

§4

Secant varieties

Secant varieties

We defined the join $J(X, Y)$ of two disjoint projective varieties $X, Y \in \mathbb{P}^n$ as the union of all lines $\overline{pq}, p \in X, q \in Y$.

If X and Y are not disjoint, we can still look at the set of lines in $\mathbb{G}(1, n)$ joining two points $x \in X$ and $y \in Y$ with $x \neq y$. Let $\mathcal{J}(X, Y)$ be the Zariski closure of that set and, as before, let

$$J(X, Y) = \bigcup_{\ell \in \mathcal{J}(X, Y)} \ell,$$

a closed subvariety of \mathbb{P}^n .

In particular, it makes sense to define $\mathcal{S}_1(X) = \mathcal{J}(X, X) \subset \mathbb{G}(1, n)$, the **variety of secant lines**, and $S_1(X) = J(X, X) \subset \mathbb{P}^n$, the **secant variety** of X .

More generally, let $\mathcal{S}_\ell(X)$ be the closure of the set of ℓ -subspaces in $\mathbb{G}(\ell, n)$ spanned by $\ell + 1$ independent points on X . This is the **variety of secant ℓ -subspaces**, and $S_\ell(X) = \bigcup_{\Lambda \in \mathcal{S}_\ell(X)} \Lambda$, is the **ℓ th secant variety** of X .

In general, it can be quite hard to say anything substantial about the secant variety of a given projective variety. For example, what can be said about its dimension? Let $X \subset \mathbb{P}^n$ be irreducible of dimension k . It is not hard to show that

$$\dim(\mathcal{S}_1(X)) = 2k,$$

unless X is itself a linear subspace. Since lines are one-dimensional and $\mathcal{S}_1(X)$ is a union of lines parametrized by $\mathcal{S}_1(X)$ we would therefore guess that the dimension of $S_1(X)$ is equal to $2k + 1$.

By the same argument, we would expect the dimension of $S_\ell(X)$ to be $k\ell + k + \ell$.

With a bit of dimension theory, one can establish the following, which confirms intuition.

Proposition 4.1. *If X is irreducible of dimension k , its secant variety $S_1(X)$ is of dimension at most $2k + 1$, with equality if and only if there exists a point on $S_1(X)$ lying on only a finite number of secant lines to X . (In fact, if this is true for a single point, it is true for a dense set of points.)* ■

The analogous statement holds for the higher secant varieties $S_\ell(X)$.

The condition in the proposition can be hard to check. Only the case of curves is easy.

Proposition 4.2. *If $X \subset \mathbb{P}^n$ is an irreducible curve, then the secant variety $S_1(X)$ is three-dimensional, unless X is contained in a plane.* ■

The case of the twisted cubic is treated in the exercises.

For surfaces, things already become more complicated.

Example 4.3. The secant variety to the Veronese surface $X = \nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ (the image of the map ν_2 sending $[Z] \in \mathbb{P}^2$ to all quadratic monomials in Z) is only four-dimensional.

The secant variety of the rational normal curve

Proposition 4.4. *Let $C \subset \mathbb{P}^n$ be a rational normal curve. The secant variety $S_\ell(C)$ has dimension $\min(2\ell + 1, n)$, for any ℓ between 1 and n .*

Sketch of proof for the case $2\ell + 1 \leq n$. Let $U \subset S_\ell(C)$ be an open subset consisting of secant ℓ -subspaces spanned by $\ell + 1$ distinct points of C . Let $\Lambda \in U$, spanned by $p_1, \dots, p_{\ell+1} \in C$. Since any $n + 1 \geq 2(\ell + 1)$ points on C are linearly independent, the intersection of Λ with any other secant ℓ -subspace $\Lambda' \in U$ is contained in a subspace of Λ spanned by some subset of the points $p_1, \dots, p_{\ell+1}$. It follows that if $p \in \Lambda$ is a point not in any such subspace, then p is contained in no other secant ℓ -subspace of C in U . By Prop. 4.1, this implies that $S_1(C)$ has the expected dimension. ■

The secant variety of the rational normal curve

Proposition 4.4. *Let $C \subset \mathbb{P}^n$ be a rational normal curve. The secant variety $S_\ell(C)$ has dimension $\min(2\ell + 1, n)$, for any ℓ between 1 and n .*

This has a neat application to sums of n th powers of linear forms.

Corollary 4.5. *Let K be an algebraically closed field of characteristic 0. For any $n \geq 1$ and d with $2d - 1 \geq n$, there is a Zariski open subset U of the space $K[X, Y]_n$ such that every $F \in U$ admits a representation*

$$F = L_1^n + \cdots + L_d^n$$

with $L_1, \dots, L_d \in K[X, Y]_1$.

Proof. Let $V = K[X, Y]_1$ and $W = K[X, Y]_n$. If we take the monomial basis X, Y on V and $X^n, X^{n-1}Y, \dots, Y^n$ on W , the Veronese map $v_n: \mathbb{P}V = \mathbb{P}^1 \rightarrow \mathbb{P}^n = \mathbb{P}W$ takes a point $[u, v]$ corresponding to a linear form $uX + vY$ to the point $[u^n, u^{n-1}v, \dots, v^n]$. Since $\text{char}(K) = 0$, the rational normal curve $v_d(\mathbb{P}V)$ is projectively equivalent to the curve

$$[u, v] \mapsto \left[u^n, nu^{n-1}v, \dots, \binom{n}{k} u^{n-k} v^k, \dots, v^n \right]$$

which sends $uX + vY$ to $(uX + vY)^n$.

Hence the set of n th powers of linear forms is a rational normal curve in $\mathbb{P}W$. By the above proposition, its $(d - 1)$ th secant variety is all of $\mathbb{P}W$. By definition, an open dense subset of that secant variety consists of sums of d n th powers. ■

The secant variety of the rational normal curve

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with $L_1, \dots, L_d \in K[X, Y]_1$.

To obtain analogous statements for polynomials in more variables, one has to understand the secant varieties of higher-dimensional Veronese varieties. As the example of the Veronese surface in \mathbb{P}^5 shows, the answer becomes more complicated.

Determinantal Varieties

Let M be the projective space $\mathbb{P}(\text{Mat}_{m \times n}(K)) \cong \mathbb{P}^{mn-1}$ of matrices. The **general determinantal variety of rank k** is the variety $M_k \subset M$ of matrices of rank at most k . It is closed since it is defined by the vanishing of all $(k+1) \times (k+1)$ -minors.

Again, it is not clear that the $(k+1) \times (k+1)$ -minors generate the radical ideal $\mathcal{I}(M_k)$. This is true, but we do not prove it.

Example 4.6. The Segre variety $\Sigma_{m,n} \subset \mathbb{P}^{mn-1} \cong \mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$ is exactly M_1 . To see this, note that a matrix $Z \in \text{Mat}_{m \times n}(K)$ has rank 1 if and only if it can be written as $Z = UV^T$ for $U \in \mathbb{K}^m \setminus \{0\}$, $V \in \mathbb{K}^n \setminus \{0\}$, i.e. if and only if it lies in the image of the Segre embedding.

The general determinantal variety M_k is the **k th secant variety to M_1** . This is because a matrix has rank at most k if and only if it is the sum of k matrices of rank 1.

One can try to use this simple characterisation of secants for the general determinantal variety to study the secant varieties of other varieties defined by the vanishing of minors. We will carry this out for the rational normal curve.

Linear Determinantal Varieties

Let $\Omega = (L_{ij})_{i,j}$ be an $m \times n$ -matrix with entries in $K[Z_0, \dots, Z_\ell]_1$. The variety

$$\Sigma_k(\Omega) = \{[Z_0, \dots, Z_\ell] : \text{rank}(\Omega(Z)) \leq k\} \subset \mathbb{P}^\ell$$

is called the **linear determinantal variety** determined by Ω . It is the pullback of M_k under the linear map $\mathbb{P}^\ell \rightarrow M$ given by the linear forms L_{ij} . (In case that map is injective, $\Sigma_k(\Omega)$ can be identified with the intersection of M_k with the image of the linear map.)

Remember from the exercises that the rational normal curve C in \mathbb{P}^d is the rank-1 determinantal variety associated with the matrix

$$\Omega_k = \begin{pmatrix} Z_0 & Z_1 & Z_2 & \cdots & Z_{d-k} \\ Z_1 & Z_2 & Z_3 & \cdots & Z_{d-k+1} \\ \vdots & & & & \\ Z_k & Z_{k+1} & \cdots & \cdots & Z_d \end{pmatrix},$$

for any k between 1 and $d-1$.

Our goal for the rest of this lecture is

Theorem 4.7. *The secant variety $S_1(C)$ to the rational normal curve $C \subset \mathbb{P}^d$ is the rank-2 determinantal variety associated with Ω_k , for k between 2 and $d-2$.*

The analogous statement is true for the higher secant varieties of C .

Lemma 4.8. Let S_d be the space of homogeneous polynomials of degree d in two variables X and Y and let $V \subsetneq S_d$ be a proper linear subspace without common zeros (i.e. $\mathcal{V}(V) = \emptyset$ in \mathbb{P}^1). Let $W = S_1 \cdot V$ be the subspace of S_{d+1} generated by all products of elements of V with linear forms. Then

$$\dim(W) \geq \dim(V) + 2$$

Proof. For any point $p \in \mathbb{P}^1$ and any $U \subset S_d$, let $\text{Ord}_p(U) \subset \mathbb{Z}_{\geq 0}$ denote the set of all vanishing orders of elements in U at the point p .

Note first that if U is a subspace of dimension k , then $|\text{Ord}_p(U)| = k$. (Exercise).

Now suppose $\dim(V) = k$ and $\dim(W) < k + 2$, then

$$\text{Ord}_p(W) \supset \text{Ord}_p(V) \cup (\text{Ord}_p(V) + 1),$$

together with the fact that the polynomials in V have no common zeros, implies

$$(i) \text{Ord}_p(V) = \{0, 1, \dots, k-1\} \quad \text{and} \quad (ii) \text{Ord}_p(W) = \{0, 1, 2, \dots, k\}.$$

By (i), we can find a basis $\{F_1, \dots, F_k\}$ of V with $\text{ord}_p(F_i) = k - i$ for all $i = 1, \dots, k$, where we take $p = [0, 1]$, the zero of X .

Now the three polynomials $XF_1, YF_1, XF_2 \in W$ all vanish to order at least $k - 1$ at p , so by (ii), there must be a non-trivial linear relation between them. On the other hand, XF_1 and YF_1 are linearly independent, hence there are $a, b \in K$ such that

$$XF_2 = aXF_1 + bYF_1 = (aX + bY)F_1,$$

so F_1 and F_2 have a common divisor of degree $d - 1$.

Lemma 4.8. Let S_d be the space of homogeneous polynomials of degree d in two variables X and Y and let $V \subsetneq S_d$ be a proper linear subspace without common zeros (i.e. $\mathcal{V}(V) = \emptyset$ in \mathbb{P}^1). Let $W = S_1 \cdot V$ be the subspace of S_{d+1} generated by all products of elements of V with linear forms. Then

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Proof (continued).

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By (i), we can find a basis $\{F_1, \dots, F_k\}$ of V with $\text{ord}_p(F_i) = k - i$ for all $i = 1, \dots, k$, where we take $p = [0, 1]$, the zero of X .

We proceed to show by induction that F_1, \dots, F_j have a common factor of degree at least $d - j + 1$, for $j \in \{2, \dots, k\}$. Let $j \geq 3$ and assume that G is a common factor of degree $d - j + 2$ of F_1, \dots, F_{j-1} , say $F_i = GF'_i$. The $2j - 1$ polynomials

$$XF_1, \dots, XF_{j-1}, YF_1, \dots, YF_{j-1}, XF_j$$

vanish to order at least $k - j + 1$ at p . By (ii), they span a space of dimension at most j . On the other hand, $XF_1, \dots, XF_{j-1}, YF_1, \dots, YF_{j-1}$ span a space of dimension at least j , so there is an expression

$$XF_j = \sum_{i=1}^{j-1} a_i XF_i + \sum_{i=1}^{j-1} b_i YF_i = \sum_{i=1}^{j-1} (a_i X + b_i Y) F_i = G \cdot \sum_{i=1}^{j-1} (a_i X + b_i Y) F'_i,$$

which shows what we want. In conclusion, f_1, \dots, f_j have at least $d - j + 1$ zeros in common. Since V has no common zeros, we conclude $d - k + 1 \leq 0$, hence $k = d + 1$ and $V = S_d$. \blacksquare

Proof of Thm. 4.7. First note that any point $(Z_0, \dots, Z_d) \in K^{d+1}$ can be viewed as a linear functional φ_Z on the space S_d of polynomials of degree d in X and Y , via the rule $\varphi_Z(X^{d-i}Y^i) = Z_i$. Thus we have an identification $\mathbb{P}^d \cong \mathbb{P}(S_d^*)$.

Let C be the rational normal curve. If $[Z] = [X^d, X^{d-1}Y, \dots, Y^d] \in C$, then φ_Z is just evaluation of polynomials at the point $[X, Y] \in \mathbb{P}^1$. Conversely, if all polynomials in $\ker(\varphi_Z)$ have a common zero $[X, Y] \in \mathbb{P}^1$, it follows that φ_Z is equal to evaluation at $[X, Y]$ and $[Z] = v_d[X, Y] \in C$.

Now we consider the matrix

$$\Omega_k = \begin{pmatrix} Z_0 & Z_1 & Z_2 & \cdots & Z_{d-k} \\ Z_1 & Z_2 & Z_3 & \cdots & Z_{d-k+1} \\ \vdots & & & & \\ Z_k & Z_{k+1} & \cdots & \cdots & Z_d \end{pmatrix}.$$

It follows from the case of the secant variety of the general determinantal variety M_1 that the rank-2 determinantal variety associated with Ω_k contains $S_1(C)$. We have to show the converse. So suppose that $[Z_0, \dots, Z_d] \in \mathbb{P}^d$ is a point where $\Omega_k(Z)$ has rank at most 2. The matrix $\Omega_k(Z)$ represents the bilinear map

$$S_k \times S_{d-k} \xrightarrow{m} S_d \xrightarrow{\varphi_Z} K, (X^{k-i}Y^i, X^{d-k-j}Y^j) \mapsto X^{d-i-j}Y^{i+j} \mapsto Z_{i+j}$$

where m is the multiplication map.

That $\Omega_k(Z)$ has rank at most 2 means that there exist subspaces $V_1 \subset S_k$ and $V_2 \subset S_{d-k}$ of codimension 2 such that $W_1 = V_1 \cdot S_{d-k}$ and $W_2 = S_k \cdot V_2$ are contained in $V = \ker(\varphi) \subset S_d$. By our lemma above, both V_1 and V_2 must have a common zero, since otherwise we would have $W_1 = W_2 = S_d$. Hence if $W_1 = V$ or $W_2 = V$, then V has a common zero, hence $[Z] \in C$.

Proof (continued).

$$\Omega_k = \begin{pmatrix} Z_0 & Z_1 & Z_2 & \cdots & Z_{d-k} \\ Z_1 & Z_2 & Z_3 & \cdots & Z_{d-k+1} \\ \vdots & & & & \\ Z_k & Z_{k+1} & \cdots & \cdots & Z_d \end{pmatrix}.$$

That $\Omega_k(Z)$ has rank at most 2 means that there exist subspaces $V_1 \subset S_k$ and $V_2 \subset S_{d-k}$ of codimension 2 such that $W_1 = V_1 \cdot S_{d-k}$ and $W_2 = S_k \cdot V_2$ are contained in $V = \ker(\varphi) \subset S_d$. By our lemma above, both V_1 and V_2 must have a common zero, since otherwise we would have $W_1 = W_2 = S_d$. Hence if $W_1 = V$ or $W_2 = V$, then V has a common zero, hence $[Z] \in C$.

Otherwise, both W_1 and W_2 have codimension 2 in S_d . It follows from the lemma that each must have a common divisor P_i of degree 2 (check!) and thus

$$W_1 = P_1 S_{d-2} \quad \text{and} \quad W_2 = P_2 S_{d-2}.$$

Now there are two cases to consider:

- (1) If $W_1 \neq W_2$, then $W_1 + W_2 = V$ and $W_1 \cap W_2$ has codimension 2 in V and thus codimension 3 in S_d . It follows that P_1 and P_2 must have a common zero which is then a common zero of V . So we again conclude $[Z] \in C$.