Universität Konstanz
Fachbereich Mathematik und Statistik
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Summer 2015

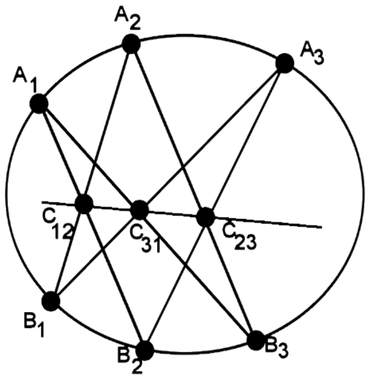
CLASSICAL ALGEBRAIC GEOMETRY

11th problem sheet
Tutorial on 30 June 2015

1. Use the Cayley-Bacharach theorem to deduce *Pascal's Theorem*: Given an irreducible conic Q in $\mathbb{P}^2$ and six distinct points $A_1, A_2, A_3, B_1, B_2, B_3 \in Q$, show that the points

$$C_{12} = \overline{A_1B_2} \cap \overline{A_2B_1}, \quad C_{13} = \overline{A_1B_3} \cap \overline{A_3B_1}, \quad C_{23} = \overline{A_2B_3} \cap \overline{A_3B_2}$$

are collinear.



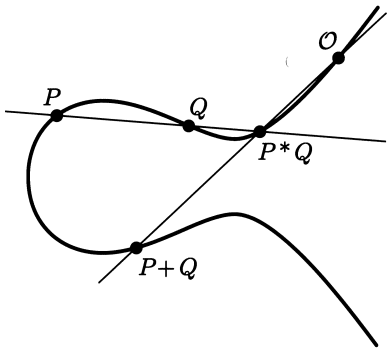
from https://terrytao.wordpress.com

(turn page)

2. Let C be a smooth cubic curve in $\mathbb{P}^2$. We will introduce the *group law on C*:
Given $P, Q \in C$, let $\ell_{P,Q}$ be the line $\overline{PQ}$ if $P \neq Q$ and let $\ell_{P,P}$ be the tangent line $\mathbb{T}_P C$. Then $\ell_{P,Q}$ intersects C in P, Q and one further point which we denote $P * Q$. Assume that $\mathcal{O}$ is an inflection point of C, which means that the tangent line $\mathbb{T}_{\mathcal{O}} C$ intersects C only at $\mathcal{O}$, so that $\mathcal{O} * \mathcal{O} = \mathcal{O}$. Define an operation

$$P + Q = (P * Q) * \mathcal{O}$$

for $P, Q \in C$.



taken from *Silverman, Tate: Rational Points on Elliptic Curves, Fig. 1.6*

Show that $(C, +)$ is an abelian group with neutral element $\mathcal{O}$, by proving the following for all $P, Q, R \in C$.

(a) $P + Q = Q + P$;
(b) $\mathcal{O} + P = P + \mathcal{O} = P$;
(c) Put $-P = P * \mathcal{O}$. Then $(-P) + P = P + (-P) = \mathcal{O}$;
(d) $(P + Q) + R = P + (Q + R)$.

Hint for (d): Assume that the nine points

$$P, Q, R, \mathcal{O}, P * Q, P + Q, Q * R, Q + R, (P + Q) * R$$

are all distinct. Use the Cayley-Bacharach theorem to verify that $(P + Q) * R$ is the same point as $P * (Q + R)$.

Remark. We did not show that every smooth cubic possesses an inflection point. Suppose that C is given by an equation of the form

$$Y^2 Z = aX^3 + bX^2 Z + cXZ^2 + dZ^3$$

(in what is called the *Weierstraß form*). Then $[0, 1, 0] \in C$ is an inflection point.