

TWENTY POINTS IN THREE-SPACE

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ABSTRACT. We show that every finite subscheme of length twenty in complex affine three-space is a limit of reduced subschemes. Hence the Hilbert scheme of twenty points in $\mathbb{A}^3$ is irreducible, and a reducible Hilbert scheme of points in $\mathbb{A}^3$ has at least twenty-one points. The proof continues the induction of *Nineteen points in three-space*, where a scheme of length d is often linked to a shorter one through a complete intersection of length less than $2d$. Our main tools are two bounds for complete intersections through a scheme whose tangent cone lies on a double plane and whose cubic initial forms share the linear factor defining that plane, so that excess linkage by three cubics does not apply. The first is a computation on the blow-up of the origin, in which each special direction costs according to the first degree that separates it; the second is a weighted Bézout theorem. When the tangent cone lies on no quadric, we choose the higher terms of three equations by varying the inverse system.

1. INTRODUCTION

A finite subscheme of $\mathbb{A}^n$ is *smoothable* if it is a limit of reduced subschemes, that is, of collections of distinct points. The finite subschemes of length d are parametrized by the Hilbert scheme $\text{Hilb}_d(\mathbb{A}^n)$, and the smoothable ones form an irreducible component of dimension nd . Thus $\text{Hilb}_d(\mathbb{A}^n)$ is irreducible exactly when every finite subscheme of $\mathbb{A}^n$ of length d is smoothable.

On a smooth surface this is always the case, by a theorem of Fogarty [4]. In higher dimensions it fails: Iarrobino showed that the Hilbert scheme of d points in $\mathbb{A}^n$, $n \geq 3$, is reducible for large d [8], namely for $d \geq 78$ [9, Example 4.3]. For every n it is irreducible for $d \leq 7$, by Mazzola [13], and when $n \geq 4$ reducibility starts at $d = 8$ [10, 1]. When $n = 3$, the Hilbert scheme of d points is irreducible for $d = 8$ by Cartwright, Erman, Velasco and Viray [1], for $d = 9$ and 10 by Henni and Jardim [6], building on work of Šivic [16], for $d = 11$ by Douvropoulos, Jelisiejew, Nødland and Teitler [3], for $d = 12, 13$ and 14 by [14], and for $15 \leq d \leq 19$ by [15]. So the smallest d for which $\text{Hilb}_d(\mathbb{A}^3)$ is reducible lies between 20 and 78; see [11, Problem V], where the bounds $12 \leq d \leq 78$ are recorded. In this paper we settle the next case.

Theorem 1.1. *Every finite subscheme of $\mathbb{A}^3$ of length at most twenty is smoothable. In particular, the Hilbert scheme $\text{Hilb}_{20}(\mathbb{A}^3)$ is irreducible, of dimension 60.*

Consequently, the smallest d for which $\text{Hilb}_d(\mathbb{A}^3)$ is reducible lies between 21 and 78. The question is local, so Theorem 1.1 holds for any smooth irreducible quasi-projective threefold in place of $\mathbb{A}^3$, see [3, Section 1].

1.2. The proof is by induction on the length, and it uses [15] for all tools and for the lengths up to nineteen. Let $S = \mathbb{C}[x, y, z]$ and $\mathfrak{m} = (x, y, z)$. It suffices to show that every local algebra $A = S/I$ of length twenty with $I \subseteq \mathfrak{m}^2$, supported at the origin, is smoothable [15, Proposition 2.2(3)]. Such an algebra is organized by its Hilbert function $H_A = (1, 3, H_A(2), \dots)$, the Hilbert function of its tangent cone. There are 199 such Hilbert functions, beginning with $(1, 3)$ and summing to twenty. For 166 of them a general result of [14] or [15] applies verbatim; the remaining 33 need new arguments. Tables 1 and 2 in Section 4 form the map of all 199.

1.3. Why is twenty harder than nineteen? The three ways to smooth a scheme Z are those of [15]: find a surface through Z with a rational double point; split off a point along a curve; or link Z , through a complete intersection $Z' \supseteq Z$, to the residual scheme, of length $\ell(Z') - \ell(Z)$. Linkage shortens Z when $\ell(Z') < 2\ell(Z)$. As shorter schemes are smoothable, a scheme of length twenty is smoothable as soon as it lies in a complete intersection of local length at most 39. Staying within this budget is the main difficulty.

Three equations whose cubic leading forms share only finitely many directions contribute 27, plus a contribution of each common direction [15, Theorem 3.12]; this leaves 12 for the common directions. In length nineteen, a global complete intersection of degrees 3, 3, 4, of length 36 by Bézout's theorem, often removes the need to count them [15, Proposition 3.18]. It requires at least three cubics in the tangent cone without a common factor, and an inverse system that leaves room to add a quartic term to one of them. The Hilbert functions left at length twenty are those where these devices fail: there are fewer than three cubics, or the cubics share a linear factor, or their common directions carry a heavy fat point, or the tangent cone lies on a quadric. On a double plane we use equations of orders 2, 3 and 4 instead. Their initial contribution is 24, which leaves 15.

1.4. The first new tool measures how much of this margin is used. Suppose that the tangent cone lies on exactly one quadric, the double plane $x^2 = 0$, that its cubics are exactly the multiples of x , and that the equation of Z with leading form x^2 has no cubic terms in y and z alone. Let p be a nonzero quartic of the tangent cone in y and z alone. Its roots are points of the line $x = 0$ in the plane $\mathbb{P}^2$ of directions. At a root v of p let $\kappa(v) \geq 4$ be the first degree in which the tangent cone has an equation that does not vanish at v . Then three suitable equations of Z , of orders 2, 3 and 4, define a complete intersection of local length at most

$$(1) \quad 24 + \sum_{p(v)=0} (\kappa(v) - 2) \mu_v,$$

where μ_v is the multiplicity of the root (Theorem 3.15). This is the analogue for one quadric of the excess linkage of [15, Theorem 3.12], where three cubics cost 27 and a common direction p costs $(k(p) - 3)e_p$. The proof is a computation on the blow-up of the origin, where the proper transforms of the three surfaces meet only over the roots of p . Over a root v they meet in a scheme of length at most $(\kappa(v) - 2)\mu_v$: the quadric costs two layers of thickness μ_v , and each degree $k \geq 4$ in which every form of degree k of the tangent cone vanishes at v costs one more.

For example, let A have Hilbert function $(1, 3, 5, 4, 3, 3, 1)$ and satisfy the hypotheses above. Then the binary quartics of the tangent cone are the multiples of one binary cubic h (Theorem 6.1). Take $p = hl$ with a general linear form l . At the root of l some binary quartic of the tangent cone does not vanish, so $\kappa = 4$. At a root of h every quartic and quintic of the tangent cone vanishes. If the top form is not a sixth power, some sextic does not, so $\kappa = 6$ at the roots of h , and (1) is $24 + 2 \cdot 1 + 4 \cdot 3 = 38 < 40$, wherever these roots lie (Figure 1). If it is a sixth power, h yields a turning operator instead.

1.5. The second tool is a weighted Bézout theorem. Give x the weight two and y, z the weight one, so that x^2 , the cubics xm with m a binary quadric, and the binary quartics all have weighted degree four. If three equations of Z have weighted initial forms of weighted degree four with no common zero other than the origin, then near the origin they cut out a complete intersection of length at most $4^3/2 = 32$. Suppose that the tangent cone lies on exactly one quadric $x^2 = 0$, that $\mathfrak{m}^6 \subseteq I$, and that the cubics and quartics of the tangent cone are as in Proposition 3.12. Then such equations exist for the general member of a flat one-parameter family through Z that *turns* the equation with leading form x^2 , unless there is a *bad* direction on the line $x = 0$. Under the additional local hypotheses of Corollary B.2(c), a single bad direction can still be handled: the general member of the turning family lies in a complete intersection of local length at most 36.

1.6. The third tool is old but used more systematically: the *free tails* of an inverse system. The tangent cone fixes the leading forms of the equations of Z , and the tails of the inverse system decide their higher terms. Suppose

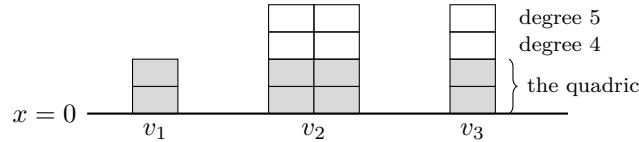


FIGURE 1. The bound (1) for $(1, 3, 5, 4, 3, 3, 1)$ and $p = hl$, where v_1 is the root of l , and h has a double root v_2 and a simple root v_3 . The column over a root v has width μ_v and height $\kappa(v) - 2$. Its 14 cells, added to 24, give 38.

that $H_A(2) = 6$ and that the socle degree is five. For fixed graded data the algebras are parametrized by a vector space of tails, the solutions of the linear equations (4). Moving in this space bends the surfaces through Z without changing their tangent cones, and we bend three of them until they meet in a short complete intersection. Precisely, some member contains three equations with any three independent cubic parts from the tangent cone, any quartic parts that annihilate the top forms of degree five, and any quintic parts (Lemma 3.5). If they cut out a complete intersection of local length less than 40, linkage smooths every algebra with these graded data. For instance, when the cubics of the tangent cone are the multiples of x , and a cubic γ in y, z annihilates the top forms of degree five, the equations $x^3 + y\gamma + z^5$, $xy^2 + z\gamma$, $xz^2 + y\gamma + y^5$ can be placed in the ideal, and they have local length 39 (Proposition 3.6).

1.7. Several steps are exact computer calculations in SageMath, with rational coefficients: local lengths of explicit complete intersections, closedness conditions for inverse systems with all tails as indeterminates, and dimensions of parameter spaces at explicit members. Each is cited where it is used, and Appendix D lists them.

Section 2 recalls the tools of [15], and Section 3 proves the new general results. Section 4 gives the map of length twenty. Sections 5–7 treat the three groups of remaining Hilbert functions, each with one case in full; the other cases are in Appendices A–C. In Sections 5–7 and in the appendices, a statement about local algebras of length d assumes, without further mention, that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable; for $d = 20$ this is the main theorem of [15].

2. TOOLS RECALLED

This section fixes notation and recalls the results of [15] that we use, with their numbers there: inverse systems, families, rays, linkage, and surfaces with rational double points.

2.1. We work over $\mathbb{C}$, with $S = \mathbb{C}[x, y, z]$, $\mathfrak{m} = (x, y, z)$ and S_k the forms of degree k . The ring S acts on $T = \mathbb{C}[X, Y, Z]$ by differentiation, $g \circ F$, and $\langle g, F \rangle = (g \circ F)(0)$. A local algebra $A = S/I$ is supported at the origin; its *inverse system* $M = \{F \in T : I \circ F = 0\}$ is closed under differentiation, of dimension $\ell(A)$, and I is its orthogonal complement. The leading forms of the elements of I span the ideal I^* of the tangent cone, $H_A(k) = \dim S_k/I_k^*$, and the *socle degree* s is the largest k with $H_A(k) \neq 0$. The top forms of degree k of the elements of M form the orthogonal complement $D_k \subseteq T_k$ of I_k^* , the *graded data* of A [15, Lemma 2.5]. A nonzero $P \in T_1$ is a point $p = [P]$ of the plane $\mathbb{P}^2 = \mathbb{P}(T_1)$ of directions, $N_P \subseteq S_1$ is the plane of linear forms vanishing at p , and

$$(2) \quad P^k \in D_k \iff \text{all forms of degree } k \text{ in } I^* \text{ vanish at } p \quad [15, (1)].$$

A form has r *essential variables* if its first derivatives span a space of dimension r ; it is *concise* if $r = 3$ and a *power* if $r = 1$.

When $H_A(2) = 6$, we have $I \subseteq \mathfrak{m}^3$ and $T_{\leq 2} \subseteq M$, and we drop terms of degree at most two. An element of M with top form F is a *lift* of F , and its homogeneous parts below F are its *tails*. In socle degree five,

$$(3) \quad M = T_{\leq 2} \oplus D_3 \oplus \{u + \tau(u) : u \in D_4\} \oplus \{F + R(F) + E(F) : F \in D_5\}$$

with linear maps $\tau : D_4 \rightarrow T_3$, $R : D_5 \rightarrow T_4$, $E : D_5 \rightarrow T_3$; given $S_1 \circ D_5 \subseteq D_4$ and $S_1 \circ D_4 \subseteq D_3$, this space is closed under differentiation if and only if

$$(4) \quad v \circ R(F) - \tau(v \circ F) \in D_3 \quad \text{for all } v \in S_1 \text{ and } F \in D_5,$$

and the cubic tails E are free [15, Lemma E.8 and its proof]. So in socle degree five, for fixed graded data the algebras are parametrized by a vector space of tails.

For a finite subscheme $\Gamma \subset \mathbb{P}^2$, $\nu_k(\Gamma) \subseteq T_k$ is the orthogonal complement of $(I_\Gamma)_k$, spanned by the powers P^k of the points of Γ when Γ is reduced [15, Section 2.13]. If $H_A(k) = H_A(k+1) = n \leq k$, then $D_k = \nu_k(\Gamma)$, $D_{k+1} = \nu_{k+1}(\Gamma)$ and $D_j \subseteq \nu_j(\Gamma)$ for $j \geq k$, for a subscheme Γ of length n , by Gotzmann's persistence theorem [15, Lemma 2.16].

2.2. Smoothability passes to limits [15, Proposition 2.2(1)], and families of inverse systems with polynomial bases of constant dimension give morphisms to the Hilbert scheme [15, Lemma 2.3(2)]. Being Gorenstein [15, Proposition 2.20] and lying in a short complete intersection (Section 2.4) are open conditions, so one member suffices in an irreducible family. Irreducibility is proved by an *expected-dimension count*: if the data are cut out by N equations in an irreducible variety $\mathcal{Q}$, every component has dimension at least $\dim \mathcal{Q} - N$, so smaller strata are not components [15, Lemmas 2.17 and 2.18]; tails solving a linear system of constant rank, equal to the rank of the augmented matrix, form an affine bundle [15, Lemma 5.16].

2.3. A *ray* is a curve along which one point leaves the scheme. Precisely, for polynomial coordinates a, b, c centered at the origin, a *ray* of I is an ideal $L = (b, c)$ with $(I : L) \not\subseteq I + L$. Then $V(I)$ is a flat limit of a reduced point moving along the curve $V(L)$ and a scheme of length $\ell(S/I) - 1$ [15, Theorem 2.11(1)]. The ideal (N_P) is a ray, along the line of p , if $P^k \in M$ but $P^k \notin N_P \circ M$ for some k [15, Theorem 2.11(2)]; and (b, c) is a ray if $a^k b, a^k c \in I$ and some $\Psi \in M$ has $b \circ \Psi = c \circ \Psi = 0 \neq \langle a^k, \Psi \rangle$ [15, Lemma 5.4], a ray along a parabola or a curve of higher order [15, Lemmas 5.11 and 5.13]. The coordinate changes $y \mapsto y + a x^m$, $z \mapsto z + a' x^m$, $m \geq 2$, keep the graded data and remove the terms $X^{s-m}Y$, $X^{s-m}Z$ from the tail of degree $s - m + 1$ of a lift of X^s [15, Section 5.17].

2.4. Linkage replaces a scheme by the residual scheme in a complete intersection. A finite subscheme Z' of an open set $X \subseteq \mathbb{A}^3$ is a *local complete intersection* if its ideal is generated by three elements at each of its points.

Theorem 2.5 (Linkage, [15, Corollary 2.23 and Proposition 2.24], from [12]). *Suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. If a scheme of length d lies, in some open $X \subseteq \mathbb{A}^3$, in a finite local complete intersection of length less than $2d$, then it is smoothable, and so is every member of an irreducible family of schemes of length d that contains it.*

The residual scheme has length less than d , and linkage preserves smoothability [15, Theorem 2.22]. We call such a member a *linkage anchor* of the family. Elements of I whose leading forms of degrees k_1, k_2, k_3 have no common zero in $\mathbb{P}^2$ cut out such a complete intersection of length at most $k_1 k_2 k_3$ [15, Lemma 3.3]. Local lengths are certified by stabilization, $\dim S/(J + \mathfrak{m}^N) = \dim S/(J + \mathfrak{m}^{N+1})$, and a bound on them is an open condition on parameters [15, Section 2.21].

2.6. Suppose $H_A(2) = 6$. The *cubic base scheme* $\mathfrak{B} \subset \mathbb{P}^2$ is defined by the cubics of I_3^* . At a point p of a finite $\mathfrak{B}$, the *multiplicity* e_p is the Hilbert–Samuel multiplicity of the local ideal of $\mathfrak{B}$, the intersection number of two general curves of that ideal; it is the length when the ideal is generated by two elements [15, Section 3.11]. The *separation degree* $k(p)$ is the least k with $4 \leq k \leq s$ and $P^k \notin D_k$, and s if there is none [15, (4)]: the degree in which the surfaces through Z are first separated over the direction p . (The cap at s comes from the deformation of the cubic tail in the proof of Theorem 2.7; the degree $\kappa(v)$ of Section 3.13 is not capped, and can be $s + 1$.)

Theorem 2.7 (Excess linkage, [15, Theorem 3.12]). *Let A be a local algebra of length d with $H_A(2) = 6$ and socle degree $s \geq 4$, and suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. If $\dim I_3^* \geq 3$, $\mathfrak{B}$ is finite, and*

$$27 + \sum_{p \in \mathfrak{B}} (k(p) - 3) e_p < 2d,$$

then A is smoothable.

The proof deforms the cubic tail of the lift of the top forms and measures three surfaces on the blow-up of the origin by the formula of Fulton and Lazarsfeld [5, Theorem 3(A) and its proof]. We use two refinements. First, *free quartic tails* [15, Lemma D.3]: let $s \geq 6$ and $D_s = \mathbb{C}X^s$, and, in coordinates dual to X, Y, Z , let $\mathcal{K}$ be the space of cubics $x \circ R$ with $R \in T_4$ and $y \circ R, z \circ R \in D_3$. If $\dim(\mathcal{K} + D_3) \geq \dim D_3 + 3$, every $k(p)$ in Theorem 2.7 may be replaced by $\min(k(p), s - 1)$. Second, *global linkage* [15, Proposition 3.18]: if $d \geq 19$, $\mathfrak{m}^5 \subseteq I$, $\dim I_3^* \geq 3$ and the cubics of I_3^* have no common factor, then A is smoothable, through a global complete intersection of degrees 3, 3, 4 and length 36; a variant holds in socle degree five [15, Lemma E.8].

2.8. When the tangent cone lies on a quadric, we look for a surface through the scheme with a rational double point. Suppose that $I_2^* = \mathbb{C}x^2$, and let $f \in I$ have leading form x^2 . Its *transverse cubic* γ_f is the cubic part of $f(0, y, z)$. We write $\bar{T}_k = \mathbb{C}[Y, Z]_k$ and $\bar{S}_k = \mathbb{C}[y, z]_k$ for the binary forms on

the line $x = 0$ of $\mathbb{P}^2$, and π for the restriction to the plane $x = 0$. As in [15], we regard $0 = 0^3$ as a cube.

Theorem 2.9 (Surfaces, [14], [15, Theorem 2.9]). *A is smoothable if $H_A(2) \leq 4$, or $H_A(2) = 5$ and $H_A(3) \leq 2$, or the tangent cone lies on a quadric that is not a square, or $I_2^* = \mathbb{C}x^2$ and γ_f is not a cube.*

In these cases $V(I)$ lies on a surface with a rational double point, and Zheng's theorem [17, Theorem 3.3] applies; see also [2]. When the transverse cubics are cubes, they can be *turned*. Let $u \in \mathfrak{m}^3$ with $\mathfrak{m}^2 u \subseteq I$. Then I is the special member of a flat family I_t whose members contain $f + tu$ and $I \cap \mathfrak{m}^4$ [15, Lemma 4.2], and, by its proof, every element of $I \cap \mathfrak{m}^3$ whose cubic part lies in a fixed complement of $x^2 S_1$ in I_3^* . For general t , an equation of type D or E appears in I_t when the jets of u satisfy explicit conditions [15, Theorem 4.5]. We call u a *turning operator*, of type (a) or (b), if

- (a) $u \in (I : \mathfrak{m}^2) \cap \mathfrak{m}^3$ and π of its cubic part is not a cube [15, Corollary 4.6];
- (b) $\mathfrak{m}^7 \subseteq I$, $u \in (I : \mathfrak{m}^2) \cap \mathfrak{m}^3$ and π of its cubic part is not zero [15, Corollary 4.8].

In both cases A is smoothable. An element $u \in \mathfrak{m}^3$ lies in $(I : \mathfrak{m}^2)$ if and only if $u \circ M \subseteq T_{\leq 1}$; in socle degree five, every cubic h with $h \circ D_5 = 0$ does [15, remark before Corollary 4.9]. By type (b), if $\mathfrak{m}^7 \subseteq I$, only the *binary case* $I_3^* \subseteq xS_2$ remains; then $\pi(I_3^*) = 0$, and γ_f does not depend on f . If moreover $H_A(3) = 4$, then $I_3^* = xS_2$ and $D_k \subseteq \bar{T}_k$ for all $k \geq 3$ [15, remark after Corollary 4.8].

3. NEW GENERAL RESULTS

This section proves the general statements used at length twenty. When $H_A(2) = 6$ there are two: short cubic base schemes have small multiplicity (Section 3.1), and free tails prescribe the higher terms of three equations (Section 3.4). When the tangent cone lies on one quadric, a transverse cubic that is a nonzero cube can be turned when $\mathfrak{m}^7 \subseteq I$ (Section 3.8), and two bounds for complete intersections replace excess linkage: a weighted Bézout theorem (Section 3.10) and radial costs on the blow-up (Section 3.13). Finally, a bound for the multiplicity of an ideal containing the equations can replace the choice of the equations themselves (Section 3.16).

3.1. The weights of Theorem 2.7 are at most $s - 3$, so in socle degree five each point of $\mathfrak{B}$ costs at most $2e_p$. For short base schemes the multiplicities exceed the length by at most one.

Lemma 3.2. *Let $\mathfrak{B} \subset \mathbb{P}^2$ be a finite subscheme of length at most five. Then $\sum_{p \in \mathfrak{B}} e_p \leq \ell(\mathfrak{B}) + 1$.*

Proof. Only a point that is not curvilinear can have e_p larger than its length, and for such a point we compare with monomial ideals. At a curvilinear point the local ideal is generated by two elements, so e_p is the local length.

A point that is not curvilinear has local ideal $\mathfrak{b} \subseteq \mathfrak{m}_p^2$ and length at least three, so there is at most one such point p , and it suffices to show $e_p \leq \ell_p + 1$, where $\ell_p = \ell(\mathcal{O}_p/\mathfrak{b}) \leq 5$. For $\ell_p \leq 4$ this is [15, Lemma A.4]: $e_p = 4$ for $\mathfrak{m}_p^2$ and $e_p = 5$ for (u^2, uv, v^3) , in local coordinates u, v . Let $\ell_p = 5$. The associated graded ring of $\mathcal{O}_p/\mathfrak{b}$ has Hilbert function $(1, 2, 2)$ or $(1, 2, 1, 1)$, as $\mathfrak{b} \subseteq \mathfrak{m}_p^2$.

In the first case $\mathfrak{m}_p^3 \subseteq \mathfrak{b} + \mathfrak{m}_p^4$, so $\mathfrak{m}_p^3 \subseteq \mathfrak{b}$ by Nakayama's lemma, and $\mathfrak{b} = (q) + \mathfrak{m}_p^3$ for a quadric q , which is u^2 or uv in suitable coordinates. In the second case let $\mathfrak{b}^*$ be the ideal of leading forms. Its quadrics form a pencil with a common linear factor, since two coprime quadrics define a scheme of length four; so $\mathfrak{b}_2^* = \langle u^2, uv \rangle$, then $\mathfrak{b}_3^* = u\mathfrak{m}_p^2$ and $\mathfrak{b}^* = (u^2, uv, v^4)$. Since $(\mathfrak{b}^n)^* \supseteq (\mathfrak{b}^*)^n$ and $\ell(\mathcal{O}_p/\mathfrak{b}^n) = \ell(\mathcal{O}_p/(\mathfrak{b}^n)^*)$, we have $e(\mathfrak{b}) \leq e(\mathfrak{b}^*)$. It remains to see that the three monomial ideals (u^2, uv^2, v^3) , (uv, u^3, v^3) and (u^2, uv, v^4) have multiplicity at most 6. They contain the parameter ideals (u^2, v^3) , $(uv, u^3 + v^3)$ and $(uv, u^2 + v^4)$, of colengths $2 \cdot 3$, $3 + 3$ and $4 + 2$, and the multiplicity of an ideal is at most that of a parameter ideal it contains, which is its colength [7, Corollary 1.2.5, Propositions 11.2.1 and 11.1.10(2)]. $\square$

Corollary 3.3. *Let A be a local algebra of length $d \geq 20$ with $H_A(2) = 6$ and socle degree five, and suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. If $\dim I_3^* \geq 3$ and the cubic base scheme $\mathfrak{B}$ is finite of length at most five, then A is smoothable.*

Proof. Every weight $k(p) - 3$ is at most 2, so $27 + \sum_p (k(p) - 3)e_p \leq 27 + 2(\ell(\mathfrak{B}) + 1) \leq 39 < 2d$. Theorem 2.7 applies. $\square$

3.4. When $H_A(2) = 6$ and the socle degree is five, changing the tails, subject to (4), changes the higher terms of the equations, and often we can prescribe them.

Lemma 3.5 (Prescribed equations). *Let $A = S/I$ be a local algebra with $H_A(2) = 6$ and socle degree five, with graded data D_3, D_4, D_5 . Let $w_1, w_2, w_3 \in I_3^*$ be linearly independent, let $p_1, p_2, p_3 \in S_4$ satisfy $p_i \circ F = 0$ for all $F \in D_5$, and let $r_1, r_2, r_3 \in S_5$. Then some algebra with the graded data of A has the three elements $f_i = w_i + p_i + r_i$ in its ideal. Consequently, if $V(f_1, f_2, f_3)$ has local length less than $2\ell(A)$ at the origin, and every finite subscheme of $\mathbb{A}^3$ of length less than $\ell(A)$ is smoothable, then every algebra with the graded data of A is smoothable.*

Proof. The independent cubic parts provide three dual cubic tails, and these tails absorb the prescribed quartic and quintic terms. As the w_i are linearly independent, there are $C_1, C_2, C_3 \in T_3$ with $\langle w_i, C_j \rangle = \delta_{ij}$. In (3) put $R = 0$ and

$$\tau(u) = - \sum_j \langle p_j, u \rangle C_j \quad (u \in D_4), \quad E(F) = - \sum_j \langle r_j, F \rangle C_j \quad (F \in D_5).$$

For $v \in S_1$ and $F \in D_5$ we have $\langle p_j, v \circ F \rangle = \langle v, p_j \circ F \rangle = 0$, so $\tau(v \circ F) = 0$, and (4) holds. Each f_i is orthogonal to M : to $D_3 = (I_3^*)^\perp$; to $u + \tau(u)$, as $\langle p_i, u \rangle + \langle w_i, \tau(u) \rangle = 0$; and to $F + E(F)$, as $\langle r_i, F \rangle + \langle w_i, E(F) \rangle = 0$; the other pairings vanish for reasons of degree. So the f_i lie in the ideal, the orthogonal complement of M . For the last assertion, the algebras with the graded data of A are parametrized by a vector space of tails, irreducible and with a morphism to the Hilbert scheme, and Theorem 2.5 applies to it. $\square$

The lemma is most useful when the cubic base scheme is a line, so that the cubics of the tangent cone cannot cut out a finite scheme.

Proposition 3.6. *Let A be a local algebra of length $d \geq 20$ with $H_A(2) = 6$ and socle degree five, and suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. Suppose that $D_3 = \bar{T}_3$ in suitable coordinates, and that some nonzero binary cubic $\gamma \in \bar{S}_3$ satisfies $\gamma \circ D_5 = 0$. Then A is smoothable.*

Proof. We normalize γ and exhibit three equations for Lemma 3.5. If $u \in D_4$, then $x \circ u$ is a binary cubic all of whose derivatives $x \circ (v \circ u)$ vanish, so $x \circ u = 0$; hence $D_4 \subseteq \bar{T}_4$, and likewise $D_5 \subseteq \bar{T}_5$. The group GL_2 acting on y, z preserves $\bar{T}_k$ and $I_3^* = xS_2$, and a nonzero binary cubic has three simple roots, a double and a simple root, or a triple root; so we may assume $\gamma \in \{yz(y+z), y^2z, y^3\}$. Put

$$f_1 = x^3 + y\gamma + z^5, \quad f_2 = xy^2 + z\gamma, \quad f_3 = xz^2 + y\gamma + y^5.$$

Their cubic parts are linearly independent elements of $xS_2 = I_3^*$, and their quartic parts $l\gamma$ satisfy $(l\gamma) \circ F = l \circ (\gamma \circ F) = 0$ for $F \in D_5$. For each of the three cubics γ , the local length of $V(f_1, f_2, f_3)$ at the origin is $39 < 40 \leq 2d$ (script 120_136451.sage, part (P); certified by $\dim S/((f_1, f_2, f_3) + \mathfrak{m}^N) = 39$ for $N = 10, 11$). Lemma 3.5 concludes. $\square$

The hypothesis holds when $\dim D_5 = 1$, because $\gamma \mapsto \gamma \circ F$ maps $\bar{S}_3$ to $\bar{T}_2$, and when $D_5 = \nu_5(\Gamma)$ for a subscheme $\Gamma = V(x, \gamma)$ of the line: then $\langle w, \gamma \circ F \rangle = \langle w\gamma, F \rangle = 0$ for $w \in S_2$.

Remark 3.7 (Adding a cubic). Let M' be the inverse system of an algebra with $H(2) = 6$, let $f_1, f_2, f_3 \in \mathrm{Ann}(M')$, and let $G \in T_3$ be a cubic outside the cubic top forms of M' and orthogonal to the cubic parts of the f_i . Then $M' \oplus \mathbb{C}G$ is the inverse system of an algebra [15, Lemma A.14], and its ideal still contains f_1, f_2, f_3 , since $\langle f_i, G \rangle$ is the pairing of G with the cubic part of f_i . So an anchor of length d in a complete intersection of length e gives an anchor of length $d+1$ in the same complete intersection. We use this for $(1, 3, 6, 6, 2, 2)$, starting from the anchor of [15, Theorem C.10].

3.8. We turn to a tangent cone on one quadric, $I_2^* = \mathbb{C}x^2$, and keep the notation of Section 2.8. A transverse cubic that is a nonzero cube can always be turned in socle degree at most six.

Lemma 3.9. *Let $A = S/I$ be a local algebra with $I_2^* = \mathbb{C}x^2$ and $\mathfrak{m}^7 \subseteq I$. If some $f \in I$ with leading form x^2 has $\gamma_f \neq 0$, then A is smoothable.*

Proof. If γ_f is not a cube, [15, Theorem 2.9(2)] applies; so we may assume $\gamma_f = \alpha y^3$ with $\alpha \neq 0$. The element $u = z^5$ lies in $(I : \mathfrak{m}^2)$, as $\mathfrak{m}^2 u \subseteq \mathfrak{m}^7 \subseteq I$. In [15, Theorem 4.5] the coordinate change fixes y and z , so $u(0, y, z) = z^5$ and $\partial_x u = 0$; hence $G_t = g + tz^5$, with cubic part αy^3 , and with the coefficients of z^4 and yz^3 of g . If one of these is nonzero, case (2) or (3) of the theorem holds for all t . Otherwise the coefficient of z^5 in case (4) is $gz^5 + t$, nonzero for all but one t . So A is smoothable. $\square$

This is the final z^5 -perturbation in the proof of [15, Corollary 4.8], applied directly once the transverse cubic is nonzero. So from now on, in the binary case with $\mathfrak{m}^7 \subseteq I$, we may assume $\gamma_f = 0$.

3.10. Give x the weight two and y, z the weight one. For a nonzero power series g let $\text{in}_w(g)$ be its weighted initial form, the part of lowest weighted degree; the $\text{in}_w(g)$, $g \in I$, span a weighted homogeneous ideal $\text{in}_w(I)$.

Lemma 3.11 (Weighted Bézout). (a) *Let $I \subseteq S$ be an $\mathfrak{m}$ -primary ideal and $F_1, F_2, F_3 \in I$, and suppose that their weighted initial forms h_i , of weighted degrees a_i , have only the origin as common zero in $\mathbb{A}^3$. Then the local length of $V(F_1, F_2, F_3)$ at the origin is at most $a_1 a_2 a_3 / 2$, and $V(I)$ lies in a finite complete intersection of length at most $a_1 a_2 a_3 / 2$ in an open set $X \ni 0$.*
 (b) *If Λ is a space of weighted forms of degree a with only the origin as common zero, three general elements of Λ have only the origin as common zero.*

Proof. (a) Let $J = (F_1, F_2, F_3)$ in $\hat{S} = \mathbb{C}[[x, y, z]]$. As in the proof of [15, Lemma 3.3], the filtration by weighted order induces on $\hat{S}/J$ a filtration whose associated graded ring is $S/\text{in}_w(J)$; it is finite and exhaustive once $\text{in}_w(J)$ has finite colength, which holds because $\text{in}_w(J)$ contains h_1, h_2, h_3 . These form a regular sequence, and $S/(h_1, h_2, h_3)$ has Hilbert series $\prod_i (1 - t^{a_i}) / ((1 - t^2)(1 - t)^2)$, of value $a_1 a_2 a_3 / 2$ at $t = 1$. So $\ell(\hat{S}/J) \leq a_1 a_2 a_3 / 2$, and we take X as in [15, Lemma 3.3]. (b) For general $h_1 \in \Lambda$, a general h_2 vanishes on no component of $V(h_1)$, so $V(h_1, h_2)$ is a finite union of curves stable under the action $t \cdot (x, y, z) = (t^2 x, ty, tz)$, each the closure of one orbit. A general h_3 does not vanish at a chosen point of each orbit, hence on none of them. $\square$

In the binary case $I_3^* = x^2 S_1 \oplus x W_2$ with $W_2 \subseteq \bar{S}_2$, and we assume $\gamma_f = 0$. For $m \in W_2$ choose $K_m \in I$ with cubic part exactly xm , and let $\theta_m \in \bar{S}_4$ be the binary part of its quartic part. Let $J_4 = \pi(I_4^*) \subseteq \bar{S}_4$ be the space of binary parts of the quartic leading forms of I ; then θ_m is well defined modulo J_4 . We call a point v of the line $x = 0$ *bad* if $g(v) = 0$ for all $g \in J_4$ and $m(v) = \theta_m(v) = 0$ for all $m \in W_2$.

Proposition 3.12 (Weighted linkage). *Let $A = S/I$ be a local algebra of length $d \geq 17$ with $I_2^* = \mathbb{C}x^2$ and $\mathfrak{m}^6 \subseteq I$, in the binary case with $\gamma_f = 0$, and suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. If $J_4 \neq 0$ and there is no bad point, then A is smoothable. This holds in particular if $J_4 \neq 0$ and W_2 has no common zero on the line $x = 0$, for instance if $I_3^* = xS_2$ and $H_A(4) \leq 4$.*

Proof. We turn the quadric equation by a fourth power and apply Lemma 3.11 to three equations of weighted degree four. Write $f = x^2 + xa + x^2l + f_4 + \dots$ with $a \in \bar{S}_2$, $l \in S_1$, and let $\beta = \pi(f_4)$. Choose $b \in \bar{S}_1$ vanishing at no point of the finite set $V(J_4)$ of the line. As $\mathfrak{m}^6 \subseteq I$, $u = b^4$ lies in $(I : \mathfrak{m}^2)$, and [15, Lemma 4.2 and its proof], with the complement xW_2 of x^2S_1 , give a flat family I_t with $I_0 = I$ whose members contain $f + tb^4$, the K_m , and $I \cap \mathfrak{m}^4$. Their weighted initial forms of degree four are

$$x^2 + xa + \beta + tb^4, \quad xm + \theta_m \quad (m \in W_2), \quad g \quad (g \in J_4),$$

all other terms having weight at least five; let Λ_t be their span. A common zero $(x_0, v) \neq 0$ of Λ_t has $v \neq 0$, $g(v) = 0$ for all $g \in J_4$, $x_0m(v) + \theta_m(v) = 0$ for all m , and $x_0^2 + x_0a(v) + \beta(v) + tb(v)^4 = 0$. For each of the finitely many points v of $V(J_4)$, the middle conditions have no solution x_0 , or exactly one, or, if v is bad, every x_0 . Without bad points, the last condition fails for all t outside a finite set, since $b(v) \neq 0$. Lemma 3.11 then gives, for general t , a complete intersection of length at most $4^3/2 = 32 < 2d$ containing $V(I_t)$, so I_t is smoothable, and so is I [15, Proposition 2.2(1)]. If W_2 has no common zero on the line, no point is bad. If $I_3^* = xS_2$, then $W_2 = \bar{S}_2$, $D_4 \subseteq \bar{T}_4$, and $J_4 = D_4^\perp \cap \bar{S}_4 \neq 0$ when $H_A(4) \leq 4$. $\square$

When J_4 contains two coprime quartics, the proposition is [15, Theorem 3.6] with $k_1 = k_2 = 4$. At a bad point v , the quadratic equation $x_0^2 + x_0a(v) + \beta(v) + tb(v)^4 = 0$ has two distinct roots for general t , which give two orbits of common zeros of Λ_t . Under the additional hypotheses of Corollary B.2(c), the local contributions on the blow-up give a total length of at most 36.

3.13. Excess linkage [15, Theorem 3.12] measures three surfaces with cubic tangent cones on the blow-up of the origin. On one quadric, three equations have orders 2, 3 and 4, and the analogous measure has weights $\kappa - 2$ instead of $k - 3$. All our local computations on the blow-up reduce to one lemma.

Lemma 3.14 (Lengths on the blow-up). (a) *Let $F_1, F_2, F_3 \in S$ have orders k_1, k_2, k_3 at the origin, and suppose that their tangent cones meet in $\mathbb{P}^2$ only at finitely many points v . If the proper transforms of the surfaces $V(F_i)$ meet over each v with finite local length λ_v , then the origin is an isolated point of $V(F_1, F_2, F_3)$, of local length $k_1k_2k_3 + \sum_v \lambda_v$.*

(b) *Let B be a $\mathbb{C}\{s\}$ -algebra that is free of rank n as a $\mathbb{C}\{s\}$ -module, let $a, c_1, \dots, c_m \geq 0$, and let $h, u_1, \dots, u_m \in B$, where u_1 is a unit.*

Then for $\sigma \in \mathbb{C}^m$ outside a proper Zariski-closed subset, $\ell(B/s^a(h + \sum_j \sigma_j s^{c_j} u_j)B) \leq (a + c_1)n$.

Proof. (a) A curve common to the three surfaces through the origin would give a point of infinite length on the exceptional divisor; so the origin is isolated, and the formula of Fulton and Lazarsfeld [15, (3)] gives the length. (b) As B is free over the discrete valuation ring $\mathbb{C}\{s\}$, the length of B/gB is the order in s of the determinant of multiplication by g , when this determinant is nonzero. So the length in question is $an + \text{ord}_s \det(H_\sigma)$, with $H_\sigma = h + \sum_j \sigma_j s^{c_j} u_j$. The determinant is a polynomial in σ with coefficients in $\mathbb{C}\{s\}$. On the line $\sigma = (\sigma_1, 0, \dots, 0)$,

$$\det(h + \sigma_1 s^{c_1} u_1) = \det(u_1) \cdot \left(\sigma_1^n s^{c_1 n} + \sum_{t < n} \sigma_1^t s^{c_1 t} \gamma_t(s) \right)$$

by the characteristic polynomial of $-u_1^{-1}h$, with $\gamma_t(s) \in \mathbb{C}\{s\}$. In the coefficient of $s^{c_1 n}$, the term σ_1^n appears only with the constant term of $\det(u_1)$, which is nonzero. So the coefficient of $s^{c_1 n}$ in $\det(H_\sigma)$ is a nonzero polynomial in σ , and outside its zero set $\text{ord}_s \det(H_\sigma) \leq c_1 n$. $\square$

For a local algebra in the binary case with $I_3^* = xS_2$ we have $D_k \subseteq \bar{T}_k$ and $I_k^* = xS_{k-1} \oplus J_k$ for $k \geq 4$, where $J_k = D_k^\perp \cap \bar{S}_k$; for $k = 4$ this is the space $J_4 = \pi(I_4^*)$ of Section 3.10. For a point $v = [V]$ of the line $x = 0$ let

$$\kappa(v) = \min\{k \geq 4 : V^k \notin D_k\},$$

and $\kappa(v) = s + 1$ if $V^k \in D_k$ for all $4 \leq k \leq s$. By (2), $\kappa(v)$ is the first degree $k \geq 4$ in which some form of J_k does not vanish at v , or $s + 1$.

Theorem 3.15 (Radial costs). *Let $A = S/I$ be a local algebra of length d with $I_2^* = \mathbb{C}x^2$, $I_3^* = xS_2$, $\gamma_f = 0$ and socle degree s , and let $p \in J_4$ be nonzero. Then $V(I)$ lies, in an open set $X \ni 0$, in a finite complete intersection cut out by three elements of I of orders 2, 3 and 4, of length at most*

$$24 + \sum_{p(v)=0} (\kappa(v) - 2) \mu_v,$$

where μ_v is the multiplicity of the root v . Consequently, if this number is less than $2d$ and every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable, then A is smoothable.

Proof. The cubic equation eliminates the transverse coordinate. Over a root v of p , the quartic equation then leaves a module of rank μ_v , on which the quadric equation is divisible by the square of the radial parameter; an equation that first separates v , in degree $\kappa(v)$, bounds the remaining contribution by $(\kappa(v) - 4)\mu_v$.

Choose $r \in \bar{S}_2$ not vanishing at the roots of p , and $K, P \in I$ with leading forms xr and p . For each root v choose $Q_v \in I$ whose leading form q_v lies in $J_{\kappa(v)}$ and does not vanish at v , or $Q_v = l_0^{s+1}$ with $l_0(v) \neq 0$ if $\kappa(v) = s + 1$; it lies in $\mathfrak{m}^{s+1} \subseteq I$. Let k_v be the order of Q_v , so $k_v = \kappa(v) \geq 4$. Let $f \in I$

have leading form x^2 ; its cubic part lies in xS_2 , as $\gamma_f = 0$. The tangent cones of $G = f + \sum_v \sigma_v Q_v$, K and P are x^2 , xr and p , which meet in $\mathbb{P}^2$ exactly at the roots of p . By Lemma 3.14(a) it suffices to show $\lambda_v \leq (\kappa(v) - 2)\mu_v$ for general σ ; then the origin is an isolated point of $V(G, K, P)$, and we take for X the complement of the other components of $V(G, K, P)$.

Choose y, z with $y(v) \neq 0$ at all roots, so $v = [0 : 1 : c]$, and use the chart $x = \rho u$, $y = \rho$, $z = \rho w$, where ρ is the radial parameter. For g of order k write $g(\rho u, \rho, \rho w) = \rho^k g^\natural(u, w, \rho)$; so $g^\natural(u, w, 0)$ is the leading form of g at $(u, 1, w)$, and $V(g^\natural)$ is the proper transform of $V(g)$. Since $K^\natural = ur(1, w) + \rho(\cdots)$ and $r(1, c) \neq 0$, the implicit function theorem gives $K^\natural = (\text{unit}) \cdot (u - \rho\varphi(w, \rho))$ near $(0, c, 0)$. For g as above let $g'' = g^\natural(\rho\varphi, w, \rho)$, always normalized by the order of g . So the local ring at v is $\mathbb{C}\{w - c, \rho\}/(G'', P'')$. As $P''(w, 0) = p(1, w)$ vanishes to order μ_v at c , the ring $B = \mathbb{C}\{w - c, \rho\}/(P'')$ is free of rank μ_v over $\mathbb{C}\{\rho\}$, by the Weierstrass preparation theorem. Since the cubic part of f is divisible by x , every term of $f^\natural = u^2 + \rho(\cdots)u + \rho^2(\cdots)$ is divisible by ρ^2 after $u = \rho\varphi$, so $f'' = \rho^2\psi$. The term $\sigma_{v'}Q_{v'}$ of G , of order $k_{v'}$, contributes $\sigma_{v'}\rho^{k_{v'}-2}Q_{v'}''$ to G'' . Hence

$$G'' = \rho^2\psi + \sum_{v'} \sigma_{v'} \rho^{k_{v'}-2} Q_{v'}'' = \rho^2 H, \quad H = \psi + \sum_{v'} \sigma_{v'} \rho^{k_{v'}-4} Q_{v'}'',$$

and Q_v'' is a unit of B , as its value at the closed point is $q_v(0, 1, c) \neq 0$. Lemma 3.14(b), with $a = 2$ and $c_1 = k_v - 4 = \kappa(v) - 4$, gives $\lambda_v \leq (2 + \kappa(v) - 4)\mu_v$ for general σ . The last assertion follows from Theorem 2.5. $\square$

The bound is attained, already for $f = x^2$, $K = xr$, $P = p$ and $Q = l_0^k$ (script 120_135_lemmaR.sage). Three further instances of Lemma 3.14 (a turning quartic with a simple zero, two quartics with one common root, and the bad point of Proposition 3.12) are stated and proved as Corollary B.2 in Appendix B.

3.16. To find a short complete intersection we need not choose its equations at once. We first bound the multiplicity of an ideal that contains them; three general elements then realize the bound.

Proposition 3.17 (Linkage by multiplicity). *Let $L \subseteq S$ be an $\mathfrak{m}$ -primary ideal containing elements a, b, c that generate an ideal of finite colength e in the local ring $S_{\mathfrak{m}}$.*

- (a) *If $N \subseteq L$ is a reduction of L , then three general elements of N generate an ideal of colength at most e in $S_{\mathfrak{m}}$.*
- (b) *Let $A = S/I$ be a local algebra of length d , and suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. If L is homogeneous, $I^* \subseteq L$, $L^2 = I^*L$ and $e < 2d$, then A is smoothable.*

Proof. Let $e(\cdot)$ denote the Hilbert–Samuel multiplicity in $S_{\mathfrak{m}}$. As $\ell(S_{\mathfrak{m}}/L^n) \leq \ell(S_{\mathfrak{m}}/(a, b, c)^n)$ for all n , we have $e(L) \leq e((a, b, c)) = e$ [7, Corollary 1.2.5, Propositions 11.2.1 and 11.1.10(2)]. (a) As N is a reduction of L , $e(N) = e(L)$. Three general elements of N generate a reduction J of N [7, Theorem 8.6.6

and Corollary 8.3.9], and $\ell(S_{\mathfrak{m}}/J) = e(J) = e(N) \leq e$ [7, Corollary 1.2.5, Propositions 11.2.1 and 11.1.10(2)]. (b) Since $(I^*)^n \subseteq (I^n)^*$ and $\ell(S_{\mathfrak{m}}/I^n) = \ell(S/(I^n)^*)$, we have $e(I) \leq e(I^*)$, and $e(I^*) = e(L)$, as I^* is a reduction of L . As in (a), three general elements of I generate an ideal J with $\ell(S_{\mathfrak{m}}/J) = e(I) \leq e < 2d$. So $V(I)$ lies in a finite complete intersection of local length less than $2d$, and Theorem 2.5 applies. $\square$

4. THE MAP OF LENGTH TWENTY

This section reduces Theorem 1.1 to 33 Hilbert functions and sorts them by method.

TABLE 1. General results and the number of Hilbert functions of length twenty that they cover. Here $H = H_A$, s is the socle degree, and each Hilbert function is counted in the first row that applies.

Hilbert functions	result	$d = 20$
$H(2) \leq 4$, or $H(2) = 5$ and $H(3) \leq 2$	[15, Thm. 2.9(1)]	110
$H(2) = 5$, $H(3) = 3$	[15, Thm. 4.12]	10
$H(2) = 5$, $s \leq 4$	[15, Cor. 4.7]	3
$H(2) = 5$, $s = 5$, $H(5) = 1$	[15, Cor. 4.9]	3
$H(2) = 5$, $s = 5$, $H(4) = H(5) \leq 3$	[15, Cor. 4.10]	2
$H(2) = 5$, and either $s = 6$ and $(H(5) = H(6) = 1$ or $H(4) = H(5) = 2)$, or $s = 7$, $H(6) = H(7) = 1$ and $(H(5) = 1$ or $H(4) = H(5) = 2)$	[15, Prop. 4.11]	11
$H(2) = 6$, $H(4) = \dots = H(s) = 1$, $s \geq 5$	[15, Prop. 2.12(1)]	8
$s \geq 2k_0$, where k_0 is the largest k with $H(k) > 1$	[15, Prop. 2.12(2)]	10
$H(2) = 6$, $s = 7$, $H(5) = H(6) = H(7) = 1$	[15, Lemma 5.14]	2
$H(2) = 6$, $H(3) = H(4) = H(5) = 2$	[15, Cor. 3.14]	3
$H(2) = 6$, $H(3) = H(4) = 3$, $s = 6$	[15, Cor. 3.15]	2
$(1, 3, 6, 10)$	[15, Thm. G.4]	1
$(1, 3, 6, c, 2, 1, 1)$, $c \geq 2$	[15, Thm. 5.21]	1
individual	Table 2	33
total		199

4.1. There are 199 Hilbert functions of local algebras of length twenty and embedding dimension three, that is, sequences beginning with $(1, 3)$, satisfying Macaulay's growth inequalities and summing to twenty (scripts `enumerate.sage` and `classify.sage`, Appendix D). Table 1 lists the general results of [14] and [15] that apply in length twenty, with the number of Hilbert functions each of them covers. We checked the hypotheses of each row at $d = 20$: they need no induction, or only that schemes of length at most nineteen are smoothable, or, for [15, Corollaries 3.14 and 3.15], excess linkage with a bound of at most 39.

TABLE 2. The Hilbert functions of length twenty not covered by Table 1, the result that treats them (its number gives the section or appendix), and its main method, with at most one secondary method.

Hilbert function	result	methods
(1, 3, 6, 9, 1)	Thm. 5.1	rays
(1, 3, 6, 8, 2)	Thm. 5.1	anchor (36)
(1, 3, 6, 7, 3)	Thm. 5.1	global linkage; expected dimension
(1, 3, 6, 6, 4)	Thm. 5.1	global linkage; expected dimension
(1, 3, 6, 5, 5)	Thm. 5.1	anchor (36); rays
(1, 3, 6, 7, 2, 1)	Thm. A.13	global linkage; cubic reduction
(1, 3, 6, 6, 3, 1)	Thm. A.14	global linkage; expected dimension
(1, 3, 6, 6, 2, 2)	Thm. A.15	anchor (32)
(1, 3, 6, 5, 4, 1)	Thm. A.10	line plus point; prescribed equations
(1, 3, 6, 5, 3, 2)	Thm. A.11	line plus point; prescribed equations
(1, 3, 6, 4, 5, 1)	Thm. A.5	prescribed equations (39)
(1, 3, 6, 4, 4, 2)	Thm. A.7	prescribed equations; anchors (36)
(1, 3, 6, 4, 3, 3)	Thm. A.6	prescribed equations; excess
(1, 3, 5, 6, 3, 2)	Thm. B.8	turning; weighted linkage
(1, 3, 5, 5, 4, 2)	Thm. B.7	weighted linkage
(1, 3, 5, 5, 3, 2, 1)	Thm. B.12	turning; multiplicity
(1, 3, 5, 4, 5, 2)	Thm. B.6	anchor (32)
(1, 3, 5, 4, 4, 3)	Thm. 6.2	weighted linkage
(1, 3, 5, 4, 4, 2, 1)	Thm. B.10	radial costs; turning
(1, 3, 5, 4, 3, 3, 1)	Thm. 6.1	radial costs; turning
(1, 3, 5, 4, 3, 2, 2)	Thm. B.11	radial costs; turning
(1, 3, 5, 4, 3, 2, 1, 1)	Thm. B.16	radial costs; turning
(1, 3, 5, 4, 2, 2, 2, 1)	Thm. B.14	turning (no induction)
(1, 3, 5, 4, 2, 2, 1, 1, 1)	Thm. B.14	turning (no induction)
(1, 3, 6, 3, 3, 2, 1, 1)	Thm. C.2	excess
(1, 3, 6, 4, 4, 1, 1)	Thm. C.3	excess; rays
(1, 3, 6, 3, 2, 2, 2, 1)	Thm. C.4	excess; anchors
(1, 3, 6, 3, 2, 2, 1, 1, 1)	Thm. 7.1	excess; rays
(1, 3, 6, 4, 3, 2, 1)	Thm. C.11	anchors; excess
(1, 3, 6, 4, 2, 2, 2)	Thm. C.8	few bad flags; rays
(1, 3, 6, 4, 2, 2, 1, 1)	Thm. C.9	few bad flags; anchors
(1, 3, 6, 5, 3, 1, 1)	Thm. C.13	excess at one member; rays
(1, 3, 6, 5, 2, 2, 1)	Thm. C.10	few bad flags; rays

Methods. An *anchor* is a member of an irreducible family in a complete intersection of the stated local length (Theorem 2.5); *rays* are dense in an irreducible family (Section 2.3). *Global linkage*: complete intersections of degrees 3, 3, 4 ([15, Proposition 3.18 and Lemma E.8], Lemma A.2); *excess*: Theorem 2.7, with free quartic tails [15, Lemma D.3]; *excess at one member*: Section C.12; *expected dimension*: Section 2.2; *few bad flags*: Lemma C.6; *prescribed equations*: Lemma 3.5; *line plus point*: Section A.8; *cubic reduction*: [15, Lemmas A.14 and A.15]; *turning*: the turning operators of Section 2.8 and [15, Theorem 4.5]; *weighted linkage*: Proposition 3.12 and Corollary B.2(c); *radial costs*: Theorem 3.15 and Corollary B.2; *multiplicity*: Proposition 3.17. “No induction” marks proofs that do not use shorter schemes.

4.2. The remaining 33 Hilbert functions fall into three groups, treated in Sections 5, 6 and 7: 13 with $H(2) = 6$ and socle degree at most five, 11 with $H(2) = 5$, and 9 with $H(2) = 6$ and socle degree at least six. Table 2 lists them with the result that proves them, whose number gives its place, and with its main method and at most one secondary method, explained below the table.

4.3. *Proof of Theorem 1.1.* By [15] and [15, Proposition 2.2(3)], it suffices to treat local algebras of length twenty with $I \subseteq \mathfrak{m}^2$. Their Hilbert functions are covered by Tables 1 and 2. $\square$

5. SOCLE DEGREE AT MOST FIVE

In this section $H_A(2) = 6$ and the socle degree is at most five. Thirteen Hilbert functions of length twenty remain; we prove the five of socle degree four here, in one theorem, and the eight of socle degree five in Appendix A. Socle degree four shows the division of labour cleanly. Three or more cubics without a common factor are linked globally, with no condition at their common directions. When three or four cubics have a common factor, a dimension count shows that these algebras are limits of algebras whose cubics have none. When five cubics have a common factor, it is a line in the cubic base scheme, and an anchor, a ray and a limit handle it. Two cubics are still enough for a complete intersection of degrees 3, 3, 4, but one cubic is not: $3 \cdot 4 \cdot 4 = 48 > 40$, and there a point leaves along a line of the cubic cone.

Theorem 5.1. *Every local algebra of length twenty with $H_A(2) = 6$ and socle degree four is smoothable. These are the Hilbert functions $(1, 3, 6, 9, 1)$, $(1, 3, 6, 8, 2)$, $(1, 3, 6, 7, 3)$, $(1, 3, 6, 6, 4)$ and $(1, 3, 6, 5, 5)$.*

The Hilbert functions are $(1, 3, 6, c, e)$ with $c + e = 10$ and $1 \leq e \leq c^{(3)}$; every $c \leq 4$ is excluded, as $c^{(3)} < 10 - c$ for $c \leq 4$. Let $W = I_3^*$, of dimension $10 - c$. By [15, Lemma 2.7], $M = T_{\leq 2} \oplus D_3 \oplus \{F + \tau(F) : F \in D_4\}$, where $D_3 = W^\perp$, $D_4 \subseteq D'_3 = (S_1 W)^\perp$, and the tail $\tau : D_4 \rightarrow T_3/D_3$ is free. We organize the proof by W .

5.2. *Three or more cubics without a common factor.* This covers $c \leq 7$ when W has no common factor: global linkage [15, Proposition 3.18] applies, as $d = 20 \geq 19$.

5.3. *A common factor, $\dim W \in \{3, 4\}$: not a component.* We need not classify the common-factor locus; it suffices that its dimension is smaller than that of every component of the incidence. For $(1, 3, 6, 7, 3)$, the pairs $(D_3, D_4) \in \text{Gr}(7, T_3) \times \text{Gr}(3, T_4)$ with $S_1 \circ D_4 \subseteq D_3$ form the zero set of a section of a bundle of rank $3 \cdot 3 \cdot 3 = 27$ on a variety of dimension $21 + 36 = 57$, so every component has dimension at least 30 [15, Lemma 2.17]. If W has a common factor, then either $W = lV$ with $l \in S_1$ and $V \in \text{Gr}(3, S_2)$, or $W = qS_1$ with $q \in S_2$. In the first case Macaulay's bound $3^{(2)} = 4$ for $S/(V)$ gives $\dim S_1 V \geq 6$, so $\dim(S_1 W)^\perp = 15 - \dim S_1 V \leq 9$, and the

data (l, V, D_4) have dimension at most $2 + 9 + 18 = 29$. In the second, $\dim S_1 W = 6$ and the data have dimension $5 + 18 = 23$. For $(1, 3, 6, 6, 4)$ the same count gives at least $68 - 48 = 20$ for every component, and a common factor of the four cubics is linear, $W = lV$ with $V \in \text{Gr}(4, S_2)$; Macaulay's bound $2^{(2)} = 2$ gives $\dim S_1 V \in \{8, 9, 10\}$. If $\dim S_1 V = 10$, the data have dimension at most $2 + 8 + 4 = 14$. If $\dim S_1 V = 9$, then $(S_1 V)^\perp = \mathbb{C}G$ with $S_1 \circ G \subseteq V^\perp$, so G has at most two essential variables; if it had two, say X, Y , then $V^\perp = S_1 \circ G$ and $V \supseteq zS_1$, so $\dim S_1 V = 8$. Hence $G = P^3$ and $P^2 \in V^\perp$, and V moves in a family of dimension at most six, and the data have dimension at most $2 + 6 + 8 = 16$. If $\dim S_1 V = 8$, then $V = (I_\Gamma)_2$ for a subscheme Γ of length two [15, Lemma 2.16 with $k = n = 2$]: at most $2 + 4 + 12 = 18$. In both Hilbert functions the common-factor locus has smaller dimension than every component. So the general member of every component has no common factor, and is smoothable by the first step; over each component the tails form a vector bundle, and [15, Proposition 2.2(1)] concludes.

5.4. *A common factor*, $\dim W = 5$. This is $(1, 3, 6, 5, 5)$, where a count is useless: the bundle has rank 75, the dimension of $\text{Gr}(5, T_3) \times \text{Gr}(5, T_4)$. A common factor is linear, as a quadric factor allows only three cubics. In suitable coordinates $W = xV$, where $V = Q^\perp$ is a hyperplane of S_2 for a nonzero quadric $Q \in T_2$. Then

$$D_3 = \{G \in T_3 : x \circ G \in \mathbb{C}Q\} = \bar{T}_3 \oplus \mathbb{C}G_Q,$$

$$D'_3 = \{F \in T_4 : x \circ F \in (S_1 V)^\perp\},$$

where $x \circ G_Q = Q$, and $(S_1 V)^\perp$ consists of the cubics G with $S_1 \circ G \subseteq \mathbb{C}Q$. Such a $G \neq 0$ has one essential variable, $G = P^3$, and then $Q \in \mathbb{C}P^2$. So the cubic base scheme is the line $x = 0$, together with the point $[P]$ if $Q = P^2$ is a square.

(a) *Q is not a square*. Then $D'_3 = \bar{T}_4$, so $D_4 = \bar{T}_4$. Consider all algebras with $D_3 = \bar{T}_3 \oplus \mathbb{C}G$ and $D_4 = \bar{T}_4$, as the line $x = 0$ moves in $\mathbb{P}^2$, $[G]$ moves in $\mathbb{P}(T_3/\bar{T}_3) \cong \mathbb{P}^5$, and τ in $\text{Hom}(\bar{T}_4, T_3/D_3) \cong \mathbb{C}^{25}$: an irreducible family that contains all algebras of type (a). Its member with $G = X^2Y$, $\tau(Y^4) = -4X^3 + 12XZ^2$, $\tau(Z^4) = -4X^3 + 12XY^2$ and $\tau = 0$ on Y^3Z, Y^2Z^2, YZ^3 has

$$x^3 + y^4 + z^4, \quad xy^2 - z^4, \quad xz^2 - y^4$$

in its ideal. The last two define a curve germ, and modulo them the first is $x(x^2 + y^2 + z^2)$; so the local length is the sum of those of the two factors. The factor x gives $\ell(S_m/(x, z^4, y^4)) = 16$. For the other, the tangent cones $x^2 + y^2 + z^2, xy^2, xz^2$ meet only at $[0 : 1 : \pm i]$, and in the chart $x = \rho u, y = \rho, z = \rho w$ the proper transforms $u^2 + 1 + w^2, u - \rho w^4, uw^2 - \rho$ meet transversally at $(0, \pm i, 0)$; so it gives $2 \cdot 3 \cdot 3 + 1 + 1 = 20$ (Lemma 3.14(a)). The complete intersection has local length $36 < 40$ (script `120_r2_anchors.sage`, part (S4)), and Theorem 2.5 covers the family.

(b) $Q = P^2$ with $x(P) \neq 0$. In coordinates with $P = X$ we have $D_3 = \bar{T}_3 \oplus \mathbb{C}X^3$ and $D'_3 = \bar{T}_4 \oplus \mathbb{C}X^4$. So $N_X \circ D_4 \subseteq \langle y, z \rangle \circ D'_3 = \bar{T}_3$ does not contain X^3 , and [15, Lemma 5.2] gives a ray along the line of X , for every D_4 and every tail.

(c) $Q = P^2$ with $x(P) = 0$: the point is embedded in the line. For $t \in \mathbb{C}$ let $P_t = P + tX$ and

$$g_t = \frac{(P + tX)^3 - P^3}{3t} = XP^2 + tX^2P + \frac{t^2}{3}X^3, \quad h_t = \frac{(P + tX)^4 - P^4}{4t},$$

polynomial in t , with $g_0 = XP^2$ and $h_0 = XP^3$. For every t the spaces $D_3(t) = \bar{T}_3 \oplus \mathbb{C}g_t$ and $D'_3(t) = \bar{T}_4 \oplus \mathbb{C}h_t$ have dimensions five and six, $S_1 \circ D'_3(t) \subseteq D_3(t)$, and for $t \neq 0$ they are the spaces of case (b) for the point $[P_t]$, which is off the line. Write $D_4 = \langle c_j h_0 + B_j \rangle_{j \leq 5}$ with $B_j \in \bar{T}_4$, and let $b_j(t) = c_j h_t + B_j$. With fixed representatives $\tau_j \in T_3$ of the tails $\tau(b_j(0))$, the spaces $T_{\leq 2} \oplus D_3(t) \oplus \langle b_j(t) + \tau_j \rangle$ are closed under differentiation [15, Lemma 2.7] and form a family through A near $t = 0$, whose members with $t \neq 0$ have rays by (b). [15, Proposition 2.2(1)] concludes.

5.5. *Two cubics.* This is $(1, 3, 6, 8, 2)$. We adapt the proof of [15, Theorem 3.8, $c = 8$], which treats $(1, 3, 6, 8, 1)$, replacing the single quartic top form by a pencil. The pairs (D_3, D_4) form the zero set $\mathcal{J}$ of a section of a bundle of rank $3 \cdot 2 \cdot 2 = 12$ on $\text{Gr}(8, T_3) \times \text{Gr}(2, T_4)$, of dimension 42, so every component of $\mathcal{J}$ has dimension at least 30. If the pencil W has no common factor, its members have no linear syzygy, $\dim S_1 W = 6$, and D_4 moves in $\text{Gr}(2, 9)$: an irreducible stratum of dimension $16 + 14 = 30$. If $W = l\langle a, b \rangle$ with coprime quadrics a, b , the stratum has dimension at most $2 + 8 + 14 = 24$, and if $W = q\langle l_1, l_2 \rangle$, at most $5 + 2 + 16 = 23$. So $\mathcal{J}$ is irreducible [15, Lemma 2.18], and with the tails, a vector bundle of rank four, the algebras form one irreducible family. Its member

$$M_0 = T_{\leq 2} \oplus \langle \text{the cubic monomials other than } X^3, Y^3 \rangle \oplus \langle X^2 Y^2, X^2 Y Z \rangle$$

is closed under differentiation, with Hilbert function $(1, 3, 6, 8, 2)$, and x^3, y^3, z^4 lie in its annihilator. The complete intersection $V(x^3, y^3, z^4)$ has length $36 < 40$, and the residual scheme has length 16 (script `120_socle4.sage`, part (B)). Theorem 2.5 concludes.

5.6. *One cubic.* This is $(1, 3, 6, 9, 1)$, the most instructive case. The tangent cone lies on exactly one cubic, $W = \mathbb{C}w$, so $D_3 = w^\perp$ is a hyperplane of T_3 and $D_4 = \mathbb{C}F$. Since $\langle w, v \circ F \rangle = \langle v, w \circ F \rangle$, the condition $S_1 \circ F \subseteq D_3$ says $w \circ F = 0$. By (2), D_3 contains the cube P^3 of every point p of the plane cubic curve $V(w)$. As $\dim I_3^* = 1 < 3$ and $\mathfrak{B} = V(w)$ is a curve, neither linkage by three cubics nor excess linkage applies, and a complete intersection inside I has degrees at least 3, 4, 4. Instead, a point leaves along a line of the cubic cone.

Let $\mathcal{Y}$ be the set of tuples (w, F, G, p) with $w \in S_3$ and $F \in T_4$ nonzero, $G \in T_3$, $p \in \mathbb{P}^2$, $w \circ F = 0$ and $w(p) = 0$. The pairs (w, p) form the

complement of the zero section in a vector bundle of rank nine over $\mathbb{P}^2$. For fixed w the map $T_4 \rightarrow T_1$, $F \mapsto w \circ F$, is onto, as its transpose $v \mapsto vw$ is injective; so F moves in a space of dimension twelve. Hence $\mathcal{Y}$ is irreducible, and the algebras with inverse system $T_{\leq 2} \oplus w^\perp \oplus \mathbb{C}(F + G)$ form a family over it [15, Lemma 2.3(2)], which contains every algebra in question, as every plane cubic has a zero. Consider the condition

(5) the map $N_P \oplus \mathbb{C} \rightarrow T_3$, $(v, \lambda) \mapsto v \circ F - \lambda P^3$, is injective.

It is open on $\mathcal{Y}$. If it holds, the line of p is a ray. Indeed, $P^3 \in D_3 \subseteq M$, while the elements of $N_P \circ M$ are combinations of elements of $N_P \circ T_{\leq 2} \subseteq T_{\leq 1}$, of $N_P \circ D_3 \subseteq T_2$, and of $v \circ F + v \circ G$ with $v \in N_P$; so their cubic parts lie in $N_P \circ F$, which does not contain P^3 by (5), and [15, Theorem 2.11(2)] gives the ray. The condition is not empty: for $w = x^3$, $F = Y^4 + Z^4 + X^2YZ$ and $p = [0 : 1 : 0]$, we have $w \circ F = 0$, $w(p) = 0$, $N_P = \langle x, z \rangle$, and the cubics $x \circ F = 2XYZ$, $z \circ F = 4Z^3 + X^2Y$ and Y^3 are linearly independent. So the algebras with a ray are dense in an irreducible family that contains all algebras with Hilbert function $(1, 3, 6, 9, 1)$, and [15, Proposition 2.2(1)] concludes the proof of Theorem 5.1. $\square$

6. ONE QUADRIC IN THE TANGENT CONE

In this section $H_A(2) = 5$. Eleven Hilbert functions of length twenty remain; we prove $(1, 3, 5, 4, 3, 3, 1)$ and $(1, 3, 5, 4, 4, 3)$ here, and the others in Appendix B. By Theorem 2.9 we may assume $I_2^* = \mathbb{C}x^2$. When $\mathfrak{m}^7 \subseteq I$, that is, in socle degree at most six, turning operators of type (b) (Section 2.8) leave only the binary case $I_3^* \subseteq xS_2$, and Lemma 3.9 leaves only $\gamma_f = 0$. In this situation the three equations $x^2 + \dots$, $xr + \dots$ and $p + \dots$, with r and p binary, are the natural candidates for a complete intersection. Their orders 2, 3, 4 contribute 24 before the excess over their common tangent directions is counted, and the question is how much the roots of p on the line $x = 0$ cost. Weighted linkage (Proposition 3.12) settles the cases where this cost is invisible in weight four; radial costs (Theorem 3.15) measure it in general. Where neither suffices, the tails of the top forms point to a turning operator with a binary cubic part.

Theorem 6.1. *Every local algebra with Hilbert function $(1, 3, 5, 4, 3, 3, 1)$ is smoothable.*

Proof. We first show that the binary quartics and quintics of the tangent cone are the multiples of one binary cubic h . If the top form is not a sixth power, radial costs give a complete intersection of local length at most 38; if it is, h itself, corrected by a quartic, is a turning operator.

Reduction. As $\dim I_3^* = 6 = \dim xS_2$, the binary case is $I_3^* = xS_2$; so, as above, we may assume $I_3^* = xS_2$ and $\gamma_f = 0$. Then $D_k \subseteq \bar{T}_k$ for $k \geq 3$, the spaces $U = D_4$ and $V = D_5$ have dimension three, and $D_6 = \mathbb{C}F$. Let $J_k = D_k^\perp \cap \bar{S}_k$, so that $I_k^* = xS_{k-1} \oplus J_k$.

A common cubic factor. The space J_4 is a pencil $\langle p, q \rangle$, and J_5 has dimension three. Since $S_1 \circ V \subseteq U$, we have $\bar{S}_1 J_4 \subseteq J_5$, so the four products yp, zp, yq, zq are linearly dependent: $l_1 p = l_2 q$ for some $l_1, l_2 \in \bar{S}_1$, not both zero. Let g be the greatest common divisor of p and q , and $p = gp', q = gq'$. Then $l_1 p' = l_2 q'$ with p', q' coprime, so q' divides l_1 , and g has degree at least three; it has degree three, as p and q are independent. So $J_4 = h\bar{S}_1$ for a binary cubic h , and $J_5 = h\bar{S}_2$ by dimensions. Hence U and V are the binary quartics and quintics killed by h , and $h \circ F = 0$, because the first derivatives of $h \circ F \in \bar{T}_3$ are $h \circ (S_1 \circ F) \subseteq h \circ V = 0$.

F is not a sixth power: radial costs. Let $p = hl$ with $l \in \bar{S}_1$ general, so that its root v_l is not a root of h . At v_l , some $hl' \in J_4$ does not vanish, so $\kappa(v_l) = 4$. At a root $v = [V]$ of h , all forms of J_4 and J_5 vanish, so $V^4 \in D_4$ and $V^5 \in D_5$, while $V^6 \notin D_6 = \mathbb{C}F$, as F is not a power; so $\kappa(v) = 6$. The roots of h have multiplicities adding up to three, so

$$24 + (4 - 2) \cdot 1 + (6 - 2) \cdot 3 = 38 < 40,$$

for every root pattern of h , and Theorem 3.15 applies.

$F = Y^6$: a turning operator. Now $h \circ Y^6 = 120 h(Y)Y^3 = 0$, so h vanishes at $[Y]$; choose z with $z(Y) = 0$. Let $\Phi = Y^6 + F_5 + F_4 + \dots$ be the lift of Y^6 . As $x \circ Y^6 = z \circ Y^6 = 0$, the elements $x \circ \Phi$ and $z \circ \Phi$ of M have degree at most four, so their quartic parts $x \circ F_5$ and $z \circ F_5$ lie in $U \subseteq \bar{T}_4$. The first gives $F_5 = XG_4 + B_5$ with $G_4 \in \bar{T}_4$, $B_5 \in \bar{T}_5$, and the second $z \circ G_4 = 0$ and $z \circ B_5 \in U$. So $G_4 = \alpha Y^4$, and $z \circ (h \circ B_5) = h \circ (z \circ B_5) = 0$, so $h \circ B_5 = cY^2$ for some $c \in \mathbb{C}$. Put

$$u = h - \frac{c}{360} y^4.$$

The cubic part of $u \circ \Phi$ is $h \circ Y^6 = 0$, and its quadratic part is $\alpha X h \circ Y^4 + h \circ B_5 - cY^2 = 0$, as $h \circ Y^4 = 24h(Y)Y = 0$. It remains to check u on $M \cap T_{\leq 5}$, since $M = \mathbb{C}\Phi \oplus (M \cap T_{\leq 5})$: for an element of degree five, the part of degree two of its image under u is h applied to its top form in V , hence zero, and every element of degree at most four maps into $T_{\leq 1}$, as u has order three. Hence $u \in (I : \mathfrak{m}^2) \cap \mathfrak{m}^3$, and its cubic part h does not vanish on the plane $x = 0$: u is a turning operator of type (b), and A is smoothable. (Script 120_135_s6.sage, part 331, checks the identity with all tails as indeterminates.) $\square$

The next case is a direct application of weighted linkage.

Theorem 6.2. *Every local algebra with Hilbert function $(1, 3, 5, 4, 4, 3)$ is smoothable.*

Proof. As in Theorem 6.1, we may assume $I_3^* = xS_2$ and $\gamma_f = 0$. As $\mathfrak{m}^6 \subseteq I$ and $H_A(4) = 4$, Proposition 3.12 applies: there is no bad point, and for general t the turned ideal I_t lies in a complete intersection of local length at most $4^3/2 = 32 < 40$. $\square$

7. SOCLE DEGREE AT LEAST SIX

In this section $H_A(2) = 6$ and the socle degree is at least six. Nine Hilbert functions of length twenty remain. In most of them a plateau of the Hilbert function, or the space of cubics, makes the graded data those of a scheme Γ of length two, three or four in the plane, and the top form is often a power X^s . Excess linkage now has weights up to $s - 3$, and the work is to keep the weighted multiplicity of the cubic base scheme at most 12. For $(1, 3, 6, 3, 3, 2, 2)$, which is in Table 1, this is immediate: by [15, Lemma 2.16], $D_3 = \nu_3(\Gamma)$ and $D_4 = \nu_4(\Gamma)$ for a subscheme Γ of length three, which is the cubic base scheme, and [15, Corollary 3.15] gives $27 + 4(s - 3) = 39 < 40$. In the other cases the cubic base scheme can be heavier, and three devices reduce its weight: free quartic tails [15, Lemma D.3], rays along straight lines or curves, and irreducible families whose general member has a lighter base scheme. The following case shows the last two.

Theorem 7.1. *Every local algebra with Hilbert function $(1, 3, 6, 3, 2, 2, 1, 1, 1)$ is smoothable.*

Proof. We normalize the lift of the top form by coordinate changes. Then either a point leaves along the line of the top form, or the algebras form two explicit irreducible families, and a general member of each is linked by excess linkage.

Setup. By [15, Lemma 2.16] with $k = 4$ and $n = 2$, $D_4 = \nu_4(\Gamma)$ and $D_5 = \nu_5(\Gamma)$ for a subscheme $\Gamma \subset \mathbb{P}^2$ of length two. The form spanning D_8 has all its first derivatives in the line D_7 , so it is an eighth power; choose coordinates with $D_8 = \mathbb{C}X^8$, so $D_7 = \mathbb{C}X^7$ and $D_6 = \mathbb{C}X^6$. As $X^5 \in \nu_5(\Gamma)$, the point $[X]$ lies in Γ , and in coordinates fixing X either $\Gamma = \{X, Y\}$ and $D_k = \langle X^k, Y^k \rangle$, or Γ is the double point $\Gamma_0 = V(z, y^2)$ and $D_k = \langle X^k, X^{k-1}Y \rangle$, for $k = 4, 5$. Moreover $D_3 = \nu_3(\Gamma) \oplus \mathbb{C}G$ for a cubic G . Let $\Phi = X^8 + A_7 + B_6 + C_5 + H_4 + J_3$ be the lift of X^8 . As $M \cap T_{\leq 5}$ has dimension 17, $M = (M \cap T_{\leq 5}) \oplus \langle \Phi, x \circ \Phi, x^2 \circ \Phi \rangle$.

Normalization. The elements $y \circ \Phi$ and $z \circ \Phi$ of M have degree at most six, and their parts of degree six lie in $D_6 = \mathbb{C}X^6$; so $A_7 = \alpha X^7 + \alpha_1 X^6 Y + \alpha_2 X^6 Z$. Replacing Φ by $\Phi - \frac{\alpha}{8} x \circ \Phi$, and changing coordinates by $y \mapsto y + \lambda_1 x^2$, $z \mapsto z + \lambda_2 x^2$ [15, Section 5.17], we may assume $A_7 = 0$. Then $y \circ \Phi$ and $z \circ \Phi$ have degree at most five, so their quintic parts $y \circ B_6$ and $z \circ B_6$ lie in D_5 . This puts B_6 in $\langle X^6, X^5 Y, X^5 Z, Y^6 \rangle$ if $\Gamma = \{X, Y\}$, and in $\langle X^6, X^5 Y, X^5 Z, X^4 Y^2 \rangle$ if $\Gamma = \Gamma_0$ (script 120_136322111.sage, part (A)). Replacing Φ by $\Phi - \frac{\alpha'}{56} x^2 \circ \Phi$, where α' is the coefficient of X^6 in B_6 , and changing coordinates by $y \mapsto y + \lambda_3 x^3$, $z \mapsto z + \lambda_4 x^3$, which does not affect the degrees seven and eight, we reach $B_6 = \delta Y^6$, respectively $B_6 = \delta X^4 Y^2$.

A straight ray. Suppose that $x^3 \circ B_6 = 0$; this holds if $\Gamma = \{X, Y\}$, and if $\Gamma = \Gamma_0$ and $\delta = 0$. Then $x^3 \circ \Phi / 336$ equals X^5 modulo $T_{\leq 2}$, so $X^5 \in M$. The operators $x^5 y$ and $x^5 z$ have order six, so they kill $M \cap T_{\leq 5}$; they kill Φ , $x \circ \Phi$ and $x^2 \circ \Phi$, as $A_7 = 0$, X^8 has no term with Y or Z , and B_6 has no term

X^5Y, X^5Z . So $x^5y, x^5z \in I$, while $y \circ X^5 = z \circ X^5 = 0$ and $\langle x^5, X^5 \rangle \neq 0$. By [15, Lemma 5.4] with $k = 5$, the ideal (y, z) is a ray: a point leaves along the line of X .

Two families. Let $\Gamma = \Gamma_0$ and $\delta \neq 0$. Then $y \circ \Phi$ has top form $2\delta X^4Y$, and all of $D_4, \dots, D_8$ consist of top forms of derivatives of Φ . Replacing Φ by Φ minus a multiple of $x^3 \circ \Phi/336 = X^5 + \frac{\delta}{14}XY^2 + \dots$, and changing coordinates by $y \mapsto y + \lambda_5x^4, z \mapsto z + \lambda_6x^4$, we remove the terms X^5, X^4Y, X^4Z from C_5 . The derivatives

$$\begin{aligned} \Phi, \quad x \circ \Phi/8, \quad x^2 \circ \Phi/56, \quad x^3 \circ \Phi/336, \\ y \circ \Phi/(2\delta), \quad x^4 \circ \Phi/1680, \quad xy \circ \Phi/(8\delta), \end{aligned}$$

with top forms $X^8, X^7, X^6, X^5, X^4Y, X^4, X^3Y$, together with $T_{\leq 2}, E_0 = \langle X^3, X^2Y \rangle$ and G , span a space of dimension 20 contained in M , hence equal to M . So an algebra with these data exists if and only if this space is closed under differentiation. We reduce every first derivative of every element of the list, with all coefficients of C_5, H_4, J_3 as indeterminates over $\mathbb{Q}(\delta)$ (scripts 120_136322111.sage, part (C), and 120_136322111_rho.sage). The remainders of degree at least four vanish exactly when $z \circ C_5 \in \langle X^4, X^3Y \rangle$; with the normalization already imposed this is $C_5 = c_1X^3YZ + c_2X^3Y^2 + c_3X^2Y^3 + c_4XY^4 + c_5Y^5$. After imposing this condition, all remaining remainders vanish except those of $y^2 \circ \Phi/(2\delta)$ and $z \circ \Phi$ (reduced by $x^4 \circ \Phi/1680$ and $\frac{c_1}{8\delta}xy \circ \Phi$, respectively). Modulo E_0 , these are

$$\rho_1 = \frac{6c_4XY^2 + 10c_5Y^3}{\delta}, \quad \rho_2 = z \circ H_4 - \frac{c_1}{8\delta}xy \circ C_5,$$

and the space is closed if and only if $\rho_1, \rho_2 \in D_3$. As $z \circ: T_4 \rightarrow T_3$ is onto with a kernel of dimension five, for given δ, C_5 and D_3 the admissible H_4 form an affine space of dimension eight, and J_3 is free. So the algebras form two irreducible families [15, Lemma 5.16]:

- (U_0) $\rho_1 = 0$, that is, $c_4 = c_5 = 0$, and $D_3/E_0 = \mathbb{C}G$ arbitrary: a bundle over $\mathbb{C}^* \times \mathbb{C}^3 \times \mathbb{P}(T_3/E_0)$;
- (U_1) $\rho_1 \neq 0$, and then $D_3 = E_0 \oplus \mathbb{C}\rho_1$ is forced: a bundle over $\mathbb{C}^* \times \mathbb{C}^3 \times (\mathbb{C}^2 \setminus 0)$.

Excess linkage at a general member. As $\dim I_3^* = 7 > 6 = \dim lS_2$ for every $l \in S_1$, the cubics of I_3^* have no common factor, and the cubic base scheme $\mathfrak{B}$ is finite. It contains Γ_0 , and it has length at most three, as $\nu_3(\mathfrak{B}) \subseteq D_3$ has dimension at least $\min(\ell(\mathfrak{B}), 4)$. At $[X]$ the separation degree is 8, as $X^k \in D_k$ for $4 \leq k \leq 8$ (the capped value of Section 2.6: every graded equation through degree 8 vanishes at $[X]$, and the separation used by Theorem 2.7 comes from varying the cubic tail), and $e_{[X]} = 2$ when $\mathfrak{B}$ has length two at $[X]$. At a point $p = [P] \neq [X]$ we have $P^4 \notin D_4$, so $k(p) = 4$. In (U_0), $\mathfrak{B}$ has length three at $[X]$ only if D_3 contains ν_3 of a subscheme of length three at $[X]$ containing Γ_0 , that is, X^2T_1 or $\langle X^3, X^2Y, XY^2 + \xi X^2Z \rangle$ for some $\xi \in \mathbb{C}$; this means $G \in \langle XY^2, X^2Z \rangle$ modulo E_0 , a proper closed condition.

In (U_1) , $D_3 = \langle X^3, X^2Y, 3c_4XY^2 + 5c_5Y^3 \rangle$, so $I_3^* = zS_2 \oplus \mathbb{C}y^2(5c_5x - c_4y)$, and $\mathfrak{B} = V(z, y^2(5c_5x - c_4y))$ is Γ_0 together with the point $[c_4 : 5c_5 : 0]$ when $c_5 \neq 0$. So a general member of each family has a cubic base scheme consisting of Γ_0 and at most one further reduced point, and

$$27 + (8 - 3) \cdot 2 + (4 - 3) \cdot 1 = 38 < 40.$$

Theorem 2.7 shows that these members are smoothable; so are all members of the two irreducible families [15, Proposition 2.2(1)]. (Script 120_136322111.sage, part (D).) Undoing the coordinate changes, every algebra with this Hilbert function is smoothable. $\square$

A member of (U_1) with $c_5 = 0$, for example $M = T_{\leq 2} + S \circ \Phi$ with $\Phi = X^8 + X^4Y^2 + XY^4$, has the curvilinear cubic base scheme $V(z, y^3)$, where Theorem 2.7 only gives $27 + 15 = 42$. Its ideal contains x^2z , $q = y^3 - \frac{1}{2}x^3y$ and $r = z^3 + x^2y^2 - \frac{1}{840}x^6$. On the curve germ $R = \mathbb{C}[[x, y, z]]/(q, r)$ the functions x and z are parameters, so the local length of $V(x^2z, q, r)$ is $2\ell(R/x) + \ell(R/z)$. Here $\ell(R/x) = \ell(\mathbb{C}[[y, z]]/(y^3, z^3)) = 9$, and $\ell(R/z)$ is the intersection number of $y(y^2 - \frac{1}{2}x^3)$ and $x^2(y^2 - \frac{1}{840}x^4)$, which is $2 + 4 + 4 + 6 = 16$, as the difference of the two quadrics in y is $x^3(\frac{1}{2} - \frac{x}{840})$. So the complete intersection has local length $2 \cdot 9 + 16 = 34$ (script 120_r2_anchors.sage, part (U1)), and this member is linked directly, although Theorem 2.7 does not reach it.

APPENDIX A. SOCLE DEGREE FIVE

A.1. This appendix proves the eight Hilbert functions of Section 5 of socle degree five. Here $H_A(2) = 6$, the socle degree is five, $\mathfrak{m}^6 \subseteq I$, and the inverse system is written as in (3), with the closedness conditions (4). As everywhere in the appendices, the length is twenty and all shorter schemes are smoothable.

Two further lemmas are used only in this appendix. First, with a single top form, the condition of [15, Lemma E.8] for global linkage can be read off from the cubics of the tangent cone and the essential variables of the top form. For a concise top form it says that three general cubics of I_3^* have no linear syzygy [15, remark after Lemma E.8]. For a top form in two variables:

Lemma A.2. *Let A be a local algebra of length $d \geq 19$ with $H_A(2) = 6$ and socle degree five, and suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable. Let $D_5 = \mathbb{C}F$ with $F \in \mathbb{C}[X, Y]_5$ not a fifth power, and $W = I_3^*$. Suppose that two general cubics of W have no common factor, and that for three general cubics $w_1, w_2, w_3 \in W$ the six quartics xw_i, yw_i are linearly independent modulo zW . Then every algebra with the graded data of A is smoothable. If $\dim W = 3$, the second condition says that the nine quartics vw_i , $v \in \{x, y, z\}$, are linearly independent.*

Proof. By (4) with $v = z$, $z \circ R(F) \in D_3$, that is, $R(F) \in (zW)^\perp$; with $v = x, y$ it determines τ on $S_1 \circ F = \langle x \circ F, y \circ F \rangle$ by $\tau(v \circ F) \equiv v \circ R(F)$ modulo D_3 , and τ is free on a complement of $S_1 \circ F$ in D_4 . The map

Θ of [15, Lemma E.8] sends (τ, R) to the values $\langle w_i, \tau(u) \rangle$, $u \in D_4$; on $u = v \circ F$ these are $\langle w_i, v \circ R(F) \rangle = \langle vw_i, R(F) \rangle$. So Θ is onto if and only if $R \mapsto (\langle xw_i, R \rangle, \langle yw_i, R \rangle)_i$ is onto from $(zW)^\perp$, that is, if the six quartics are independent modulo zW . [15, Lemma E.8] concludes. $\square$

Both conditions are open on the graded data, so in an irreducible family of graded data with tails of constant dimension one example suffices.

Second, when the cubic base scheme is a line together with a point off the line, the point carries a ray, whatever the tails.

Lemma A.3. *Let A be a local algebra with $H_A(2) = 6$ and socle degree five, and suppose that $D_3 = \bar{T}_3 \oplus \mathbb{C}X^3$ in suitable coordinates. Then I has a ray along the line of X or along a parabola tangent to it. In particular, A is smoothable if all finite subschemes of length $\ell(A) - 1$ are.*

Proof. We show that x^3y and x^3z annihilate M , or a hyperplane of it, and apply the ray criteria of [15]. The cubic base scheme is $\mathfrak{B} = V(xy, xz)$, the line $x = 0$ with the point $[X]$, and its ideal is generated by quadrics; so $D_4 \subseteq \nu_4(\mathfrak{B}) = \bar{T}_4 \oplus \mathbb{C}X^4$ and $D_5 \subseteq \nu_5(\mathfrak{B}) = \bar{T}_5 \oplus \mathbb{C}X^5$, because $\langle ql, u \rangle = \langle q, l \circ u \rangle = 0$ for $q \in (I_{\mathfrak{B}})_{k-1}$ and $u \in D_k$. In the notation of (3), the quartic part of an element m of M is $u + R(F)$ with $u \in D_4$ and $F = G + cX^5 \in D_5$ its quintic part, $G \in \bar{T}_5$, $c = \langle x^5, m \rangle / 120$. Since u has no term X^3Y , $\langle x^3y, m \rangle = \langle x^2y, x \circ R(F) \rangle$, and x^2y annihilates D_3 . If $x \circ D_5 = 0$, then (4) gives $x \circ R(F) \in D_3$ for all F , so x^3y and, likewise, x^3z lie in I ; as $X^3 \in M$ has $y \circ X^3 = z \circ X^3 = 0 \neq \langle x^3, X^3 \rangle$, the ideal (y, z) is a ray [15, Lemma 5.4]. Otherwise some $F \in D_5$ has $c \neq 0$, and $X^4 = x \circ F / (5c) \in D_4$. We apply [15, Lemma 5.13] with $k = 3$: let M^0 be the space of $m \in M$ with $\langle x^5, m \rangle = 0$. We show that $\langle x^3y, \cdot \rangle$ vanishes on M^0 . By (4), $x \circ R(F) \equiv \tau(x \circ F) = 5c\tau(X^4)$ modulo D_3 . So $\langle x^3y, m \rangle = 5c\langle x^2y, \tau(X^4) \rangle$ vanishes on M^0 ; likewise $\langle x^3z, \cdot \rangle$. By [15, Lemma 5.13], whose condition $E_3(\Lambda) \in M$ always holds for $k = 3$, the ideal $(y - \beta x^2, z - \gamma x^2)$ is a ray for some $\beta, \gamma \in \mathbb{C}$. [15, Theorem 2.11(1)] gives the last assertion. $\square$

A.4. We begin with the three Hilbert functions with $H_A(3) = 4$, where the cubic base scheme is short or a line, and D_3 is described by [15, Lemma A.11].

Theorem A.5. *Every local algebra with Hilbert function $(1, 3, 6, 4, 5, 1)$ is smoothable.*

Proof. Let K be the ideal generated by $D_3^\perp$. Then $H_{S/K}(3) = 4$ and $H_{S/K}(4) = \dim D'_3 \geq \dim D_4 = 5 = 4^{(3)}$, so by the proof of [15, Lemma A.11], $D_3 = \bar{T}_3$ in suitable coordinates. As in the proof of Proposition 3.6, $D_4 \subseteq \bar{T}_4$, so $D_4 = \bar{T}_4$, and $D_5 = \mathbb{C}F$ with $F \in \bar{T}_5$. The map $\bar{S}_3 \rightarrow \bar{T}_2$, $\gamma \mapsto \gamma \circ F$, has a nonzero kernel, and Proposition 3.6 applies. $\square$

Theorem A.6. *Every local algebra with Hilbert function $(1, 3, 6, 4, 3, 3)$ is smoothable.*

Proof. By [15, Lemma A.11], either the cubic base scheme $\mathfrak{B}$ is finite of length at most four, or $D_3 = \bar{T}_3$ in suitable coordinates. In the first case $\sum_p e_p \leq 5$ (Lemma 3.2), every weight is at most two, and $27 + 10 = 37 < 40$: Theorem 2.7 applies, as $\dim I_3^* = 6$. In the second, $D_4 \subseteq \bar{T}_4$ and $D_5 \subseteq \bar{T}_5$ (as in the proof of Proposition 3.6), and by [15, Lemma 2.16] with $k = 4$ and $n = 3$, $D_4 = \nu_4(\Gamma)$ and $D_5 = \nu_5(\Gamma)$ for a subscheme Γ of length three. As $xS_3 \subseteq (I_\Gamma)_4$ and I_Γ is saturated, $x \in I_\Gamma$, so $\Gamma = V(x, \gamma)$ for a binary cubic γ , and $\gamma \circ D_5 = 0$ by the remark after Proposition 3.6. Proposition 3.6 applies. $\square$

Theorem A.7. *Every local algebra with Hilbert function $(1, 3, 6, 4, 4, 2)$ is smoothable.*

Proof. If the cubic base scheme is finite, Theorem 2.7 applies as in Theorem A.6, with $27 + 10 = 37$. Otherwise $D_3 = \bar{T}_3$, $D_4 \subseteq \bar{T}_4$ and $D_5 \subseteq \bar{T}_5$, and we distinguish the dimension r of $S_1 \circ D_5 \subseteq D_4$. We have $r \geq 2$, as two independent fifth powers have independent derivatives. For $F \in D_5$, (4) with $v = x$ gives $x \circ R(F) \in \bar{T}_3$, so $R(F) = B_F + XC_F$ with $B_F \in \bar{T}_4$, relevant modulo D_4 , and $C_F \in \bar{T}_3$; for $v \in \{y, z\}$ it reads $\tau(v \circ F) \equiv X(v \circ C_F)$ modulo D_3 .

$r = 2$: *prescribed equations.* By [15, Lemma 2.16] with $k = 4$, $n = 2$, $D_5 = \nu_5(\Gamma)$ for a subscheme $\Gamma = V(x, c)$ of length two of the line, $c \in \bar{S}_2$. The binary cubic $\gamma = cy$ lies in I_Γ , so $\gamma \circ D_5 = 0$, and Proposition 3.6 applies.

$r = 4$: *one family with an anchor.* Then $D_4 = S_1 \circ D_5$, the map $\langle y, z \rangle \otimes D_5 \rightarrow D_4$ is bijective, and τ is determined on D_4 by the C_F . So for fixed graded data the tails (τ, R) form a space of constant dimension $2 \cdot (1 + 4) = 10$, and with the free cubic tails E the algebras form a vector bundle over the irreducible variety of pairs (line, D_5), with D_5 in an open subset of $\text{Gr}(2, \bar{T}_5)$. Its member with $D_5 = \langle Y^4 Z, Y Z^4 \rangle$ and $R(F) = XC_F$, where $y^2 \circ C_F = z^4 \circ F$ and $z^2 \circ C_F = y^4 \circ F$, $\tau(v \circ F) = X(v \circ C_F)$ and $E = 0$, is closed under differentiation and has

$$x^3 + y^2 z^2, \quad xy^2 - z^4, \quad xz^2 - y^4$$

in its ideal, which cut out a complete intersection of local length 36 (script 120_136442.sage, part (A4)). Theorem 2.5 concludes.

$r = 3$: *one family with an anchor.* Let $Q = S_1 \circ D_5$, so D_4 is a hyperplane of $\bar{T}_4$ containing Q . As in the proof of [15, Theorem E.10], the pairs (Q, D_5) with $S_1 \circ D_5 \subseteq Q$ form an irreducible variety of dimension six, and those with $S_1 \circ D_5 = Q$ an open subset; D_4 is a point of the line of hyperplanes containing Q . The derivatives of a basis F_1, F_2 of D_5 satisfy one relation $v_1 \circ F_1 + v_2 \circ F_2 = 0$, which imposes the three conditions $v_1 \circ C_{F_1} + v_2 \circ C_{F_2} = 0$, and τ is free on a complement of Q in D_4 . So the tails (τ, R) form a space of constant dimension $(8 - 3) + 2 + 6 = 13$ (script 120_136442.sage, part (T), confirms it at seven points), and the algebras form one irreducible family. Its member with $D_5 = \langle Y^3 Z^2, Y^2 Z^3 \rangle$ and $D_4 = Q \oplus \mathbb{C}(Y^4 + Z^4)$ given by

Lemma 3.5, whose quartics y^4 and z^4 annihilate D_5 , contains

$$x^3 + y^4 + z^4, \quad xy^2 - z^4, \quad xz^2 - y^4,$$

which cut out a complete intersection of local length 36 (part (A3)). Theorem 2.5 concludes. $\square$

A.8. When $H_A(3) = 5$, the cubic base scheme may contain a line, and excess linkage fails there; we use rays, limits and prescribed equations instead. We apply [15, Lemma A.12] to D_3 . Its cases are: (a) $\mathfrak{B}$ finite of length at most four; (b) $\mathfrak{B}$ finite of length five, $D_3 = \nu_3(\mathfrak{B})$; (c) $\mathfrak{B}$ a line with a point, isolated or embedded, $D_3 = \nu_3(\mathfrak{B})$; (d) $\mathfrak{B}$ a line Λ , $D_3 = \nu_3(\Lambda) \oplus \mathbb{C}G$, $D'_3 = \nu_4(\Lambda)$. In (a) and (b) Corollary 3.3 applies, with the bound 39. In (c) with the point off the line, $D_3 = \bar{T}_3 \oplus \mathbb{C}X^3$ in suitable coordinates, and Lemma A.3 gives a ray. In case (d), $x \circ G$ is not a square (remark after [15, Lemma A.12]), and Proposition A.9 below applies. For the embedded point we use families over the pairs (ℓ, q) of a line and a point. Let $\mathcal{P} = \mathbb{P}^2 \times \mathbb{P}^2$ be the set of such pairs and $\mathfrak{B}_{\ell, q} = V(l \mathfrak{m}_q)$, where $\ell = V(l)$. Its Hilbert function is $k + 2$ for $k \geq 1$ and all (ℓ, q) , so the spaces $\nu_k(\mathfrak{B}_{\ell, q})$ form vector bundles over $\mathcal{P}$, as in [15, Lemma 2.14], and the pairs with $q \notin \ell$ are dense. In coordinates with $\ell = V(x)$ and $q = [Z] \in \ell$ we have $I_{\mathfrak{B}} = (x^2, xy)$ and $\nu_k(\mathfrak{B}) = \bar{T}_k \oplus \mathbb{C}XZ^{k-1}$. Our aim is to place every algebra with embedded $\mathfrak{B}$ in an irreducible family, with tails of constant dimension, over a base in which the off-line pairs are dense.

In case (d) the cubic base scheme is the line alone, and neither excess linkage nor a ray along a point applies. Here the tails prescribe three equations, and their local length is controlled by a multiplicity, as in Proposition 3.17.

Proposition A.9. *Let A be a local algebra of length 20 with Hilbert function $(1, 3, 6, 5, 4, 1)$ or $(1, 3, 6, 5, 3, 2)$, and suppose that $D_3 = \bar{T}_3 \oplus \mathbb{C}G$ in suitable coordinates, where $x \circ G$ is not a square. Then A is smoothable.*

Proof. We bound the multiplicity of the ideal generated by the parts that Lemma 3.5 lets us prescribe, by degenerating to monomial ideals. Let $Q \subseteq S_2$ be the hyperplane orthogonal to $x \circ G$, so that $I_3^* = xQ$. A cubic orthogonal to S_1Q has all its first derivatives in $\mathbb{C}x \circ G$, so it is zero, as $x \circ G$ is not a square; hence $S_1Q = S_3$ and $S_2Q = S_4$. As in Theorem 5.1(a), $D'_3 = \bar{T}_4$, so $D_4 \subseteq \bar{T}_4$ and $D_5 \subseteq \bar{T}_5$. Let $K \subseteq \bar{S}_4$ be the space of binary quartics killing D_5 ; it is orthogonal to $S_1 \circ D_5$, so $\dim K \geq 2$. Let g be the greatest common divisor of K , of degree $\delta \leq 3$, and $K = gH$ with $H \subseteq \bar{S}_{4-\delta}$ without common zero, so that $H\bar{S}_{4-\delta} = \bar{S}_{8-2\delta}$. Consider the ideals

$$N = xQ + K + \mathfrak{m}^5 \subseteq L = x\mathfrak{m}^2 + K + \mathfrak{m}^5 \subseteq L' = x\mathfrak{m}^2 + g \cdot (y, z)^{4-\delta} + \mathfrak{m}^5.$$

As $(x\mathfrak{m}^2)^2 = (xQ)(x\mathfrak{m}^2)$ and $(g \cdot (y, z)^{4-\delta})^2 = K \cdot g \cdot (y, z)^{4-\delta}$, we have $L^2 = NL$ and $L'^2 = LL'$, so N is a reduction of L' . Normalize $g(0, 1) = 1$. For $t \neq 0$ the substitution $\varphi_t: y \mapsto ty$ carries L' to the isomorphic ideal $L'_t = x\mathfrak{m}^2 + g(ty, z) \cdot (y, z)^{4-\delta} + \mathfrak{m}^5$, which tends to the monomial ideal $L_\delta^0 = x\mathfrak{m}^2 + z^\delta(y, z)^{4-\delta} + \mathfrak{m}^5$ as $t \rightarrow 0$. The triples $(x^3 - z^4 - z^5, xy^2 - y^3z, xz^2 - y^4)$,

$(x^3 + y^3z - y^5, xy^2 + z^4, xz^2 - y^3z)$, $(x^3 - z^4 + y^5, xy^2 - yz^3 - z^4, xz^2 - y^2z^2 + yz^3)$ and $(x^3 - z^4 + y^5, xy^2 - yz^3 + y^5, xz^2 - yz^3 + z^4)$ for $\delta = 0, 1, 2, 3$ lie in L_δ^0 and define complete intersections of local length $36 + \delta$ (script `120_line_case_d.sage`, part A, by local standard bases). By openness of the local length, the same combinations of the generators of L'_t , for small $t \neq 0$, define a complete intersection of local length at most $36 + \delta \leq 39$, and pulling back by φ_t gives three such elements of L' . By Proposition 3.17(a), three general elements $f_i = w_i + p_i + s_i$ of N , with $w_i \in xQ$ independent, $p_i \in K$ and $s_i \in S_5$, define a complete intersection of local length at most $39 < 40$. Lemma 3.5 places them in the ideal of an algebra with the graded data of A and concludes. $\square$

Theorem A.10. *Every local algebra with Hilbert function $(1, 3, 6, 5, 4, 1)$ is smoothable.*

Proof. Types (a), (b), (d), and (c) with the point off the line, are settled above. We show that the algebras with an embedded point are limits of algebras whose point is off the line. Let $\mathfrak{B} = \mathfrak{B}_{\ell, q}$ with $q \in \ell$, and $D_5 = \mathbb{C}F$. As $I_{\mathfrak{B}}$ is generated by quadrics, $D_4 \subseteq \nu_4(\mathfrak{B})$ and $F \in \nu_5(\mathfrak{B})$ (proof of Lemma A.3). We list the types of F , with the tails over them.

- F concise: τ is determined on $S_1 \circ F$ by R , free on a complement in D_4 (5 parameters), and $R \in T_4/D_4$ is free (11); $D_4/S_1 \circ F$ is a point of $\mathbb{P}(\nu_4(\mathfrak{B})/S_1 \circ F) \cong \mathbb{P}^2$.
- $F \in \mathbb{C}[\ell]_5$, binary on the line: $R \in x^{-1}(D_3)$, of dimension 10, modulo D_4 ; τ free on a complement of $S_1 \circ F$ (10); $D_4/S_1 \circ F \in \text{Gr}(2, 4)$.
- $F = P^5$ with $P \in \ell$, $P \neq q$: $N_P \circ R \subseteq D_3$, a space of dimension $15 - \dim N_P(I_{\mathfrak{B}})_3 = 7$, as $(I_{\mathfrak{B}} : \mathfrak{m}_P) = I_{\mathfrak{B}}$.
- $F = q^5$: here $(I_{\mathfrak{B}} : \mathfrak{m}_q) = (l)$, and the space of R has dimension 8.

In each of these four types the tails have the same dimension at embedded and at off-line pairs, so the algebras form an irreducible family over $\mathcal{P}$ (over the pairs (ℓ, q) with a point $P \in \ell \setminus q$ for the third type), in which the off-line members, which have rays, are dense; [15, Proposition 2.2(1)] applies. The remaining top forms in two variables are $F = Z^4(c(X + \alpha Y) + bZ)$ with $c \neq 0$, the forms of ν_5 of a double point at q transverse to ℓ . Such an algebra is a limit of concise embedded ones: $S_1 \circ F = \langle Z^4, (X + \alpha Y)Z^3 \rangle$, and $D_4 = S_1 \circ F + \langle u, u' \rangle$ with $u \in \bar{T}_4 \setminus \mathbb{C}Z^4$. Choose $G \in \bar{T}_5$ with $y \circ G = u$, and let $F_t = F + tG$ and $D_4(t) = \langle Z^4, u, z \circ F_t, u' \rangle \subseteq \nu_4(\mathfrak{B})$. For $t \neq 0$ the form F_t is concise, $S_1 \circ F_t \subseteq D_4(t)$, and $D_4(0) = D_4$. The tails (τ, R) over $(D_4(t), F_t)$ are the kernel of a linear system depending polynomially on t , of dimension 16 for every t ; so they form a vector bundle near $t = 0$, the tails of A extend, and A is a limit of concise embedded algebras. (Script `120_1365_lineplus.sage`, parts (B) and (C).) $\square$

Theorem A.11. *Every local algebra with Hilbert function $(1, 3, 6, 5, 3, 2)$ is smoothable.*

Proof. Types (a), (b), (d), and (c) with the point off the line, are settled above. It remains to treat $\mathfrak{B} = \mathfrak{B}_{\ell,q}$ with $q \in \ell$, and, as in Theorem A.10, we show that these algebras are limits of algebras whose point is off the line. Then $D_3 = \nu_3(\mathfrak{B})$, $D_4 \subseteq \nu_4(\mathfrak{B})$ has dimension three, $D_5 \subseteq \nu_5(\mathfrak{B})$ is a pencil, and $r = \dim S_1 \circ D_5 \in \{2, 3\}$. For fixed graded data the tails (τ, R) , modulo those with values in D_3 and D_4 , form a space of dimension $24 - \text{rank } \rho$ if $r = 3$, and $29 - \text{rank } \rho$ if $r = 2$, where ρ sends R to the values $\sum_j v_j \circ R(F_j)$ modulo D_3 on the relations $\sum_j v_j \otimes F_j$ in the kernel Syz of $S_1 \otimes D_5 \rightarrow S_1 \circ D_5$. Dually, $\text{rank } \rho = \dim(W \cdot \text{Syz})$, the span of the pairs $(a_1 w, a_2 w)$ with $(a_1, a_2) \in \text{Syz}$ and $w \in W = I_3^*$. The cubic tails E are free.

Classification. In coordinates with $\ell = V(x)$ and $q = [Z]$, write $F = G + cXZ^4$ for $F \in D_5$, with G binary; then $x \circ F = cZ^4$. If $r = 2$, then $D_5 = \nu_5(\Gamma)$ for a subscheme $\Gamma \subseteq \mathfrak{B}$ of length two [15, Lemma 2.16]. If $r = 3$ and D_5 is neither binary nor contains a fifth power, then $D_5 = \langle XZ^4 + G_1, G_2 \rangle$ with G_1, G_2 binary, and $r = 3$ forces $\dim \langle Z^4, y \circ G_1, y \circ G_2, z \circ G_2 \rangle = 2$; hence $D_5 \subseteq \nu_5(\Gamma_3)$ for a subscheme $\Gamma_3 \subseteq \mathfrak{B}$ of length three, which is (i) a point of ℓ and a double point at q transverse to ℓ , (ii) a curvilinear scheme $\Gamma_c = V(xz - cy^2, y^3, x^2, xy)$ with $c \neq 0$, or (iii) the fat point $V(\mathfrak{m}_q^2)$. This gives six types, each the set of embedded points of an irreducible family over a base in which the off-line pairs are dense.

- (1) $r = 2$, $\Gamma \subset \ell$ not containing q .
- (2) $r = 2$, $\Gamma \ni q$: the base is the closure of $\{(\ell, q, \{N, q\}) : q \notin \ell, N \in \ell\}$, whose embedded points, as q moves into ℓ , are the pairs $\{N, q\}$ with $N \in \ell$ and the double points at q in every direction.
- (3) $r = 3$, $D_5 \subseteq \mathbb{C}[\ell]_5$ binary, $q^5 \notin D_5$.
- (4) $r = 3$, $q^5 \in D_5$.
- (5) $r = 3$, $D_5 \subseteq \nu_5(\Gamma_3)$ without fifth powers: the base is the closure of $\{(\ell, q, \Gamma_2 \cup \{q\}) : q \notin \ell, \Gamma_2 \subset \ell\}$; it contains (i)–(iii), as in the chart $z = 1$ the points $(x, y) = (0, t), (0, -t), (3ct^2, 2t)$ lie on $x = c(y^2 - t^2)$, two of them on ℓ , with limit Γ_c , and $(0, t), (0, -t), (t, 0)$ tend to $V(\mathfrak{m}_q^2)$.
- (6) $r = 3$, D_5 not binary, $N^5 \in D_5$ for a point $N \in \ell \setminus q$.

Tail dimensions. In types (3) and (4) the pencils D_5 have moduli, and we compute $\dim(W \cdot \text{Syz})$ uniformly. At every pair (ℓ, q) , $W = x \cdot (I_q)_2$, where I_q is the ideal of q . If $D_5 = \langle F_1, F_2 \rangle$ is binary, then Syz is spanned by $(x, 0)$, $(0, x)$ and (a, b) , where $a \circ F_1 + b \circ F_2 = 0$ with $a, b \in \langle y, z \rangle$. If $(w_1, w_2, w_3) \in W^3$ and $xw_1 + aw_3 = xw_2 + bw_3 = 0$, then x divides w_3/x , so $w_3 = x^2 l$ with $xl \in I_q$, and $w_1 = -axl$, $w_2 = -bxl$ lie in W if and only if $al, bl \in I_q$. So the kernel of $W^3 \rightarrow W \cdot \text{Syz}$ has dimension two, or three if $q \in \ell$ and $a(q) = b(q) = 0$, that is, if $q^5 \in D_5$. If $q^5 \in D_5$ and D_5 is not binary, the kernel has dimension three as well: for $q = [Z] \in \ell$, $D_5 = \langle Z^5, cXZ^4 + G \rangle$ with $c \neq 0$ and G binary, Syz is spanned by $(x, 0)$, $(y, 0)$, $(-\frac{c}{5}z, x)$, and the kernel is $\{(xyl, -x^2l, 0)\}$; for $q = [X] \notin \ell$, $D_5 = \langle X^5, G \rangle$, Syz is spanned by $(y, 0)$, $(z, 0)$, $(0, x)$, and the kernel is $\{(xzl, -xyl, 0)\}$ (script 120_r2_136532.sage).

In types (1) and (2), $\text{Syz} = N_{P_1} \otimes F_1 \oplus N_{P_2} \otimes F_2$ for $\Gamma = \{P_1, P_2\}$ and $F_i = P_i^5$, with $\dim N_P W = 8$ for $P \in \ell \setminus q$ and $\dim N_q W = 7$, as $(I_{\mathfrak{S}} : \mathfrak{m}_q) = (x)$. At general points of the six types:

type	(1)	(2)	(3)	(4)	(5)	(6)
$\dim(W \cdot \text{Syz})$	8 + 8	8 + 7	15 - 2	15 - 3	15 - 1	15 - 2
tails	13	14	11	12	10	11

For (5) and (6) these are the values at off-line points. At the special points of types (1), (2), (5) and (6) (double points, and (i)–(iii)) the members form finitely many orbits of the stabilizer of (ℓ, q) , and the tail dimension is computed at one representative of each orbit, by hand or by the script `120_136532_embedded.sage`. So every embedded algebra is a limit of off-line ones, which have rays by Lemma A.3, and [15, Proposition 2.2(1)] concludes. $\square$

A.12. The remaining three Hilbert functions have $H_A(3) \geq 6$, so only three or four cubic equations remain. We use global linkage when their linear syzygies allow the required quartic terms; otherwise we split off a cubic, count dimensions, or use an anchor.

Theorem A.13. *Every local algebra with Hilbert function $(1, 3, 6, 7, 2, 1)$ is smoothable.*

Proof. Let $D_5 = \mathbb{C}F$; as $\dim D_4 = 2$, F has at most two essential variables.

F binary: global linkage. If F has two essential variables, $F \in \mathbb{C}[X, Y]_5$ in suitable coordinates, and $D_4 = S_1 \circ F$. The tails are $R \in (zW)^\perp$ modulo D_4 and E , of constant dimension $10 + 3$. The space $S_2 \circ F$ has dimension three or two, and the graded data form two irreducible varieties: $D_3 = S_2 \circ F \oplus K$ with $K \in \text{Gr}(4, T_3/S_2 \circ F)$, or $F \in \nu_5(\Gamma)$ for Γ of length two on the line $z = 0$ and $D_3 = \nu_3(\Gamma) \oplus K$ with $K \in \text{Gr}(5, T_3/\nu_3(\Gamma))$. As $\dim W = 3$, the condition of Lemma A.2 is that W has no linear syzygy, together with two coprime members. It holds for one member of each variety, $F = X^5 + Y^5 + (X + Y)^5$ and $F = X^5 + 2Y^5$ with random K , where the construction of [15, Lemma E.8] gives a complete intersection of length 36, hence of local length at most 36 (script `120_136721.sage`). So the algebras over a dense open subset of the graded data are smoothable, and so are all of them.

F = X⁵: cubic reduction. Let $\Phi = X^5 + R + E$ be the lift of F . Then $D_4 = \langle X^4, Q \rangle$, $5\tau(X^4) \equiv x \circ R$, and $y \circ R, z \circ R \in D_3$. The space $C_0 = S_1 \circ D_4 + \langle y \circ R, z \circ R \rangle \subseteq D_3$ has dimension at most six, so $D_3 = C \oplus \mathbb{C}G$ for a six-dimensional $C \supseteq C_0$, and $M = M' \oplus \mathbb{C}G$, where $M' = T_{\leq 2} \oplus C \oplus \langle x \circ \Phi/5, Q + \tau(Q) \rangle \oplus \mathbb{C}\Phi$ is closed under differentiation, with Hilbert function $(1, 3, 6, 6, 2, 1)$. By the proof of [15, Theorem E.12], case $r = 1$, M' lies in a family of algebras with this Hilbert function in which rays given by operators of order four are dense; by [15, Lemma A.14] and the remark after it, so does M , and A is smoothable. $\square$

Theorem A.14. *Every local algebra with Hilbert function $(1, 3, 6, 6, 3, 1)$ is smoothable.*

Proof. Let $D_5 = \mathbb{C}F$ and $W = I_3^*$, of dimension four.

F concise: anchors and global linkage. Then $D_4 = S_1 \circ F$ and $R \in T_4$ is free. The concise quintics fall into five irreducible families according to $h = \dim S_2 \circ F \in \{3, 4, 5, 6\}$, two of them with $h = 4$, as in the proof of [15, Theorem E.11], with $D_3 = S_2 \circ F \oplus K$, and the tails have constant dimension over them. For $h = 6$ and $h = 5$, the members with zero tails and $F = X^2Y^2Z + X^2YZ^2 + XY^2Z^2$, respectively $F = X^2Y^2Z$ and $K = \mathbb{C}XZ^2$, are annihilated by x^3, y^3, z^3 : they lie in a complete intersection of length 27 (script `120_r2_anchors.sage`, part (A14)), and Theorem 2.5 applies. For $h \leq 4$, each family contains a member satisfying the condition of [15, Lemma E.8], that three general cubics of W have no linear syzygy and two of them no common factor (sums of four general fifth powers, $X^5 + Y^5 + Z^5 + (Y + Z)^5$, and $X^5 + Y^5 + Z^5$, with random K ; script `120_136631.sage`). As the condition is open, all algebras with concise F are smoothable.

F binary: global linkage. In coordinates with $z \circ F = 0$, $D_4 = S_1 \circ F \oplus \mathbb{C}u$, $R \in (zW)^\perp$, and $\tau(u), E$ are free: tails of constant dimension $8 + 4 + 4 = 16$. Let $s' = \dim S_2 \circ F \in \{2, 3\}$; then $S_2 \circ F = \nu_3(\Gamma)$ for a subscheme Γ of length s' of the line $z = 0$ (for $s' = 3$, defined by the cubic of the complete intersection of degrees 3, 4 that annihilates F in $\mathbb{C}[x, y]$). Let $m = \dim(S_2 \circ F + S_1 \circ u)$. The graded data are the plane $z = 0$ and F ($2 + 5$, respectively $2 + 2 + 1$ parameters), $[u] \in \mathbb{P}(T_4/S_1 \circ F)$ (12) and $D_3/S_2 \circ F$ (12, respectively 16), and $S_1 \circ u \subseteq D_3$ is a section of a bundle of rank 12; so every component has dimension at least 19 if $s' = 3$, and 21 if $s' = 2$. Stratifying by m , with $\dim\{u : N_L \circ u \subseteq \nu_3(\Gamma)\} = 15 - \dim N_L(I_\Gamma)_3$, which is $s' + 1$ for $L \notin \Gamma$, as $(I_\Gamma : \mathfrak{m}_L) = I_\Gamma$, and $s' + 2$ for $L \in \Gamma$ (the proof of [15, Lemma A.3]), and parametrizing a stratum by the plane and F , by $[u]$, and by $D_3 \supseteq S_2 \circ F + S_1 \circ u$ in a Grassmannian of dimension $4(6 - m)$, the strata have dimensions $19, \leq 18, \leq 18, 19$ for $s' = 3$, $m = 6, 5, 4, 3$, and $21, \leq 19, \leq 19$ for $s' = 2$, $m = 5, 4, 3$. The strata $(s', m) = (3, 6)$ and $(2, 5)$ are open in bundles over the choices of the plane, F and $[u]$, and in $(3, 3)$ the form u lies in $\nu_4(\Gamma)$ [15, Lemma 2.16 with $k = n = 3$]; so these three strata are irreducible, of dimensions 19, 21 and 19, the lower bounds for the components. Hence the general member of every component lies in one of them. One member of each satisfies the open condition of Lemma A.2 (script `120_136631_binary.sage`), and all these algebras are smoothable.

F = X^5 : expected dimension. The data $(p; D_3 \ni X^3; D_4 \ni X^4; \tau; R; E)$, with $p = [X]$, range over a smooth irreducible variety of dimension $2 + 20 + 24 + 12 + 12 + 4 = 74$ and are cut out by $24 + 12 = 36$ equations: $S_1 \circ D_4 \subseteq D_3$ and (4). So every component of the zero set has dimension at least 38. Over fixed graded data the tails form a space of dimension $16 + r'$, where $r' = \dim\{q \in S_2 : yq, zq \in W\} \leq 3$. If W has no common factor,

its base scheme Z is finite, Z is not collinear of length at least four (the line would divide every cubic of W), and $h_Z(3) \leq \dim D_3 = 6$. Write the first differences of h_Z as $(1, 2, d_2, d_3, \dots)$, the Hilbert function of an Artinian quotient of $\mathbb{C}[s, t]$. If $d_2 \leq 1$, then $d_3 = 0$: a pencil $(I_Z)_2$ without common factor cuts out at most four points, and one with a common factor l makes l divide $(I_Z)_3 \supseteq W$. If $d_2 = 2$ and $d_3 = 1$, then $W = (I_Z)_3 = qS_1 + \mathbb{C}f$ with q, f coprime, and $\ell(Z) \leq 6$; if $d_2 = 3$, then $d_3 = 0$. So $\ell(Z) \leq 6$, and if $\ell(Z) = 6$, then $W = (I_Z)_3$, I_Z is generated in degrees at most three, and $\dim S_1W = 9$. The algebras with $\ell(Z) = 6$ form a set of dimension at most $12 + 6 + 19 = 37$ (Z ; then $D_4 \ni X^4$ in the six-dimensional $(S_1W)^\perp$; then the tails). If $W = lV$ has a common factor, with $\dim V = 4$ and $\dim S_1V = 10$, 9 or 8 as in Section 5, the pairs (l, V) have dimension at most 10, 8, 6, the point $[X]$ at most two, and $D_4 \ni X^4$ in $(S_1W)^\perp$, of dimension 5, 6, 7, gives 4, 6, 8: at most $10 + 2 + 4 + 19 = 8 + 2 + 6 + 19 = 6 + 2 + 8 + 19 = 35$ in all. So the general member of every component has a finite cubic base scheme of length at most five, and is smoothable by Corollary 3.3. $\square$

Theorem A.15. *Every local algebra with Hilbert function $(1, 3, 6, 6, 2, 2)$ is smoothable.*

Proof. We repeat the proof of [15, Theorem C.10] with one more cubic. By [15, Lemma 2.16], $D_4 = \nu_4(\Gamma)$ and $D_5 = \nu_5(\Gamma)$ for a unique $\Gamma \in \text{Hilb}_2(\mathbb{P}^2)$, and D_3 is determined by the four-dimensional space $D_3/\nu_3(\Gamma)$. The algebras are given by data $(\Gamma, D_3/\nu_3(\Gamma), R_1, R_2, E_1, E_2)$ in a smooth irreducible variety of dimension $4 + 16 + 26 + 8 = 54$, subject to 16 equations: four *columns* $\sum_k v_k \circ R_k$ must lie in D_3 . So every component has dimension at least 38. The bounds of [15, Theorem C.10] for the pairs (R_1, R_2) whose columns have rank r modulo $\nu_3(\Gamma)$ do not involve D_3 ; adding the Grassmannian $\text{Gr}(4 - r, 8 - r)$ of $D_3/\nu_3(\Gamma)$, the strata with $r = 3, 2, 1, 0$ have dimensions at most 37, 35, 34, 32, and for $r = 4$ the space D_3 is determined by the columns: an irreducible open set $\mathcal{U}$ of dimension 38 over reduced Γ , and 37 over double points. By [15, Lemma 2.18], all these algebras form one irreducible family, the closure of $\mathcal{U}$. The anchor of [15, Theorem C.10], with Hilbert function $(1, 3, 6, 5, 2, 2)$, lies in the complete intersection cut out by

$$x(y^2 + yz + z^2) - \frac{1}{15}x^4, \quad y(x^2 - z^2) + \frac{1}{120}y^5, \quad z(x^2 - y^2) + \frac{1}{120}z^5$$

of local length 32. The cubic $XZ^2 - XY^2$ is orthogonal to the three cubic parts and is not in the anchor's D_3 ; by Remark 3.7, adding it gives a member of our family in the same complete intersection, with residual scheme of length 12 (script 120_136622.sage). Theorem 2.5 concludes. $\square$

APPENDIX B. ONE QUADRIC

B.1. This appendix proves the nine Hilbert functions of Section 6 not treated there. Here $I_2^* = \mathbb{C}x^2$, $f \in I$ has leading form x^2 , and we use the notation of Sections 2.8, 3.10 and 3.13. We first prove three variants of Theorem 3.15

and a normal form for tails. The notation g^h, g'' and the charts are those of the proof of Theorem 3.15.

Corollary B.2. *Let $A = S/I$ be a local algebra of length d with $I_2^* = \mathbb{C}x^2$, in the binary case $I_3^* = x^2S_1 \oplus xW_2$, with $\gamma_f = 0$, and suppose that every finite subscheme of $\mathbb{A}^3$ of length less than d is smoothable.*

- (a) A turning quartic with a simple zero. *Let $d \geq 18$ and $\mathfrak{m}^7 \subseteq I$. Suppose that I_4^* contains a nonzero binary quartic p , that some $r \in W_2$ vanishes at no root of p , and that $(I : \mathfrak{m}^2)$ contains a binary quartic w_4 that has a simple zero at one root of p and vanishes at no other root of p . Then A is smoothable: it belongs to a turning family whose general member lies in a complete intersection of local length at most 35.*
- (b) Two quartics. *Let $d \geq 20$ and $\mathfrak{m}^7 \subseteq I$. If I_4^* contains binary quartics p, q whose only common root is a simple root of p , then A is smoothable, through a complete intersection of local length at most 38.*
- (c) A bad point. *Let $d \geq 19$ and $\mathfrak{m}^6 \subseteq I$. Let $g \in J_4$ be nonzero, with roots v_i of multiplicities μ_i , let $m \in W_2$ be nonzero, fix $K_m \in I$ with cubic part xm , and let θ_m be the binary part of its quartic part. Suppose that at every root v_i except possibly one, v_0 , either $m(v_i) \neq 0$ or $\theta_m(v_i) \neq 0$, and that at v_0 either (i) m has a simple zero, or (ii) $\mu_0 = 1$ and I_4^* contains xb_3 for some $b_3 \in \tilde{S}_3$ with $b_3(v_0) \neq 0$. Then A is smoothable: it belongs to a turning family whose general member lies in a complete intersection of local length at most 36.*

Proof of Corollary B.2. In each part we exhibit three elements of I , or of a nearby member of a turning family, apply Lemma 3.14(a), and bound the local length at each point of the exceptional divisor by Lemma 3.14(b).

(a) Let v_0 be the root of p at which w_4 vanishes. In the turning family of w_4 [15, Lemma 4.2 and its proof], with the complement xW_2 of x^2S_1 in I_3^* , the members contain $f + tw_4$, an element K with cubic part xr , an element P with leading form p , and l_0^7 for $l_0 \in S_1$ with $l_0(v_0) \neq 0$. We use $f + tw_4$, K and $P + \nu l_0^7$ in the chart of the proof of Theorem 3.15, where now $(f + tw_4)'' = \rho^2 H$ with $H = \psi + tw_4(1, w)$, as w_4 is a binary quartic. At a root $v \neq v_0$, H is a unit for t outside one value, so $\lambda_v = 2\mu_v$; this does not depend on ν , which changes P'' by $\nu\rho^3 L$, where $L = (l_0^7)''$ has $L(c_0, 0) = l_0(0, 1, c_0)^7 \neq 0$, but not $P''(w, 0)$. At v_0 , if $\psi(c_0, 0) \neq 0$, again $\lambda_{v_0} = 2\mu_{v_0}$. Otherwise $\partial_w H(c_0, 0) \neq 0$ for t outside one value, as w_4 has a simple zero, so H is a unit times $w - c_0 - g(\rho)$ with $g(0) = 0$, and $\lambda_{v_0} = 2\mu_{v_0} + \ell(\mathbb{C}\{\rho\} / (P''(c_0 + g(\rho), \rho) + \nu\rho^3 L(c_0 + g(\rho), \rho)))$; by Lemma 3.14(b), with $B = \mathbb{C}\{\rho\}$ of rank $n = 1$, $a = 0$, $c_1 = 3$ and the unit $u_1 = L(c_0 + g(\rho), \rho)$, the last term is at most three for general ν . In total the length is at most $24 + 2 \cdot 4 + 3 = 35 < 2d$, so the members of the turning family with general t are smoothable, and so is A .

(b) Here we use f , P and $Q + \lambda l_0^7$, where $P, Q \in I$ have leading forms p, q and $l_0(v_0) \neq 0$ at the common root v_0 . The tangent cones x^2, p, q meet only at v_0 , so the length is $2 \cdot 4 \cdot 4 + \lambda_{v_0}$. In the chart with $v_0 = [0 : 1 : 0]$, since p

has a simple root at v_0 , $P^\natural$ is a unit times $w - \rho g(u, \rho)$, and the local ring at v_0 is $\mathbb{C}\{u, \rho\}/(f'', (Q + \lambda l_0^7)'')$, where now $''$ denotes the substitution $w = \rho g$. As $f''(u, 0) = u^2$, the ring $B = \mathbb{C}\{u, \rho\}/(f'')$ is free of rank two over $\mathbb{C}\{\rho\}$; as $q(v_0) = 0$, $Q'' = \rho h$ for some $h \in B$, and $(Q + \lambda l_0^7)'' = \rho h + \lambda \rho^3 U = \rho(h + \lambda \rho^2 U)$, where $U = (l_0^7)^\natural|_{w=\rho g}$ is a unit, as $l_0(v_0) \neq 0$. Lemma 3.14(b) with $n = 2$, $a = 1$ and $c_1 = 2$ gives $\lambda_{v_0} \leq 6$, and the length is at most $32 + 6 = 38 < 2d$.

(c) Choose z with $z(v_i) \neq 0$ for all i , and use $b = z$ in the turning family I_t of the proof of Proposition 3.12. It contains $F = f + tz^4$, $K = K_m$, and $G \in I \cap \mathfrak{m}^4$ with binary quartic part g ; in case (i) also $G + \nu z^6$, and in case (ii) $K + \eta H'$ for $H' \in I$ with leading form xb_3 , as $\mathfrak{m}^6 \subseteq I_t$. Their tangent cones meet in $\mathbb{P}^2$ exactly at the roots v_i . In the chart $x = \rho u$, $y = \rho v$, $z = \rho$ the proper transforms are

$$\begin{aligned} F' &= u^2 + \rho u a(v) + \rho^2(\beta(v) + t) + \cdots, \\ K' &= u m(v) + \rho \theta_m(v) + \cdots, \quad G' = g(v) + \cdots, \end{aligned}$$

with terms of higher order in u, ρ omitted. If $m(v_i) \neq 0$, then $K' = 0$ gives $u = \rho \varphi$, and $F' = \rho^2 \psi$ with $\psi(v_i, 0)$ a nonzero value of $x^2 + a(v_i)x + \beta(v_i) + t$ for general t : the length is $2\mu_i$. If $m(v_i) = 0 \neq \theta_m(v_i)$, then $K' = 0$ gives $\rho = u\chi$ with $\chi(0, v_i) = 0$, F' is u^2 times a unit, and again the length is $2\mu_i$. This also covers v_0 unless $m(v_0) = \theta_m(v_0) = 0$. Then the quadratic part $u^2 + a(v_0)u\rho + (\beta(v_0) + t)\rho^2$ of F' has distinct roots for general t , so F' is a unit times $(u - \rho\alpha_+)(u - \rho\alpha_-)$, and the length is the sum over the two branches $u = \rho\alpha_\pm$, on which $K' = \rho H_\pm(v, \rho)$ with $H_\pm(v_0, 0) = 0$. In case (i), $\partial_v H_\pm(v_0, 0) = \alpha_\pm(v_0, 0)m'(v_0) + \theta'_m(v_0) \neq 0$ for general t , as $\alpha_\pm(v_0, 0)$ moves with t ; so $H_\pm = 0$ is a curve $v = \psi_\pm(\rho)$, and the branch contributes $\mu_0 + \ell(\mathbb{C}\{\rho\}/(G'(\psi_\pm(\rho), \rho) + \nu\rho^2)) \leq \mu_0 + 2$ for general ν , by Lemma 3.14(b). In case (ii), $G' = 0$ is a curve $v = \phi_\pm(\rho)$, and $\eta H'$ adds $\eta\rho(\alpha_\pm(v_0, 0)b_3(v_0) + h_5)$ plus terms of order two to $H_\pm$, where h_5 does not depend on t : the branch contributes at most $1 + 1$ for general t and η . In both cases the length is at most $24 + 2\sum_{i \neq 0} \mu_i + 2\mu_0 + 4 \leq 36 < 2d$. $\square$

B.3. Normal forms of tails come from two operations: *shears*, which add to one coordinate a form in the other two, and *gauges*, which replace a lift by itself minus a derivative of lower degree.

Lemma B.4 (Shears). *Let $v \in \{x, y, z\}$ with dual variable V , let q be a form of degree at least two in the two other variables, and let φ_q be the automorphism of S with $v \mapsto v + q$ fixing the other variables. Then $\langle \varphi_q(g), F \rangle = \langle g, \exp(Vq(\partial))F \rangle$ for $g \in S$ and $F \in T$. So $\exp(Vq(\partial))M$ is the inverse system of $\varphi_q^{-1}(I)$, which defines an isomorphic scheme with the same tangent cone and the same graded data.*

Proof. For $P = aX + bY + cZ$ and the formal exponential e^P , $\langle g, e^P \rangle = g(a, b, c)$, and $\exp(Vq(\partial))e^P = e^{P+q(a,b,c)V}$, whose pairing with g is the value of g at the point with v -coordinate increased by q . The coefficients of e^P

span T . Since $\exp(Vq(\partial))F = F + (\text{terms of degree at most } \deg F - 1)$, the top forms do not change. $\square$

Shears and gauges remove tails of the top forms, as the coordinate changes of [15, Section 5.17] do. Neither changes the cubic part of an element of $(I : \mathfrak{m}^2)$, so turning operators are preserved.

B.5. Socle degree five. Here $\mathfrak{m}^6 \subseteq I$, and a cubic h with $h \circ D_5 = 0$ lies in $(I : \mathfrak{m}^2)$.

Theorem B.6. *Every local algebra with Hilbert function $(1, 3, 5, 4, 5, 2)$ is smoothable.*

Proof. We show that these algebras form one irreducible family, and link one member by weighted Bézout. As before $I_3^* = xS_2$ and $\gamma_f = 0$; now $D_4 = \bar{T}_4$, so $J_4 = 0$ and Proposition 3.12 does not apply, and $D_5 = V$ is an arbitrary plane in $\bar{T}_5$. The normal form of the proof of [15, Theorem F.2], with $\gamma_f = 0$, shows that for fixed (D_4, V) of dimensions (w, v) these algebras form an affine space of dimension $4w + v(7 - w)$: the free data are c (w parameters), η ($v(5 - w)$), the coefficients of $X\bar{T}_2$ in the cubic tails of the lifts of V ($3v$) and of X^2 in their quadratic tails (v), and the solutions of $6v$ linear equations of rank $6v$ in the $3w + 4v$ coefficients of (α_2, T) , by the dual argument there, which uses only $V \subseteq \bar{T}_5$. Here $(w, v) = (5, 2)$: an affine bundle of rank 24 over the choices of the line $x = 0$ and of $V \in \text{Gr}(2, \bar{T}_5)$, irreducible (script `120_135_tails.sage` computes the fibre with all closedness equations). One member is

$$I_0 = (x^2 + y^4 + z^4, xy^2 + y^4, xyz, xz^2 + z^4, y^4z, y^3z^2, y^2z^3, yz^4),$$

of length 20 with this Hilbert function and $\gamma_f = 0$. Its elements $x^2 + y^4 + z^4, xy^2 + y^4, xz^2 + z^4$ are weighted homogeneous of degree four with only the origin as common zero, so they generate a complete intersection of length 32 (Lemma 3.11); the residual scheme has length 12 (script `120_135_checks.sage`, part (A)). Theorem 2.5 concludes; only the residual scheme, of length twelve, has to be smoothable [15, Theorem 2.22 and Proposition 2.24]. $\square$

Theorem B.7. *Every local algebra with Hilbert function $(1, 3, 5, 5, 4, 2)$ is smoothable.*

Proof. By a turning operator of type (b) (Section 2.8) and Lemma 3.9, $I_3^* = x^2S_1 \oplus xW_2$ with a pencil $W_2 \subseteq \bar{S}_2$, and $\gamma_f = 0$. As $\dim D_4 = 4 < 5$, $J_4 \neq 0$. If W_2 has no common zero on the line $x = 0$, Proposition 3.12 applies. Otherwise $W_2 = l\bar{S}_1$ for a linear form l with zero v_0 , the only possible bad point. If it is not bad, Proposition 3.12 applies. If it is, choose $g \in J_4$ nonzero and $m = ll' \in W_2$ with l' vanishing at no root of g ; then m has a simple zero at v_0 and does not vanish at the other roots, and Corollary B.2(c)(i) applies. $\square$

Theorem B.8. *Every local algebra with Hilbert function $(1, 3, 5, 6, 3, 2)$ is smoothable.*

Proof. As before, $I_3^* = x^2S_1 \oplus \mathbb{C}xr_2$ with $0 \neq r_2 \in \bar{S}_2$, $\gamma_f = 0$, and $D_3 = \bar{T}_3 \oplus X \cdot (r_2^\perp \cap \bar{T}_2)$. Let $V = D_5$ and $U = D_4$, of dimensions two and three. As x^2S_3 and xr_2S_2 lie in I_5^* , every $F \in V$ is $F_0 + XF_1$ with binary F_0, F_1 and $r_2 \circ F_1 = 0$. We find either a cubic p with $p \circ V = 0$ and $\pi(p) \neq 0$, which is a turning operator, or show that $U \subseteq \bar{T}_4$ and that J_4 is a pencil with at most one common root.

V not binary. If the F_1 span two dimensions, then U contains them and the elements $y \circ F$, $z \circ F$, whose X -parts span at least two dimensions, so $\dim U \geq 4$, a contradiction. So $V = \langle F, G_5 \rangle$ with $G_5 \in \bar{T}_5$ and $F = F_5 + XG_4$, $G_4 \neq 0$; and G_4 is a fourth power, say Y^4 , as otherwise $U \cap \bar{T}_4$ would be a line containing G_4 and the derivatives of G_5 . If $G_5 \notin \mathbb{C}Y^5$, then in suitable coordinates $G_5 \in \langle Y^5, Z^5 \rangle$ and $F_5 \in \langle Y^5, Y^4Z, Z^5 \rangle$, or $G_5 \in \langle Y^5, Y^4Z \rangle$ and $F_5 \in \langle Y^5, Y^4Z, Y^3Z^2 \rangle$; the cubics yz^2 , respectively $yz^2 - \frac{\gamma}{2}xy^2$ (γ the coefficient of Y^3Z^2 in F_5), kill V . If $G_5 = Y^5$, the map $z\bar{S}_2 \rightarrow \bar{T}_2/\mathbb{C}Y^2$, $q \mapsto q \circ F_5$, has a kernel element $q \neq 0$ with $q \circ F_5 = cY^2$, and $q - \frac{c}{12}xy^2$ kills V .

V binary. Let $U' \subseteq U \cap \bar{T}_4$ be the span of the first derivatives of V . If $\dim U' = 2$, then by [15, Lemma 2.16] $V \subseteq \nu_5(\Gamma)$ for $\Gamma = V(x, q)$ of length two, and $p = ql$ kills V for every $l \in \bar{S}_1$. If $\dim U' = 3$, then $U = U' \subseteq \bar{T}_4$, and J_4 is a pencil with $\bar{S}_1J_4 \subseteq V^\perp \cap \bar{S}_5$, a space of dimension four. Let δ be the degree of the common factor of J_4 . If $\delta = 0$, [15, Theorem 3.6] applies with two quartics. If $\delta = 2$, $J_4 = g_2P_2$ with a pencil P_2 of coprime quadrics, $\bar{S}_1P_2 = \bar{S}_3$, so g_2l kills V . If $\delta = 3$, $J_4 = h\bar{S}_1$ and h kills V . If $\delta = 1$, $J_4 = l_0P_3$ with a pencil P_3 of cubics without common zero, and $v_0 = V(l_0)$ is the only common root of J_4 and the only possible bad point. If it is not bad, Proposition 3.12 applies. If it is, $r_2(v_0) = 0$, and a general $g \in J_4$ has a simple root at v_0 and its other roots off the zeros of r_2 : if r_2 has a simple zero at v_0 , Corollary B.2(c)(i) applies with $m = r_2$; if $r_2 = l_0^2$, Corollary B.2(c)(ii) applies, as $xb_3 \in I_4^*$ for every $b_3 \in \bar{S}_3$, since $U \subseteq \bar{T}_4$.

In the cases with a cubic p killing V and $\pi(p) \neq 0$, a turning operator of type (b) (Section 2.8) applies. $\square$

B.9. Socle degree six. Here $\mathfrak{m}^7 \subseteq I$, and as in the proof of Theorem 6.1, a turning operator of type (b) (Section 2.8) and Lemma 3.9 reduce to the binary case with $\gamma_f = 0$; if $H_A(3) = 4$, then $I_3^* = xS_2$ and $D_k \subseteq \bar{T}_k$ for $k \geq 3$.

Theorem B.10. *Every local algebra with Hilbert function $(1, 3, 5, 4, 4, 2, 1)$ is smoothable.*

Proof. Here $I_3^* = xS_2$, $J_4 = \mathbb{C}p_4$, $V = D_5$ has dimension two, J_5 dimension four, and $D_6 = \mathbb{C}F$. Let $\Phi = F + F_5 + \dots$ be the lift of F . As $x \circ F = 0$, $x \circ \Phi$ has degree at most four, so $A_4 = x \circ F_5 \in D_4 \subseteq \bar{T}_4$, and $F_5 = XA_4 + G_5$ with $G_5 \in \bar{T}_5$.

F not a power, J_5 without base point: radial costs. Then $V = S_1 \circ F$, and $P^5 \notin D_5$ for every point $[P]$ of the line, so $\kappa \leq 5$ at every root of p_4 : $24 + 3 \cdot 4 = 36 < 40$, and Theorem 3.15 applies.

F not a power, J_5 with a base point $[P]$: a turning operator. Then $P^5 = \delta \circ F$ for some $\delta \in \bar{S}_1$, and integrating, $F = Y^6 + Z^6$ or $F = Y^5 Z$ in suitable coordinates. Let $g_2 = yz$, respectively z^2 , be the quadratic annihilator of F . Then $g_2 \circ \Phi$ has degree at most three, so its cubic part $X(g_2 \circ A_4) + g_2 \circ G_5$ lies in $D_3 = \bar{T}_3$, and $g_2 \circ A_4 = 0$. Choose $l \in \bar{S}_1$ nonzero with $(g_2^2 l) \circ G_5 = 0$, one linear condition, and $v_4 \in \bar{S}_4$ with $v_4 \circ F = -g_2 l \circ G_5$, possible as $g_2 l \circ G_5$ is orthogonal to g_2 . Then $u = g_2 l + v_4$ satisfies $u \circ \Phi \in T_{\leq 1}$. Its cubic part $g_2 l$ annihilates $V = S_1 \circ F$, so u sends every element of M of degree at most five into $T_{\leq 1}$; as $M = \mathbb{C}\Phi \oplus (M \cap T_{\leq 5})$, $u \in (I : \mathfrak{m}^2)$, with binary cubic part $g_2 l \neq 0$ (script 120_135_s6.sage, part 421). A turning operator of type (b) (Section 2.8) applies.

$F = Y^6$: a turning quartic. Then $p_4(Y) = 0$; choose z with $z(Y) = 0$, and $h_3 \in \bar{S}_3$ nonzero at Y and at the other roots of p_4 . The quartic $w_4 = zh_3$ kills Y^6 and maps every element of M of degree at most five into $T_{\leq 1}$, so $w_4 \in (I : \mathfrak{m}^2)$. It has a simple zero at the root $[Y]$ of p_4 and vanishes at no other root of p_4 , and Corollary B.2(a) applies, with $W_2 = \bar{S}_2$. $\square$

Theorem B.11. *Every local algebra with Hilbert function $(1, 3, 5, 4, 3, 2, 2)$ is smoothable.*

Proof. Here $I_3^* = xS_2$, $U = D_4$ has dimension three, and by [15, Lemma 2.16] $D_5 = \nu_5(\Gamma)$ and $D_6 = \nu_6(\Gamma)$ for Γ of length two on the line $x = 0$: in binary coordinates $D_k = \langle Y^k, Z^k \rangle$ (secant) or $\langle Y^k, Y^{k-1}Z \rangle$ (tangent). The pencil J_4 consists of quartics vanishing on Γ ; let h be its greatest common divisor, of degree two or three.

$\deg h = 3$: a turning operator. Then h kills U , D_5 and D_6 . Let $F = Y^6 + F_5 + \dots$ and T be the lifts of the two top forms. The elements $x \circ F$, $z \circ F$ of M have degree at most four, so $F_5 = AXY^4 + B_5$ with B_5 binary and $z \circ B_5 \in U$; similarly for T , and in the tangent case $z \circ T - \frac{1}{6}y \circ F$ has degree at most four. Then $h \circ B_5 = \alpha Y^2$, and $u = h - \frac{\alpha}{360}y^4 - \frac{\beta}{360}z^4$ (secant), respectively $u = h - \frac{\alpha}{360}y^4 - \frac{\beta}{60}y^3z$ (tangent), for suitable β , satisfies $u \circ F, u \circ T \in T_{\leq 1}$ (script 120_135_s6.sage, part 322). So u is a turning operator.

$\deg h = 2$: radial costs. Then $J_4 = hP_2$ with a base-point-free pencil P_2 of binary quadrics, and $h = yz$ or z^2 . For general $w_2 \in P_2$, $p = hw_2$ has two simple roots with $\kappa = 4$, and the roots of h are points of Γ , with $\kappa = 7 = s + 1$. So $24 + 2 + 2 + (7 - 2) \cdot 2 = 38 < 40$, and Theorem 3.15 applies. $\square$

Theorem B.12. *Every local algebra with Hilbert function $(1, 3, 5, 5, 3, 2, 1)$ is smoothable.*

Proof. We sort the algebras by the sextic top form F_6 . If it is not a sixth power, the algebra is Gorenstein, or has a turning operator, or [15, Theorem 3.6]

applies; if $F_6 = Y^6$, Corollary B.2 applies except in one configuration, which we sort by the second quintic top form. Here $I_3^* = x^2 S_1 \oplus x W_2$ with a pencil W_2 , so $D_3 = \bar{T}_3 \oplus \mathbb{C} X Q_D$ with $Q_D \in \bar{T}_2$ spanning $W_2^\perp$, and $\gamma_f = 0$. A shear $x \mapsto x + a$ with $a \in \bar{S}_2$ makes $f = x^2 + x^2 l + (\text{order} \geq 4)$. Let $U = D_4$, $V = D_5$, $D_6 = \mathbb{C} F_6$ and $\Phi = F_6 + F_5 + \cdots$; then $x^2 \circ F_6 = 0$, and $F_5 = X A_4 + H_5$ with binary A_4, H_5 .

F_6 not a sixth power. Writing $F_6 = X A_5 + G_6$, the third derivatives of F_6 show $A_5 = a L^5$; if $a \neq 0$, say $L = Y$, then $\dim S_1 \circ F_6 = 2$ forces $z \circ G_6 \in \mathbb{C} Y^5$, so $F_6 = Y^5(a X + b Y + c Z)$. So a linear change of X gives F_6 binary or $F_6 = X Y^5$. If F_6 is binary with $\dim \bar{S}_2 \circ F_6 = 3$ and no cubic annihilator, two coprime quartics annihilate F_6 and [15, Theorem 3.6] applies with $k_1 = k_2 = 4$. If it has a cubic annihilator p_3 , then p_3 kills $U = \bar{S}_2 \circ F_6 \ni A_4$, so $p_3 \circ F_5 \in \bar{T}_2 = \bar{S}_4 \circ F_6$, and $p_3 + u_4$ with $u_4 \circ F_6 = -p_3 \circ F_5$ is a turning operator. If $F_6 = Y^6 + Z^6$ or $Y^5 Z$, with quadratic annihilator g_2 , let $Q = g_2 \circ A_4$; then $Q \in \mathbb{C} Q_D$, as $g_2 \circ \Phi$ has top form $X Q + g_2 \circ H_5 \in D_3$. If $\text{rank } Q = 2$, the top forms of the derivatives of Φ are F_6 , $S_1 \circ F_6$, $U = \bar{S}_2 \circ F_6 \oplus \mathbb{C} A_4$, $\bar{T}_3 \oplus \mathbb{C} X Q$ (which uses $\text{rank } Q = 2$), $\bar{T}_2 \oplus X \bar{T}_1$, T_1 and T_0 , of total dimension $1+2+3+5+5+3+1 = 20$; so the derivatives of Φ span M , and A is Gorenstein (script 120_1355321.sage, part G) and smoothable; if $\text{rank } Q \leq 1$, a turning operator is obtained as follows (part Q). If $Q = 0$, choose $l' \in \bar{S}_1$ nonzero with $(g_2^2 l') \circ H_5 = 0$, and put $c = 0$; if $Q = L^2 \neq 0$, choose $l' \neq 0$ with $l' \circ Q = 0$, a binary quadric h_2 with $h_2 \circ Q \neq 0$, and $c = -(g_2^2 l') \circ H_5 / (h_2 \circ Q)$. Then the quadratic part of $(g_2 l' + c x h_2) \circ \Phi$ is binary and killed by g_2 , hence lies in $\bar{S}_4 \circ F_6$, and a quartic correction cancels it; the result lies in $(I : \mathfrak{m}^2)$ and has the nonzero binary cubic part $g_2 l'$, a turning operator of type (b). If $F_6 = X Y^5$, a cubic in $z \bar{S}_2$ corrected by a quartic is a turning operator (part X).

$F_6 = Y^6$. Suppose that some nonzero binary quartic $p \in I_4^* \cap \bar{S}_4$ and some $r \in W_2$ satisfy $r(v) \neq 0$ at all roots of p . As $Y^4 \in U$, $p(Y) = 0$. Choose z with $z(Y) = 0$ and $h_3 \in \bar{S}_3$ nonzero at $[Y]$ and at the other roots of p ; the quartic $w_4 = z h_3$ lies in $(I : \mathfrak{m}^2)$ for every tail, and it has a simple zero at $[Y]$ and vanishes at no other root of p . So Corollary B.2(a) applies (a member of the turning family lies in a complete intersection of length at most 35). This fails only if $Q_D = B^2$ is a square and B^4 lies in the binary part of U , the space of the X -free parts of its elements. In that case we sort the algebras by the second quintic top form T_5 , with $D_5 = \langle Y^5, T_5 \rangle$.

(R1) If T_5 is binary and $\dim \langle Y^4, S_1 \circ T_5 \rangle = 3$, either a cubic of $z \bar{S}_2$ annihilating T_5 gives a turning operator, or the binary annihilator of T_5 is a complete intersection (r_3, s_4) with $z \nmid r_3$, the binary quartics of I_4^* form the pencil $\langle z r_3, s_4 - \xi y r_3 \rangle$ with greatest common divisor z , and two general members share only the root $[Y]$ (a common root $v \neq [Y]$ would be a common root of r_3 and s_4), which is simple for one of them; then Corollary B.2(b) gives 38.

(R2) If $T_5 \in \{Z^5, Y^4Z\}$, let $g_2 = yz$, respectively z^2 , and $U = \langle Y^4, S_1 \circ T_5 \rangle \oplus \mathbb{C}R_4$. A cubic $c_3 + xq_2$ with $c_3 \in g_2\bar{S}_1$ is a turning operator, except when R_4 is binary, the coefficient ρ of R_4 in $z \circ H_5$ is nonzero, and $\text{rank } g_2 \circ R_4 = 2$. Then U is binary and $B^4 \in \langle Y^4, S_1 \circ T_5 \rangle$ (if $B^4 = u_0 + cR_4$ with $u_0 \in \langle Y^4, S_1 \circ T_5 \rangle$ and $c \neq 0$, then $g_2 \circ R_4$ is proportional to $g_2 \circ B^4$, of rank at most one): either $Q_D \in \{Y^2, Z^2\}$ with $T_5 = Z^5$, where I^* lies in the monomial ideal $L = (x^2, xW_2, x\bar{S}_3, yz\bar{S}_2, z^6, y^7)$ with $L^2 = I^*L$, and three elements of L have local length 39 (part R2), so Proposition 3.17(b) applies; or $Q_D = Y^2$ with $T_5 = Y^4Z$. In the last case write $z \circ H_5 = \mu Y^4 + \nu Y^3Z + \rho R_4$ and replace R_4 by $R_4 + \frac{\nu}{\rho} Y^3Z$, which keeps U , keeps R_4 binary and does not change $g_2 \circ R_4$. Now $D_3 = \bar{T}_3 \oplus \mathbb{C}XY^2$, $D_4 = \langle Y^4, Y^3Z, R_4 \rangle$, $D_5 = \langle Y^5, Y^4Z \rangle$, $D_6 = \mathbb{C}Y^6$, and, as the cubic part of f lies in x^2S_1 ,

$$N := M \cap T_{\leq 3} = T_{\leq 1} \oplus \langle XY, XZ, Y^2, YZ, Z^2 \rangle \oplus \bar{T}_3 \oplus \mathbb{C}XY^2.$$

Let Φ, Ψ be the lifts of Y^6, Y^4Z , and $Q_1 = Y^4 + \beta X^2$, Q_2, Q_R lifts of Y^4, Y^3Z, R_4 (closedness and binarity of D_4 give $A_4 = aY^4$). Modulo N ,

$$\begin{aligned} x \circ \Phi &= aQ_1, & z \circ \Phi &= \mu Q_1 + \rho Q_R, & y^2 \circ \Phi &= 30Q_1, \\ x \circ \Psi &= 0, & z \circ \Psi &= Q_1, & y \circ \Psi &= 4Q_2. \end{aligned}$$

Here the analogous monomial ideal has multiplicity 42 (part R2), so Proposition 3.17 gives no bound below 40. We instead move the second quintic top form out of the binary plane; its quartic derivative moves with it, and all other generators stay fixed. By the first row of relations, and as the derivatives of Q_1 and Q_R lie in N , the space $N_0 = N + \langle Q_1, Q_R, y \circ \Phi, \Phi \rangle$, of dimension 18, is closed under differentiation. Put $P = XY^4 + \frac{\beta}{3}X^3$, so that $x \circ P = Q_1$, $y \circ P = 4XY^3$ and $z \circ P = 0$, and

$$M_t = N_0 + \langle Q_2 + tXY^3, \Psi + tP \rangle.$$

By the second row, the derivatives of $\Psi + tP$ are $tQ_1, 4(Q_2 + tXY^3)$ and Q_1 modulo N , and those of $Q_2 + tXY^3$ lie in N ; so M_t is closed under differentiation. Its top forms are independent, so $\dim M_t = 20$, and M_t has the Hilbert function of A for every t , while D_4 and D_5 move: the second quintic top form is $Y^4Z + tXY^4$. As $M_0 = M$, and for $t \neq 0$ this form is not binary, these algebras are smoothable by (R3), and so is A (part D).

(R3) If T_5 is not binary, its X -part is a fourth power L^4 , and after a linear change of y, z by multiples of x (which keeps $\mathbb{C}x^2, x^2S_1 + xW_2$ and Y^6) and a gauge, either $T_5 = XZ^4$ and yz^2 is a turning operator, or $L = Y$ and $T_5 = XY^4 + G_5$ with G_5 binary. In the second case, for a suitable $c \in \mathbb{C}$, a turning operator is $z^3 + cy^4$ if $A_4 \notin \mathbb{C}Y^4$, and $p + cy^4$ with $p \in z\bar{S}_2$ nonzero and $p \circ R_4 = 0$ if $A_4 \in \mathbb{C}Y^4$ and $z \circ G_5 \in \mathbb{C}Y^4$, where such a change gives $T_5 = XY^4$ (part R3 (iii)–(v), with G_5, R_4 and all tails symbolic, and the loci where $p \mapsto p \circ R_4$ drops rank treated separately). In the remaining subcase, $z \circ G_5 \notin \mathbb{C}Y^4$ and $A_4 = aY^4$, closedness gives $z \circ H_5 \in \langle Y^4, z \circ G_5 \rangle$, that is, $H_5 = h_0Y^5 + \mu Y^4Z + \nu G_5$, and the following operator u is a turning operator (part R3 (ii), with G_5 and all tails symbolic, and the loci where

the map below has rank one treated separately). Choose $p \neq 0$ in the kernel of the map $z\bar{S}_2 \rightarrow \bar{T}_2/\mathbb{C}Y^2$, $p \mapsto p \circ G_5$, from a space of dimension three to one of dimension two, and define k by $p \circ G_5 = kY^2$. Then $u_3 = p - \frac{k}{12}xy^2$ kills T_5 , as $xy^2 \circ XY^4 = 12Y^2$; the quadratic part of $u_3 \circ \Phi$ is λY^2 for some $\lambda \in \mathbb{C}$, and, as $y^4 \circ Y^6 = 360Y^2$,

$$u = p - \frac{k}{12}xy^2 - \frac{\lambda}{360}y^4$$

sends Φ and the lift of T_5 into $T_{\leq 1}$. The subcase is needed: for $M = N + \langle Y^4, XY^3, R_4, y \circ \Phi, XY^4, \Phi \rangle$ with N as in (R2), $R_4 = Y^2Z^2 + Z^4$ and $\Phi = Y^6 + \frac{1}{3}Y^2Z^3 + \frac{1}{5}Z^5$, we have $T_5 = XY^4$, so every $p \in z\bar{S}_2$ lies in the kernel, but $(y^2z) \circ \Phi = 2Z^2$; here the turning operator is $z^3 - 6y^2z - \frac{1}{180}y^4$ (script 120_r2_anchors.sage, part (B12)). In all cases A is smoothable. $\square$

B.13. *Socle degree seven and eight.*

Theorem B.14. *Let A be a local algebra with $H_A(2) = 5$, $H_A(4) = H_A(5) = 2$ and $H_A(6) + H_A(7) + \dots \leq 3$. Then A is smoothable. No induction on the length is used. In length twenty this covers $(1, 3, 5, 4, 2, 2, 2, 1)$ and $(1, 3, 5, 4, 2, 2, 1, 1, 1)$.*

Proof. Apart from cases settled in [15], we exhibit a turning operator. By Macaulay's bound the tail $(H_A(6), H_A(7), \dots)$ is empty, (1) , (2) , $(1, 1)$, $(2, 1)$ or $(1, 1, 1)$. The first is [15, Corollary 4.10], the next three [15, Proposition 4.11(2),(4)]. For $(2, 1)$ and $(1, 1, 1)$: by [15, Lemma 2.16], $D_4 = \nu_4(\Gamma)$ and $D_5 = \nu_5(\Gamma)$ for Γ of length two supported on the line $x = 0$, which is secant, $I_\Gamma = (x, yz)$, tangent inside, (x, z^2) , or tangent outside, (z, x^2) , and the top form is $Y^7 + Z^7$, Y^6Z , XY^6 or Y^7 , respectively Y^8 . Let Φ be its lift. Shears and gauges remove most terms of the tails of degrees seven and six; in each case at most one coefficient, λ , δ or σ , remains (script 120_13542221.sage, respectively 120_135422111.sage, part (0)). An element $u = u_3 + u_4 + \dots$ of $\mathfrak{m}^3$ lies in $(I : \mathfrak{m}^2)$ if $u_3 \in (I_\Gamma)_3$ and $u \circ \Phi \in T_{\leq 1}$ (and $u \circ \Psi \in T_{\leq 1}$ for the lift Ψ of a second sextic top form, when the top form Y^7 has a one-dimensional space of first derivatives). In each case an explicit u with $u_3 \in \{yz^2, p_1yz^2 + p_2y^2z, p_0z^3 + p_1yz^2\}$ and corrections of degrees four to six in y, z satisfies this, with all lower tails arbitrary (part (2) of the scripts, exact over $\mathbb{Q}$ with all tail coefficients symbolic); its restriction to the plane $x = 0$ is yz or z^2 times a linear form, and [15, Corollary 4.6] applies. For instance, for the secant Γ with top form Y^7 , second sextic Z^6 and lifts $\Phi = Y^7 + Q_5 + \dots$, $\Psi = Z^6 + R_5 + \dots$, closedness gives $z \circ Q_5, y \circ R_5 \in \langle Y^4, Z^4 \rangle$, and $u = yz^2 - \frac{g}{30}z^4$, with g the coefficient of YZ^4 in R_5 , satisfies $u \circ \Phi, u \circ \Psi \in T_{\leq 1}$: the quadratic part of $u \circ \Psi$ is $12gZ^2 - 12gZ^2 = 0$. In the two sub-cases, one for each Hilbert function, where only $u_3 = z^3$ is available, the transverse cubic γ_f has no y^3 term; if it is not a cube, [15, Theorem 2.9(2)] applies, and otherwise [15, Theorem 4.5](2), with y and z exchanged, applies to $u + V'$ for an element $V' = xy^2 + \dots$ of $(I : \mathfrak{m}^2)$: the y^4 -coefficient of G_t is $c_0 + c_1t - \frac{t^2}{4}$. $\square$

B.15. The last Hilbert function of this group needs induction, through radial costs, and a refinement of them at one double root.

Theorem B.16. *Every local algebra with Hilbert function $(1, 3, 5, 4, 3, 2, 1, 1)$ is smoothable.*

Proof. The transverse cubics leave two branches. In the cube branch, where $\pi(I_3^*) = \mathbb{C}z^3$, jets give a turning operator. In the binary branch, radial costs apply except in two configurations, (M2) and (M1), where we find a turning operator or refine the radial count at a double root. As $H_A(6) = H_A(7) = 1$, $D_7 = \mathbb{C}L^7$ and $D_6 = \mathbb{C}L^6$, and $M = \mathbb{C}\Phi + \mathbb{C}y \circ \Phi + (M \cap T_{\leq 5})$ for the lift $\Phi = L^7 + P_6 + P_5 + \dots$, in coordinates with $L = Y$ (below).

Reduction. Here $\dim I_3^* = 6$. The transverse cubics of the elements of I with leading form x^2 form the affine space $\gamma_f + \pi(I_3^*)$. If it contains a non-cube, [15, Theorem 2.9(2)] applies. Otherwise, as in the proof of [15, Theorem 4.12], $\pi(I_3^*)$ is 0 or a line $\mathbb{C}z^3$ containing γ_f . In the second case $I_3^* \cap xS_2 = x^2S_1 + x\Pi$ for a plane $\Pi \subseteq \bar{S}_2$, and we may take $\gamma_f = \lambda z^3$ with $\lambda \neq 0$. If some $q \in \Pi$ has a term y^2 , [15, Theorem 4.5](2), with y and z exchanged, applies to an element $u \in I$ with leading form $xq + x^2l$: the y^4 -coefficient of G_t is $c_0 + c_1t - \frac{t^2}{4}q(0, 1, 0)^2$. The same holds for the element with leading form $z^3 + x(ay^2 + \dots)$ if $a \neq 0$. There remains the *cube branch* $I_3^* = x^2S_1 + \langle xyz, xz^2, z^3 \rangle$, $D_3 = \langle Y^3, Y^2Z, YZ^2, XY^2 \rangle$, $L = Y$.

Cube branch (no induction). The elements of I with leading forms x^2 , xyz , xz^2 , z^3 annihilate Φ , which forces $P_6 = \alpha X^2Y^4 + \rho XY^5 + \sigma XY^4Z + \beta Y^6 + \gamma Y^5Z + \delta Y^4Z^2 + \varepsilon Y^3Z^3$, and a gauge and two shears remove β, ρ, γ . If $\alpha \neq 0$, then γ_f has a term y^3 , which was excluded. If $\sigma \neq 0$ or $\varepsilon \neq 0$, [15, Theorem 4.5](2) applies to the element of I with leading form xyz , respectively z^3 , whose y^4 -coefficient is forced to be $-\sigma/210$, respectively $-\varepsilon/140$. If $\alpha = \sigma = \varepsilon = 0$, the element y^3z (if $\delta = 0$) or $y^2z^2 - \frac{\delta}{105}y^5$ lies in $(I : \mathfrak{m}^2)$, and [15, Theorem 4.5](2), (3) or (4) applies to it (script 120_13543211.sage, part R).

Binary branch. Now $I_3^* = xS_2$, $D_k \subseteq \bar{T}_k$ for $k \geq 3$, $L = Y$, $D_5 = \langle Y^5, G \rangle$, and every quartic of the pencil J_4 vanishes at $[Y]$; let h be the greatest common divisor of J_4 , so z divides h . If γ_f is not a cube, [15, Theorem 2.9(2)] applies; if it is a nonzero cube αl^3 , [15, Lemma F.4] applies to an element of I with cubic part xq , q not vanishing at the root of l . Let $\gamma_f = 0$. We use Theorem 3.15 with $p = h \cdot g$, g a general member of the residual pencil. At a root of g , $\kappa = 4$; at $[Y]$, $\kappa = 8$; at another root $v = [V]$ of h , $\kappa(v) = 5$ if $V^5 \notin D_5$ and 6 otherwise, which happens for at most one v . So the bound of Theorem 3.15 is $24 + 6 + 3 \cdot 2 = 36$ for $h = z$; at most $24 + 6 + 4 + 2 \cdot 2 = 38$ for $h = zl$, $l \neq z$; at most $24 + 6 + 4 + 3 + 2 = 39$ for $h = zl_1l_2$ with distinct $l_i \neq z$; and $24 + 6 + 2 \cdot 3 + 2 = 38$ for $h = zl^2$ with $l^5 \notin D_5$. In the remaining cases a finer argument is needed.

(M2) *h of degree three, otherwise.* Then h kills D_4 , D_5 and D_6 . After a shear and a gauge, $P_6 = B_6$ is binary without Y^6 , Y^5Z , and $P_5 = XA_4 + B_5$ with $A_4 \in D_4$; put $G' = z \circ B_6 \in D_5$ and $W' = \mathbb{C}Y^2 + \bar{S}_3 \circ G'$. Every element

of I with cubic part xr_2 shows $r_2 \circ A_4 \in W'$. Write $h \circ B_6 = \mu Y^3$. If $W' = \bar{T}_2$, the element $u = h - \frac{\mu}{840}y^4 + zc_3 + cy^5$, with $c_3 \in \bar{S}_3$ and $c \in \mathbb{C}$ chosen so that $h \circ B_5 - \frac{\mu}{840}y^4 \circ B_6 + c_3 \circ G' + 2520cY^2 = 0$ (possible as $W' = \bar{T}_2$), has $u \circ \Phi \in T_{\leq 1}$; if $G' \in \mathbb{C}Y^5$, then after normalizing $B_6 = 0$, $u = h - \frac{\mu'}{2520}y^5$ does. If $G' \notin \mathbb{C}Y^5$ and $W' \neq \bar{T}_2$, then $D_5 = \langle Y^5, Z^5 \rangle$ or $\langle Y^5, Y^4Z \rangle$ in suitable coordinates, and an element with cubic part $p_1yz^2 + p_2y^2z$, respectively $p_0z^3 + p_1yz^2$, and explicit corrections of degrees four and five, of the same shape as in the proof of Theorem B.14, works. In each case $u \in (I : \mathfrak{m}^2) \cap \mathfrak{m}^3$, with cubic part h or a binary cubic with a square factor; if it is not a cube, [15, Corollary 4.6] applies, and if it is z^3 , [15, Theorem 4.5](2) applies to $u + v$ with $v = xy^2 + \cdots \in I$ (script 120_13543211_m.sage, parts M2, M2-E, and 120_13543211.sage, part Z). If $h = z^2$, then $\bar{S}_1J_4 = z^2\bar{S}_3 = J_5$, so $D_5 = \langle Y^5, Y^4Z \rangle$, and after the normalization $P_6 = \delta Y^4Z^2$; the same argument covers $\delta \neq 0$, and $\delta = 0$ is (M1).

(M1) $h = z^2$, $P_6 = 0$. Then $A_4 = \alpha Y^4$ and $p \circ B_5 = \lambda(p)Y$ for $p \in J_4$, with a linear form λ on J_4 . If $\lambda = 0$, yz^2 lies in $(I : \mathfrak{m}^2)$ and [15, Corollary 4.6] applies. If $\lambda \neq 0$, choose $p_4 = z^2q_2 \in J_4$ with $\lambda(p_4) \neq 0$ and q_2 with two simple roots $v_1, v_2 \neq [Y]$, an independent $p' \in J_4$, and elements K, P_v, Q of I with leading forms xr_2 (r_2 not vanishing at $[Y], v_1, v_2$), p_4 and p' ; also $y^5z \in I$, as $P_6 = 0$. Closedness forces three jet identities (part M1, all tails symbolic): the coefficient of y^4 in the quartic part of K vanishes, and the coefficients of y^5 and y^6 in the quintic and sextic parts of P_v are 0 and $-\lambda(p_4)/5040$. Theorem 3.15 would give $24 + (8 - 2) \cdot 2 + 2 + 2 = 40$ at the double root $[Y]$. Instead, at $[Y]$, in the chart $x = At, y = t, z = wt$, the first identity gives $A(t, 0) = O(t^2)$ when $K = 0$ is solved for A , and the other two give $V(t, 0) = -\frac{\lambda(p_4)}{5040}t^2 + O(t^3)$ for the proper transform $V(t, w)$ of P_v restricted to $K = 0$; moreover $V(0, w) = w^2 \cdot (\text{unit})$. So $B = \mathbb{C}[[t, w]]/(V)$ is free of rank two over $\mathbb{C}[[t]]$, with $V = (\text{unit})(w^2 + a_1w + a_0)$ and $\text{ord}_t a_0 = 2$, and the determinant of multiplication by t^4w on B has order $8 + \text{ord}_t a_0 = 10$. The element $f + \sigma Q + \nu y^5z$ restricts to $G + \nu t^4w$ on B , and in $\det(G + \nu t^4w)$ the coefficient of ν^2 is this determinant; so for general ν the local length at $[Y]$ is at most 10. With 2 at each of v_1, v_2 , as in Lemma 3.14, the complete intersection has local length at most $24 + 10 + 2 + 2 = 38 < 40$; one example attains 38 (script 120_13543211_m.sage, parts M1 and M1-E). Theorem 2.5 concludes. $\square$

APPENDIX C. SOCLE DEGREE AT LEAST SIX

C.1. This appendix proves the eight Hilbert functions of Section 7 not treated there. Here $H_A(2) = 6$ and the socle degree s is at least six. When $H_A(s - 1) = H_A(s) = 1$, the form spanning D_s is a power, and we choose coordinates with $D_s = \mathbb{C}X^s$.

Theorem C.2. *Every local algebra with Hilbert function $(1, 3, 6, 3, 3, 2, 1, 1)$ is smoothable.*

Proof. By [15, Lemma 2.16], $D_3 = \nu_3(\Gamma)$ and $D_4 = \nu_4(\Gamma)$ for a subscheme Γ of length three, which is the cubic base scheme, as in the proof of [15, Corollary 3.15]. Moreover $D_7 = \mathbb{C}X^7$, $D_6 = \mathbb{C}X^6$, and $[X] \in \Gamma$. If Γ is curvilinear, its multiplicities add up to three, every weight is at most $s-3=4$, and $27 + 4 \cdot 3 = 39 < 40$: Theorem 2.7 applies. Otherwise $\Gamma = V(\mathfrak{m}_X^2)$, $D_3 = X^2T_1$, $e_X = 4$ and $k(X) = 7$, and Theorem 2.7 only gives 43. But D_3 contains X^2T_1 and no nonzero element of $X \cdot \bar{T}_2$, so [15, Lemma D.3] applies by the remark after it, with the weight $(6-3) \cdot 4$: $27 + 12 = 39 < 40$. $\square$

Theorem C.3. *Every local algebra with Hilbert function $(1, 3, 6, 4, 4, 1, 1)$ is smoothable.*

Proof. Excess linkage, sharpened by free quartic tails, settles the cases (i) and (ii) with $\sigma = 2$ below; in the others a point leaves along the line of X , on a dense subset of one irreducible family. Here $D_6 = \mathbb{C}X^6$, $D_5 = \mathbb{C}X^5$, $X^4 \in D_4$ and $X^3 \in D_3$. Let $\Phi = X^6 + P + Q + R$ be the lift of X^6 . By [15, Lemma A.11], either (i) the cubic base scheme $\mathfrak{B}$ is finite of length at most three, or (ii) $\mathfrak{B}$ has length four, $D_3 = \nu_3(\mathfrak{B})$ and $D_4 = \nu_4(\mathfrak{B})$, or (iii) $D_3 = \mathbb{C}[X, L]_3$ is the space of binary cubics of a line through $[X]$, D_4 is a hyperplane of $\mathbb{C}[X, L]_4$, and $D_3 = \nu_3(\Gamma)$, $D_4 = \nu_4(\Gamma)$ for a collinear subscheme Γ of length four. In (ii) let $\Gamma = \mathfrak{B}$. In Theorem 2.7, $k(X) = 6$, and $k(p) \leq 5$ at the other points, as $P^5 \notin D_5$.

(i) Here $e_X \leq \sum_p e_p \leq 4$ (Lemma 3.2), so $27 + \sum_p (k(p) - 3)e_p \leq 27 + 2 \cdot 4 + e_X \leq 39 < 40$, and Theorem 2.7 applies.

(ii) Suppose that the socle of the local ring of Γ at $[X]$ has dimension $\sigma = 2$; then Γ is $V(\mathfrak{m}_X^2) \cup \{q\}$ or has local ideal $(\eta^2, \eta\zeta, \zeta^3)$ at $[X]$, and $\sum_p e_p = 5$. The space K of [15, Lemma D.3] is $\mathcal{K}_4(\Gamma, X)$, of dimension $4 + \sigma + 1 = 7$ by [15, Lemma A.3], and an element $R' \in K$ with $x \circ R' \in D_3$ has all its derivatives in $\nu_3(\Gamma)$, so it lies in $D'_3 = \nu_4(\Gamma)$ [15, Lemma A.11], of dimension four. So the rank condition of [15, Lemma D.3] holds, all weights are at most 2, and $27 + 2 \cdot 5 = 37 < 40$.

(ii) and (iii) with $\sigma = 1$: one family with rays. We follow case (c) of the proof of [15, Theorem D.7], with $U = \nu_4(\Gamma)$ instead of a hyperplane; no incidence is needed, as U is determined by Γ . By dimensions, $M = T_{\leq 2} \oplus \nu_3(\Gamma) \oplus \{u + \tau(u) : u \in \nu_4(\Gamma)\} \oplus \mathbb{C}x \circ \Phi \oplus \mathbb{C}\Phi$, and this space is closed under differentiation if and only if

$$\begin{aligned} y \circ P, z \circ P &\in \nu_4(\Gamma), & \tau(X^4) &\equiv \frac{1}{30}x^2 \circ P, \\ y \circ Q &\equiv \tau(y \circ P), & z \circ Q &\equiv \tau(z \circ P) \end{aligned}$$

modulo D_3 . The first condition puts P in $\mathcal{K}_5(\Gamma, X)$, of dimension 6, which contains X^4T_1 . As in the proof of [15, Theorem D.7(c)], the subschemes Γ of length four through $[X]$ with $\sigma = 1$ form an irreducible variety of dimension six, over which the $\mathcal{K}_5(\Gamma, X)$ form a vector bundle of rank six, with total space $\mathcal{K}$. If $\beta_P: N_X \rightarrow \nu_4(\Gamma)/\mathbb{C}X^4$, $d \mapsto d \circ P$, vanishes, then $P \in X^4T_1$, a coordinate change of [15, Section 5.17] with $m = 2$ reduces P to cX^5 , then $\tau(X^4) \in D_3$, $X^4 \in M$, and $X^4 \notin N_X \circ M$, which has degree at

most three: a ray. On the open set $\mathcal{K}^+$ where $\beta_P \neq 0$ (nonempty by script 120_1364411.sage), the remaining conditions are linear in (τ, Q) and onto $(T_3/D_3)^3$, so the algebras form an affine bundle over $\mathcal{K}^+$, with R free: an irreducible family. In it, the reduced $\Gamma = \{X, L_1, L_2, L_3\}$ are dense, and for them $\mathcal{K}_5(\Gamma, X) = \nu_5(\Gamma) + X^4 T_1$; after the coordinate change that removes $X^4 T_1$ from P , we get $X^4 \in M$ and $N_X \circ M$ with quartic parts in $\langle L_1^4, L_2^4, L_3^4 \rangle$, so the line of X is a ray, as in [15, Theorem D.7]. The script checks the dimensions and the rank of the map on the tails on fourteen types of Γ , and the rays for reduced Γ . [15, Proposition 2.2(1)] concludes. $\square$

Theorem C.4. *Every local algebra with Hilbert function $(1, 3, 6, 3, 2, 2, 2, 1)$ is smoothable.*

Proof. Excess linkage applies unless the cubic base scheme is a fat point; then the algebras form three explicit families, each with an anchor. By [15, Lemma 2.16], $D_k = \nu_k(\Gamma)$ for $k = 4, 5, 6$ and $\Gamma \in \text{Hilb}_2(\mathbb{P}^2)$, $D_7 = \mathbb{C}F$ with $F \in \nu_7(\Gamma)$, and $D_3 = \nu_3(\Gamma) \oplus \mathbb{C}G$. The seven cubics of I_3^* have no common factor, as cubics with a common factor span at most six dimensions, so $\mathfrak{B}$ is finite; it contains Γ , and has length at most three, since $\min(4, \ell(\mathfrak{B})) \leq h_{\mathfrak{B}}(3) \leq \dim D_3 = 3$. If $\mathfrak{B}$ is not the fat point, then $\sum_p e_p \leq 3$, every weight is at most $s - 3 = 4$, and $27 + 12 = 39 < 40$: Theorem 2.7 applies.

If $\mathfrak{B} = V(\mathfrak{m}_X^2)$, then $D_3 = X^2 T_1$, and Γ is a double point at $[X]$, $\Gamma = V(z, y^2)$ in coordinates fixing X . So $D_k = \langle X^k, X^{k-1}Y \rangle$ for $k = 4, 5, 6$, and $F = X^6 Y$ or $F = X^7$ after a coordinate change. We write the lifts of the top forms with all tail coefficients as indeterminates, in these fixed graded data, and compute the closedness conditions (script 120_13632221.sage). For $F = X^6 Y$, together with $T_{\leq 2} \oplus D_3$, the derivatives of the lift span M , and the closedness conditions are solved by Gaussian elimination with constant pivots: the algebras form one family over an affine space. For $F = X^7$, with lifts of X^7 and $X^5 Y$, elimination with constant pivots leaves the two equations $uv = uw = 0$, where u is a tail coefficient and v, w can be solved for two further coefficients with constant pivots. So the parameter space is $V(u) \cup V(v, w)$, and each component is the graph of a polynomial map over an affine space. The nonpure family and the components $u = 0$ and $v = w = 0$ contain members in complete intersections of local length 38, 39 and 36, respectively, certified by stabilization (script, mode AN). Theorem 2.5 concludes. $\square$

C.5. For three Hilbert functions over a double point we use the following form of the expected-dimension count: the exceptional algebras need not form a family, they only have to be few.

Lemma C.6. *Let $\mathcal{Q}$ be an irreducible variety, $\mathcal{E}$ a vector bundle of rank N on $\mathcal{Q}$, ψ a section with zero set $\mathcal{Z}$, and $\mathcal{Z} \rightarrow \text{Hilb}_d(\mathbb{A}^3)$ a morphism. Let $\mathcal{Z}_{\text{bad}} \subseteq \mathcal{Z}$ be a constructible subset of dimension less than $\dim \mathcal{Q} - N$. If the schemes parametrized by the points of $\mathcal{Z} \setminus \mathcal{Z}_{\text{bad}}$ are smoothable, then so are all schemes parametrized by $\mathcal{Z}$.*

Proof. Every irreducible component $\mathcal{C}$ of $\mathcal{Z}$ has dimension at least $\dim \mathcal{Q} - N$ [15, Lemma 2.17], so it is not contained in the closure of $\mathcal{Z}_{\text{bad}}$, and $\mathcal{C} \setminus \overline{\mathcal{Z}_{\text{bad}}}$ is dense in $\mathcal{C}$. [15, Proposition 2.2(1)] concludes. $\square$

In all three applications the morphism $\mathcal{Z} \rightarrow \text{Hilb}_d(\mathbb{A}^3)$ comes from [15, Lemma 2.3(2)] on the charts of the Grassmannian, and Γ is a double point at $[X]$, $\Gamma_0 = V(z, y^2)$, so that $D_4 = U_4 = \langle X^4, X^3Y \rangle$, $D_5 = \langle X^5, X^4Y \rangle$, and $D_3 \supseteq E_0 = \nu_3(\Gamma_0) = \langle X^3, X^2Y \rangle$; we call the subspace D_3/E_0 of T_3/E_0 the *flag*. For the four-dimensional spaces D_3 of Theorems C.8 and C.9, [15, Lemma A.11] shows that the cubic base scheme $\mathfrak{B} \supseteq \Gamma_0$ is finite of length at most four, or $D_3 = \mathbb{C}[X, Y]_3$; the five-dimensional spaces D_3 of Theorem C.10 are treated with [15, Lemma A.12]. In Theorem 2.7 the weight of $[X]$ is $(s-3)e_X$, as $X^k \in D_k$ for $4 \leq k \leq s$, and every other point P of $\mathfrak{B}$ has weight one, as $P^4 \notin U_4$. For a four-dimensional space $D \supseteq E_0$ of cubics let $\mathcal{A}_D = \{A \in T_4 : y \circ A, z \circ A \in D\}$.

Lemma C.7. $\dim \mathcal{A}_D \leq 7$ for every four-dimensional $D \supseteq E_0$.

Proof. The map $\mathcal{A}_D \rightarrow (D/E_0)^2$, $A \mapsto (y \circ A, z \circ A)$, has kernel $\langle X^4, X^3Y, X^3Z, X^2Y^2 \rangle$, and its image lies in the kernel of $(u, v) \mapsto z \circ u - y \circ v \in T_2/\mathbb{C}X^2$, which is not zero, as $D \not\subseteq \langle X^3, X^2Y, X^2Z \rangle$. $\square$

Theorem C.8. *Every local algebra with Hilbert function $(1, 3, 6, 4, 2, 2, 2)$ is smoothable.*

Proof. Excess linkage settles the good flags, and the algebras over the bad flags are too few to fill a component (Lemma C.6). By [15, Lemma 2.16], $D_k = \nu_k(\Gamma)$ for $k = 4, 5, 6$ and a subscheme Γ of length two through $[X]$. If $\Gamma = \{X, Y\}$, [15, Proposition C.6](b) gives a ray. Let $\Gamma = \Gamma_0$, so $D_6 = \langle X^6, X^5Y \rangle$.

Good flags. With $s = 6$, Theorem 2.7 gives at most $27 + 6 + 2 = 35$ if $\mathfrak{B}$ has length two at $[X]$; $27 + 9 + 1 = 37$ if it is curvilinear of length three there; and 39 if it is $V(\mathfrak{m}_X^2)$ alone or two-generated of length four at $[X]$ ($e_X = 4$ in both cases, [15, Lemma A.4] and Section 2.6). The *bad* flags are those with $\mathfrak{B} = V(\mathfrak{m}_X^2) \cup \{P\}$ (40), with a local ideal (u^2, uv, v^3) at $[X]$ (42), or with $D_3 = \mathbb{C}[X, Y]_3$. They are determined by $P \in \mathbb{P}^2$, by a point of $\mathbb{P}^1$, respectively unique: a constructible set of flags of dimension at most two.

The incidence. Let $F = X^6 + A_5 + B_4 + H_3$ and $G = X^5Y + K_5 + C_4 + J_3$ be lifts of the top forms. Then $M = T_{\leq 2} + D_3 + \langle x^2 \circ F, x^2 \circ G, x \circ F, x \circ G, F, G \rangle$, and M is closed under differentiation if and only if: (Q5) the quartic parts $y \circ A_5, z \circ A_5, z \circ K_5, y \circ K_5 - \frac{1}{6}x \circ A_5$ lie in U_4 , a linear condition on (A_5, K_5) with an eight-dimensional solution space $\mathcal{S}_5$; and (Q4) four cubics $y \circ B_4, z \circ B_4, z \circ C_4, y \circ C_4 - \frac{1}{6}x \circ B_4$, corrected by constants depending on (A_5, K_5) , lie in D_3 (script 120_1364222.sage checks this criterion against direct closedness). The data $(D_3/E_0, (A_5, K_5), (B_4, C_4), (H_3, J_3))$ range over $\mathcal{Q} = \text{Gr}(2, 8) \times \mathcal{S}_5 \times T_4^2 \times T_3^2$, of dimension $12 + 8 + 30 + 20 = 70$, and (Q4) is a section of a bundle of rank $4 \cdot 6 = 24$; so every component of its zero set $\mathcal{Z}$ has dimension at least 46. Over a flag, for fixed (A_5, K_5) , the

solutions (B_4, C_4) of (Q4) are empty or a translate of the kernel, which maps to $\mathcal{A}_D$ with fibres translates of $\mathcal{A}_D$; so it has dimension at most 14 by Lemma C.7. Hence the part of $\mathcal{Z}$ over the bad flags has dimension at most $2 + 8 + 14 + 20 = 44 < 46$. The algebras over good flags are smoothable by Theorem 2.7, and Lemma C.6 concludes. $\square$

Theorem C.9. *Every local algebra with Hilbert function $(1, 3, 6, 4, 2, 2, 1, 1)$ is smoothable.*

Proof. The coefficient δ of X^4Y^2 in the lift of X^7 organizes the tails. When $\delta \neq 0$, a derivative of the lift of X^7 is a lift of X^4Y , and one cubic obstruction separates two irreducible families. When $\delta = 0$ we need no explicit deformation: the bad flags support at most 44 parameters, while every component has dimension at least 45. Here $D_7 = \mathbb{C}X^7$, $D_6 = \mathbb{C}X^6$, $D_4 = \nu_4(\Gamma)$ and $D_5 = \nu_5(\Gamma)$ with $[X] \in \Gamma$. If $\Gamma = \{X, Y\}$, [15, Proposition C.6](a) gives a ray. Let $\Gamma = \Gamma_0$. Now $s = 7$: the good flags are those where $\mathfrak{B}$ has length two at $[X]$ ($27 + 8 + 2 = 37$) or $\mathfrak{B}$ is curvilinear of length three at $[X]$ and nothing else ($27 + 12 = 39$); the bad flags are those with $D_3 \supseteq X^2T_1$, that is, $D_3 = X^2T_1 \oplus \mathbb{C}(Xg_2 + g_3)$ with binary g_2, g_3 , a $\mathbb{P}^6$; those where $\mathfrak{B}$ is curvilinear of length three at $[X]$ plus a point (dimension at most three) or of length four at $[X]$ (at most two); and $D_3 = \mathbb{C}[X, Y]_3$.

Let $\Phi = X^7 + \Phi_6 + \dots$ and $\Psi = X^4Y + \Psi_4 + \Psi_3$ be lifts of X^7 and X^4Y . Closedness forces $\Phi_6 = \alpha X^6 + \beta X^5Y + \gamma X^5Z + \delta X^4Y^2$ (script 120_13642211.sage, part A).

$\delta \neq 0$: *two families with anchors.* A gauge and the shears $y \mapsto y + \lambda x^2$, $z \mapsto z + \lambda' x^2$ [15, Section 5.17] remove α, β, γ , so $\Phi = X^7 + \delta X^4Y^2 + C_5 + R_4 + E_3$; write $\Psi = X^4Y + A_4 + B_3$. The lifts $x^3 \circ \Phi/210$ and $x \circ \Psi/4$ of X^4 and X^3Y have the cubic parts $h_X = \frac{4\delta}{35}XY^2$ and $h_Y = \frac{1}{4}\partial_X A_4$. Closedness in degree four says that $\partial_Z C_5 = aX^4 + bX^3Y$ and $\partial_Y C_5 - 2\delta A_4 = rX^4 + sX^3Y$ for some $a, b, r, s \in \mathbb{C}$; so $C_5 \in \mathbb{C}[X, Y]_5 \oplus \langle X^4Z, X^3YZ \rangle$, with a, b the coefficients of X^4Z, X^3YZ , and A_4 is determined by C_5 modulo U_4 . In degree three, closedness says exactly that the three cubics

$$\begin{aligned} \partial_Z R_4 - ah_X - bh_Y, \quad \partial_Y R_4 - rh_X - sh_Y - 2\delta B_3, \\ \bar{w} = ((12c_4 - \frac{8}{35}\delta^2)XY^2 + 20c_5Y^3)/(2\delta) \end{aligned}$$

lie in D_3 modulo E_0 , where c_4, c_5 are the coefficients of XY^4 and Y^5 in C_5 (script 120_r2_13642211.sage, all tails symbolic over $\mathbb{Q}(\delta)$). The first is affine in R_4 , with linear part $R_4 \mapsto \partial_Z R_4$ onto T_3/D_3 ; the second determines B_3 modulo D_3 ; and E_3 is free. So the algebras form affine bundles of constant rank over the set $Y_0 \sqcup Y_1$ of data $(\delta, C_5, D_3/E_0)$, where Y_0 is the locus $\bar{w} = 0$ and Y_1 the locus where $\bar{w} \neq 0$ and $\bar{w} \in D_3/E_0$. Both are irreducible. The member of the first family with $D_3 = \langle X^3, X^2Y, XZ^2, YZ^2 \rangle$ and $\Phi = X^7 + X^4Y^2 + X^3Y^2 + \frac{2}{105}XY^4$ lies in a complete intersection of local length 32, and the member of the second with $D_3 = \langle X^3, X^2Y, XY^2, XZ^2 \rangle$ and $\Phi = X^7 + X^4Y^2$ in one of local length 33 (part E). Theorem 2.5 concludes.

$\delta = 0$: *few bad flags*. Then the conditions in degree four give

$$\Phi_5 = \frac{10\beta\gamma}{21}X^3YZ + \frac{5\gamma^2}{21}X^3Z^2 + a_1X^5 + a_2X^4Y + a_3X^4Z + \varepsilon X^3Y^2$$

with free $a_1, a_2, a_3, \varepsilon$ (part C), and the remaining conditions are affine in (Φ_4, Ψ_4) , with linear part $L_\omega(\Phi_4, \Psi_4) = (y \circ \Phi_4 - \frac{\omega}{4}x \circ \Psi_4, z \circ \Phi_4, y \circ \Psi_4, z \circ \Psi_4)$ modulo D_3 , where $\omega = 2\varepsilon - \frac{10}{21}\beta^2$; Φ_3, Ψ_3 are free. The data (flag, $\alpha, \beta, \gamma, a_1, a_2, a_3, \varepsilon, \Phi_4, \Psi_4, \Phi_3, \Psi_3$) range over a variety of dimension $12 + 3 + 4 + 30 + 20 = 69$ and are cut out by a section of rank 24, so every component of the zero set has dimension at least 45. For $\omega = 0$ the kernel of L_0 is $\mathcal{A}_D \times \mathcal{A}_D$; for $\omega \neq 0$, $(\Phi_4, \Psi_4) \mapsto (B_4, C_4) = (\Psi_4, \frac{2}{3\omega}\Phi_4)$ identifies it with the space $\text{Sol}(D)$ of solutions of the homogeneous conditions (Q4) of Theorem C.8, of dimension $r(D) \leq 14$. As $\omega = 0$ is a hypersurface in the seven parameters, the part of the zero set over a set Σ of flags has dimension at most $\max(\dim \Sigma + 7 + r + 20, \dim \Sigma + 6 + 2 \dim \mathcal{A}_D + 20)$, and it suffices that $\dim \Sigma + r \leq 17$ and $\dim \Sigma + 2 \dim \mathcal{A}_D \leq 18$ on each set of bad flags. This holds when $\dim \Sigma \leq 3$. For $D = X^2T_1 \oplus \mathbb{C}(Xg_2 + g_3)$, write $A \in T_4$ as $X^2Q + Xa_3 + a_4$ with binary a_3, a_4 . Then $y \circ A, z \circ A \in D$ if and only if $(\partial_Y a_3, \partial_Y a_4) = \lambda_1(g_2, g_3)$ and $(\partial_Z a_3, \partial_Z a_4) = \lambda_2(g_2, g_3)$; a binary form of positive degree is determined by its gradient, and a pair is a gradient if and only if it is closed, so $\mathcal{A}_D \cong X^2T_2 \oplus \{(\lambda_1, \lambda_2) : (\lambda_1\partial_Z - \lambda_2\partial_Y)g_2 = (\lambda_1\partial_Z - \lambda_2\partial_Y)g_3 = 0\}$. A nonzero operator $\lambda_1\partial_Z - \lambda_2\partial_Y$ kills only powers of the linear form it kills, so $\dim \mathcal{A}_D = 7$ if $g_2 \in \mathbb{C}L^2$ and $g_3 \in \mathbb{C}L^3$ for one linear form L , and 6 otherwise. Similarly, for $(B_4, C_4) \in \text{Sol}(D)$, with $B_4 = X^2Q + Xa_3 + a_4$, $C_4 = X^2Q' + Xc_3 + c_4$ and $q_2 = Q(0, Y, Z)$, the conditions say that $y \circ B_4, z \circ B_4 \in D$ and that $\text{grad } c_3 = (\frac{1}{3}q_2 + \lambda_4g_2, \lambda_3g_2)$ and $\text{grad } c_4 = (\frac{1}{6}a_3 + \lambda_4g_3, \lambda_3g_3)$ for scalars λ_3, λ_4 . Closedness of the first determines q_2 up to $\mathbb{C}Y^2$, and that of the second is $(\lambda_4\partial_Z - \lambda_3\partial_Y)g_3 = -\lambda_2g_2/6$. The part of Q divisible by X , the form Q' and the multiple of Y^2 are free, so $r(D) = 3 + 6 + 1 + \dim \Lambda$, where Λ is the space of $(\lambda_1, \dots, \lambda_4)$ satisfying these conditions. Its dimension is computed as follows. If g_3 is not a cube, no nonzero operator kills g_3 , so $\Lambda = 0$. If $g_3 = \nu L^3 \neq 0$, the pairs (λ_1, λ_2) form a line if $g_2 \in \mathbb{C}L^2$ and are zero otherwise, and for each of them the last condition is one linear equation on (λ_3, λ_4) with nonzero linear part: $\dim \Lambda = 2$, respectively 1. If $g_3 = 0$, then $g_2 \neq 0$, the last condition says $\lambda_2g_2 = 0$, so $\lambda_2 = 0$ and λ_3, λ_4 are free, and $\lambda_1 \neq 0$ is possible only if $\partial_Z g_2 = 0$: $\dim \Lambda = 3$ if $g_2 \in \mathbb{C}Y^2$, and 2 otherwise. The strata of $[Xg_2 + g_3] \in \mathbb{P}^6$ are (part C checks the values at a representative of each stratum):

stratum of $[Xg_2 + g_3]$	$\dim \Sigma$	$\dim \mathcal{A}_D$	r	$\dim \Sigma + r$	$\dim \Sigma + 2 \dim \mathcal{A}_D$
g_3 not a cube	6	6	10	16	18
$g_3 \in \mathbb{C}^* L^3, g_2 \notin \mathbb{C} L^2$	4	6	11	15	16
$g_3 \in \mathbb{C}^* L^3, g_2 \in \mathbb{C} L^2$	2	7	12	14	16
$g_3 = 0, g_2$ not a square	2	6	12	14	14
$g_3 = 0, g_2 = L^2, L \notin \mathbb{C} Y$	1	7	12	13	15
$g_3 = 0, g_2 \in \mathbb{C} Y^2$	0	7	13	13	14

So the part of the zero set over the bad flags has dimension at most $17 + 7 + 20 = 44 < 45$, the algebras over good flags are smoothable by Theorem 2.7, and Lemma C.6 concludes. $\square$

Theorem C.10. *Every local algebra with Hilbert function $(1, 3, 6, 5, 2, 2, 1)$ is smoothable.*

Proof. By [15, Lemma 2.16], $D_4 = \nu_4(\Gamma)$, $D_5 = \nu_5(\Gamma)$ and $D_6 = \mathbb{C}F$ with $F \in \nu_6(\Gamma)$, for a subscheme Γ of length two; $D_3 \supseteq \nu_3(\Gamma)$ has dimension five. In suitable coordinates, F is X^6 (pure), $X^6 + Y^6$ (nonpure, $\Gamma = \{X, Y\}$) or X^5Y (nonpure, $\Gamma = \Gamma_0$).

Two points, pure top: a ray. [15, Proposition C.6](b) gives a ray.

Nonpure top: excess at a general member. Then $D_5 = S_1 \circ F$ and $D_4 = S_2 \circ F$, and $I_\Gamma = (z, q)$ with $q = xy$, respectively y^2 . Let $\Phi = F + P + R + E$ be the lift of F . As $z \circ F = 0$, $z \circ \Phi$ has degree at most four, so $z \circ P \in D_4 = S_2 \circ F = \mathbb{C}[x, y]_2 \circ F$, and a shear $z \mapsto z + h(x, y)$ (Lemma B.4) makes P binary. Then $M = T_{\leq 2} + D_3 + S \circ \Phi$, and this space has the right graded data if and only if the cubic parts $q \circ P$ of $q \circ \Phi$ and $z \circ R$ of $z \circ \Phi$ lie in D_3 . Fix F and P , and let $C = \nu_3(\Gamma) + \mathbb{C}q \circ P$, of dimension $n \in \{2, 3\}$; its cubic base scheme is a curvilinear subscheme of length n of the line $z = 0$. The algebras with $D_3 \supseteq C$, $R \in z^{-1}(D_3)$ and E free form an irreducible family, and for a general $D_3 \supseteq C$ the cubic base scheme is still that of C , by [15, Lemma A.16] applied $5 - n$ times. All weights are at most three, so excess linkage gives at most $27 + 3n \leq 36$ for the general member, and all members are smoothable.

Pure top over a double point: few bad flags. Let $F = X^6$ and $\Gamma = \Gamma_0$, with lifts $\Phi = X^6 + \Phi_5 + \Phi_4 + \Phi_3$ and $\Psi = X^4Y + \Psi_4 + \Psi_3$. As in the proof of Theorem C.9, closedness forces $\Phi_5 = a_1X^5 + a_2X^4Y + a_3X^4Z + \varepsilon X^3Y^2$, and the remaining four conditions are affine in (Φ_4, Ψ_4) , with linear part $(\Phi_4, \Psi_4) \mapsto (y \circ \Phi_4 - \frac{\varepsilon}{2}x \circ \Psi_4, z \circ \Phi_4, y \circ \Psi_4, z \circ \Psi_4)$ modulo D_3 , whose kernel has dimension at most $2 \dim \mathcal{A}_{D_3}$. As in Lemma C.7, now with $\dim D_3/E_0 = 3$, $\dim \mathcal{A}_{D_3} \leq 9$. If $D_3 = X^2T_1 \oplus \langle Xg_{2,i} + g_{3,i} \rangle_{i=1,2}$, then, writing $A = X^2Q + Xa_3 + a_4$ as in the proof of Theorem C.9, $A \in \mathcal{A}_{D_3}$ if and only if $p = (\partial_Y a_3, \partial_Y a_4)$ and $q = (\partial_Z a_3, \partial_Z a_4)$ lie in the plane W spanned by the pairs $(g_{2,i}, g_{3,i})$ and $\partial_Z p = \partial_Y q$. The map $W \times W \rightarrow \bar{T}_1 \oplus \bar{T}_2$, $(p, q) \mapsto \partial_Z p - \partial_Y q$, has rank at least two: otherwise $\partial_Y W$ and $\partial_Z W$ lie on one line, which forces $W = \langle (aY^2, bY^3), (cZ^2, dZ^3) \rangle$, whose images $\partial_Z W$ and $\partial_Y W$ are independent. So $\dim \mathcal{A}_{D_3} \leq 6 + 2 = 8$ if $D_3 \supseteq X^2T_1$. The weight of $[X]$ in Theorem 2.7 is $3e_X$, and every other point has weight one.

By [15, Lemma A.12], $\mathfrak{B}$ is finite of length at most five, or a line, possibly with a point. Excess linkage gives at most 39 when $\mathfrak{B}$ is finite and its part at $[X]$ is Γ_0 ($27 + 6 + 4$, by Lemma 3.2), curvilinear of length three ($27 + 9 + 2$), $V(\mathfrak{m}_X^2)$ alone, or curvilinear of length four alone. The other flags $D_3/E_0 \in \text{Gr}(3, T_3/E_0)$ have $\mathfrak{B}$ infinite (dimension at most five: the line through Γ_0 and $[G] \in \mathbb{P}^5$, or fewer), or $\mathfrak{B} \supseteq V(\mathfrak{m}_X^2)$ of length four or five (here $D_3 \supseteq X^2T_1$; choose $\mathfrak{B}_4$ of length four with $V(\mathfrak{m}_X^2) \subset \mathfrak{B}_4 \subseteq \mathfrak{B}$, either the fat point with a further point, two parameters, or a punctual extension, given by a hyperplane of $\mathfrak{m}_X^2/\mathfrak{m}_X^3$, a point of $\mathbb{P}^2$; then $D_3 = \nu_3(\mathfrak{B}_4) \oplus \mathbb{C}G$ with $[G] \in \mathbb{P}(T_3/\nu_3(\mathfrak{B}_4)) \cong \mathbb{P}^5$: at most $2 + 5 = 7$), or $\mathfrak{B}$ curvilinear of length four at $[X]$ plus a point, or of length five at $[X]$ (at most four, with $D_3 = \nu_3(\mathfrak{B})$). The data range over a variety of dimension $15 + 4 + 30 + 20 = 69$, cut out by a section of rank 20, so every component has dimension at least 49, while the part over the bad flags has dimension at most $5 + 4 + 18 + 20 = 47$, $7 + 4 + 16 + 20 = 47$ and $4 + 4 + 18 + 20 = 46$. Lemma C.6 concludes. $\square$

Theorem C.11. *Every local algebra with Hilbert function $(1, 3, 6, 4, 3, 2, 1)$ is smoothable.*

Proof. Here $D_6 = \mathbb{C}F$, with F in one or two variables, and $N = \mathfrak{m} \circ M + T_{\leq 2}$. All weights in Theorem 2.7 are at most $s - 3 = 3$.

A free cubic: cubic reduction. If $D_3 \not\subseteq N$, then $M = M' \oplus \mathbb{C}G$ for an algebra M' with Hilbert function $(1, 3, 6, 3, 3, 2, 1)$ [15, Lemma A.15], whose cubics and quartics are those of a subscheme Γ' of length three [15, Lemma 2.16], and whose cubic base scheme is Γ' . The algebras $M' \oplus \mathbb{C}G'$ form an irreducible family whose general member has the cubic base scheme Γ' [15, Lemma A.16], so excess linkage gives $27 + 3 \cdot 4 = 39$ for it, and A is smoothable. Let $D_3 \subseteq N$.

Finite cubic base scheme: excess. By [15, Lemma A.11], either $\mathfrak{B}$ is finite of length at most four, with $D_3 = \nu_3(\mathfrak{B})$ and $D_k \subseteq \nu_k(\mathfrak{B})$ if the length is four (for $k \geq 5$ because $I_{\mathfrak{B}}$ is generated in degrees at most four), or $D_3 = \mathbb{C}[X, Y]_3$ in suitable coordinates. In the first case $\sum_p e_p \leq 5$, and the bound $27 + 3 \cdot 4 = 39$ holds unless $\sum_p e_p = 5$. Then $\mathfrak{B}$ has one point $p = [P]$ that is not Gorenstein, with local ideal $\mathfrak{m}_p^2$ (and one further point) or (u^2, uv, v^3) [15, Lemma A.4]. If $P^5 \notin D_5$, the weight at p is at most two, and the bound is at most 38. If $P^5 \in D_5$ and $D_6 = \mathbb{C}P^6$, take $P = X$; then [15, Lemma D.3] applies, as the quartics R with $y \circ R, z \circ R \in D_3$ form a space of dimension seven [15, Lemma A.3], which meets $x^{-1}(D_3)$ in $\nu_4(\mathfrak{B})$, of dimension four: all weights become at most two, and the bound is 37. There remains the stratum (O2): $\sum_p e_p = 5$, $P^5 \in D_5$ and $D_6 \neq \mathbb{C}P^6$.

The stratum (O2): families with anchors. With $p = [X]$, either $\mathfrak{B} = V(\mathfrak{m}_X^2) \cup \{q\}$ (type A) or $\mathfrak{B}$ is supported at $[X]$ with local ideal $(\eta^2, \eta\zeta, \zeta^3)$, $\eta = y/x$, $\zeta = z/x$ (type B), and $X^5 \in D_5$. In type A, with $q = [Y]$, $F = X^5L + eY^6$; if $e \neq 0$, then $D_5 = \langle X^5, Y^5 \rangle$, and $x \circ F \in D_5$ forces $L \in \mathbb{C}X$, so $F = Y^6$ (type A1) or $F = X^6 + Y^6$ (A2) after scaling; if $e = 0$, then $F = X^5L$ with $L \notin \mathbb{C}X$ (A3). In type B, $F = X^5L$, as a term X^4Z^2

would give $\dim D_5 \geq 3$. Except in A1, $D_5 = S_1 \circ F$ and $D_4 = S_2 \circ F \oplus \mathbb{C}H$ with $H \in \nu_4(\mathfrak{B})$. The linear changes of coordinates preserving $\mathfrak{B}$ are, in type A, $X \mapsto aX$, $Y \mapsto bY$ and $Z \mapsto cZ + \lambda X + \lambda'Y$, which include the shears of Z , and in type B, $X \mapsto aX$, $Y \mapsto tY + \xi Z + \rho X$ and $Z \mapsto uZ + wX$. They rescale and shear the coefficients of H modulo $S_2 \circ F$, and bring the graded data to one of 18 normal forms (in A3 we normalize F instead of q): in A1 and A2, $D_4 = \langle X^4, Y^4, X^3l \rangle$ with $l \in \{Y, Z\}$; in A3, $F = X^5Y$, $q = [Q]$ with $Q \in \{Y, X + Y, Z\}$, and $H \in \{X^3Z, Q^4, X^3Z + Q^4\}$; in B, $F = X^5Y$ with $H \in \{X^3Z, X^2Z^2\}$, or $F = X^5Z$ with $H \in \{X^3Y, X^2Z^2, X^3Y + X^2Z^2\}$. If $D_5 = S_1 \circ F$, let z' span the linear forms killing F . Then the algebras with these graded data are exactly those with inverse system $M = T_{\leq 2} \oplus D_3 + \mathbb{C}(H + \tau_2) + S \circ \Phi$, $\Phi = F + R_5 + R_4 + R_3$, such that (a) $z' \circ R_5 = g \circ F + \mu H$ for some $g \in S_2$ and $\mu \in \mathbb{C}$, (b) $n \circ R_5 \in D_3$ for every $n \in S_2$ with $n \circ F = 0$, and (c) $z' \circ R_4 - g \circ R_5 - \mu \tau_2 \in D_3$: these conditions make the span of $T_{\leq 2}$, D_3 , $H + \tau_2$ and Φ with its derivatives closed, and they are necessary, as $z' \circ \Phi$ and $n \circ \Phi$ have low degree. Here R_5 ranges over a linear space, τ_2 and R_3 are free, and (c) is an affine condition on R_4 whose linear part $z' \circ T_4 \rightarrow T_3/D_3$ is onto; so these algebras form one family over an affine space. In A1, where $H = X^3l$, let $\Phi = Y^6 + R_5 + R_4 + R_3$, $X^5 + P_4 + P_3$ and $H + \tau_2$ be the lifts of Y^6 , X^5 and H , and write $x \circ R_5 = b_0X^4 + b_1Y^4 + \mu H$. Closedness confines R_5 and P_4 to fixed spaces of dimensions five and seven, and then reduces to twelve conditions, affine in (R_4, τ_2) , with linear part $(R_4, \tau_2) \mapsto (x \circ R_4 - \mu \tau_2, z \circ R_4)$ modulo $D_3 \oplus D_3$. This linear part has rank 12 if $\mu \neq 0$; if $\mu = 0$, it has rank 9, does not involve τ_2 , and the conditions are consistent (script 120_r2_1364321_A1.sage, all tails symbolic). So for each l the algebras of type A1 form two irreducible affine bundles, over the parameter strata $\mu \neq 0$ and $\mu = 0$. In all,

type	A1	A2	A3	B	total
graded normal forms	2	2	9	5	18
families	4	2	9	5	20

Each of the 20 families has a member in a complete intersection of local length 33, 34 and 33, 34 (A1, $l = Y, Z$); 33, 33 (A2); 33, 34, 34, 33, 35, 34 and 35, 34, 33 (A3, $Q = Y, X + Y, Z$, with H in the order above); 37, 33 and 33, 33, 34 (B) (script 120_1364321_02.sage), and Theorem 2.5 concludes.

The common line: families with anchors. Let $D_3 = \mathbb{C}[X, Y]_3$, so that all D_k consist of binary forms. If F has two essential variables, then $D_5 = S_1 \circ F$, and the closedness conditions say that $z \circ R_5 \in S_2 \circ F$ and that a cubic affine in R_4 , with linear part $z \circ R_4$, lies in $\mathbb{C}[X, Y]_3$. If $\dim S_2 \circ F = 2$, then $D_4 = S_2 \circ F \oplus \mathbb{C}Q_4$; writing $z \circ R_5 = q \circ F + \mu Q_4$, the cubic part of $g \circ \Phi$, for the quadric $g \in \mathbb{C}[x, y]_2$ killing F , is a binary cubic plus $\mu Z(g \circ Q_4)$ with $g \circ Q_4 \neq 0$, so $\mu = 0$. With $\dim S_2 \circ F = 3$, respectively 2, the algebras form one irreducible family each. If $F = X^6$, $D_5 = \langle X^5, Q \rangle$, and the algebras form one irreducible family when $\dim \langle X^4, x \circ Q, y \circ Q \rangle = 3$, and two for

each of the two types $Q = L^5$ and $Q = X^4L$ when this dimension is two: here $D_4 = \langle X^4 \rangle + S_1 \circ Q + \mathbb{C}Q_4$, and the compatibility condition reads $\mu\tau_{XZ^2} = \mu\tau_{YZ^2} = \mu\tau_{Z^3} = 0$, where μ is the coefficient of Q_4 in $y \circ R_5$ and $\tau_{XZ^2}, \tau_{YZ^2}, \tau_{Z^3}$ are coefficients of the cubic tail of the lift of Q_4 . Each of these seven families has a member in a complete intersection of local length 36, 38, 38, respectively 39, 38, 39, 38 (scripts 120_1364321_01.sage and 120_1364321_c2.sage). Theorem 2.5 concludes. $\square$

C.12. A successful excess test need not itself exhibit a linkage anchor. Its proof [15, Theorem 3.12, Step 1] produces one by changing the cubic tail E of the lift Φ of the top form by an element of T_3 , which changes the derivatives of Φ only modulo $T_{\leq 2}$. If E is a free coordinate of an irreducible family of algebras with $D_s = \mathbb{C}X^s$, this deformation stays in the family. So a member satisfying the inequality of Theorem 2.7 supplies a linkage anchor for the whole family, and Theorem 2.5 smooths it. We call this *excess at one member*.

Theorem C.13. *Every local algebra with Hilbert function $(1, 3, 6, 5, 3, 1, 1)$ is smoothable.*

Proof. Here $D_6 = \mathbb{C}X^6$, $D_5 = \mathbb{C}X^5$, $X^4 \in U = D_4$, $\dim U = 3$ and $\dim D_3 = 5$. Let $\Phi = X^6 + P + R + E$ be the lift of X^6 ; subtracting a multiple of $x \circ \Phi$ and changing coordinates as in [15, Section 5.17], we may assume that P has no terms X^5 , X^4Y , X^4Z . Closedness gives $P_Y, P_Z \in U$, where $P_Y = y \circ P$, and neither contains X^4 . Let $t = \dim \langle P_Y, P_Z \rangle$, and let $N = \mathfrak{m} \circ M + T_{\leq 2}$.

A binary quartic space gives a ray. If $\mathbb{C}[X, Y]_3 \subseteq D_3$ and $U \subseteq \mathbb{C}[X, Y]_4$ in coordinates fixing X , then $P \in \mathbb{C}[X, Y]_5$ after the normalization, redone in these coordinates (as P_Z is binary, $P = P_b + ZQ'$ with P_b binary, and as P_Y is binary, $Q' \in \mathbb{C}X^4$, which the normalization removes), $\frac{1}{30}x^2 \circ \Phi \equiv X^4$ modulo $D_3 + T_{\leq 2}$, so $X^4 \in M$, and the quartic parts of the elements of $\langle y, z \rangle \circ M$ lie in $\mathbb{C}P_Y \not\equiv X^4$; so the line of X is a ray [15, Theorem 2.11(2)]. The same holds if $t = 0$.

Four cubics: excess. Let $C \subseteq T_3$ be four-dimensional with $X^3 \in C$ and a finite cubic base scheme $\mathfrak{B}_C$. Then $\mathfrak{B}_C$ has length at most four, and a general five-dimensional $D \supseteq C$ has the same base scheme [15, Lemmas A.11 and A.16]; so $\sum_p e_p \leq 5$, $k(X) = 6$ and $k(p) \leq 5$ elsewhere. If $\sum_p e_p \leq 4$, excess linkage gives at most $27 + 3 \cdot 4 = 39$. If $\sum_p e_p = 5$, $\mathfrak{B}_C$ is a fat point and a point, or has a local ideal (u^2, uv, v^3) [15, Lemma A.4]; a fat point away from $[X]$ gives $27 + 3 + 2 \cdot 4 = 38$, and in the two remaining normal forms, $C = \langle X^3, X^2Y, X^2Z, Y^3 \rangle$ and $C = \langle X^3, X^2Y, X^2Z, XZ^2 \rangle$, the quartics X^2Y^2, X^2YZ, X^2Z^2 for the first and X^2Y^2, X^2YZ, XZ^3 for the second satisfy the rank condition of [15, Lemma D.3] for a general $D \supseteq C$, which gives $27 + 2 \cdot 5 = 37$. So every algebra with this Hilbert function and $D_3 = D$ is smoothable. If $D_3 \not\subseteq N$, then $M = M_0 \oplus \mathbb{C}G$ for an algebra M_0 with $D_3(M_0) = C$ of dimension four [15, Lemma A.15], and $X^3 \in C$, as it is the top form of $x^3 \circ \Phi \in \mathfrak{m} \circ M \subseteq M_0$; if C has a finite base scheme,

the algebras $M_0 \oplus \mathbb{C}G'$ form an irreducible family whose general member has a general $D_3 \supseteq C$, and otherwise $C = \mathbb{C}[X, L]_3$ [15, Lemma A.11], $U \subseteq \mathbb{C}[X, L]_4$ as $S_1 \circ U \subseteq C$, and the ray above applies. So we may assume $D_3 \subseteq N$.

$t = 2$: *excess at one member*. Then $U = \langle X^4, P_Y, P_Z \rangle$, $M = T_{\leq 2} + D_3 + \langle \Phi, x \circ \Phi, x^2 \circ \Phi, y \circ \Phi, z \circ \Phi \rangle$, and $D_3 \subseteq N$ means that $D_3 = \mathbb{C}X^3 + W_0 \circ P$ has dimension five, where $W_0 = \langle xy, xz, y^2, yz, z^2 \rangle$; then M is closed for all R and E . The pairs $([h], P)$ with $h \circ P \in \mathbb{C}X^3$ form a vector bundle of rank nine over $\mathbb{P}(W_0) \cong \mathbb{P}^4$, as $h \circ : T_5/X^4T_1 \rightarrow T_3/\mathbb{C}X^3$ is onto; the bundle contains $P = 0$, and the admissible P , those with $\dim \langle P_Y, P_Z \rangle = 2$ and $\dim(\mathbb{C}X^3 + W_0 \circ P) = 5$, form a nonempty open subset, on which $[h]$ is unique. With R and E free, these algebras form one irreducible family. Its member with $P = Y^5 + X^2Y^3 + Z^5 + X^2Z^3$ has five reduced base points $[X]$, $[X \pm \sqrt{10}Y]$, $[X \pm \sqrt{10}Z]$, of weights 3, 1, 1, 1, 1: $27 + 3 + 4 = 34$. Excess at one member concludes.

$t = 1$. In coordinates with $P_Z = 0$, put $Q = P_Y$ and $U = \langle X^4, Q, H \rangle$. Then $M = T_{\leq 2} + D_3 + \langle \Phi, x \circ \Phi, x^2 \circ \Phi, y \circ \Phi, H + \tau \rangle$ with $\tau \in T_3$, closed if and only if $C = S_1 \circ U \subseteq D_3$ and $z \circ R \in D_3$; and $D_3 \subseteq N$ means $D_3 = C + \mathbb{C}z \circ R$. If $\dim C = 4$, the algebras with fixed (P, U) , $D_3 \supseteq C$ and $z \circ R \in D_3$ form an irreducible family whose general member has a general $D_3 \supseteq C$; it is smoothable by the four-cubic paragraph if $\mathfrak{B}_C$ is finite, and by the ray if $C = \mathbb{C}[X, L]_3$, as then $U \subseteq \mathbb{C}[X, L]_4$. If $\dim C = 5$, then $D_3 = C$; let $C_Q = \langle X^3, x \circ Q, y \circ Q \rangle$, of dimension n . If $n = 3$, the data $(P, [w], H)$, where P is a binary quintic and $[w]$ spans the kernel of $S_1 \rightarrow T_3/C_Q$, $w \mapsto w \circ H$, form the open subset $\dim(C_Q + S_1 \circ H) = 5$ of a vector bundle over the pairs $(P, [w])$, whose fibre is the kernel, of dimension eight, of the surjection $w \circ : T_4 \rightarrow T_3/C_Q$; so they form one irreducible family, nonempty by its member $P = Y^5 + X^2Y^3$, $H = Z^4 + X^2Z^2$ of excess 34. If $n = 2$, then by [15, Lemma 2.16], $C_Q = \nu_3(\Gamma)$ and $\langle X^4, Q \rangle = \nu_4(\Gamma)$ for a subscheme $\Gamma \ni [X]$ of length two, and after the normalization $P = Y^5$ or $P = X^3Y^2$; as H ranges over the open set where $\dim C = 5$, these form two irreducible families, with the members $H = Y^4 + Z^4 + XYZ^2$ of excess $27 + 3 + 2 = 32$ (base $\{[X], [Y]\}$) and $27 + 3 \cdot 2 = 33$ (a double point at $[X]$). Excess at one member concludes (script 120_1365xx_check.sage). $\square$

APPENDIX D. COMPUTATIONS

D.1. This appendix lists the computer calculations, in SageMath 10.7. All are exact, over $\mathbb{Q}$, polynomial rings over $\mathbb{Q}$, explicitly specified localizations of them, or function fields over $\mathbb{Q}$ such as $\mathbb{Q}(\delta)$; whenever a parameter is inverted, the excluded locus is treated separately. Local lengths are certified by stabilization (Section 2.4) or by local standard bases; identities “with all tails symbolic” hold in a polynomial ring in all tail coefficients. We mark *universal* calculations, valid for all values of the parameters or exhaustive (U), *exact anchors and representatives*, which certify one member

while exhaustion and irreducibility are proved in the text (A), and *sanity checks*, not used in the proofs (S). The scripts, their outputs and a manifest accompany this paper as the archive `twenty_points_scripts.tar.gz` on the author's web page, with SHA-256 checksum `d28a0135d6b51a2f0aca6e0847ce8b0f0161842ddcc202396616829efc1c060c`; a reader verifies them from the extracted archive with `./run_all.sh`, which first checks every file against `SHA256SUMS` (`./run_all.sh --full` also runs the three 120_1364321 scripts, about three hours in all); `test_release.sh` is the author-side test run in the source repository. Scripts are named without the suffix `.sage`.

script	result	establishes
<code>classify</code>	Tables 1, 2	U: the 199 Hilbert functions and their rows
<code>120_r2_anchors</code>	3.2, 3.5, 5.1, 7.1, A.14, B.12	U: colengths in 3.2 (L5); A: anchors of length 36 in 5.1 (S4), 34 in 7.1 (U1) and 27 in A.14 (A14), with the pieces of the hand proofs; S: dual tails (R); the example in case (R3) of B.12 (B12)
<code>120_socle4</code>	5.1	A: anchor of length 36 (B), deformation (D)
<code>120_136451</code>	3.6, A.5	A: the three triples, length 39; realizations
<code>120_136433</code>	A.6	A: realizations, length 39
<code>120_136442</code>	A.7	A: anchors of length 36; tail dimensions 10, 13
<code>120_1365_lineplus</code>	A.3, A.10, A.11	A: tail dimensions; deformation; S: rays (A.3 is proved by hand)
<code>120_136532_embedded,</code> <code>120_r2_136532</code>	A.11	A: tail dimensions at orbit representatives; limits; U: the kernels for types (3), (4)
<code>120_line_case_d</code>	A.9	A: bounds $36 + \delta$; members over 36 data
<code>120_136721</code>	A.13	A: two members, length 36
<code>120_136631, _binary, _pure</code>	A.14	A: members, length 36; S: dimensions
<code>120_136622</code>	A.15	A: anchor plus a cubic, length 32
<code>120_135_lemmaR</code>	3.14, 3.15, B.2	S: the bounds are attained
<code>120_135_s6</code>	6.1, B.10, B.11	U: turning operators, all tails symbolic (in part 331 for sampled h)
<code>120_135_tails, _checks</code>	B.6	A: fibre dimension 24; anchor I_0 (32)
<code>120_1355321</code>	B.12	U: parts G, Q, X, R3, D; A: part R2
<code>120_13542221, _135422111</code>	B.14	U: normal forms; elements u
<code>120_13543211, _m</code>	B.16	U: cube branch, turning, jets of (M1); A: (M1), 38
<code>120_136322111, _rho</code>	7.1	U: rays, families, the remainders ρ_1, ρ_2 ; A: base schemes, anchors
<code>120_13632221</code>	C.4	U: closedness; A: anchors 38, 39, 36
<code>120_1364222, _13642211,</code> <code>120_r2_13642211</code>	C.8, C.9	U: the quintic space S_5 of C.8; the closedness conditions of C.9, all tails symbolic (Φ_6 ; $\delta = 0$ in degree four; $\delta \neq 0$); A: ranks at representatives of the bad flags; anchors 32, 33; S: closedness on sampled members
<code>120_1364321_01, _c2, _02,</code> <code>120_r2_1364321_A1</code>	C.11	A: anchors 36–39; (O2) anchors 33–37; U: the ranks of type A1
<code>120_1365xx_check</code>	C.13, C.10	A: test members; S: random members
<code>120_1364411</code>	C.3	A: dimensions, tail rank 18, rays
<code>120_free_tail</code>	C.2, C.3	S: lengths along [15, Lemma D.3]

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