e + 2f \geq 1$, the northeast gap has table

	M	N	Z
O	$2(e + 2f - 1)$	0	$2(e + f)^2/k$
E	$4R$	$4U$	$4RS/k$
F	$4R(e + f)/\ell$	$4U(e + f)/\ell$	$4RS(e + f)/(k\ell)$

so $G_{NE} \geq 0$.

Suppose $e + 2f \leq 1$ and $2e + 3f \leq 1$. Use

$$(A.98) \quad Q = \frac{e + f}{2}G_{NE} + \frac{R}{2}G_{NW} + \frac{U}{2}G_{SE}.$$

Its table is

	M	N	Z
O	A	$U(1 - 2e - 3f)$	B/k
E	$2R^2$	$2U^2$	$2R(U^2 + f^2 - e^2)/k$
F	$2RC/\ell$	$2UD/\ell$	$2RE_*/(k\ell)$

where

$$(A.100) \quad \begin{aligned} A &= 1 - 2e + f^2 + 2e(e + f), \\ B &= 1 - e - 2f + 2e^3 + 2e^2f + ef^2 + 2ef + 2f^3 + f^2, \\ C &= 1 - e + f + 2e^2 + ef, \\ D &= 1 - f^2 + 2e^2 + 2ef, \\ E_* &= 1 - f + 2e^3 + e^2f + e^2 + 2ef + 2f^3. \end{aligned}$$

All entries are nonnegative. For A , use $e \leq f$ and $e + 2f \leq 1$, hence $e \leq 1/3$; the other displayed terms are manifestly nonnegative.

The remaining wedge is

$$(A.101) \quad e + 2f \leq 1 \leq 2e + 3f.$$

Set

$$(A.102) \quad \delta_0 = f - e, \quad s_0 = 1 - e - 2f, \quad r_0 = 2e + 3f - 1,$$

and use the convex average

$$(A.103) \quad W = \frac{3e + 4f - 1}{2} G_{NE} + \frac{R}{2} G_{NW} + s_0 G_{SE}.$$

Its coefficients are nonnegative and sum to one. The vertex table is

		M	N	Z
(A.104)	O	P_{OM}	$(r_0 + \delta_0)r_0$	P_{OZ}/k
	E	$2R^2$	$4(e f + e + 2f^2)$	$2R(2e + 2ef + 5f^2 - e^2)/k$
	F	$4RP_{FM}/\ell$	$2P_{FN}/\ell$	$2RP_{FZ}/(k\ell)$

The five remaining numerators have the exact nonnegative slack representations

$$(A.105) \quad 3P_{OM} = 4s_0 + 5s_0^2 + 2\delta_0 + 5\delta_0 s_0 + 2\delta_0^2,$$

$$(A.106) \quad \begin{aligned} 375P_{OZ} = & 180 + 193r_0^2 + 73r_0^3 + 75s_0^2 + 324\delta_0 r_0 + 55\delta_0^2 + 343\delta_0^2 r_0 \\ & + 120e + 40er_0^2 + 70e\delta_0^2, \end{aligned}$$

$$(A.107) \quad 105P_{FM} = 38 + 15r_0^2 + 52s_0 + 89\delta_0 s_0 + 43\delta_0^2 + 9er_0,$$

$$(A.108) \quad \begin{aligned} 750P_{FN} = & 408 + 137r_0^2 + 130s_0 r_0^2 + 75s_0^2 + 546\delta_0 r_0 + 458\delta_0 r_0^2 \\ & + 253\delta_0^2 + 406\delta_0^2 r_0 + 1260e + 40e\delta_0^2, \end{aligned}$$

$$(A.109) \quad \begin{aligned} 75P_{FZ} = & 52 + 28r_0 + 56r_0^2 + 5r_0^3 + 18\delta_0 r_0 + 32\delta_0^2 + 80\delta_0^2 r_0 \\ & + 100e + 20er_0^2 + 20e\delta_0^2. \end{aligned}$$

Thus (A.104) is nonnegative. The three regions exhaust $0 \leq e \leq f < 1$, so the cell is proved.

A.9. The 1000 cell for $0 \leq f < e < 1$. Again $J \geq 0 \geq K, J_2, K_2$, now in the opposite sector. We divide the cell into the three closed pieces $d \geq 0$, $d \leq 0, b \leq 0$, and $d \leq 0 \leq b$, which cover it and are settled respectively by a crossing determinant, a Bernstein expansion and two vertex tables.

A.9.1. *The part $d \geq 0$.* Put

$$(A.110) \quad M = 1 + 2e - f.$$

By Corollary A.4, the active (a, b) -cut is $J_2 \leq 0$, that is,

$$(A.111) \quad Sa + (e - f)b \leq V,$$

so

$$(A.112) \quad (a, b) \in \text{conv}\{O = (0, 0), E = (1, -1), F = (V/M, V/M)\}.$$

Let

$$(A.113) \quad n = 2\Delta - \Phi_{NE}, \quad w = 2\Delta - \Phi_{NW}.$$

Both are affine in (a, b) for each fixed (p, d) , so by Remark A.6, applied with (p, d) frozen and the second factor a point, each of n and w is the convex combination, with the barycentric coordinates on the triangle (A.112), of its three values at O, E, F . Their values at O, E are

$$(A.114) \quad \begin{aligned} n_O &= -de - 2df + d + e + fp + f - p, \\ w_O &= de - 2df + d - e - fp + f + p, \\ n_E &= 2\{d(e - f) + e + f + 2fp\}, \\ w_E &= 2\{d(2e - f + 1) + pV\}. \end{aligned}$$

Thus $n_E, w_E \geq 0$, and

$$(A.115) \quad n_O + w_O = 2\{f + d(1 - 2f)\} \geq 0.$$

For $f \leq 1/2$ this is immediate; for $f \geq 1/2$, its minimum on $0 \leq d \leq 1$ is attained at $d = 1$ and equals $2(1 - f)$. For the first coordinate at F , direct expansion gives

$$(A.116) \quad \frac{M}{2}n_F = c_0 + c_p p + c_d d,$$

where

$$(A.117) \quad c_0 = e^2 - ef - e - f^2 + f + 1, \quad c_p = (f - 1)(e - 2f - 1), \quad c_d = (e - f)(2 - e - f).$$

Here $c_p, c_d > 0$, and $c_0 > 0$: its minimum as a quadratic in $e \in [f, 1]$ is either the endpoint value $1 - f^2$, or $(1 - f)(5f + 3)/4$. Hence $n_F > 0$.

The crossing determinant of the gap vectors (n_O, w_O) and (n_F, w_F) factors as

$$(A.118) \quad w_O n_F - n_O w_F = \frac{2V}{M} \left[3\{Up + ed - e\}^2 + (d + f - 2)(2df - d - f) \right] \geq 0.$$

Indeed, both factors in the last product are nonpositive:

$$(A.119) \quad d + f - 2 \leq 0, \quad 2df - d - f = -\{d(1 - f) + f(1 - d)\} \leq 0.$$

If $n_O \geq 0$ then $n \geq 0$ on $[O, F]$, since $n_F > 0$; so let $n_O < 0$, whence $w_O > 0$ by (A.115), and $w \geq 0$ on $[O, F]$ if $w_F \geq 0$. When $w_F < 0$, the zero times on the segment $[O, F]$ are

$$(A.120) \quad t_n = \frac{-n_O}{n_F - n_O}, \quad t_w = \frac{w_O}{w_O - w_F},$$

and (A.118) is exactly $t_n \leq t_w$. Thus the segment never enters the open negative quadrant. Every point of the triangle is $\lambda E + (1 - \lambda)y$ with $y \in [O, F]$ and $\lambda \in [0, 1]$; the gap vector (n, w) being affine, its value there is the same convex combination of its values at E and at y . Both coordinates are nonnegative at E ; whichever is nonnegative at y stays nonnegative in the combination. Hence $\max(n, w) \geq 0$ throughout (A.112).

A.9.2. *The part $d \leq 0, b \leq 0$.* The two closed parameter triangles are

$$(A.121) \quad X = \text{conv}\{O = (0, 0), E = (1, -1), W = (V/S, 0)\},$$

$$(A.122) \quad Y = \text{conv}\{C = (1, -1), Q = (1, 0), N = (R/V, 0)\}.$$

They follow from Corollary A.4: writing $b = -c, a = c + x$ and $d = -h, p = h + q$, the active cuts $J_2 \leq 0$ and $K_2 \leq 0$ become

$$(A.123) \quad Vc + Sx \leq V, \quad Rh + Vq \geq R.$$

Use the vertical-edge gap

$$(A.124) \quad G = 2\Delta - \frac{1 + d}{2}\Phi_{NE} - \frac{1 - d}{2}\Phi_{SE}.$$

It is affine in (a, b) for fixed (p, d) , but only quadratic in (p, d) , because the coefficients $(1 \pm d)/2$ depend on d ; so in the second block a Bernstein expansion replaces the vertex values.

Let $(\lambda_C, \lambda_Q, \lambda_N)$ be the barycentric coordinates on Y and write G_Z for the restriction of G to $\{Z\} \times Y$. For each $Z \in \{O, E, W\}$, write its degree-two Bernstein expansion as

$$(A.125) \quad G_Z = \sum_{T \in \{C, Q, N\}} B_{TT}^Z \lambda_T^2 + 2 \sum_{\{T, T'\} \subset \{C, Q, N\}} B_{TT'}^Z \lambda_T \lambda_{T'}.$$

Abbreviate the four long numerators occurring among these coefficients by

$$\begin{aligned} \mathcal{E}_1 &= e + 3f + 1 - ef, & \mathcal{E}_2 &= ef + 3e + 3f^2 + 7f + 2, \\ \mathcal{W}_1 &= e^2 - ef - e + 3f + 2, & \mathcal{W}_2 &= e^2 - ef - e + 2f^2 + 7f + 4. \end{aligned}$$

In the column order CC, QQ, NN, CQ, CN, QN , the exact coefficients are

$$(A.126) \quad \begin{array}{c|cccccc} & CC & QQ & NN & CQ & CN & QN \\ \hline O & 2V & 0 & \frac{e+f}{V} & \frac{V}{2} & \frac{e+2f+1}{2} & \frac{e+f}{2V} \\ E & 4V & 2e+3f+1 & \frac{\mathcal{E}_1}{V} & e+3f+2 & 2V & \frac{\mathcal{E}_2}{2V} \\ W & \frac{4V}{S} & \frac{2V}{S} & \frac{\mathcal{W}_1}{SV} & \frac{(5-e)V}{2S} & \frac{e^2-2e+4f+5}{2S} & \frac{\mathcal{W}_2}{2SV} \end{array}$$

Every coefficient is nonnegative. The only two less immediate numerators are

$$\mathcal{W}_1 = e(e-f) + (2-e) + 3f > 0,$$

$$(A.127) \quad \mathcal{W}_2 = e(e-f) + (4-e) + 2f^2 + 7f > 0.$$

Being affine in (a, b) , the gap G equals $\lambda_O G_O + \lambda_E G_E + \lambda_W G_W$ in the barycentric coordinates of (a, b) on X , and the Bernstein basis functions λ_T^2 and $2\lambda_T \lambda_{T'}$ are nonnegative on Y . Thus $G \geq 0$ on $X \times Y$.

A.9.3. *The part $d \leq 0 \leq b$.* Now the domains are

$$(A.128) \quad X = \text{conv}\{O = (0, 0), B = (V/M, V/M), H = (V/S, 0)\},$$

and the same triangle Y in (A.122). The corner gaps G_{NE}, G_{SE} are bi-affine in the blocks $(a, b) \in X$ and $(p, d) \in Y$, so by Remark A.6 nonnegativity on $X \times Y$ follows from nonnegativity at the nine vertex pairs.

If $e + 2f \leq 1$, the table of G_{SE} , with rows O, B, H and columns C, Q, N , is

$$(A.129) \quad \begin{array}{c|ccc} & C & Q & N \\ \hline O & 2V & 1-e-2f & U \\ B & \frac{4UV}{M} & \frac{2S(1-e-2f)}{M} & \frac{2U^2}{M} \\ H & \frac{4V}{S} & 2-e-f & \frac{2U}{S} \end{array}$$

It is nonnegative, so $G_{SE} \geq 0$.

If $e + 2f \geq 1$, the table of G_{NE} is

$$(A.130) \quad \begin{array}{c|ccc} & C & Q & N \\ \hline O & 2(e+2f-1) & e+2f-1 & \frac{2e+f^2+2f-1}{V} \\ B & \frac{4(R^2+2fU)}{M} & \frac{2(R^2+U(1+3f))}{M} & \frac{2\mathcal{B}}{VM} \\ H & \frac{2(R^2+4f)}{S} & \frac{e^2+ef-e+5f+2}{S} & \frac{2\{e(e-f)+R+f^2+3f\}}{SV} \end{array}$$

where

$$(A.131) \quad \mathcal{B} = 2e^2 + ef^2 - 2ef - 3e - f^3 - 2f^2 + 3f + 2$$

has the exact positive decomposition

$$(A.132) \quad \mathcal{B} = 2 \left(e - \frac{3 + 2f - f^2}{4} \right)^2 + \frac{(1-f)(f^3 + 5f^2 + 19f + 7)}{8} > 0.$$

Also $2e + f^2 + 2f - 1 = (e + 2f - 1) + e + f^2 \geq 0$, and the remaining less immediate entry satisfies

$$e^2 + ef - e + 5f + 2 = (2 - e) + e^2 + ef + 5f > 0.$$

All remaining entries are manifestly positive. Thus $G_{NE} \geq 0$. The two threshold regions cover this final sign piece.

A.10. The 1010 and 1011 cells, $0 \leq f < e < 1$. Both cells have

$$(A.133) \quad J \geq 0, \quad J_2 \geq 0.$$

We show that the northeast corner alone works, using the nine vertex values (A.140). Put

$$(A.134) \quad \delta_1 = e - f, \quad W_1 = 1 + 2e - f.$$

By Corollary A.4, the inequality $J_2 \geq 0$ is

$$(A.135) \quad Sa + \delta_1 b \geq V.$$

Inside the dominance triangle for (a, b) , this is the triangle with vertices

$$(A.136) \quad E = (1, 1), \quad F = (1, -1), \quad Q = (V/W_1, V/W_1).$$

The other inequality in (A.133) is automatic there, since

$$(A.137) \quad Ra + \delta_1 b \leq (R + \delta_1)a = Ua \leq U.$$

Thus both coordinates of X_{NE} are nonnegative and

$$(A.138) \quad G = 2\Delta - (1 - b)(A_0R + B_0U) - (1 - d)(-P_0R + D_0U)$$

is the northeast gap, bi-affine in the blocks (a, b) and (p, d) . The first ranges over the triangle (A.136), the second over the dominance triangle with vertices

$$(A.139) \quad O = (0, 0), \quad M = (1, -1), \quad N = (1, 1).$$

By Remark A.6 the sign of (A.138) on the product of the two triangles is therefore decided at the nine products of (A.136) and (A.139), where it has the table

$$(A.140) \quad \begin{array}{c|ccc} & O & M & N \\ \hline E & 2R & 4R & 4U \\ F & 2(e + f) & 8f & 4(e + f) \\ Q & \frac{2\nu_0}{W_1} & \frac{4\nu_1}{W_1} & \frac{4UV}{W_1} \end{array}$$

where

$$(A.141) \quad \nu_0 = e^2 - ef - e - f^2 + f + 1, \quad \nu_1 = e^2 - 2e - 2f^2 + 2f + 1.$$

Writing $r = e - f$,

$$(A.142) \quad \nu_0 = \left(r - \frac{1-f}{2} \right)^2 + \frac{(1-f)(3+5f)}{4} \geq 0, \quad \nu_1 = (1-e)^2 + 2f(1-f) \geq 0.$$

Hence $G \geq 0$, proving both words.

A.11. Exhaustion and conclusion. The first-quadrant table (A.41) is now covered as follows:

	word	sector	certificate
	0000	$e < f$	Section A.5
	1001	all $e, f \geq 0$	Section A.6
(A.143)	0001	$e < f$	Section A.7
	1000	$e \leq f$	Section A.8
	1000	$f < e$	Section A.9
	1010, 1011	$f < e$	Section A.10

Section A.9 itself covers every sign possibility of b, d : $d \geq 0$, or $d \leq 0$ together with $b \leq 0$ or $b \geq 0$. Section A.7 similarly covers $b \geq 0$, or $b \leq 0$ together with $d \leq 0$ or $d \geq 0$. Those pieces are closed and overlap, so the walls $b = 0$ and $d = 0$ lie in both of the pieces meeting there. The line $e = f$ is included in the 1000 proof of Section A.8 and in the 1001 proof, and a vanishing ambiguous form is included by the weak closed-cell inequalities and continuity. Hence no word, sector, sign subcase, or wall is omitted.

For every opposite two-cycle matrix (A.22), the cell analyses of Sections A.4 to A.10 therefore supply a corner $q \in \{\pm 1\}^2$ with $\Phi_q \leq 2\Delta$. By (A.30),

$$(A.144) \quad \|L^{-T}(q_1, q_2, 1)^T\|_1 \leq 3.$$

Together with the three-cycle, tied, and zero-row branches of Section A.2, this proves (A.3) for every matrix satisfying (A.1)–(A.2). Finally, in the two-cycle branch set $s = (q_1, q_2, 1)$; in the other branches use the displayed sign witness. This completes the proof of Lemma 9.1.

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