§2
Review:
Morphisms & Rational Maps, Products & Projections

Morphisms: Affine vs. projective

If $V \subset \mathbb{A}^m$ and $W \subset \mathbb{A}^n$ are affine varieties, a **morphism** $V \to W$ is just given by an n-tuple of polynomials $f_1, \dots, f_n \in K[x_1, \dots, x_m]$ such that

$$\big(f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m)\big) \in W$$

for all $(x_1, \dots, x_m) \in V$.

If $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$ are projective varieties, the following seems the most natural: We should take homogeneous polynomials $F_0, \dots, F_n \in K[Z_0, \dots, Z_m]$ such that

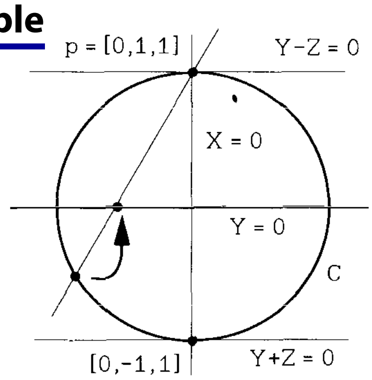
$$[F_0(Z_0, \dots, Z_m), \dots, F_n(Z_0, \dots, Z_m)] \in Y$$

for all $[Z_0, \dots, Z_m] \in X$. For this to be well-defined, we must have

$$\deg(F_0) = \dots = \deg(F_n) \quad \text{and} \quad \mathcal{V}(F_0, \dots, F_n) \cap X = \varnothing.$$

This is straightforward, but it turns out to be a little too restrictive, so it is not yet the 'correct' notion of morphism of projective varieties.

An instructive example



Let $C = \mathcal{V}(X^2 + Y^2 - Z^2) \subset \mathbb{P}^2$ and consider

$$\varphi \colon \begin{cases} C & \to & \mathbb{P}^1 \\ [X,Y,Z] & \mapsto & [X, Y-Z] \end{cases}$$

This is undefined if $X = 0$ and $Y = Z$, i.e. at the point $p = [0,1,1]$. It is the **stereographic projection** from the point p, sending $r \in C, r \neq p$ to the intersection point of the lines $\overline{pr}$ and $\{Y = 0\}$.

Consider the open subset $U_0 = \{[S,T] \in \mathbb{P}^1 : S \neq 0\}$. For $[X,Y,Z] \in \varphi^{-1}(U_0)$, we can write

$$\varphi[X,Y,Z] = [X, Y-Z] = \left[1, \frac{Y-Z}{X}\right],$$

which of course still appears to be undefined at the point $p = [0,1,1]$. But note that

$$\frac{Y-Z}{X} = \frac{Y^2 - Z^2}{X(Y+Z)} = \frac{-X^2}{X(Y+Z)} = \frac{-X}{Y+Z}$$

so the restriction of φ to $\varphi^{-1}(U_0)$ is given by

$$\varphi[X,Y,Z] = \left[1, \frac{-X}{Y+Z}\right] = [Y+Z, -X]$$

which *is* defined in the point $[0,1,1]$, but not in the point $[0,1,-1]$.
So we can put $\varphi(p) = [1,0]$ and φ is well-defined everywhere on C.

Maps on varieties in projective space

A subset W of $\mathbb{P}^n$ is a **quasi-projective variety** if it is locally closed in the Zariski topology, i.e. if it is the intersection of an open and a closed subset of $\mathbb{P}^n$.

Since $\mathbb{A}^n$ can be identified with the open subset U_0 of $\mathbb{P}^n$, any affine variety can be regarded as a quasiprojective variety.

Let $V \subset \mathbb{P}^m$ and $W \subset \mathbb{P}^n$ be quasi-projective varieties. A map $\varphi \colon V \to W$ is called a **morphism** or a **regular map** if the following condition holds: For every point $p \in V$, there is an open subset U of $\mathbb{P}^m$ containing p and homogeneous polynomials $F_0, \ldots, F_n \in K[Z_0, \ldots, Z_m]$ of the same degree such that $\mathcal{V}(F_0, \ldots, F_n) \cap U = \varnothing$ and

$$\varphi[Z_0, \ldots, Z_m] = [F_0(Z_0, \ldots, Z_m), \ldots, F_n(Z_0, \ldots, Z_m)]$$

holds for all points $[Z_0, \ldots, Z_m] \in V \cap U$.

A morphism $V \to \mathbb{A}^1$ is called a **regular function** on V.

A morphism $V_0 \to W$, where V_0 is a non-empty open subset of V, is also called a **rational map**, denoted

$$V \dashrightarrow W.$$

(More precisely, a rational map is an equivalence class of such maps for various choices of V_0, where two maps are equivalent if they agree on the intersection of their domains.)

Summary

If $V \subset \mathbb{A}^m$ and $W \subset \mathbb{A}^n$ are affine varieties, a morphism $V \to W$ is given by an n-tuple of polynomials $f_1, \ldots, f_n \in K[x_1, \ldots, x_m]$ such that

$$\big(f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m)\big) \in W$$

for all $(x_1, \ldots, x_m) \in V$.

If $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$ are projective varieties, a morphism $V \to W$ may be given by homogeneous polynomials $F_0, \ldots, F_n \in K[Z_0, \ldots, Z_m]$ of the same degree such that

$$[F_0(Z_0, \ldots, Z_m), \ldots, F_n(Z_0, \ldots, Z_m)] \in Y$$

for all $[Z_0, \ldots, Z_m] \in X \smallsetminus \mathcal{V}(F_0, \ldots, F_n)$ (which should be non-empty).

But it may not be immediately clear whether such a tuple of polynomials really induces a morphism defined on all of X or just a rational map defined on some proper subset of X. To decide this, it is necessary to examine the points where $F_0, \ldots, F_n$ vanish on X.

Projections

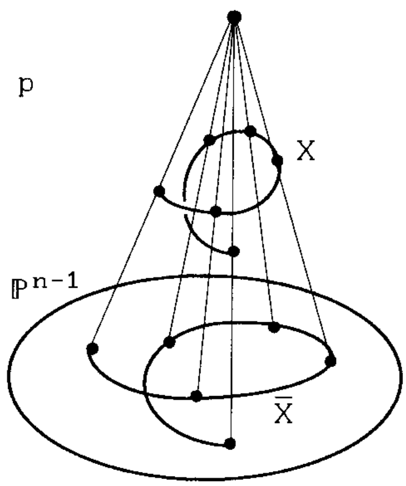
The linear **projection**

$$\pi : \mathbb{P}^n \to \mathbb{P}^{n-1}, [Z_0, \ldots, Z_n] \mapsto [Z_1, \ldots, Z_n]$$

is not a morphism, because It is undefined at the point $p_0 = [1, 0, \ldots, 0]$.
Thus π is only a rational map.

More generally, if $\varphi : V \to W$ is a linear map between finite-dimensional vector spaces, it induces a rational map $\mathbb{P}V \dashrightarrow \mathbb{P}W$, defined on $\mathbb{P}V \smallsetminus \mathbb{P}(\ker \varphi)$.

We should also think of projections in projective space differently than in affine space: The point p_0 is called the **center** of the projection. Geometrically, π maps a point $q \in \mathbb{P}^n \smallsetminus \{p_0\}$ to the intersection of the line $\overline{p_0 q}$ with the hyperplane $H_0 = \{[Z] \in \mathbb{P}^n : Z_0 = 0\}$.



For any hyperplane $H \subset \mathbb{P}^n$ and any point $p \in \mathbb{P}^n \smallsetminus H$, we may define the **projection from p onto H**, which is just π after the unique change of coordinates taking H to H_0 and p to p_0.

Theorem 2.1 (Fundamental theorem of elimination theory).

Let $X \subset \mathbb{P}^n$ be a projective variety, $p \in \mathbb{P}^n$ a point not on X and $H \subset \mathbb{P}^n$ a hyperplane not containing p. Then $\pi_p(X)$ is closed and therefore again a projective variety.

Resultants

Lemma. *Two monic polynomials $f, g \in k[t]$ (over any field k) have a common factor if and only if $R(f,g) = 0$, where $R(f,g)$ is the* ***resultant*** *of f and g.*

Explicitly, if f has roots $\lambda_1, \dots, \lambda_d \in K = \overline{k}$ and g has roots $\mu_1, \dots, \mu_e$, then $R(f,g) = \prod_{i,j}(\lambda_i - \mu_j)$ The point is that $R(f,g)$ can also be expressed in the coefficients of f and g, rather than the roots. Namely, if $f = \sum_{i=0}^d a_i z^i$, $g = \sum_{i=0}^e b_i z^i$, then

$$R(f,g) = \det \begin{bmatrix} a_d & a_{d-1} & \cdot & \cdot & a_0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & a_d & a_{d-1} & \cdots & \cdot & a_0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & & & & & & & & & & \\ \cdot & & & & & & & & & & \\ 0 & 0 & \cdot & \cdot & a_d & a_{d-1} & \cdot & \cdot & \cdot & \cdot & a_0 \\ b_e & b_{e-1} & \cdot & \cdot & b_0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & b_e & b_{e-1} & \cdots & \cdot & b_0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & & & & & & & & & & \\ \cdot & & & & & & & & & & \\ 0 & 0 & \cdot & \cdot & b_e & b_{e-1} & \cdot & \cdot & \cdot & \cdot & b_0 \end{bmatrix}$$

(This is a $(d+e) \times (d+e)$-matrix, called the **Sylvester matrix** of f and g.)

Since f, g are assumed monic, we actually have $a_d = b_e = 1$. If $a_d = 0$ or $b_e = 0$, the statement about the Sylvester matrix becomes wrong. However, this is an "affine" problem, which goes away in the projective picture (Exercise):

Lemma 2.2. *Two homogeneous polynomials $F = \sum_{i=0}^d a_i X_0^d X_1^{d-i}$ and $G = \sum_{i=0}^e b_i X_0^i X_1^{e-i}$ have a common zero on $\mathbb{P}^1$ if and only if $R(F,G) = 0$, where $R(F,G)$ is the Sylvester determinant above.*

Elimination theory

Theorem 2.1 (Fundamental theorem of elimination theory).

Let $X \subset \mathbb{P}^n$ be a projective variety, $p \in \mathbb{P}^n$ a point not on X and $H \subset \mathbb{P}^n$ a hyperplane not containing p. Then $\pi_p(X)$ is closed and therefore again a projective variety.

Sketch of proof. Let I be a homogeneous ideal in $K[Z_0, \dots, Z_n]$ defining X and assume again $p = [1, 0, \dots, 0]$ and $H = \{[Z] \in \mathbb{P}^n : Z_0 = 0\}$. For homogeneous polynomials $F, G \in K[Z_0, \dots, Z_n]$, we let $R(F,G)$ denote the resultant with respect to Z_0. This means that we regard F, G as polynomials in Z_0 with coefficients in $K[Z_1, \dots, Z_n]$ and define $R(F,G)$ via the Sylvester matrix.

For $q \in H$, we claim that the following are equivalent:

(1) The line $\ell = \overline{pq}$ meets X, i.e. $q \in \pi(X)$.

(2) Every pair of polynomials $F, G \in I$ has a common zero on ℓ.

(3) The resultant $R(F,G)$ vanishes at q for all $F, G \in I$.

The theorem follows, since $\pi(X)$ is then defined by all $R(F,G)$, for homogeneous $F, G \in I$.

Note that if $q = [0, Z_1, \dots, Z_n]$, then $\ell = \{[\lambda, \mu Z_1, \dots, \mu Z_n] : [\lambda, \mu] \in \mathbb{P}^1\}$.

The implication (1)⇒(2) is clear and (2)⇔(3) follows from the properties of the resultant of two homogeneous polynomials in two variables.

(2)⇒(1): If $\ell \cap X = \emptyset$, then we can first find $F \in I$ that does not vanish identically on ℓ. For each of the finitely many points $r \in \ell \cap \mathcal{V}(F)$, we can find $G_r \in I$ with $G_r(r) \neq 0$, by hypothesis. Now the space $\{\sum_{r \in \ell \cap \mathcal{V}(F)} \alpha_r G_r : \alpha_r \in K\}$ contains some G with $\ell \cap \mathcal{V}(F) \cap \mathcal{V}(G) = \emptyset$. ■

The Segre embedding

Recall that cartesian products of projective spaces and varieties are more subtle than in the affine case:
The product $\mathbb{P}^m \times \mathbb{P}^n$ is **not** $\mathbb{P}^{m+n}$. For example, if $m \geqslant n$, any two m-subspaces in $\mathbb{P}^{m+n}$ have non-empty intersection, while

$$\mathbb{P}^m \times \{p\} \quad \text{and} \quad \mathbb{P}^m \times \{q\}$$

for $p, q \in \mathbb{P}^n$, $p \neq q$, are clearly disjoint in $\mathbb{P}^m \times \mathbb{P}^n$.

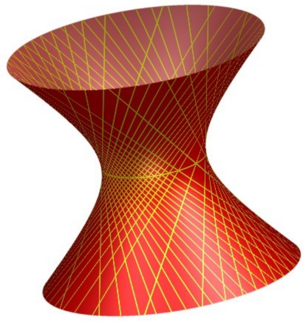
Instead, the cartesian product is realized via the **Segre embedding**

$$\sigma_{m,n}: \begin{cases} \mathbb{P}^m \times \mathbb{P}^n & \to & \mathbb{P}^{(m+1)(n+1)-1} \\ [X_0, \ldots, X_m], [Y_0, \ldots, Y_n] & \mapsto & [X_0Y_0, X_0Y_1, X_0Y_2, \ldots, X_1Y_0, X_1Y_1, \ldots, X_mY_n] \end{cases}.$$

The image $\Sigma_{m,n}$ of $\sigma_{m,n}$ is closed and $\mathbb{P}^m \times \mathbb{P}^n$ as a projective variety is *defined* as $\Sigma_{m,n}$.

For example, the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ is the quadric surface in $\mathbb{P}^3$.

The product of two projective varieties $X \subset \mathbb{P}^m$, $Y \subset \mathbb{P}^n$ is defined as the image of $X \times Y$ in $\mathbb{P}^m \times \mathbb{P}^n$. Such subvarieties of $\mathbb{P}^m \times \mathbb{P}^n$ are defined by **bi-homogeneous polynomials**, i.e. polynomials $F \in K[X_0, \ldots, X_m, Y_0, \ldots, Y_n]$ that are homogeneous of degree d in X and homogeneous of degree e in Y, so that $F(\lambda X, Y) = \lambda^d F(X, Y)$ and $F(X, \lambda Y) = \lambda^e F(X, Y)$.



[taken from R. Vakil's homepage

More elimination theory

Theorem 2.3 (Fundamental theorem of elimination theory, second version).
Let Y be any (quasi-projective) variety. Then for any $n \geqslant 0$, the projection

$$Y \times \mathbb{P}^n \to Y, \ (y, z) \mapsto y$$

is a closed map, i.e. it takes Zariski-closed subsets of $Y \times \mathbb{P}^n$ to Zariski-closed subsets of Y.

The proof can again be carried out using resultants and induction on n (see [Ha, Thm. 3.12).

Remark. In topology, a Hausdorff-space P has the property that the projection $Y \times P \to Y$ is closed for all spaces Y if and only if P is compact. The Zariski-topology is not Hausdorff and every quasi-projective variety is (quasi-)compact. Also, the Zariski topology on $X \times Y$ is *not* the product topology. Nevertheless, the property expressed in the fundamental theorem of elimination theory can be seen as an analogue of compactness in algebraic geometry.
Furthermore, the complex projective space $\mathbb{P}^n(\mathbb{C})$ is a compact complex manifold, from which it follows that a quasi-projective variety over $\mathbb{C}$ is compact (in the Euclidean topology) if and only if it is projective.

Theorem 2.3 (Fundamental theorem of elimination theory, second version).

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Corollary 2.4. *If $X \subset \mathbb{P}^m$ is a projective variety and $\varphi \colon X \to \mathbb{P}^n$ any morphism, then $\varphi(X)$ is closed.*

Proof. First, one checks that the graph map $X \to \mathbb{P}^m \times \mathbb{P}^n, x \mapsto (x, \varphi(x))$ is an isomophism from X onto its image Γ_φ. Then $\varphi(X)$ is the image of Γ_φ under the projection onto the second factor, so it is closed. ■

Corollary 2.5. *A connected projective variety does not admit any non-constant regular function.*

Proof. A regular function on a projective variety X is a morphism $f \colon X \to \mathbb{A}^1$. Composing with the inclusion $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$, we can view this as a morphism $X \to \mathbb{P}^1$. By the above corollary, the image of f in $\mathbb{P}^1$ is closed. Since the image of f is contained in $\mathbb{A}^1$, it is not all of $\mathbb{P}^1$, so it can be only finitely many points. If X is connected, so is $f(X)$, hence it is only a single point. ■

Corollary 2.6. *If $X \subset \mathbb{P}^n$ is a hypersurface and $Y \subset \mathbb{P}^n$ any closed subvariety of positive dimension, then $X \cap Y \neq \varnothing$.*

Proof. Since X is a hypersurface, there is $F \in K[Z_0, \dots, Z_n]_d$ such that $X = \mathcal{V}(F)$. Suppose that $X \cap Y = \varnothing$. Then

$$Z \mapsto \frac{G(Z)}{F(Z)}$$

is a regular function on Y, for any $G \in K[Z_0, \dots, Z_n]_d$. Since Y is projective, this map is constant on every connected component of Y. We deduce that Y consists of finitely many points. ■