Universität Konstanz
Fachbereich Mathematik und Statistik
Daniel Plaumann
Summer 2015

CLASSICAL ALGEBRAIC GEOMETRY

3rd problem sheet
Tutorial on 5 May 2015

These problems study the Grassmannian $\mathbb{G} = \mathbb{G}(1,3) = G(2,4)$ parametrizing lines in $\mathbb{P}^3$.

1. The Plücker embedding puts $\mathbb{G}$ into $\mathbb{P}(\bigwedge^2 K^4) \cong \mathbb{P}^5$. Writing $z_{ij} = v_i \wedge v_j$, $0 \leqslant i < j \leqslant 3$, show that the image is the quadratic hypersurface

$$\mathcal{V}(z_{01}z_{23} - z_{02}z_{13} + z_{03}z_{12})$$

called the *Plücker quadric*.

2. For any point $p \in \mathbb{P}^3$ and plane $H \subset \mathbb{P}^3$ with $p \in H$, let $\Sigma_{p,H} \subset \mathbb{G}$ be the set of lines in $\mathbb{P}^3$ passing through p and lying in H. Show that, under the Plücker embedding, $\Sigma_{p,H}$ is a line in $\mathbb{P}^5$, and that, conversely, every line in $\mathbb{G} \subset \mathbb{P}^5$ is of the form $\Sigma_{p,H}$ for some choice of p, H. *[Ha], Ex. 6.4*

3. For any point $p \in \mathbb{P}^3$, let $\Sigma_p \subset \mathbb{G}$ be the set of lines in $\mathbb{P}^3$ passing through p; for any plane $H \subset \mathbb{P}^3$, let $\Sigma_H \subset \mathbb{G}$ be the locus of lines lying in H. Show that, under the Plücker embedding, both Σ_p and Σ_H are carried into planes in $\mathbb{P}^5$. Show that, conversely, any plane $\Lambda \subset \mathbb{G} \subset \mathbb{P}^5$ is either of the form Σ_p for some point p or of the form Σ_H for some plane H. *[Ha], Ex. 6.5*

4. Let $\ell_1, \ell_2 \subset \mathbb{P}^3$ be skew lines (i.e. $\ell_1 \cap \ell_2 = \varnothing$). Show that the set $Q \subset \mathbb{G}$ of lines in $\mathbb{P}^3$ meeting both is the intersection of $\mathbb{G}$ with a three-dimensional subspace of $\mathbb{P}^5$. Deduce that $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. What happens if ℓ_1 and ℓ_2 meet? *[Ha], Ex. 6.6*