§5
Rational Functions and Maps

Coordinate rings: affine and projective

Let $V \subset \mathbb{A}^n$ be an affine variety with vanishing ideal $I(V) \subset K[x_1, \ldots, x_n]$. The **coordinate ring** of V is the residue class ring

$$A(V) = K[x_1, \ldots, x_n]/I(V).$$

Its elements are exactly the **regular functions** on V, i.e. the morphisms $V \to \mathbb{A}^1$.

Next, if $X \subset \mathbb{P}^n$ is a projective variety with vanishing ideal $I(X) \subset K[Z_0, \ldots, Z_n]$, the ring

$$S(X) = K[Z_0, \ldots, Z_n]/I(X)$$

is the **homogeneous coordinate ring** of X. Unlike the coordinate ring of an affine variety, its elements are not functions on X.

Write $I(X)_d$ for $I(X) \cap K[Z_0, \ldots, Z_n]_d$, the **$d$th homogeneous part of $I(X)$**.

Since $I(X)$ is generated by homogeneous polynomials, a polynomial $F = \sum_{d=1}^{k} F_d$ with homogeneous parts F_d is contained in $I(X)$ if and only if each F_d is contained in $I(X)_d$.

Therefore, the homogeneous coordinate ring $S(X) = K[Z_0, \ldots, Z_n]/I(X)$ inherits the grading from $K[Z_0, \ldots, Z_n]$. In other words, an element $\overline{F} \in S(X)$ is homogeneous of degree d if and only if F is homogeneous of degree d, and this definition is independent of the choice of representative F of $\overline{F}$. So $S(X)$ has a decomposition into homogeneous parts

$$S(X) = \bigoplus_{d \geqslant 1} S(X)_d$$

The function field of a variety

An affine variety $V \subset \mathbb{A}^n$ is irreducible if and only if its vanishing ideal $I(V)$ is prime if and only if the coordinate ring $A(V)$ is a domain. In this case, we can form the field of fractions of $A(V)$, denoted $K(V)$, whose elements are all fractions f/g, $f, g \in A(V)$, $g \neq 0$. This is the **function field of V**.

Similarly, $X \subset \mathbb{P}^n$ is irreducible if and only if its homogeneous vanishing ideal $I(X)$ is prime, and we define the **function field of X**, again denoted $K(X)$, to be the field of all fractions

$$\frac{F}{G}, \text{ where } F, G \in S(X) \text{ homogenoeus with } \deg(F) = \deg(G).$$

Unlike the elements of the homogeneous coordinate ring $S(X)$, the elements of $K(X)$ can be viewed as functions with target $\mathbb{A}^1$, but only where the denominator does not vanish. More precisely, given $F/G \in K(X)$, consider the open subset $U = \{[Z] \in X : G(Z) \neq 0\}$, then

$$U \to \mathbb{A}^1, [Z] \mapsto \frac{F(Z)}{G(Z)}$$

is a regular function on U. (Note that the fraction is independent of any scaling of Z and thus well-defined at $[Z] \in \mathbb{P}^n$.)

Rational Maps

Now if X is an affine or projective variety, we could define a rational map from X to $\mathbb{A}^n$ to be an n-tuple $(h_1, \ldots, h_n)$ with $h_i \in K(X)$. As a map, this is defined in all points in which none of the denominators of the entries $h_1, \ldots, h_n$ vanish.

However, the issue of a rational function or map not being defined on all of X should be a taken more seriously, leading to the following more technically precise definition:
Given two quasi-projective varieties X and Y, a **rational map from X to Y** is an equivalence class of pairs (U, φ), where $U \subset X$ is an open-dense subvariety and $\varphi: U \to Y$ is a morphism of varieties. Two pairs (U, φ), (U', φ') are equivalent if φ and φ' agree on $U \cap U'$. We usually drop U from the notation and write

$$\varphi: X \dashrightarrow Y$$

to denote the rational map defined by such φ.

If $X \subset \mathbb{P}^m$ is irreducible, a rational map $\varphi: X \dashrightarrow \mathbb{P}^n$ may be given more concretely by an $(n+1)$-tuple of homogeneous polynomials $F_0, \ldots, F_n \in K[Z_0, \ldots, Z_m]$ of the same degree, not all identically zero on X, via the rule

$$\varphi[Z] = [F_0(Z), \ldots, F_n(Z)].$$

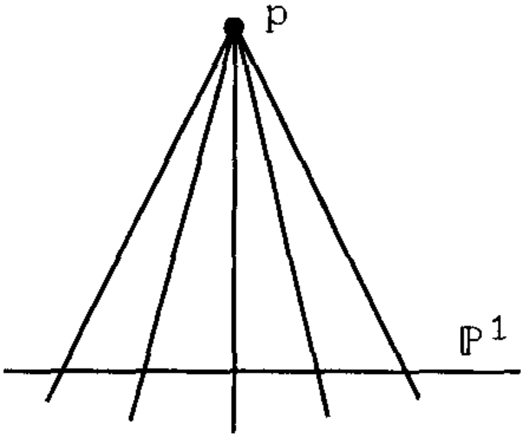
By our general definition of morphism, this will define a morphism $X \smallsetminus \mathcal{V}(F_0, \ldots, F_n) \to \mathbb{P}^n$ and conversely, any rational map $X \to \mathbb{P}^n$ has a representative of this form. It is also clear that the map φ depends only on the class of $F_0, \ldots, F_n$ in $S(X)$, so rational maps to $\mathbb{P}^n$ may be given by $(n+1)$-tuples of homogeneous elements of $S(X)$ of the same degree.

Rational maps

A simple (but important) example of a rational map is the map

$$\varphi: \begin{cases} \mathbb{A}^2 & \dashrightarrow & \mathbb{P}^1 \\ (x,y) & \mapsto & [x,y]. \end{cases}$$

Note that φ is defined on $\mathbb{A}^2 \setminus \{0,0\}$.



[Ha], p. 75

Back to the function field

If $X \subset \mathbb{P}^n$ is projective and irreducible, let us check that the rational maps $X \dashrightarrow \mathbb{A}^1$ indeed correspond to the elements of the function field $K(X)$. It is clear that an element $F/G \in K(X)$ defines a morphism from $\{[Z] \in X : G(Z) \neq 0\}$ to $\mathbb{A}^1$.

Conversely, if V is a non-empty open subset of X and $\Phi: V \to \mathbb{A}^1$ a morphism, this means there exists a non-empty open subset U of V and homogeneous elements $F, G \in S(X)$ of the same degree such that

$$\Phi([Z]) = [F(Z), G(Z)] \in \mathbb{P}^1$$

for all $[Z] \in U$. Since the image of Φ is contained in $\mathbb{A}^1 = U_1$, we must have $G(Z) \neq 0$ for all $[Z] \in U$. Hence

$$\Phi([Z]) = \left[\frac{F(Z)}{G(Z)}, 1\right]$$

for all $[Z] \in U$, which is the rational function given by F/G.

It does not matter whether we regard rational functions as rational maps $X \dashrightarrow \mathbb{A}^1$ or $X \dashrightarrow \mathbb{P}^1$. For if $\Phi: U \to \mathbb{P}^1$ is a morphism, we can restrict Φ to $\Phi^{-1}(U_1)$ and obtain a morphism to $U_1 \cong \mathbb{A}^1$.

Back to the function field

We may define the function field of any irreducible quasi-projective variety $W \subset \mathbb{P}^n$ to be the field of all rational maps $W \dashrightarrow \mathbb{A}^1$.

An alternative definition of $K(W)$ that is often more explicit is as follows: Let $U \subset W$ be any non-empty open affine subvariety of W. Then define $K(W)$ as $K(U)$, which in turn is the field of fractions of the affine coordinate ring $A(U)$.

Let us see how this plays out for $\mathbb{P}^n$. The function field $K(\mathbb{P}^n)$ is given by all fractions F/G with $F, G \in K[Z_0, \ldots, Z_n]$ of the same degree, $G \neq 0$. If we dehomogenize with respect to Z_0, i.e. we pass to the open affine subset U_0 of $\mathbb{A}^n$, then F/G goes to $F(1, Z_1, \ldots, Z_n)/G(1, Z_1, \ldots, Z_n)$ and this establishes an isomorphism with the field of fractions of $K[Z_1, \ldots, Z_n]$, which is the function field of U_0.

Summary

- If $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$ are quasi-projective varieties, a rational map $\varphi: X \dashrightarrow Y$ is given by an $(n+1)$-tuple of homogeneous polynomials in $[Z] = [Z_0, \ldots, Z_m]$ of the same degree via the rule

  $$\varphi([Z]) = [F_0(Z), \ldots, F_n(Z)].$$

- Two such $(n+1)$-tuples give the same map if they agree on some open-dense subset of X.
- We can also divide by one of the entries and regard

  $$\varphi([Z]) = \left[1, \frac{F_1(Z)}{F_0(Z)}, \ldots, \frac{F_n(Z)}{F_0(Z)}\right]$$

  as a map to $\mathbb{A}^n$. Working modulo the homogeneous ideal of X (or its closure), we can interpret the entries F_i/F_0 as elements of the function field of X.

- If a rational map $\varphi: X \dashrightarrow Y$ is given by $F_0, \ldots, F_n$, then it corresponds to a morphism $\varphi: X \setminus \mathcal{V}(F_0, \ldots, F_n) \to Y$.
  That morphism may or may not extend to a morphism defined on a larger subset of X, bearing in mind our general definition of morphism.

Graphs of rational maps

Let X be a projective variety and $\varphi: X \dashrightarrow \mathbb{P}^n$ a rational map defined on an open subset U of X. We have seen in the exercises that the graph of $\varphi|_U$ is a closed subset of $U \times \mathbb{P}^n$. We define the graph Γ_φ of φ to be the closure of the graph of $\varphi|_U$ in $X \times \mathbb{P}^n$. Note that this is independent of the choice of U. In particular, if φ is regular on X, this is just the ordinary graph.

Since the graph Γ_φ is closed by definition, the projection $\pi_2(\Gamma_\varphi)$ is closed in $\mathbb{P}^n$ and we define this variety to be the **image** of φ. (This is common but slightly dangerous terminology: It is important never to confuse the image of the rational map with the image of a morphism representing it.)

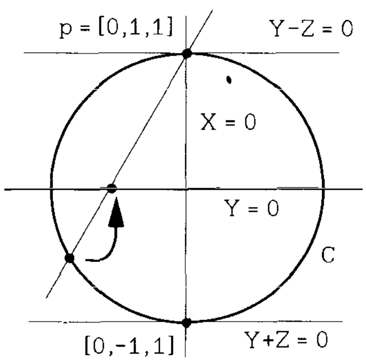
A morphism $\varphi: X \to Y$ is called **dominant** if its image is dense in Y. A rational map $\varphi: X \dashrightarrow Y$ is called dominant if it is represented by a dominant morphism, i.e. if its image in the sense above is all of Y.

It is clear that if $\varphi: X \dashrightarrow Y$ and $\Psi: Y \dashrightarrow Z$ are rational maps and φ is dominant, then the composition $\psi \circ \varphi$ is well-defined (which is not true in general).

If we think of a rational map $\varphi: X \dashrightarrow Y$ as an equivalence class of pairs (U, φ), we can take the union of all open sets U showing up in this same equivalence class and obtain a representative of φ defined on the biggest possible subset of X. This subset is called the **domain** of φ. By definition, φ is a morphism if and only if its domain is all of X.

The domain of a rational map

For example, remember our discussion of the stereographic projection in the plane in §2:



Let $C = \mathcal{V}(X^2 + Y^2 - Z^2) \subset \mathbb{P}^2$, a conic in the projective plane. What we showed is that the domain of the rational map

$$\varphi: \begin{cases} C & \dashrightarrow & \mathbb{P}^1 \\ [X, Y, Z] & \mapsto & [X, Y - Z] \end{cases}$$

is all of C, so that it is really a morphism. This is despite the fact that the projection $[X, Y, Z] \mapsto [X, Y - Z]$ chosen to *represent* the map φ is undefined at the point $[0, 1, 1]$.

Relation between rational maps and function fields

A morphism $\varphi: V \to W$ of affine varieties is dominant if and only if the induced ring homomorphism $\varphi^*: A(W) \to A(V)$ given by $f \mapsto f \circ \varphi$ is injective. It follows that any dominant rational map $\varphi: X \dashrightarrow Y$ between two irreducible quasi-projective varieties induces an injection $\varphi^*: K(Y) \to K(X)$ of their function fields.

Conversely, if $\iota: K(Y) \to K(X)$ is an inclusion of function fields, it yields a rational map $Y \dashrightarrow X$ as follows: Suppose $Y \subset \mathbb{P}^n$ and consider the open affine subvariety $W = Y \cap U_0$. We may assume that W is non-empty and thus dense in Y, so that $K(Y)$ is the field of fractions of the coordinate ring $A(W)$. The ring $A(W)$ is generated by the residue classes of $y_i = Z_i/Z_0$ ($i = 1, \ldots, n$), which we may therefore regard as elements of $K(Y)$. Put $h_i = \iota(y_i) \in K(X)$. Then

$$\begin{cases} X \dashrightarrow & Y \\ x \mapsto & [1, h_1(x), \ldots, h_n(x)] \end{cases}$$

is the rational map corresponding to ι.

(This is essentially the same proof that shows how a homomorphism $A(W) \to A(V)$ between coordinate rings induces a morphism $V \to W$ of affine varieties.)

Birational isomorphism

A rational map $\varphi: X \dashrightarrow Y$ is called **birational** or a **birational isomorphism** if there exists $\psi: Y \dashrightarrow X$ such that $\psi \circ \varphi$ resp. $\psi \circ \varphi$ are both defined and equal to the identity on X resp. Y. In this case, the varieties X and Y themselves are called **birational** or **birationally isomorphic**.

Proposition 5.1. *For two irreducible quasi-projective varieties X and Y, the following statements are equivalent:*

(1) X and Y are birational.
(2) The function fields $K(X)$ and $K(Y)$ are isomorphic (as K-algebras).
(3) There exist isomorphic non-empty open subvarieties $U \subset X$ and $V \subset Y$.

Proof. (1)⇔(2) follows from our discussion of the correspondence between dominant rational maps and inclusions of function fields. And (1)⇔(3) is clear by definition. ∎

An interesting observation is the following: If $\varphi: X \to Y$ is any rational map with graph $\Gamma_\varphi \subset X \times Y$, then the natural map $X \to \Gamma_\varphi$ is birational and the projection $\Gamma_\varphi \to Y$ is a morphism. The conclusion is that a rational map from X to Y is the same as a morphism $X' \to Y$ where X' is a variety birational to X.

The quadric surface

Let Q be the quadric surface in $\mathbb{P}^3$, that is $Q = \mathcal{V}(Z_0Z_3 - Z_1Z_2)$. (We say *the* quadric surface, because there is only one non-degenerate quadratic form of a given dimension over an algebraically closed field, up to a change of coordinates.)

Let $p = [0,0,0,1] \in Q$ and consider the projection

$$\pi_p : \begin{cases} Q & \dashrightarrow & \mathbb{P}^2 \\ [Z_0, Z_1, Z_2, Z_3] & \mapsto & [Z_0, Z_1, Z_2] \end{cases}$$

The rational map π_p is birational. The inverse is given by

$$\pi_p^{-1} : \begin{cases} \mathbb{P}^2 & \dashrightarrow & Q \\ [Z_0, Z_1, Z_2] & \mapsto & [Z_0^2, Z_0Z_1, Z_0Z_2, Z_1Z_2] \end{cases}$$

In fact, the projection π_p is an isomorphism from $\{[Z] \in Q : Z_0 \neq 0\}$ onto $\{[Z] \in \mathbb{P}^2 : Z_0 \neq 0\}$.

Note that we already knew that Q is birational to $\mathbb{P}^2$: It is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$, and since $\mathbb{P}^1 \times \mathbb{P}^1$ contains $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$ as an open subset, the function field of Q must be $K(x, y)$, which is the function field of $\mathbb{A}^2$ and $\mathbb{P}^2$.

Rational varieties

An irreducible variety is called **rational** if it is birational to $\mathbb{P}^n$ (or $\mathbb{A}^n$).

For example, the Grassmannians are rational, since we showed that they are irreducible and contain open subsets isomorphic to affine space.

A variety X is called **unirational** if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$ for some n. In other words, a variety is unirational if its function field is a subfield of the rational function field $K(t_1, \ldots, t_n)$ for some n.

A classical result due to Lüroth says that every unirational curve is rational. The same is true for surfaces, which was proved by Castelnuovo and Enriques.

Whether all unirational varieties are rational was an open question until the 1970s, when Clemens and Griffiths showed the existence of unirational cubic threefolds that are not rational. Independently and around the same time, Iskovskih and Manin showed the same for certain quartic threefolds.

Questions of rationality can be tricky and somewhat unexpected: Consider for example cubic hypersurfaces: Smooth cubic curves in $\mathbb{P}^2$ are not rational (elliptic curves), while cubic surfaces in $\mathbb{P}^3$ are rational, as we will see later. Cubic threefolds in $\mathbb{P}^4$ are in general not rational (see above). The general case is in fact unknown. On the other hand, all cubic hypersurfaces are unirational.

Hypersurfaces

Theorem 5.2. *Every irreducible variety X is birational to a hypersurface.*

Sketch of proof. There are two ways to prove this. If K is of characteristic 0, we may simply invoke the Theorem of the primitive element: If $x_1, \ldots, x_n$ are a transcendence basis of the field $K(X)$, then the algebraic extension $K(X)/K(x_1, \ldots, x_n)$ is generated by a single element x_{n+1} satisfying a polynomial equation

$$F(x_{n+1}) = a_d x_{n+1}^d + a_{d-1} x_{n+1}^{d-1} + \cdots + a_0$$

with $a_0, \ldots, a_d \in K(x_1, \ldots, x_n)$. Clearing denominators, we obtain an irreducible polynomial in $x_1, \ldots, x_{n+1}$ and X is birational to the hypersurface in $\mathbb{A}^{n+1}$ defined by that polynomial.

A more geometric proof goes as follows: If $X \subset \mathbb{P}^n$ has codimension at least 2, show that there exists a point $p \in \mathbb{P}^n$ such that the projection $\pi_p : X \to \mathbb{P}^{n-1}$ is birational onto its image. This is true in any characteristic, but again much easier to show in characteristic 0. ■

The degree of a rational map

Let $\varphi : X \dashrightarrow Y$ be a dominant rational map between irreducible varieties, corresponding to an inclusion $\varphi^* : K(Y) \to K(X)$ of function fields. We say that **the generic fibre has d points** if there exists a non-empty Zariski open subset U of Y such that the fibre $\varphi^{-1}(y)$ consists of exactly d points, for all $y \in U$. In this case, the map φ is called **generically finite**.

Theorem 5.3. *The rational map φ is generically finite if and only if the field extension $K(X)/K(Y)$ given by φ^* is finite. In this case, if the field extension has degree d and $\operatorname{char}(K) = 0$, then the generic fibre of φ has d points.*

If $K(X)/K(Y)$ is finite, the degree of this field extension is also referred to as the degree of the rational map φ.

Example. Let $X = Y = \mathbb{A}^1$ and consider the morphism φ defined by

$$\varphi(x) = x^n.$$

If the characteristic of K is zero (or at least does not divide n), the fibre $\varphi^{-1}(x)$ over any point $x \in \mathbb{A}^1 \setminus \{0\}$ has exactly n points, while $\varphi^{-1}(0) = \{0\}$. The corresponding field extension is just $K(t)/K(t^n)$, which has degree n. We say that the map φ **ramifies** over the point 0.

Note also that φ extends to a morphism $\mathbb{P}^1 \to \mathbb{P}^1$ given by $[X, Y] \mapsto [X^n, Y^n]$. This map ramifies over the two points $[0,1]$ and $[1,0]$ (corresponding to 0 and ∞ in affine coordinates).

The degree of a rational map

Let $\varphi: X \dashrightarrow Y$ be a dominant rational map between irreducible varieties, corresponding to an inclusion $\varphi^*: K(Y) \to K(X)$ of function fields. We say that **the generic fibre has d points** if there exists a non-empty Zariski open subset U of Y such that the fibre $\varphi^{-1}(y)$ consists of exactly d points, for all $y \in U$. In this case, the map φ is called **generically finite**.

Theorem 5.3. *The rational map φ is generically finite if and only if the field extension $K(X)/K(Y)$ given by φ^* is finite. In this case, if the field extension has degree d and $\operatorname{char}(K) = 0$, then the generic fibre of φ has d points.*

If $K(X)/K(Y)$ is finite, the degree of this field extension is also referred to as the degree of the rational map φ.

Proof in the case $\operatorname{char}(K) = 0$. Since the statement is only about rational maps and the existence of some non-empty open subset, we may assume that X and Y are affine varieties. Since X is birational to the graph Γ_φ, we may further replace φ by the projection of Γ_φ onto Y. In other words, we have reduced to the situation where X is closed in some affine space $\mathbb{A}^n$ and φ is a linear projection $\mathbb{A}^n \to \mathbb{A}^m$. By induction on $m - n$, we may further assume that $m = n - 1$, hence

$$\varphi: \begin{cases} X & \to & Y \\ (z_1, \ldots, z_n) & \mapsto & (z_1, \ldots, z_{n-1}) \end{cases}.$$

In this case, the function field $K(X)$ is generated over $K(Y)$ by the element z_n. Now there are two cases to consider:

The degree of a rational map

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Proof (continued).

$$\varphi: \begin{cases} X & \to & Y \\ (z_1, \ldots, z_n) & \mapsto & (z_1, \ldots, z_{n-1}) \end{cases}.$$

In this case, the function field $K(X)$ is generated over $K(Y)$ by the element z_n. Now there are two cases to consider:

(1) Suppose z_n is algebraic over $K(Y)$. Then let

$$G(z_1, \ldots, z_n) = a_d z_n^d + a_1 z_n^{d-1} + \cdots + a_0$$

be the minimal polynomial of z_n over $K(Y)$, where $a_i \in K(Y)$. After clearing denominators, we may assume that $a_0, \ldots, a_d \in A(Y)$ are regular functions on Y given by (residue classes of) polynomials in $z_1, \ldots, z_{n-1}$. Let $\Delta(z_1, \ldots, z_{n-1})$ be the discriminant of G as a polynomial in z_n. Since G is irreducible in $K(Y)[z_n]$ and $\operatorname{char}(K) = 0$, the polynomial Δ cannot vanish identically on Y. It follows that $\{(z_1, \ldots, z_{n-1}) \in Y : a_d(z_1, \ldots, z_{n-1}) = 0\}$ and $\{(z_1, \ldots, z_{n-1}) \in Y : \Delta(z_1, \ldots, z_{n-1}) = 0\}$ are proper subvarieties of Y and on the complement of their union, the fibres of φ consist of exactly d points.

The degree of a rational map

Theorem 5.3. *The rational map φ is generically finite if and only if the field extension $K(X)/K(Y)$ given by φ^* is finite. In this case, if the field extension has degree d and $\operatorname{char}(K) = 0$, then the generic fibre of φ has d points.*

Proof (continued).

$$\varphi\colon \begin{cases} X & \to & Y \\ (z_1, \dots, z_n) & \mapsto & (z_1, \dots, z_{n-1}) \end{cases}.$$

In this case, the function field $K(X)$ is generated over $K(Y)$ by the element z_n. Now there are two cases to consider:

(2) Suppose z_n is transcendental over $K(Y)$. Then for any polynomial $G(z_1, \dots, z_n) \in I(X)$, which we may write in the form

$$G(z_1, \dots, z_n) = a_d z_n^d + a_1 z_n^{d-1} + \cdots + a_0$$

with $a_i \in K[z_1, \dots, z_{n-1}]$, the coefficients a_i must vanish identically on Y. It follows that X contains the entire fibre of the projection φ over any point $p \in Y$. Hence φ is not generically finite.

∎

The degree of a rational map

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The first statement remains true in prime characteristic, but the second does not.
To see this, let K be an algebraically closed field of characteristic p and consider again the map $\varphi\colon \mathbb{A}^1 \to \mathbb{A}^1$ given by $x \mapsto x^p$ (the Frobenius morphism).

Then φ is bijective, so in particular it is generically finite with every fibre consisting of one point. But the corresponding field extension is $K(t)/K(t^p)$, which is still of degree p. The problem lies in the fact that this field extension is not separable.

The degree of a rational map

Theorem 5.3. *The rational map φ is generically finite if and only if the field extension $K(X)/K(Y)$ given by φ^* is finite. In this case, if the field extension has degree d and $\mathrm{char}(K) = 0$, then the generic fibre of φ has d points.*

Corollary 5.4. *Let X and Y be irreducible quasi-projective varieties over a field of characteristic 0 and let $\Gamma \subset X \times Y$. Then Γ is the graph of a rational map $X \dashrightarrow Y$ if and only if it is closed, irreducible and the generic fibre of the projection of Γ onto X consists of one point.*

Proof. If Γ is the graph of a rational map, then it is closed by definition and there is an open dense subset U of X such that $(U \times Y) \cap \Gamma$ is the graph of a morphism $U \to Y$. Since this morphism sends every point of U to a unique point of Y, the fibre of a point of U under the projection of Γ consists of a single point. Also, since U is irreducible, so is Γ.

Conversely, suppose that Γ is closed and irreducible and that the generic fibre of the projection of Γ onto X consists of a single point. Since K is of characteristic 0, it follows from the Theorem that the projection $\pi_1 \colon \Gamma \to X$ is birational. Hence there is an open-dense subset of Γ which agrees with the graph of the rational map $\pi_2 \circ \pi_1^{-1}$. Since Γ is closed and irreducible, it must coincide with that graph. ∎

A direct consequence is that, in characteristic 0, a rational map is birational if and only if it is dominant and generically injective.

Blowing Up

Blowing up is a rather general method for constructing varieties birational to a given variety. It serves two main purposes:

- Removing undeterminacy of rational maps (i.e. turning rational maps into morphisms)
- Resolving singularities

While we will not discuss either of these in detail, we will study the construction itself.

The simplest case is the blow-up of $\mathbb{A}^2$ in a point, defined as follows: Consider the rational map

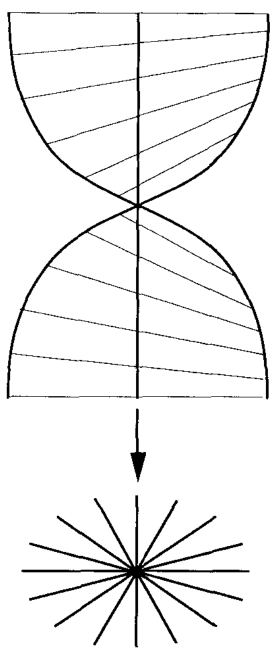
$$\varphi \colon \begin{cases} \mathbb{A}^2 & \dashrightarrow & \mathbb{P}^1 \\ (x, y) & \mapsto & [x, y] \end{cases}$$

that we looked at earlier. The **blow-up** of $\mathbb{A}^2$ in the point $(0, 0)$ is the graph of φ, denoted $\widetilde{\mathbb{A}^2}$. Explicitly, this means

$$\widetilde{\mathbb{A}^2} = \Big\{ \big((x, y), [s, t]\big) : tx = sy \Big\} \subset \mathbb{A}^2 \times \mathbb{P}^1.$$

The blow-up comes equipped with a morphism $\pi \colon \widetilde{\mathbb{A}^2} \to \mathbb{A}^2$, which is projection onto the first factor.

If $(x, y) \neq (0, 0)$, there is a unique point in $\widetilde{\mathbb{A}^2}$ mapping to (x, y), while the fibre of π over $(0, 0)$ is $\{(0, 0)\} \times \mathbb{P}^1$. This fibre is called exceptional or the **exceptional divisor**.



[Ha], p. 81

Blowing Up

$$\widetilde{\mathbb{A}^2} = \Big\{ \big((x,y),[s,t]\big) : tx = sy \Big\} \subset \mathbb{A}^2 \times \mathbb{P}^1.$$

The open subset W_0 of $\widetilde{\mathbb{A}^2}$ where $s \neq 0$ (in other words $W_0 = \widetilde{\mathbb{A}^2} \cap (\mathbb{A}^2 \times U_0)$) consists of all points $\big((x,y),[1,t]\big)$ such that $y = tx$.

Therefore, the map

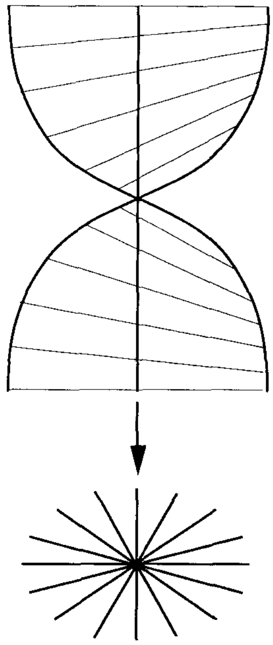
$$\begin{cases} W_0 & \to \quad \mathbb{A}^2 \\ \big((x,y),[1,t]\big) & \mapsto \quad (x,t) \end{cases}$$

is an isomorphism. The same works of course when $t \neq 0$. Hence $\widetilde{\mathbb{A}^2}$ is covered by two copies of $\mathbb{A}^2$.

In these coordinates, the map π is simply the map

$$\pi(x,t) = (x, xt).$$

Under this map, horizontal lines are mapped into lines through the origin, with every line through the origin covered except for the vertical line, the t-axis. That line is collapsed to the point $(0,0)$.



[Ha], p. 81

General blow-ups

The blow-up of $\mathbb{A}^2$ in a point is the simplest case which we can picture geometrically. General blow-ups are defined in a very similiar way, but are not as intuitive.

First, if $\varphi\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ is projection from a point $p \in \mathbb{P}^n$, we put $\widetilde{\mathbb{P}^n} = \Gamma_\varphi$ and again call $\widetilde{\mathbb{P}^n}$ together with the projection map $\pi\colon \widetilde{\mathbb{P}^n} \to \mathbb{P}^n$ the **blow-up** of $\mathbb{P}^n$ at the point p. As before, the map π is an isomorphism $\widetilde{\mathbb{P}^n} \smallsetminus \pi^{-1}(p) \xrightarrow{\sim} \mathbb{P}^n \smallsetminus \{p\}$, while the exceptional divisor is the fibre $\pi^{-1}(p) \cong \mathbb{P}^{n-1}$.

More generally, if $X \subset \mathbb{P}^n$ is a quasi-projective variety and $p \in X$ a point, let $\widetilde{X}$ be the graph of the projection $X \dashrightarrow \mathbb{P}^{n-1}$ from p. Then $\widetilde{X}$ together with the projection $\pi\colon \widetilde{X} \to X$ is called the blow-up of X at p.

To make this more explicit, it is helpful to realise it as a subvariety of $\widetilde{\mathbb{P}^n}$. Suppose that $X \subset \mathbb{P}^n$ is closed. Let $p \in X$ and let $\pi\colon \widetilde{\mathbb{P}^n} \to \mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ at p. Let $\widetilde{X}$ be the closure of $\pi^{-1}(X \smallsetminus \{p\})$ in $\widetilde{\mathbb{P}^n}$, called the **proper transform** (sometimes also called *strict transform*). This is isomorphic to the blowup of X at p as defined above.

Example

Let $X = \mathcal{V}(y^2 - x^2(x+1))$, a nodal curve in the affine plane $\mathbb{A}^2$. We compute the proper transform of X in $\widetilde{\mathbb{A}}^2$.

Thus we have to look at the equations $y^2 = x^2(x+1)$ and $xt = ys$ in $\mathbb{A}^2 \times \mathbb{P}^1$. If $s \neq 0$, we put $s = 1$ and obtain the two equations

$$\begin{aligned} y^2 &= x^2(x+1) \\ y &= tx \end{aligned}$$

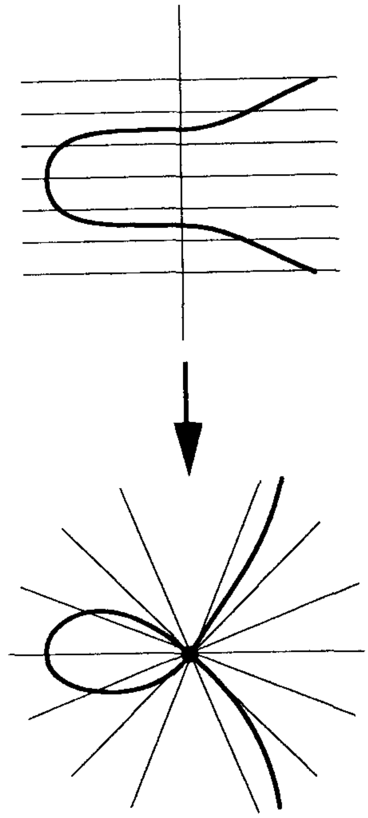
in $\mathbb{A}^2 \times \mathbb{A}^1 = A^3$ with coordinates x, y, t.

Substituting, we find $x^2t^2 - x^2(x+1) = 0$, which factors

$$x^2(t^2 - x - 1) = 0.$$

Since $x = 0$ corresponds to the exceptional divisor, the proper transform $\widetilde{C}$ is given by $t^2 = x + 1$ in $\mathbb{A}^2$.

Note that $\widetilde{C}$ meets the exceptional divisor $E = \{(0,t) \in \mathbb{A}^2\}$ in the two points $(0,1)$ and $(0,-1)$. Since E is contracted to $(0,0)$ under the projection $\pi\colon \widetilde{\mathbb{A}^2} \to \mathbb{A}^2$, both of these points are mapped to the node. Thus the two branches of the nodal curve have been separated.



[Ha], p. 82

Blowing up subvarieties

Varieties can be blown up not only in points but also in subvarieties.

The definition itself is quite straightforward: For the affine case, let $X \subset \mathbb{A}^n$ be an affine variety and $Y \subset X$ a closed subvariety with vanishing ideal $I(Y) \subset A(X)$. Let $f_0, \ldots, f_n \in A(X)$ be some set of generators of $I(Y)$. Now consider the rational map

$$\varphi\colon \begin{cases} X & \dashrightarrow & \mathbb{P}^n \\ x & \mapsto & [f_0(x), \ldots, f_n(x)] \end{cases}.$$

The map φ is regular on $X \setminus Y$ but is generally not a morphism on all of X. The graph $\Gamma_\varphi \subset X \times \mathbb{P}^n$, denoted $\mathrm{Bl}_Y(X)$, together with the projection $\pi\colon \mathrm{Bl}_Y(X) \to X$ is the **blow-up of X along Y**.

While the definition is simple enough, it is not so easy to work with this general blow-up or develop any kind of geometric intuition for it. (If X and Y are smooth, one can think of it in a differential-geometric kind of way.)

The very first step, which is already quite technical, is to show that the construction is independent of the choice of generators $f_0, \ldots, f_n$, up to isomorphism.

Example: The quadric surface

We return to the example of the quadric surface in $\mathbb{P}^3$:
Let $Q = \mathcal{V}(Z_0Z_3 - Z_1Z_2)$, $p = [0,0,0,1] \in Q$ and consider the projection

$$\pi_p\colon \begin{cases} Q & \dashrightarrow & \mathbb{P}^2 \\ [Z_0, Z_1, Z_2, Z_3] & \mapsto & [Z_0, Z_1, Z_2] \end{cases}$$

We have seen that π_p is birational. We will describe its structure in terms of blow-ups as follows. First, let $\widetilde{Q} \subset Q \times \mathbb{P}^2$ be the graph of π_p, which is exactly the blow-up of Q at the point p.
A direct computation shows that

$$\widetilde{Q} = \left\{ ([Z],[W]) \in \mathbb{P}^3 \times \mathbb{P}^2 :\ Z_0Z_3 - Z_1Z_2,\ \begin{matrix} W_0Z_1 = W_1Z_0, \\ W_0Z_2 = W_2Z_0, \\ W_1Z_2 = W_2Z_1, \end{matrix}\ W_0Z_3 - Z_1W_2 \right\}.$$

The first equation is just the equation of Q, the second corresponds to the equation $\pi_p[Z] = [W]$, while the third comes from the fact that we took the closure of the graph of the restriction of π_p to $Q \setminus \{p\}$.

The exceptional divisor under the blow-up map π_1 is $E = \pi_1^{-1}(p) = \{(p, [0, W_1, W_2]) \in Q \times \mathbb{P}^2\}$.

Now consider the other projection, $\pi_2\colon \widetilde{Q} \to \mathbb{P}^2$ taking $\widetilde{Q}$ to the image of π_p. Note first that while π_p is undefined at p, π_2 maps the exceptional divisor E to the line $\{[W] \in \mathbb{P}^2 :\ W_0 = 0\}$. The map π_2 is injective, except over the two points $q = [0,0,1]$ and $r = [0,1,0]$. The fibre $\pi_2^{-1}(q)$ is the line $\ell \times \{q\}$, where $\ell = \mathcal{V}(Z_0, Z_1) \subset Q$. Similarly, $\pi_2^{-1}(r)$ is $\ell' \times \{r\}$ with $\ell' = \mathcal{V}(Z_0, Z_2)$. Note that ℓ,ℓ' are the two lines on Q through the point p.

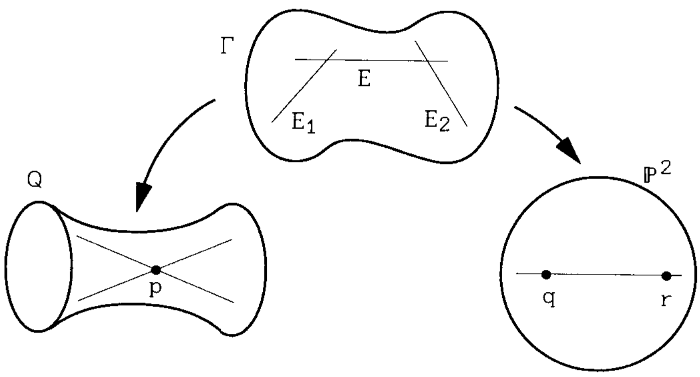
Example: The quadric surface

$$\widetilde{Q} = \left\{ ([Z],[W]) \in \mathbb{P}^3 \times \mathbb{P}^2 :\ Z_0Z_3 - Z_1Z_2,\ \begin{matrix} W_0Z_1 = W_1Z_0, \\ W_0Z_2 = W_2Z_0, \\ W_1Z_2 = W_2Z_1, \end{matrix}\ W_0Z_3 - Z_1W_2 \right\}.$$

The inverse of π_2 is the rational map given by

$$\pi_2^{-1}[W_0, W_1, W_2] = [W_0^2, W_0W_1, W_0W_2, W_1W_2].$$

Since the ideal generated by $W_0^2, W_0W_1, W_0W_2, W_1W_2$ is exactly the vanishing ideal of the two points $\{q, r\}$, we conclude that π_2 is the blowup of $\mathbb{P}^2$ in the two-point set $\{q, r\}$.



[Ha], p. 85

Conclusion. The blow-up of Q in a point is isomorphic to the blow-up of $\mathbb{P}^2$ in two points.

The projection π_p of Q from a point onto $\mathbb{P}^2$ becomes a morphism from the blow-up $\widetilde{Q}$ to $\mathbb{P}^2$.
The inverse π^{-1} becomes a morphism from the blow-up of $\mathbb{P}^2$ in two points onto Q.

Factorization of rational maps

We mention the following general result without proof.

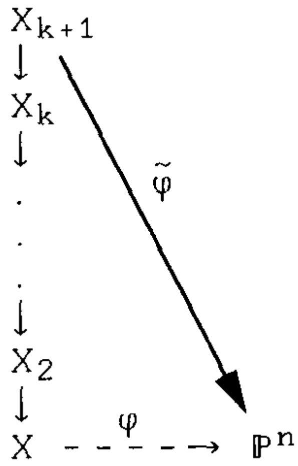
Theorem 5.5. *Let X be a quasi-projective variety and $\varphi\colon X \dashrightarrow \mathbb{P}^n$ a rational map. Then φ can be resolved via a sequence of blow-ups, which means the following:*
There is a sequence $X_1, \dots, X_{k+1}$ of varieties, where $X_1 = X$, linked by morphisms $\pi_i\colon X_{i+1} \to X_i$ such that

(1) π_i is the blow-up of X_i along a proper closed subvariety Y_i;

(2) The rational map φ has a factorization

$$\widetilde{\varphi} \circ \pi_k^{-1} \circ \cdots \circ \pi_1^{-1}$$

where $\widetilde{\varphi}\colon X_{k+1} \to \mathbb{P}^n$ is a morphism.



In other words, any rational map factors into a morphism and a sequence of blow-ups.

Obtaining more information about the blow-ups, the morphism $\widetilde{\varphi}$ and the subvarieties Y_i is quite hard. For example, the strong factorization conjecture, which characterizes birational maps between smooth projective (or complete) varieties, remains open. A weaker version, the weak factorization theorem, was proved in 1999 by Abramovich, Karu, Matsuki and Włodarczyk.