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CLASSICAL ALGEBRAIC GEOMETRY

10th problem sheet
Tutorial on 23 June 2015

1. Show that the constant term of the Hilbert polynomial of a projective variety is an integer. (*Suggestion.* Show more generally that if $p \in \mathbb{Q}[z]$ is a polynomial such that $p(m) \in \mathbb{Z}$ for all sufficiently large $m \in \mathbb{Z}$, then $p(0) \in \mathbb{Z}$.)

2. Show that a projective variety $X \subset \mathbb{P}^n$ has degree 1 if and only if X is a linear subspace.

3. Let $\varphi\colon \mathbb{P}^n \to \mathbb{P}^n$ be an automorphism, i.e. φ is bijective and φ, φ^{-1} are morphisms.

(a) Show that if $H \subset \mathbb{P}^n$ is a hyperplane, then so is $\varphi(H)$.
(*Hint.* Take any line L meeting H in a single point. Now use a version of Bézout's theorem to conclude that $\varphi(H)$ has degree 1.)

(b) Use (a) to show that φ is linear, i.e. that there exist linear forms $L_0, \dots, L_n \in K[Z_0, \dots, Z_n]$ such that $\varphi[Z] = [L_0(Z), \dots, L_n(Z)]$ for all $[Z] \in \mathbb{P}^n$.

(c) Conclude that the automorphism group of $\mathbb{P}^n$ is the linear group $\mathrm{PGL}_{n+1}(K)$.