# CLASSICAL ALGEBRAIC GEOMETRY

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## A brief inaccurate history of algebraic geometry

| | |
|---|---|
| 1800 - 1880 | **Projective geometry.** Emergence of 'analytic' geometry with cartesian coordinates, as opposed to 'synthetic' (axiomatic) geometry in the style of Euclid. (*Celebrities:* Plücker, Hesse, Cayley) |
| 1820 - 1920 | **Complex analytic geometry.** Powerful new tools for the study of geometric problems over $\mathbb{C}$. (*Celebrities:* Abel, Jacobi, Riemann) |
| 1880 - 1940 | **Classical school.** Perfected the use of existing tools without any 'dogmatic' approach. (*Celebrities:* Castelnuovo, Segre, Severi, M. Noether) |
| 1920 - 1950 | **Algebraization.** Development of modern algebraic foundations ('commutative ring theory') for algebraic geometry. (*Celebrities:* Hilbert, E. Noether, Zariski) |
| from 1950 | **Modern algebraic geometry.** All-encompassing abstract frameworks (schemes, stacks), greatly widening the scope of algebraic geometry. (*Celebrities:* Weil, Serre, Grothendieck, Deligne, Mumford) |
| from 1990 | **Computational algebraic geometry** Symbolic computation and discrete methods, many new applications. (*Celebrities:* Buchberger) |

# Literature


### Primary source

[Ha] J. Harris, *Algebraic Geometry: A first course*. Springer GTM 133 (1992)

### Classical algebraic geometry

[BCGB] M. C. Beltrametti, E. Carletti, D. Gallarati, G. Monti Bragadin. *Lectures on Curves, Surfaces and Projective Varieties. A classical view of algebraic geometry*. EMS Textbooks (translated from Italian) (2003)

[Do] I. Dolgachev. *Classical Algebraic Geometry. A modern view*. Cambridge UP (2012)

### Algorithmic algebraic geometry

[CLO] D. Cox, J. Little, D. O'Shea. *Ideals, Varieties, and Algorithms*. Springer UTM (1992)

[EGSS] D. Eisenbud, D. R. Grayson, M. Stillman, B. Sturmfels. *Computations in Algebraic Geometry with Macaulay 2*. Springer (2002).

### 'Big books'

[GH] P. Griffiths, J. Harris. *Principles of Algebraic Geometry*. John Wiley & Sons (1978)

[Hs] R. Hartshorne. *Algebraic Geometry*. Springer GTM 52 (1977)

[Sh] I. Shafarevich. *Basic Algebraic Geometry I: Varieties in projective space*. Springer (translated from Russian) (1974)


# §1 Projective Varieties

## Affine varieties

| | |
|---|---|
| $K$ | algebraically closed field |
| $\mathbb{A}^n = K^n$ | affine space |

$V \subset \mathbb{A}^n$ is an **affine variety** if there is a set of polynomials $M \subset K[x_1, \ldots, x_n]$ such that

$$V = \mathcal{V}(M) = \{p \in \mathbb{A}^n : f(p) = 0 \text{ for all } f \in M\}.$$

If $I$ is the **ideal** generated by $M$, then $\mathcal{V}(M) = \mathcal{V}(I)$. By the **Hilbert Basis Theorem**, there is a **finite** subset $M' \subset M$ that also generates $I$, so in particular $\mathcal{V}(M') = \mathcal{V}(M)$.

If $I$ and $J$ are two ideals in $K[x_1, \ldots, x_n]$, then

$$V(I) \cup V(J) = V(IJ) = \mathcal{V}(I \cap J)$$
$$V(I) \cap V(J) = V(I + J)$$

where $IJ$ is the ideal generated by all products $fg$, $f \in I$, $g \in J$.

## Projective space

Let $V$ be a $K$-vector space.

$$\mathbb{P}(V) = \{\text{one-dimensional subspaces of } V\}, \text{ the } \textbf{projective space of } V$$

$$\mathbb{P}^n = \mathbb{P}K^{n+1} = \left(K^{n+1} \setminus \{0\}\right)/\sim$$
$$\text{where } v \sim w \iff \exists \lambda \in K^\times : v = \lambda w.$$

Points of $\mathbb{P}^n$ are denoted in **homogeneous coordinates** $[Z_0, \ldots, Z_n]$ where

$$[Z_0, \ldots, Z_n] = [\lambda Z_0, \ldots, \lambda Z_n] \text{ for } \lambda \in K^\times.$$

## Projective varieties

A polynomial $F \in K[Z_0, \ldots, Z_n]$ is **not** a function on $\mathbb{P}^n$, since in general

$$F(Z_0, \ldots, Z_n) \neq F(\lambda Z_0, \ldots, \lambda Z_n).$$

If $F$ is **homogeneous** of degree $d$, then

$$F(\lambda Z_0, \ldots, \lambda Z_n) = \lambda^d F(Z_0, \ldots, Z_n).$$

So given a set $M$ of homogeneous polynomials in $K[Z_0, \ldots, Z_n]$, it makes sense to define

$$\mathcal{V}(M) = \left\{p \in \mathbb{P}^n : f(p) = 0 \text{ for all } f \in M\right\}, \text{ a } \textbf{projective variety}.$$

## The Zariski topology

The projective (resp. affine) varieties in $\mathbb{P}^n$ (resp. $\mathbb{A}^n$) form the closed sets of a topology, the **Zariski topology**. Projective space is covered by the open subsets

$$U_i = \left\{[Z_0, \ldots, Z_n] \in \mathbb{P}^n : Z_i \neq 0\right\} = \left\{[Y_0, \ldots, Y_{i-1}, 1, Y_{i+1}, \ldots, Y_n] \in \mathbb{P}^n\right\}.$$

The map

$$U_i \to \mathbb{A}^n, [Z_0, \ldots, Z_n] \mapsto (Z_0/Z_i, \ldots, Z_{i-1}/Z_i, Z_{i+1}/Z_i, \ldots, Z_n)$$

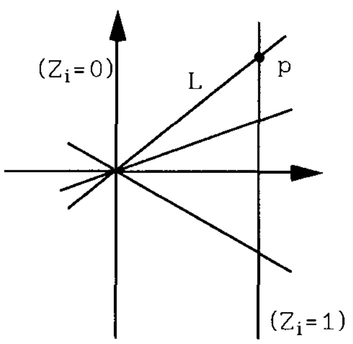
is a homeomorphism. The inverse map is

$$\mathbb{A}^n \to U_i, (z_0, \ldots, z_{i-1}, z_{i+1}, z_n) \mapsto [z_1, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n].$$

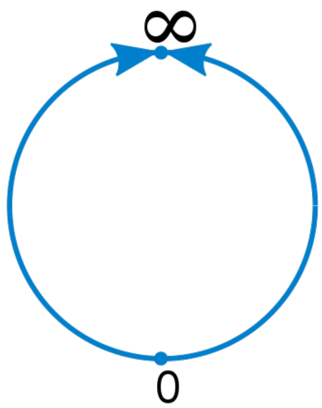
Thus $\mathbb{P}^n$ is covered by $n+1$ copies of $\mathbb{A}^n$.

## How to think about projective space

### Projective line




**Exact picture**
**[Ha, p.4]**



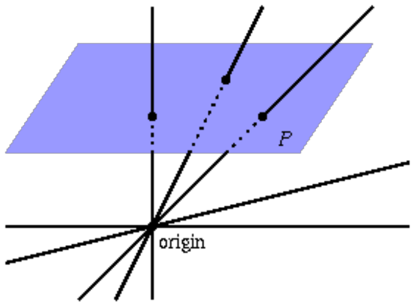

**Real topological picture**
**(O. Alexandrov - Wikimedia Commons)**



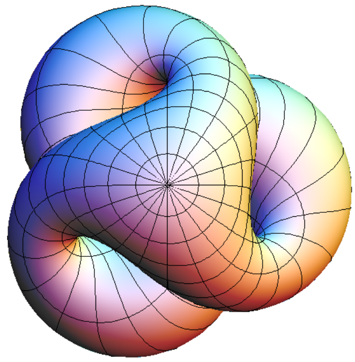
**Intuitive picture**

# How to think about projective space

## Projective plane

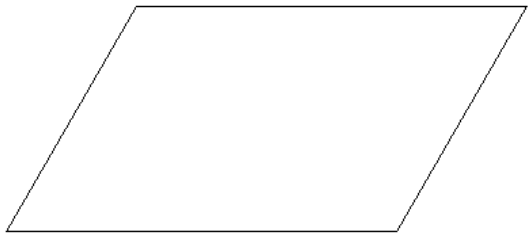



**Exact picture**
**[Univ. of Toronto Math Network]**



**Real topological picture - Boy's surface**
**(virtualmathmuseum.org)**

| |
|---|
| We think of projective space over an algebraically closed field just as real affine space, together with the *idea* that taking intersections always works perfectly. |



**Intuitive picture**
**[freehomeworkmathhelp.com]**

# Linear Spaces

If $W \subset V$ is a linear subspace, then $\mathbb{P}W \subset \mathbb{P}V$ is a projective subspace, a **linear space** of dimension $\dim \mathbb{P}W = \dim W - 1$ in $\mathbb{P}V$.

| | |
|---|---|
| $\dim \mathbb{P}W = 0$ | **point** |
| $\dim \mathbb{P}W = 1$ | **line** |
| $\dim \mathbb{P}W = 2$ | **plane** |
| $\dim \mathbb{P}W = \dim \mathbb{P}V - 1$ | **hyperplane** |

If $L = \mathbb{P}W$, $L' = \mathbb{P}W'$, write

$$\overline{LL'} = \mathbb{P}(W + W').$$

We have

$$\dim \overline{LL'} = \dim L + \dim L' - \dim L \cap L'.$$

## Dimension

Let $X$ be a variety. $X$ is **reducible** if it is the union of two proper, non-empty closed subvarieties; otherwise it is called **irreducible.**

The **Dimension of $X$** is the largest integer $k$ such that there exists a chain

$$\varnothing \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_{k-1} \subsetneq X$$

of irreducible closed subvarieties.

In particular, $\dim \mathbb{P}^n = \dim \mathbb{A}^n = n$.

If $X \subset \mathbb{P}^n$ or $X \subset \mathbb{A}^n$ is irreducible:

| | |
|---|---|
| $\dim X = 0$ | **point** |
| $\dim X = 1$ | **curve** |
| $\dim X = 2$ | **surface** |
| $\dim X = 3$ | **threefold** |
| $\dim X = n-1$ | **hypersurface** |

**Theorem.** *The hypersurfaces in $\mathbb{P}^n$ are exactly the varieties defined by a single equation.*

The hypersurfaces in $\mathbb{P}^2$ are the **plane projective curves**.

## Points

**Proposition 1.1.** *Any finite set of $d$ points in $\mathbb{P}^n$ is described by polynomials of degree at most $d$.*

*Proof.* Let $\Gamma = \{p_1, \dots, p_d\}$. For $q \notin \Gamma$, let $L_{q,i}$ be a linear form with $L_{q,i}(p_i) = 0$ and $L_{q,i}(q) \neq 0$. Put

$$F_q = L_{q,1} \cdots L_{q,d}.$$

Then $\Gamma = \mathcal{V}(F_q : q \notin \Gamma)$. ∎

**Definition.** Let $p_1, \dots, p_d \in \mathbb{P}^n$. If $d \leqslant n+1$, the points $p_i$ are **independent** if

$$\dim(\overline{p_1 \cdots p_d}) = d - 1,$$

otherwise **dependent**.

If $d > n+1$, the $p_i$ are in (linearly) **general position** if no $n+1$ of them are dependent (i.e. lie in a hyperplane).

**Theorem 1.2.** *Any collection of at most $2n$ points in general position in $\mathbb{P}^n$ can be described by quadratic forms.*

*Proof.* Let $\Gamma \subset \mathbb{P}^n$ be such a collection. We may assume that $\Gamma$ contains exactly $2n$ points.
Let $q \in \mathbb{P}^n$ be such that

$$F|_\Gamma = 0 \quad \Longrightarrow \quad F(q) = 0$$

holds for all quadratic forms $F$. We must show $q \in \Gamma$.

(1) If $\Gamma = \Gamma_1 \cup \Gamma_2$ with $|\Gamma_1| = |\Gamma_2| = n$, then $\Gamma_i$ spans a hyperplane $H_i = \mathcal{V}(L_i)$, defined by a linear form $L_i$, and $H_1 \cup H_2 = \mathcal{V}(L_1 L_2)$. So $L_1 L_2(q) = 0$ by hypothesis. Hence $q \in H_1 \cup H_2$.

(2) Let $\{p_1, \ldots, p_k\} \in \Gamma$ be a minimal subset of $\Gamma$ with the property $q \in \overline{p_1 \cdots p_k}$.
By (1), we can find such a subset with $k \leqslant n$.
**Claim:** $k = 1$ ($\iff$ $q = p_1$)
Take $\Sigma \subset \Gamma \setminus \{p_1, \ldots, p_k\}$ with $|\Sigma| = n - k + 1$. By hypothesis, the $n$ points $\{p_2, \ldots, p_k\} \cup \Sigma$ span a hyperplane $H$ that does not contain $p_1$. Since $p_1 \in \overline{q p_2 \cdots p_k}$, it follows that $q \notin H$.
By (1), $q$ lies in the hyperplane spanned by the remaining $n$ points. It follows that $q$ lies on the hyperplane spanned by $p_1$ and any $n-1$ of the points $p_{k+1}, \ldots, p_{2n}$. The intersection of all such hyperplanes is just $p_1$, hence $q = p_1$. ■

## Projective equivalence

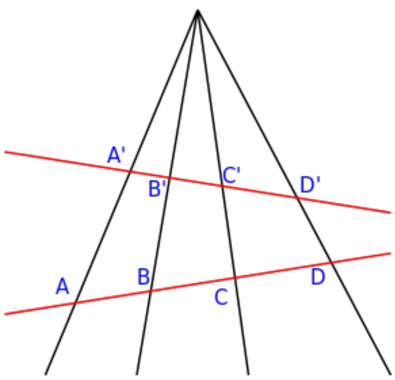
The group $\mathrm{PGL}_{n+1}K = (\mathrm{GL}_{n+1}K)/K^\times I$ acts on $\mathbb{P}^n$. Two varieties $X, Y \subset \mathbb{P}^n$ are **projectively equivalent** if there exists $A \in \mathrm{PGL}_{n+1}K$ such that $A \cdot X = Y$.
Any two ordered sets of $n+2$ points in general position in $\mathbb{P}^n$ are projectively equivalent.

The group $\mathrm{PGL}_2 K$ acts on $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$ through **Möbius transformations:**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}_2 K \text{ induces } z \mapsto \frac{az+b}{cz+d}.$$

Two sets of *four* points in $\mathbb{P}^1$ are projectively equivalent if and only if they have the same **cross-ratio**, defined by

$$\lambda(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}.$$




**Cross-ratio**
**[Krishnavedala - Wikimedia Commons]**

# The twisted cubic

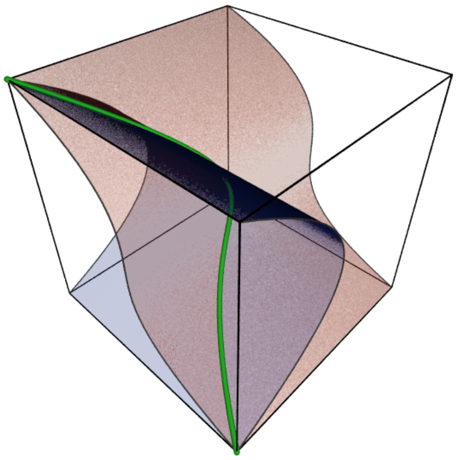
Let $\nu: \mathbb{P}^1 \to \mathbb{P}^3$, $[X_0, X_1] \mapsto [X_0^3, X_0^2X_1, X_0X_1^2, X_1^3]$.
The image $C = \nu(\mathbb{P}^1)$ is the **twisted cubic** in $\mathbb{P}^3$. It is defined by

$$C = \mathcal{V}(F_0, F_1, F_2)$$

where

$$F_0 = Z_0Z_2 - Z_1^2$$
$$F_1 = Z_0Z_3 - Z_1Z_2$$
$$F_2 = Z_1Z_3 - Z_2^2.$$

It is *not* defined by any two of these.
For example, $F_0$ and $F_1$ define the union of $C$ and the line $\{Z_0 = Z_1 = 0\}$.



Claudio Rocchini - Wikimedia Commons

**Exercise (1.11. in [Ha])**

b. More generally, for any $\lambda = [\lambda_0, \lambda_1, \lambda_2]$, let

$$F_\lambda = \lambda_0 \cdot F_0 + \lambda_1 \cdot F_1 + \lambda_2 \cdot F_2$$

and let $Q_\lambda$ be the surface defined by $F_\lambda$. Show that for $\mu \neq \nu$, the quadrics $Q_\nu$ and $Q_\mu$ intersect in the union of $C$ and a line $L_{\mu,\nu}$. (A slick way of doing this problem is described after Exercise 9.16; it is intended here to be done naively, though the computation is apt to get messy.)

**Corollary 1.3.** *If $L$ is any secant line of $C$ (i.e. $L = \overline{pq}$ with $p, q \in C$), there exist $\mu, \nu$ with*

$$Q_\mu \cap Q_\nu = C \cup L.$$

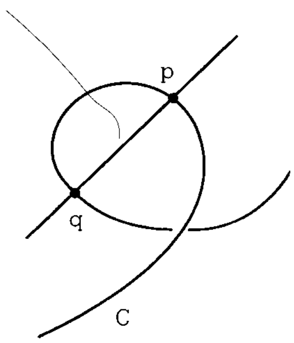
**Proof.** For any $r \in \overline{pq}$, $r \neq p$, $r \neq q$, the space

$$\{F_\lambda : F_\lambda(r) = 0\}$$

is 2-dimensional.
Let $F_\mu$, $F_\nu$ be a basis. Since $F_\mu$, $F_\nu$ vanish at the three points $p, q, r$ on $L$, we have $L \subset \mathcal{V}(F_\mu, F_\nu)$, hence

$$Q_\mu \cap Q_\nu = \mathcal{V}(F_\mu, F_\nu) = C \cup L$$

by the exercise above. ∎



## The rational normal curve

The **rational normal curve** in $\mathbb{P}^d$ is the Veronese embedding of $\mathbb{P}^1$ of degree $d$. It is the image of the map

$$\nu_d: \mathbb{P}^1 \to \mathbb{P}^d, \; [X_0, X_1] \mapsto [X_0^d, X_0^{d-1}X_1, \ldots, X_0X_1^{d-1}, X_1^d].$$

Any $d+1$ points on a rational normal curve are linearly independent. For given distinct points $p_0, \ldots, p_d \in \mathbb{P}^1$, we may assume $p_i \neq [1,0]$ for all $i$, say $p_i = [Y_i, 1]$. The matrix

$$\begin{bmatrix} Y_0^d & Y_0^{d-1} & Y_0^{d-2} & \cdot & Y_0 & 1 \\ Y_1^d & Y_1^{d-1} & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ Y_d^d & \cdot & \cdot & \cdot & Y_d & 1 \end{bmatrix}$$

is a Vandermonde matrix with determinant $\prod_{i<j}(Y_i - Y_j) \neq 0$, showing that $\nu_d(p_0), \ldots, \nu_d(p_d)$ are independent.

Any curve projectively equivalent to *the* rational normal curve is also *a* rational normal curve. In particular, if $H_0, \ldots, H_d$ is any basis of $K[X_0, X_1]_d$, then

$$\nu_d: [X_0, X_1] \mapsto [H_0(X_0, X_1), \ldots, H_d(X_0, X_1)]$$

is a rational normal curve.

**Theorem 1.4.** *Through any $d+3$ points in general position in $\mathbb{P}^d$, there passes a unique rational normal curve.*

*Construction.* Let $\mu_0, \ldots, \mu_d, \nu_0, \ldots, \nu_d \in K^\times$ with $[\mu_i, \nu_i] \neq [\mu_j, \nu_j]$ for all $i \neq j$ and consider

$$G = \prod_{i=0}^{d} (\mu_i X_0 - \nu_i X_1) \in K[X_0, X_1]_{d+1}$$

$$H_i = \frac{G}{\mu_i X_0 - \nu_i X_1}, \quad i = 0, \ldots, d.$$

Then $H_0, \ldots, H_d$ are a basis of $K[X_0, X_1]_d$. For if $\sum_{i=0}^d a_i H_i = 0$ is any linear relation, evaluation at $[\mu_i, \nu_i]$ gives $a_i = 0$.

Thus

$$\nu_d: [X_0, X_1] \mapsto [H_0(X_0, X_1), \ldots, H_d(X_0, X_1)]$$

is a rational normal curve. We find

$$\nu_d([\mu_0, \nu_0]) = [1, 0, \ldots, 0], \ldots, \nu_d([\mu_d, \nu_d]) = [0, \ldots, 0, 1]$$

$$\nu_d([1,0]) = \left[\frac{\mu_0 \cdots \mu_d}{\mu_0}, \ldots, \frac{\mu_0 \cdots \mu_d}{\mu_d}\right] = [\mu_0^{-1}, \ldots, \mu_d^{-1}] \quad \text{and} \quad \nu_d([0,1]) = [\nu_0^{-1}, \ldots, \nu_d^{-1}].$$

So let $p_0, \ldots, p_{d+2}$ be any $d+3$ points in general position in $\mathbb{P}^d$.
We can assume $p_i = [e_i]$ for $i = 0, \ldots, d$ by projective equivalence. Then $p_{d+1}$ and $p_{d+2}$ have non-zero coordinates. We can choose $[\mu_0, \nu_0], \ldots, [\mu_d, \nu_d] \in \mathbb{P}^1$ such that $\nu_d[1,0] = p_{d+1}$ and $\nu_d[0,1] = p_{d+2}$. Uniqueness is left as an exercise. ■