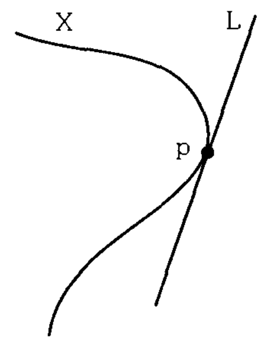
§6
Smoothness and Tangent Spaces

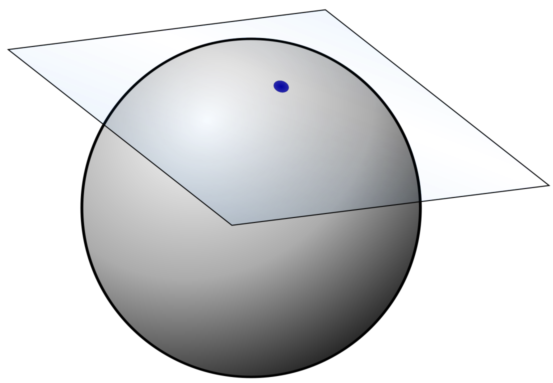
Smooth points on a hypersurface

Let $f \in K[z_1, \dots, z_n]$ be an irreducible polynomial and let $X = \mathcal{V}(f) \subset \mathbb{A}^n$ be the hypersurface defined by f. A point $p \in X$ is called **smooth** if the gradient $(\nabla f)(p)$ is non-zero.

The **tangent space** to X at a smooth point p is the orthogonal complement of the gradient.

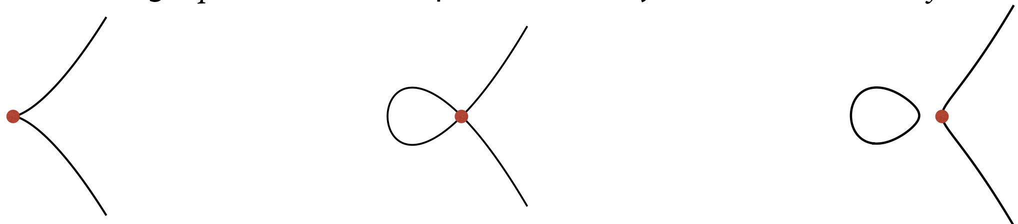


Tangent line to a curve [Ha, p. 175]



Tangent plane to a sphere [Wikimedia Commons]

In particular, the origin $p = 0$ is a smooth point if and only if the linear term of f is non-zero.



$x^3 - y^2 = 0$ $\quad$ $x^3 + x^2 - y^2 = 0$ $\quad$ $4x^3 + 5x^2 - 4y^2 + x = 0$

The tangent space to an affine variety

Let $X \subset \mathbb{A}^n$ be an affine variety with vanishing ideal $I(X) \subset K[z_1, \dots, z_n]$.
For $v \in K^n$, write $D_v = \sum_{i=1}^n v_i(\partial/\partial z_i)$ for the derivative in direction v.

The **tangent space** to X at a point $p \in X$ is the linear space

$$T_p(X) = \{v \in K^n : (D_v f)(p) = 0 \text{ for all } f \in I(X)\}.$$

In particular, $T_p(\mathbb{A}^n)$ is the space of all directional derivatives and $T_p(X)$ is a subspace of $T_p(\mathbb{A}^n)$.

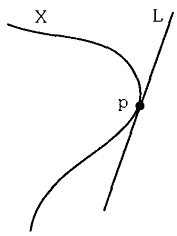
If X is a hypersurface defined by an irreducible polynomial $f \in K[z_1, \dots, z_n]$, this agrees with the previous definition:

$$T_p(X) = \{(v_1, \dots, v_n) \in K^n : D_v(f)(p) = 0\} = \left\{(v_1, \dots, v_n) \in K^n : \sum_{i=1}^n \frac{\partial f}{\partial z_i}(p) \cdot v_i = 0\right\}.$$

By definition, the tangent space is a linear subspace of $T_p(\mathbb{A}^n)$ and thus passing through the origin. However, it is often more in accordance with geometric intuition to picture the tangent space as an affine space through the point p. Thus the shifted tangent space $p + T_p(X)$ is called the **affine tangent space**.

For the hypersurface $X = \mathcal{V}(f)$, this means

$$p + T_p(X) = \left\{(v_1, \dots, v_n) \in K^n : \sum_{i=1}^n \frac{\partial f}{\partial z_i}(p) \cdot (v_i - p_i) = 0\right\}.$$



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The **tangent space** to X at a point $p \in X$ is the linear space

$$T_p(X) = \{v \in K^n : (D_v f)(p) = 0 \text{ for all } f \in I(X)\}.$$

Proposition 6.1. *Let $f_1, \dots, f_\ell$ be generators of the ideal $I(X)$ and let J be the $\ell \times n$-matrix*

$$J_{ij} = (\partial f_i/\partial z_j)_{i,j}.$$

Then $T_p(X)$ is the kernel of $J(p)$.

Proof. Given $f \in I(X)$, write $f = \sum_{j=1}^\ell g_j f_j$. Then

$$D_v(f)(p) = \sum_{j=1}^\ell g_j(p) D_v(f_j)(p)$$

by the product rule.

Thus if v is in the kernel of $J(p)$, we see that $D_v(f)(p) = \sum_{j=1}^\ell g_j(p)\left(\sum_{i=1}^n v_i(\partial f_j/\partial z_i)(p)\right) = 0$. Hence $T_p(X)$ contains the kernel of $J(p)$. Conversely, if $v \in T_p(X)$, then $D_v(f_j) = 0$ for all $j = 1, \dots, \ell$, which implies $v \in \ker J(p)$. ■

The tangent space to an affine variety

Let $X \subset \mathbb{A}^n$ be an affine variety with vanishing ideal $I(X) \subset K[z_1, \dots, z_n]$.
For $v \in K^n$, write $D_v = \sum_{i=1}^n v_i(\partial/\partial z_i)$ for the derivative in direction v.

The **tangent space** to X at a point $p \in X$ is the linear space

$$T_p(X) = \{v \in K^n : (D_v f)(p) = 0 \text{ for all } f \in I(X)\}.$$

This definition of the tangent space has two shortcomings:
(1) It does not immediately generalise to varieties that are not affine.
(2) It is not clear that it is invariant under isomorphisms of varieties.

This is fixed by the following

Proposition 6.2. *Let X be an affine variety with coordinate ring $A(X)$. Let $p \in X$ be a point and let m be the maximal ideal of p in $A(X)$. Then there is a natural isomorphism*

$$T_p(X) \to (m/m^2)^*,$$

of K-vector spaces, where $(m/m^2)^*$ denotes the dual space of m/m^2.

Proof. First, some linear algebra: Let V and W be finite-dimensional vector spaces and let

$$\alpha \colon V \times W \to K$$

be a bilinear map. If the two linear maps $\alpha_1 \colon V \to W^*, v \mapsto \alpha(v, -)$ and $\alpha_2 \colon W \to V^*, w \mapsto \alpha(-, w)$ have trivial kernel, then α is called a **perfect pairing** and α_1 and α_2 are isomorphisms. For if α_1 is injective, we must have $\dim(V) \leqslant \dim(W^*) = \dim(W)$ and the reverse for α_2, so $\dim(V) = \dim(W)$. It follows that α_1 and α_2 are also surjective.

The tangent space

Proposition 6.2. *Let X be an affine variety with coordinate ring $A(X)$. Let $p \in X$ be a point and let m be the maximal ideal of p in $A(X)$. Then there is a natural isomorphism*

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Now let $X \subset \mathbb{A}^n$ and $I(X) \subset K[z_1, \dots, z_n]$. Let $M = (z_1 - p_1, \dots, z_n - p_n)$ be the maximal ideal of p in $K[z_1, \dots, z_n]$, so that $m = M/I(X)$.

Note that $f \in M$ satisfies $D_v(f)(p) = 0$ for all $v \in T_p(\mathbb{A}^n)$ if and only if $f \in M^2$, by Taylor's formula. This implies that the bilinear map

$$\alpha \colon \begin{cases} T_p(\mathbb{A}^n) \times M/M^2 & \to \quad K \\ \quad (v, f) & \mapsto D_v(f)(p) \end{cases}.$$

is a perfect pairing. In other words, the proposition holds for $X = \mathbb{A}^n$.
We claim that when we work modulo $I(X)$, α induces a perfect pairing $\overline{\alpha} \colon T_p(X) \times (m/m^2) \to K$.

The tangent space

Proposition 6.2. *Let X be an affine variety with coordinate ring $A(X)$. Let $p \in X$ be a point and let m be the maximal ideal of p in $A(X)$. Then there is a natural isomorphism*

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of K-vector spaces, where $(m/m^2)^*$ denotes the dual space of m/m^2.

Proof (continued).
Now let $X \subset \mathbb{A}^n$ and $I(X) \subset K[z_1, \ldots, z_n]$. Let $M = (z_1 - p_1, \ldots, z_n - p_n)$ be the maximal ideal of p in $K[z_1, \ldots, z_n]$, so that $m = M/I(X)$.

$$\alpha\colon \begin{cases} T_p(\mathbb{A}^n) \times M/M^2 & \to \quad K \\ \quad (v, f) & \mapsto D_v(f)(p) \end{cases}.$$

We claim that when we work modulo $I(X)$, α induces a perfect pairing $\overline{\alpha}\colon T_p(X) \times (m/m^2) \to K$.

First, $\overline{\alpha}$ is well-defined: For $v \in T_p(X)$ and $f \in I(X)$, we have $D_v(f)(p) = 0$, by the definition of $T_p(X)$. So if f and g in M are such that $f - g \in I(X) + M^2$ and $v \in T_p(X)$, then $D_v(f)(p) = D_v(g)(p)$, so that $\overline{\alpha}$ is well-defined.

To see that $\overline{\alpha}$ is perfect, we need to check that the kernels on both sides are zero:

On the left side, this is clear, since we only restricted from $T_p(\mathbb{A}^n)$ to $T_p(X)$.

The tangent space

Proposition 6.2. *Let X be an affine variety with coordinate ring $A(X)$. Let $p \in X$ be a point and let m be the maximal ideal of p in $A(X)$. Then there is a natural isomorphism*

$$T_p(X) \to (m/m^2)^*,$$

of K-vector spaces, where $(m/m^2)^*$ denotes the dual space of m/m^2.

Proof (continued).
Now let $X \subset \mathbb{A}^n$ and $I(X) \subset K[z_1, \ldots, z_n]$. Let $M = (z_1 - p_1, \ldots, z_n - p_n)$ be the maximal ideal of p in $K[z_1, \ldots, z_n]$, so that $m = M/I(X)$.

$$\alpha\colon \begin{cases} T_p(\mathbb{A}^n) \times M/M^2 & \to \quad K \\ \quad (v, f) & \mapsto D_v(f)(p) \end{cases}.$$

We claim that when we work modulo $I(X)$, α induces a perfect pairing $\overline{\alpha}\colon T_p(X) \times (m/m^2) \to K$.

On the right side, we work directly in the finite-dimensional vector space M/M^2. Let $f_1, \ldots, f_\ell$ be generators of $I(X)$ and let U be the subspace spanned by $\overline{f_1}, \ldots, \overline{f_\ell}$ in M/M^2. Let U' be a complement of U in M/M^2, i.e. $M/M^2 = U \oplus U'$ and pick $g_1, \ldots, g_r \in M$ such that $\overline{g_1}, \ldots, \overline{g_r}$ form a basis of U'. Since α is a perfect pairing, we can pick a dual basis given by elements $v_1, \ldots, v_r \in T_p(\mathbb{A}^n)$ satisfying $D_{v_i}(g_j)(p) = \delta_{ij}$, and the subspace V spanned by $v_1, \ldots, v_r$ in $T_p(\mathbb{A}^n)$ satisfies $T_p(\mathbb{A}^n) = T_p(X) \oplus V$. Now V is exactly the kernel of the map $T_p(\mathbb{A}^n) \to (U')^*$, $v \mapsto \alpha(v, -) \in (U')^*$. Since $m/m^2 = (M/M^2)/U \cong U'$, this shows that $T_p(X) \to (m/m^2)^*$, $v \mapsto \alpha(v, -)$ is injective, as claimed. ∎

The tangent space

Proposition 6.2. *Let X be an affine variety with coordinate ring $A(X)$. Let $p \in X$ be a point and let m be the maximal ideal of p in $A(X)$. Then there is a natural isomorphism*

$$T_p(X) \to (m/m^2)^*,$$

of K-vector spaces, where $(m/m^2)^*$ denotes the dual space of m/m^2.

The description of the tangent space furnished by Prop. 6.2 is **local**, since it involves only the maximal ideal of the point in question.

To make this precise, if X is a quasi-projective variety, $p \in X$ a point and U an open-affine subvariety with $p \in X$, we define the **tangent space to X at p** to be $(m/m^2)^*$, where m is the maximal ideal of p in $K[U]$. (Of course, it has to be checked that this is independent of the choice of U. This is easy to see using the language of local rings, but we omit it here.)

Smooth points and singular points

A point $p \in X$ is called **smooth** (or X is called smooth at p) if p is contained in a single irreducible component of X and

$$\dim T_p(X) = \dim_p(X),$$

where $\dim_p(X)$ is the local dimension of X at p, i.e. the dimension of the irreducible component containing p. A point at which X is not smooth is called a **singular point** or a **singularity**.

A variety X is called smooth if it is smooth at every point.

We denote the set of smooth points of X by X_{reg} and the set of singular points by X_{sing}.

Remark. In modern algebraic geometry, what we call 'smooth' is often called 'non-singular', while the word smooth is reserved for a stronger property. The difference is only relevant in characteristic p. In many texts, the terms 'smooth' and 'non-singular' are used interchangeably, but in characteristic p, one has to exercise caution.

Smooth points and singular points

Proposition 6.3. *The set of smooth points of a variety X is open and dense in X.*

Proof. First note that the set of points in X that are contained in more than one irreducible component of X are a closed subset with dense complement. Since all those points are singular by definition, we may assume that X is irreducible. Furthermore, since X is covered by open affine subvarieties, we may also assume that X is affine.

If X is a hypersurface defined by a single irreducible polynomial $f \in K[z_1, \ldots, z_n]$, the singular points of X are the points $p \in X$ in which the gradient $(\nabla f)(p)$ vanishes. So the singular points form the subvariety $\mathcal{V}(\partial f/\partial z_1, \ldots, \partial f/\partial z_n) \cap X$ of X. This is a proper subvariety, unless the derivatives $\partial f/\partial z_1, \ldots, \partial f/\partial z_n$ are all zero in $A(X)$. Since $\partial f/\partial z_i$ has lower degree in z_i than f, it cannot be divisible by f unless it is already 0 in $K[z_1, \ldots, z_n]$.

In characteristic 0, this is only possible if the variable z_i does not occur in f. Since some variable has to occur, ∇f cannot vanish identically on X.

If $\mathrm{char}(K) = p$, then $\partial f/\partial z_i = 0$ if and only if f is a polynomial in z_i^p. If this were to happen for all $i = 1, \ldots, n$, we could take the pth root of each coefficient, since K is algebraically closed, and conclude that $f = g^p$ for some $g \in K[z_1, \ldots, z_n]$. This would contradict the fact that f is irreducible. Hence the claim is proved if X is a hypersurface.

Smooth points and singular points

Proposition 6.3. *The set of smooth points of a variety X is open and dense in X.*

Proof (continued).

If X is not a hypersurface, we can apply Thm. 5.2: Since X is birational to a hypersurface, there is an open dense subset U of X that is isomorphic to an open dense subset of a hypersurface. Therefore, U contains an open dense subset U' consisting of smooth points.

Finally, we show that the set of singular points is closed in X. Let $f_1, \ldots, f_\ell$ be generators of $I(X)$ and let J be the matrix with entries $(\partial f_i/\partial z_j)$ as before. By Prop. 6.1, the smooth points of X are exactly the points $p \in X$ in which $J(p)$ has rank $n - \dim(X)$.

Since the smooth points are dense in X, the matrix $J(p)$ can never have rank bigger than $n - \dim(X)$. For if the rank of $J(p)$ were bigger for some $p \in X$, the same would happen on some non-empty open subset of X, which is impossible. (It would mean that some minor of size $r \times r$, with $r > n - \dim(X)$ would not vanish at p. But then that same minor would be non-vanishing on some non-empty open subset of X.)

Hence the singular points of X are precisely the points at which $J(p)$ has rank less than $n - \dim(X)$. This is the closed subset given by the vanishing of all minors of size $n - \dim(X)$. ■

The proof has shown the following.

Corollary 6.4. *If X is an irreducible variety of dimension k, then*

$$\dim T_p(X) \geqslant k$$

holds for all $p \in X$, with equality on the open-dense subset of smooth points of X. ■

The projective tangent space

We have defined the tangent space to any quasi-projective variety.
But the tangent space to a projective variety should really be a projective linear space.

Here is one way to define the projective tangent space: Let $X \subset \mathbb{P}^n$ be a projective variety and $p \in X$. Then p is contained in one of the open affine sets $U_i \cong \mathbb{A}^n$. Then we can take the tangent space to $X \cap U_i$ in $\mathbb{A}^n$ and define the **projective tangent space** $\mathbb{T}_p(X)$ as the closure of the affine tangent space

$$p + T_p(X \cap U_i) \subset U_i \subset \mathbb{P}^n$$

in $\mathbb{P}^n$. It not too hard to show that this definition does not depend on the choice of U_i. But it is not a very convenient description.

We first consider again the case of a hypersurface X defined by a homogeneous polynomial $F(Z) \in K[Z_0, \dots, Z_n]$. Consider the open affine subset U_0 with affine coordinates $z_i = Z_i/Z_0$.

Then $X \cap U_0$ is defined by $f(z_1, \dots, z_n) = F(1, z_1, \dots, z_n)$.
For a point $p = (w_1, \dots, w_n) \in X \cap U_0$, the affine tangent space is given by

$$p + T_p(X) = \left\{(z_1, \dots, z_n) : \sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(p) \cdot (z_i - w_i) = 0\right\}.$$

By definition, the projective tangent space is defined by the homogenized equation:

$$\mathbb{T}_p(X) = \left\{[Z_0, Z_1, \dots, Z_n] : \sum_{i=1}^{n} \frac{\partial F}{\partial Z_i}(1, w_1, \dots, w_n) \cdot (Z_i - w_i Z_0) = 0\right\}.$$

The projective tangent space

By definition, the projective tangent space is defined by the homogenized equation:

$$\mathbb{T}_p(X) = \left\{[Z_0, Z_1, \dots, Z_n] : \sum_{i=1}^{n} \frac{\partial F}{\partial Z_i}(1, w_1, \dots, w_n) \cdot (Z_i - w_i Z_0) = 0\right\}.$$

We can further simplify this using the **Euler relation**

$$\sum_{i=0}^{n} \frac{\partial F}{\partial Z_i} \cdot Z_i = d \cdot F,$$

where $d = \deg(F)$.

Since $F(1, w_1, \dots, w_n) = 0$, it follows that

$$\sum_{i=1}^{n} \frac{\partial F}{\partial Z_i}(1, w_1, \dots, w_n) \cdot (-w_i \cdot Z_0) = \frac{\partial F}{\partial Z_0}(1, w_1, \dots, w_n) \cdot Z_0.$$

We conclude

$$\mathbb{T}_p(X) = \left\{[Z_0, \dots, Z_n] \in \mathbb{P}^n : \sum_{i=0}^{n} \frac{\partial F}{\partial Z_i}(P) \cdot Z_i = 0\right\}.$$

The point p is singular if and only if all the partial derivatives of F vanish at p, i.e. if and only if $\mathbb{T}_p(X) = \mathbb{P}^n$. In view of the Euler relation, the vanishing of all partial derivatives also implies the vanishing of F (unless $\mathrm{char} K$ divides d), so that the singular locus of $\mathcal{V}(F)$ is defined by all the partial derivatives.

Remark. Yes another way to define the projective tangent space: If $\widehat{X}$ is the cone defined by F in $\mathbb{A}^{n+1}$ and $p \neq 0$ any point on $\widehat{X}$, then $\mathbb{T}_p(X) = \mathbb{P}T_p(\widehat{X})$, by the description above.

The projective tangent space

We conclude

$$\mathbb{T}_p(X) = \left\{ [Z_0, \dots, Z_n] \in \mathbb{P}^n : \sum_{i=0}^{n} \frac{\partial F}{\partial Z_i}(P) \cdot Z_i = 0 \right\}.$$

If $X \subset \mathbb{P}^n$ is any projective variety, not necessarily a hypersurface, then $\mathbb{T}_p X$ is the intersection of all tangent spaces at p to all the hypersurfaces containing X.

In particular, if the homogeneous ideal $I(X)$ is generated by $F_1, \dots, F_\ell$, then

$$\begin{aligned} \mathbb{T}_p(X) &= \bigcap_{i=1}^{\ell} \mathbb{T}_p(\mathcal{V}(F_i)) = \left\{ [Z_0, \dots, Z_n] \in \mathbb{P}^n : \sum_{i=0}^{n} \frac{\partial F_j}{\partial Z_i}(P) \cdot Z_i = 0, j = 1, \dots, \ell \right\} \\ &= \mathbb{P}(\ker J) \end{aligned}$$

where J is the $\ell \times n$-matrix with entries $J_{ij} = (\partial F_i / \partial Z_j)(P)$.

A word on resolution of singularities

A **resolution of singularities** is a birational transformation of a variety into a smooth variety that leaves the smooth locus unchanged. The most general result is due to H. Hironaka (1964), for which he received a Fields medal.

Theorem (Hironaka).

Let X be a variety over a field of characteristic 0. Then there exists a smooth variety $\widetilde{X}$ together with a birational morphism $\varphi \colon \widetilde{X} \to X$, which is an isomorphism $\varphi^{-1}(X_{\mathrm{reg}}) \xrightarrow{\sim} X_{\mathrm{reg}}$.

This result was proved in a *very* long and technical paper:

MR0199184 (33 #7333) Reviewed
Hironaka, Heisuke
Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II.
Ann. of Math. (2) 79 (1964), 109–203; ibid. (2) **79** 1964 205–326.
14.18

The resolution is made up from successive blow-ups with centres in the singular locus. But in general it is very to hard to say (and still a subject of research) what these blow-ups look like or to find a truly constructive approach.

Resolution of singularities for curves is relatively easy and can be found in many textbooks, including [Ha], Ch. 20.

The existence of a resolution of singularities as above for varieties of any dimension over fields of prime characteristic remains unknown.

Digression: Genericity

Geometry is full of statements that hold "generically" or for a "general point/line/curve" etc.

Examples.

(1) Let $V = K^n$. A **general point** of V^n is a basis of V.

(2) The **general fibre** of a generically finite rational map of degree d has exactly d points (if $\mathrm{char}(K) = 0$).

(3) A **general hypersurface of degree d** in $\mathbb{A}^n$ is smooth.

(4) Let $X \subset \mathbb{A}^2$ be an irreducible plane curve of degree d. A **general line** in $\mathbb{A}^2$ meets X in d distinct points.

We will discuss each of these examples in turn, starting from the following definition.

Definition. Let (A) be a property of points on an irreducible variety X. Then (A) is said to **hold generically** if there exists a Zariski open subset U of X such that (A) holds for all points in U.

Terminology. Instead of saying 'property (A) holds generically' it is also common to say 'the general point of X has property (A)'.

It is also common to speak of 'the generic point'. This often has the same meaning, but also refers to an abstract concept of modern algebraic geometry. (We will briefly discuss this below.)

Example. We showed that the set of smooth points of any variety is open and dense. Thus 'the general point of a variety is smooth'.

Why this seemingly complicated terminology? The main reason is that the *existence* of an open subset of points satisfying some property is often much more significant than being able to describe the subset U explicitly.

Examples

(1) Let $V = K^n$. A **general point** of V^n is a basis of V.

An n-tuple of points $(v_1, \ldots, v_n) \in V^n$ forms a basis if and only if $v_1, \ldots, v_n$ are linearly independent. This means that the $n \times n$-matrix with row vectors v_i has non-zero determinant. We can view the determinant as a polynomial D on $V^n = K^{n \times n}$. Thus the statement '$(v_1, \ldots, v_n)$ form a basis of V' holds on the open set $\mathbb{A}^{n\times n} \setminus \mathcal{V}(D)$ and therefore generically.

Examples

(2) The **general fibre** of a generically finite rational map of degree d has exactly d points (if $\mathrm{char}(K) = 0$).

We proved in Thm. 5.3:

Theorem. *Let $\varphi: X \to Y$ be a rational map between irreducible varieties. If $\mathrm{char}(K) = 0$ and the degree of the field extension $K(X)/K(Y)$ is d, then there exists a non-empty Zariski-open subset U of Y such that the fibre $\varphi^{-1}(y)$ consists of exactly d points.*

In the proof, we also determined *in principle* equations that define the complement of U in Y. However, in general, these equations are quite complicated and we do not usually care much what they look like. It is often enough to know that the subset U exists.

Examples

(2) A **general hypersurface of degree d** in $\mathbb{P}^n$ is smooth.

Let $V = K[Z_0, \dots, Z_n]_d$. We claim that the set $\Delta \subset \mathbb{P}V$ of all $[F]$ for which the hypersurface $\mathcal{V}(F) \subset \mathbb{P}^n$ is singular is a proper closed subset of $\mathbb{P}V$.

To see this, consider the correspondence

$$\Theta = \{([p], [F]) \in \mathbb{P}^n \times \mathbb{P}V : [p] \in \mathcal{V}(F)_{\mathrm{sing}}\}.$$

The set Θ is closed in $\mathbb{P}^n \times \mathbb{P}V$, since it is defined by the equations $F(p) = 0$ and $(\nabla F)(p) = 0$, interpreted as equations in p and the coefficients of F. It follows from elimination theory that $\Delta = \mathrm{pr}_2(\Theta)$ is also closed.

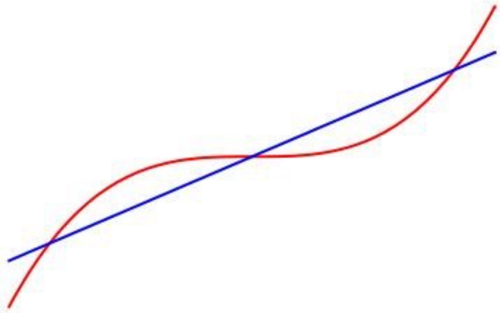
On the other hand, there clearly exists a smooth hypersurface of degree d in $\mathbb{P}^n$ for any pair (n, d). If $\mathrm{char}(K) = 0$, we may take for example the hypersurface defined by $F = \sum_{i=0}^n Z_i^d$ (Check!). (If $\mathrm{char}(K)$ if positive and divides d, one can find other examples.)

This shows that **smoothness is a generic property**.

However, it is by no means easy to actually determine equations for the variety Δ.
It turns out that Δ is a hypersurface, called the **discriminant**. It is defined by a homogeneous polynomial in the coefficients of F of degree $(n+1)(d-1)^n$. In general, no one has much of an idea as to what this polynomial looks like (c.f. the book of Gelfand, Kapranov and Zelevinsky).
For $d = 1$, we clearly have $\Delta = \varnothing$. The case $d = 2$ is also easy: A quadratic form $F \in K[Z]_d$ can be uniquely expressed as $F = Z^T A Z$, where $Z = (Z_0, \dots, Z_n)^T$ and A is a symmetric $(n+1) \times (n+1)$-matrix. Since F is singular if and only if A has rank less than $n+1$, the discriminant hypersurface is defined by the determinant, a polynomial of degree $n+1$ in the $\binom{n+2}{2}$ entries of A.

Examples

(4) Let $X \subset \mathbb{P}^2$ be an irreducible plane curve of degree d. A **general line** in $\mathbb{P}^2$ meets X in d distinct points.



Here the genericity refers to the space of lines in $\mathbb{P}^2$. This is the Grassmannian $\mathbb{G}(2,1)$, which is identified with the dual space $(\mathbb{P}^2)^*$. Explicitly, a line in $\mathbb{P}^2$ is defined by a linear form $aX+bY+cZ$ corresponding to the point $[a,b,c] \in \mathbb{P}^2$.

Assume $\mathrm{char}(K) = 0$. Let X be given by an irreducible polynomial $F \in K[X,Y,Z]_d$. We may restrict to the open set of lines as above with $a \neq 0$. Then we put $a = 1$ and substitute $X = -bY - cZ$ into F. This results in a homogeneous polynomial

$$G_{b,c}(Y,Z) = F(-bY - cZ, Y, Z)$$

of degree d. We are interested in the set of parameters b, c for which $G_{b,c}$ has no multiple roots, and thus d distinct roots in K. This corresponds to the set of b, c for which the discriminant $R(G_{b,c}, \partial G_{b,c}/\partial Y)$ is non-zero. This is a polynomial condition in b, c.

Generic vs. random

Intuitively, it makes sense to think of generic properties as properties that hold for 'random' points or objects.

In probability theory, one is often concerned with properties that hold 'almost surely' or 'with probability 1'. In other words, the set on which the property does not hold is a zero-set.
That zero-set, while small, is often very inaccessible and hard to describe explicitly.

In algebraic geometry, the intuition behind statements that hold generically is very similar.
The exceptional set is contained in some proper subvariety, but it may be quite hard to describe that subvariety explicitly.
For another example, consider the following statement:
Assume $\mathrm{char}(K) = 0$. Given finitely many points $a_1, \ldots, a_N \in K$ on the line, a generic polynomial $f \in K[t]$ of degree d will have the property

$$f^{(k)}(a_i) \neq 0 \quad \text{for all } 0 \leqslant k \leqslant d \text{ and } i = 1, \ldots, N,$$

where $f^{(k)}$ denotes the kth derivative of f.

This is easy to prove. But, given the points, how would you actually construct a polynomial that has the generic behaviour, i.e. with distinct roots and the above property?

It is not too hard to come up with *some* algorithm to do this. But for large values of d and N, it may be very hard to do in practice. Problems of this kind are frequent in certain applications.

A possible solution (in some applications) is to choose the polynomial 'at random' in some way.

An algebraic definition of genericity

Let $V \subset \mathbb{A}^n$ be an irreducible variety over $\mathbb{C}$ and $k \subset \mathbb{C}$ a subfield.
A **k-generic point** of V is a point $x \in V$ with the property that every polynomial $f \in k[x_1, \ldots, x_n]$ with $f(x) = 0$ vanishes at every point of V.

Proposition 6.5. *If $\mathbb{C}$ has infinite transcendence degree over k, then every irreducible variety possesses a k-generic point.*

Proof. Let $f_1, \ldots, f_m$ be generators of $I(V)$. Let $\widetilde{k}$ be the field extension of k obtained by adjoining all the coefficients of $f_1, \ldots, f_m$ to k. Since there are only finitely many coefficients, $\mathbb{C}/\widetilde{k}$ still has infinite transcendence degree.

Let $I_0 = I(X) \cap \widetilde{k}[x_1, \ldots, x_n]$ and let L be the field of fractions of $\widetilde{k}[x_1, \ldots, x_n]/I_0$.
Then L is a field extension of $\widetilde{k}$ of finite transcendence degree. By a theorem of Steinitz, any such field is isomorphic to a subfield of $\mathbb{C}$. More precisely, there is a $\widetilde{k}$-linear embedding $\varphi \colon L \hookrightarrow \mathbb{C}$.
Now let $\overline{x_i}$ be the image of the variable x_i in L and put

$$a = \big(\varphi(\overline{x_1}), \ldots, \varphi(\overline{x_n})\big).$$

We claim that a is a k-generic point.

Since $f_i \in I_0$, we have $f_i(\overline{x_1}, \ldots, \overline{x_n}) = 0$ in L, hence $f_i(a_1, \ldots, a_n) = \varphi(f_i(\overline{x_1}, \ldots, \overline{x_n})) = 0$, so $a \in V$. Now if $f \in k[x_1, \ldots, x_n]$ and $f \notin I(V)$, then $f \notin I_0$, hence $f(\overline{x_1}, \ldots, \overline{x_n}) \neq 0$ in L. Therefore, $f(a_1, \ldots, a_n) = \varphi(f(\overline{x_1}, \ldots, \overline{x_n})) \neq 0$. ■

An algebraic definition of genericity

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Proposition 6.5. *If $\mathbb{C}$ has infinite transcendence degree over k, then every irreducible variety possesses a k-generic point.*

For example, the variety V itself might be defined by polynomials with rational coefficients. Then we might take $k = \overline{\mathbb{Q}}$. A k-generic point of V would then be a general point with respect to any property defined (in a suitable sense) over any number field.

Bertini's theorem

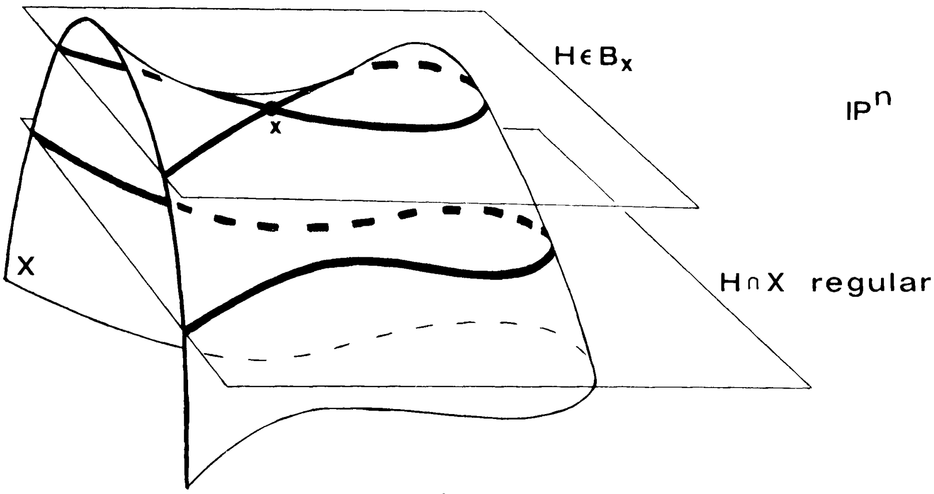
We will now see two of the most important general genericity statements. The first is Bertini's theorem.

Theorem 6.6 (Bertini).

Let $X \subset \mathbb{P}^m$ be a quasi-projective variety. The general linear subspace $L \subset \mathbb{P}^m$ satisfies

$$(X \cap L)_{\text{sing}} = X_{\text{sing}} \cap L.$$

In particular, if X is smooth, then so is the intersection of X with a general linear subspace.



From Hartshorne [Hs], p.179

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The genericity of k-dimensional linear subspaces is to be understood in the Grassmannian $\mathbb{G}(n, k)$, just like in the case of lines in $\mathbb{P}^2$.

There exist many stronger and refined versions of Bertini's theorem, like the following

Theorem 6.7. *Assume* $\mathrm{char}(K) = 0$. *Let X be a quasi-projective variety over K and $f: X \to \mathbb{P}^n$ a morphism. Then the general linear subspace $L \subset \mathbb{P}^n$ satisfies*

$$f^{-1}(L)_{\text{sing}} = X_{\text{sing}} \cap f^{-1}(L).$$

Applications of Bertini's theorem will follow later.

Dimension of fibres

The second important genericity statement is the following.

Theorem 6.8 (Fibre-dimension theorem). *Let X and Y be irreducible varieties and $\varphi\colon X \to Y$ a dominant morphism. Then the fibre of φ over a general point of Y has dimension $\dim(X) - \dim(Y)$. More precisely, the following holds: For $y \in Y$, write X_y for the fibre $\varphi^{-1}(y)$.*

(1) For every $y \in \varphi(X)$, we have $\dim(X_y) \geqslant \dim X - \dim Y$.

(2) For every $k \geqslant 0$, the set

$$\left\{y \in Y : \dim(X_y) \leqslant \dim(X) - \dim(Y) + k\right\}$$

is open in $f(X)$.

Examples 6.9.

(1) Consider the map $\varphi\colon \mathbb{A}^2 \to \mathbb{A}^2$, $(x, y) \mapsto (x, xy)$.
We find $\varphi(\mathbb{A}^2) = \{(u, v) \in \mathbb{A}^2 : u \neq 0\} \cup \{(0, 0)\}$. If $(u, v) \in \mathbb{A}^2$ with $u \neq 0$, then $\varphi^{-1}(u, v) = \{(u, v/u)\}$ has dimension 0.
The exceptional fibre $\varphi^{-1}(0, 0)$ is the line $x = 0$ in $\mathbb{A}^2$ and has dimension 1.

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is open in $f(X)$.

Examples 6.9.

(2) Let $X \subset \mathbb{A}^n$ be an irreducible affine variety of dimension k and let

$$\Theta = \{(p, v) \in X \times \mathbb{A}^n : v \in T_p(X)\}.$$

It is not hard to verify that Θ is closed in $X \times \mathbb{A}^n$. Let $\pi_1\colon \Theta \to X$ be the first projection. For $p \in X$, the fibre $\pi_1^{-1}(p)$ is exactly $T_p(X)$. From what we know about tangent spaces, it follows that the general fibre of π_1 has dimension k, so that $\dim \Theta = k + \dim X = 2k$. Furthermore, statement (2) in the fibre dimension theorem is exactly what we showed in Problem 7.2.

Dimension of fibres

Examples 6.9.

(3) In the chapter about secant varieties, we saw the following statement.

Proposition 4.1. *If $X \subset \mathbb{P}^n$ is irreducible of dimension k, its secant variety $S_1(X)$ is of dimension at most $2k+1$, with equality if and only if X is not a line and there exists a point on $S_1(X)$ lying on only a finite number of secant lines to X. (If this is true for a single point, it is true for a dense set of points.)*

We are now in a position to prove this: Let $\widehat{X} \subset \mathbb{A}^{n+1}$ be the affine cone over X and consider

$$s: \begin{cases} \widehat{X} \times \widehat{X} & \to & \mathbb{P}^n \\ (x,y) & \mapsto & [x+y] \end{cases}$$

By hypothesis, there exists a point $p \in \mathbb{P}^n$ contained in only finitely many secant lines to X. Since X is not a line, we must have $p \notin X$. Thus any secant of X containing p meets X in only finitely many points. It follows that $s^{-1}(p)$ contains only finitely many points of X, hence $\dim s^{-1}(p) = 1$ (since a point in X is a line in $\widehat{X}$).

We claim that this must be the dimension of the fibre over a general point of $\mathbb{P}^n$. This is because, by the fibre-dimension theorem, the dimension of the general fibre is always the smallest dimension that occurs anywhere over the image. Since there can be no 0-dimensional fibres (because $s(tx, ty) = s(x,y)$ for $t \in K^*$), we conclude that the general fibre must be 1-dimensional. This explains the additional claim in parenthesis.

Now we apply the fibre-dimension theorem and conclude $2\dim(\widehat{X}) - \dim S_1(X) = 1$ where $\dim(\widehat{X}) = k+1$, hence $\dim S_1(X) = 2k+1$, as claimed.

Dimension of fibres

Corollary 6.10. *Let $X \subsetneq \mathbb{P}^n$ be a projective variety. For $p \in \mathbb{P}^n \setminus X$, let π_p be the projection from p onto a hyperplane $H \cong \mathbb{P}^{n-1}$. Then $\dim(\pi_p(X)) = \dim(X)$.*

Proof. Apply the fibre-dimension theorem to the morphism $\pi_p : X \to \pi_p(X)$. For every $q \in \pi_p(X)$, the fibre $\pi_p^{-1}(q)$ consists of the intersection points of the line $\overline{pq}$ with X. Since $p \notin X$ but $q \in \pi_p(X)$, these are finitely many points. Thus every fibre is 0-dimensional, which implies $\dim(X) = \dim(\pi_p(X))$. ■

Linear spaces of complementary dimension

Theorem 6.11. *Let $X \subset \mathbb{P}^n$ be an irreducible projective variety. The dimension of X is the unique number k such the general linear subspace of dimension $n-k$ in $\mathbb{P}^n$ meets X in finitely many points.*

Proof. We first show that $k = \dim(X)$ has this property. We do induction on $\operatorname{codim}(X) = n - \dim(X)$. If $\dim(X) = n$, the claim is clear.

If $\dim(X) \leqslant n-1$, choose any point $p \in \mathbb{P}^n \setminus X$ and consider the projection $X' = \pi_p(X)$ onto $H \cong \mathbb{P}^{n-1}$. Since $\dim(X) = \dim(X')$, we have $\operatorname{codim}(X') = \operatorname{codim}(X) - 1$. By the induction hypothesis, the general subspace of H of dimension $n-k-1$ meets X' in finitely many points. If L is any such subspace, then $\overline{\pi_p^{-1}(L)}$ is a subspace of dimension $n-k$ in $\mathbb{P}^n$ (spanned by L and p) still meeting X in finitely many points. Thus the general subspace of dimension $n-k$ through p meets X in finitely many points. Since p is any point not on X, this shows the claim.

Linear spaces of complementary dimension

Theorem 6.11. *Let $X \subset \mathbb{P}^n$ be an irreducible projective variety. The dimension of X is the unique number k such the general linear subspace of dimension $n-k$ in $\mathbb{P}^n$ meets X in finitely many points.*

Proof (continued).

To show that $\dim(X)$ is the only number with this property, suppose the general subspace of dimension $n-k$ meets X in finitely many points. If $n = k$, then this implies $X = \mathbb{P}^n$, so $k = \dim X$. So we may assume $k < n$.

Let L be a subspace of dimension $n-k$ that meets X in only finitely many points. Then L contains a subspace $L^{(0)}$ of dimension $n-k-1$ which does not meet X at all. Let $p_0 \in L^{(0)}$ and let π_0 be the projection from p_0 onto $H \cong \mathbb{P}^{n-1}$. If $n-k-1 > 1$, then the image $L^{(1)} = \pi_0(L^{(0)})$ is a subspace of H of dimension $n-k-2$ which does not meet $\pi_0(X)$. Repeating this step $n-k-1$ times, we arrive at a subspace $L^{(n-k-1)} \subset \mathbb{P}^{n-(n-k-1)} = \mathbb{P}^{k+1}$ of dimension 0 which is disjoint from the image $(\pi_{n-k-1} \circ \cdots \circ \pi_0)(X)$. We can then project from this point one more time. Since the dimension of X stays the same under all these projections by Cor. 6.10 and the image of X is a subvariety of $\mathbb{P}^k$, we must have $\dim(X) \leqslant k$.

Essentially the same argument shows the converse: Let $r = \operatorname{codim}(X)$. By Cor. 6.10, we can successively project X from points $p_1, \ldots, p_r$ outside X. Then $p_1, \ldots, p_r$ span an $r-1$-dimensional subspace disjoint from X. Since each p_i can be chosen from an open subset, this shows that the general subspace of dimension $r-1$ does not meet X. So we must have $n-k > r-1$, hence $\dim(X) \geqslant k$. ∎