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7. HILBERT POLYNOMIALS

7.1. The homogeneous coordinate ring

An affine variety $V \subset \mathbb{A}^n$ with vanishing ideal $I(V) \subset K[z_1, \dots, z_n]$ is completely determined by its **coordinate ring** $A(V) = K[x_1, \dots, x_n]/I(V)$, which is the ring of regular functions on V. By Hilbert's Nullstellensatz, the points of V correspond to the maximal ideals of $A(V)$. Two affine varieties V and W are isomorphic if and only if their coordinate rings are isomorphic K-algebras.

If $X \subset \mathbb{P}^n$ is a projective variety with vanishing ideal $I(X) \subset K[Z_0, \dots, Z_n]$, the corresponding quotient is the ring

$$S(X) = K[Z_0, \dots, Z_n]/I(X),$$

the **homogeneous coordinate ring** of X. However, unlike the coordinate ring of an affine variety, its elements are not functions on X.
Furthermore, it is not invariant under isomorphisms of projective varieties. For example, the homogeneous coordinate ring of $\mathbb{P}^1$ is $K[X, Y]$, while that of the twisted cubic $C \subset \mathbb{P}^3$ is

$$\begin{aligned} S(C) = &K[Z_0, Z_1, Z_2, Z_3]/(F_0, F_1, F_2), \\ &\text{where } F_0 = Z_0Z_2 - Z_1^2, F_1 = Z_0Z_3 - Z_1Z_2, F_2 = Z_1Z_3 - Z_2^2. \end{aligned}$$

(It is not hard to check that (F_0, F_1, F_2) is indeed a radical ideal.) The twisted cubic is isomorphic to $\mathbb{P}^1$, but $S(C)$ is not isomorphic to $K[X, Y]$. That is because the affine cone $\widehat{C}$ defined by F_0, F_1, F_2 in $\mathbb{A}^4$, whose affine coordinate ring is $S(C)$, is not isomorphic to $\mathbb{A}^2$, the affine cone of $\mathbb{P}^1$; one way of showing this is to examine the tangent space of $\widehat{C}$ at the origin, which shows that $\widehat{C}$ is singular (Exercise).

So the homogeneous coordinate ring is not a ring of functions and not an invariant of a projective variety up to isomorphism. Rather, it encodes **information about the embedding** of the variety into projective space.

7.2. The Hilbert Function

Let $X \subset \mathbb{P}^n$ be a projective variety with vanishing ideal $I(X) \subset K[Z_0, \dots, Z_n]$. For $d \geqslant 1$, we denote by $I(X)_d$ the d**th homogeneous part** $I(X) \cap K[Z_0, \dots, Z_n]_d$ of $I(X)$. Since $I(X)$ is a homogeneous ideal, the homogeneous coordinate ring $S(X)$

is a graded ring with decomposition

$$S(X) = \bigoplus_{d \geqslant 0} S(X)_d$$

where $S(X)_d = K[Z_0, \dots, Z_n]_d / I(X)_d$.

Each homogeneous part $I(X)_d$ is a linear subspace of the $\binom{n+d-1}{n-1}$-dimensional K-vector space $K[Z_0, \dots, Z_n]_d$. The dimension of $I(X)_d$ is the number of independent hypersurfaces of degree d containing X. The **Hilbert Function** h_X of X counts these hypersurfaces by returning the co-dimension of $I(X)_d$, i.e.

$$h_X \colon \begin{cases} \mathbb{N} & \to & \mathbb{N} \\ m & \mapsto & \dim\bigl(S(X)_m\bigr) \end{cases} .$$

Examples 7.1. (1) Suppose X consists of three points in $\mathbb{P}^2$. Then the value $h_X(1)$ tells us exactly whether or not those three points are collinear. Namely,

$$h_X(1) = \begin{cases} 2 & \text{if the three points are collinear} \\ 3 & \text{if they are not.} \end{cases}$$

On the other hand, $h_X(2) = 3$ no matter what. To see this, let p_1, p_2, p_3 be the three points and consider the map [1]

$$\varphi \colon \begin{cases} K[Z_0, Z_1, Z_2]_2 & \to & K^3 \\ F & \mapsto & (F(p_1), F(p_2), F(p_3)) \end{cases} .$$

Now for any choice of two of the points p_1, p_2, p_3 there is some quadratic form that vanishes at these two points but not at the other one. Thus φ is surjective, and its kernel is three-dimensional. The same argument shows $h_X(m) = 3$ for all $m \geqslant 2$.

(2) If $X \subset \mathbb{P}^2$ consists of four points, there are again two possible Hilbert functions, namely

$$h_X(m) = \begin{cases} 2 & \text{for } m = 1 \\ 3 & \text{for } m = 2 \\ 4 & \text{for } m \geqslant 3 \end{cases}$$

if the points are collinear and

$$h_X(m) = \begin{cases} 3 & \text{for } m = 1 \\ 4 & \text{for } m \geqslant 2 \end{cases}$$

if they are not.

(3) We will discuss the following general statement in the exercises:

Proposition 7.2. *Let $X \subset \mathbb{P}^n$ be a set of d points. If $m \geqslant d - 1$, then $h_X(m) = d$.* ∎

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[1] There is an abuse of notation here: Since the p_i are points in the projective plane, a polynomial F has no well-defined value. However, we can pick vectors $v_i \in K^3$ with $p = [v_i]$ and take $F(p_i)$ to mean $F(v_i)$. Of course, the value depends on the choice of v_i, but as long as we only care about the dimension of kernels and images, this does not matter. Since this occurs rather frequently, it is too cumbersome to always choose representatives of points.

(4) Next, let $X \subset \mathbb{P}^2$ be a curve given by some irreducible homogeneous polynomial F of degree d. The mth homogeneous part $I(X)_m$ then consists of all polynomials of degree m divisible by F. So we can identify $I(X)_m$ with $K[Z_0, Z_1, Z_2]_{m-d}$ for $m \geqslant d$, so that

$$\dim(I(X)_m) = \binom{m-d+2}{2},$$

hence

$$h_X(m) = \binom{m+2}{2} - \binom{m-d+2}{2} = d \cdot m - \frac{d(d-3)}{2}.$$

So for $m \geqslant d$, h_X is a polynomial function of degree 1.

It turns out that the last observation comes from a general fact.

Theorem 7.3. *Let $X \subset \mathbb{P}^n$ be a projective variety with Hilbert function h_X. Then there exists a polynomial p_X in one variable with rational coefficients such that $h_X(m) = p_X(m)$ for all sufficiently large $m \in \mathbb{N}$. The degree of p_X is the dimension of X.*

The polynomial p_X is called the **Hilbert polynomial** of X.

While we will not need the result in this generality, its proof as outlined in [Ha] uses several important ideas, some of which we have already seen, and is very interesting in its own right. We will discuss it after some preparation.

Examples 7.4. (1) Let $X \subset \mathbb{P}^n$ be a finite set of d points. By Prop. 7.2, we have $h_X(m) = d$ for all $m \geqslant d-1$. So p_X is the constant polynomial d.

(2) If $X \subset \mathbb{P}^2$ is a curve given by a homogeneous polynomial F of degree d as above, we have just seen that the Hilbert function satisfies $h_X(m) = d \cdot m - d(d-3)/2$ for all $m \geqslant d$. Thus

$$p_X(t) = d \cdot t - \frac{d(d-3)}{2}.$$

(3) Let us determine the Hilbert polynomial of the rational normal curve C in $\mathbb{P}^d$. Under the Veronese map $\nu_d \colon \mathbb{P}^1 \to \mathbb{P}^d$ given by

$$[X_0, X_1] \mapsto [X_0^d, X_0^{d-1}X_1, \dots, X_0X_1^{d-1}, X_1^d]$$

the restriction of a homogeneous polynomial $F \in K[Z_0, \dots, Z_n]$ of degree m to $C = \nu_d(\mathbb{P}^1)$ is a homogeneous polynomial of degree $d \cdot m$ in X_0, X_1. It is not hard to check that this is bijection, in other words $S(C)_m$ is isomorphic to $K[X_0, X_1]_{d \cdot m}$. Thus

$$h_X(m) = p_X(m) = d \cdot m + 1.$$

7.3. Saturation of homogeneous ideals

If $X \subset \mathbb{P}^n$ is a projective variety, its vanishing ideal $I(X)$ is a homogeneous ideal. Of course, any homogeneous ideal I with $\sqrt{I} = I(X)$ also defines X. A particular such ideal arises by deleting lower homogeneous parts of X. We denote by $(Z_0, \dots, Z_n)$ the ideal generated by the variables, called the **irrelevant ideal** (because its zero set in $\mathbb{P}^n$ is empty). For $k \geqslant 0$, let $J = I(X) \cap (Z_0, \dots, Z_n)^k$, so that

$$J = \bigoplus_{\ell \geqslant k} I(X)_\ell,$$

i.e. the first $k-1$ homogeneous parts have been deleted. It is clear that $\sqrt{J} = I(X)$, since $F^k \in I$ for all $F \in I(X)$, so in particular, J still defines X.

However, it has a stronger property. Given $F \in I(X)$ and any form $G \in K[Z]_k$ of degree k, we have $FG \in J$. This has the following consequence: If we dehomogenize and pass from $\mathbb{P}^n$ to one of the standard affine-open subsets $U_i = \{[Z] \in \mathbb{P}^n : Z_i \neq 0\} \cong \mathbb{A}^n$, we in fact still obtain the radical ideal: Take $i = 0$ for simplicity, then we pass from homogeneous polynomials in $K[Z] = K[Z_0, \dots, Z_n]$ to all polynomials in $K[z] = K[z_1, \dots, z_n]$, where $z_i = Z_i/Z_0$. Now given $f \in K[z]$ with $f|_{X \cap U_0} = 0$, we have $F(Z) = Z_0^d f(Z_1/Z_0, \dots, Z_n/Z_0) \in I(X)$, hence $Z_0^k F \in J$. But the dehomogenization of $Z_0^k F$ is f. Therefore, the dehomogenisations of $I(X)$ and J are the same.

Example 7.5. For a simple example, take the ideal J generated by Z_0^2 and Z_0Z_1 in $K[Z_0, Z_1]$. Then $\mathcal{V}(I) = \{P\}$ with $P = [0,1]$, but $I(P) = (Z_0, Z_1) \neq J$. In fact, $J = I(P) \cap (Z_0, Z_1)^2$. If we dehomogenize with respect to Z_1, we send Z_0 to $z = Z_0/Z_1$ and Z_1 to 1, so that J becomes the ideal generated by z^2 and z in $K[z]$, which is the vanishing ideal (z) of $P \in U_1 \cong \mathbb{A}^1$ in $K[z]$.

To formalize this, we introduce the following notion: Let $J \subset K[Z_0, \dots, Z_n]$ be an ideal. Then

$$\mathrm{Sat}(J) = \left\{F \in K[Z_0, \dots, Z_n] : \text{there exists } k \geqslant 0 \text{ such that } (Z_0, \dots, Z_n)^k \cdot F \in J\right\}$$

is an ideal, called the **saturation of** J. If $\mathrm{Sat}(J) = J$, then J is called **saturated**.

Lemma 7.6. *For any two homogeneous ideals $I, J \subset K[Z_0, \dots, Z_n]$, the following are equivalent:*

(1) I and J have the same saturation.
(2) There exists $d \geqslant 0$ such that $I_m = J_m$ holds for all $m \geqslant d$.
(3) I and J generate the same ideal locally, i.e. they are the same in every localization $K[Z_0/Z_i, \dots, Z_n/Z_i]$, $i = 0, \dots, n$.

Proof. Exercise 9.1. ■

Examples 7.7. Let $C \subset \mathbb{P}^d$ be the rational normal curve. Its vanishing ideal $I(C) = K[Z_0, \dots, Z_n]$ is generated by the $\binom{d}{2}$ polynomials

$$F_{i,j} = Z_iZ_j - Z_{i-1}Z_{j+1}, \quad 1 \leqslant i \leqslant j \leqslant d-1.$$

We have shown that for $d = 3$, all three polynomials $F_{1,1}, F_{1,2}, F_{2,2}$ are needed to cut out the curve. But in general, one can show that the $2d-3$ polynomials

$$F_{i,i} \text{ for } i = 1, \dots, d-1 \quad \text{and} \quad F_{i,i+1} \text{ for } i = 1, \dots, d-2$$

generate an ideal with saturation $I(C)$, but fail to generate $I(C)$ if $d \geqslant 4$ (c.f. [Ha], Exercise 5.3/5.4).

We will need a strengthening of Bertini's theorem. Consider first the following

Example 7.8. Let X be the plane curve in $\mathbb{P}^2$ defined by $F = Z_0Z_2 - Z_1^2$. The line W_α defined by $L_\alpha = Z_0 - \alpha Z_2$ for $\alpha \in K$ will meet X in the two points $[\alpha, \pm\sqrt{\alpha}, 1]$. For $\alpha \neq 0$, these are two distinct points and the vanishing ideal $I(X \cap W_\alpha)$ is generated by F and L_α. (This means that $(Z_0 - \alpha Z_2, Z_1 - \sqrt{\alpha}Z_2) \cap (Z_0 - \alpha Z_2, Z_1 + \sqrt{\alpha}Z_2) = (Z_0Z_2 - Z_1^2, Z_0 - \alpha Z_2)$; check!) However, the line W_0 is tangent to X, which is reflected in the fact that $(F, L_0) = (Z_0 Z_2 - Z_1^2, Z_0)$ is not the vanishing ideal (Z_0, Z_1) of the point $\{[0,0,1]\} = X \cap W_0$, since (F, L_0) does not contain Z_1.

Theorem 7.9. *Let $X \subset \mathbb{P}^n$ be an irreducible projective variety of dimension k with vanishing ideal $I(X)$. The general linear subspace $W \subset \mathbb{P}^m$ of dimension $n-k$ satisfies*

$$\mathrm{Sat}(I(X) + I(W)) = I(X \cap W).$$

Sketch of proof. We know from Thm. 6.11 that the general linear subspace W of dimension $n-k$ intersects X in finitely many points $X \cap W = \{p_1, \dots, p_d\}$. By Bertini's theorem 6.6, $X \cap W$ will also be smooth for general W. With our definition of smoothness, this is an empty statement, since finitely many points are always a smooth variety. However, what Bertini's theorem really says in this case is that the intersection of $X \cap W$ is *transversal* for general W, which means that $p_1, \dots, p_d$ are smooth points of X and the projective tangent space $\mathbb{T}_{p_i}(X)$ meets W only at p_i, for $i = 1, \dots, d$. Using the local description of tangent space from Prop. 6.2, one can show that this implies that $I(X)$ and $I(W)$ generate $I(X \cap W)$ locally. ∎

7.4. Proof of the main theorem on Hilbert polynomials

To show that the Hilbert function eventually agrees with a polynomial, we need one more elementary lemma.

Lemma 7.10. *Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be a function and assume that there exist $m_0 \in \mathbb{Z}$ and a polynomial $p(z) \in \mathbb{Q}(z)$ such that*

$$f(m+1) - f(m) = p(m)$$

for all $m \geqslant m_0$. Then there exists a polynomial $q \in \mathbb{Q}[z]$ of degree $\deg(p) + 1$ *such that $f(m) = q(m)$ for all $m \geqslant m_0$.*

Proof. For every $k \in \mathbb{N}$, let

$$F_k(z) = \binom{z}{k} = \frac{1}{k!} z(z-1)\cdots(z-k+1).$$

Since $F_k(z)$ has degree exactly k, it is clear that $1 = F_0, F_1, \dots, F_k$ form a basis of $\mathbb{Q}[z]_{\leqslant k}$. So write $p = \sum_{i=0}^{k} c_{k-i} F_i$ with $c_i \in \mathbb{Q}$.
Using the notation $(\Delta s)(z) = s(z+1) - s(z)$ for $s \in \mathbb{Q}[z]$, we find

$$\Delta F_k = \binom{z+1}{k} - \binom{z}{k} = \binom{z}{k-1} = F_{k-1}.$$

Thus $p = \Delta \widetilde{q}$, where

$$\widetilde{q} = c_0 \binom{z}{k+1} + \cdots + c_k \binom{z}{1}.$$

It follows that $\Delta(f - \widetilde{q})(m) = 0$ for all $m \geqslant m_0$. This means that $(f - \widetilde{q})(m)$ is a constant c_{k+1} for all $m \geqslant m_0$, hence

$$f(m) = \widetilde{q}(m) + c_{k+1}$$

for all $m \geqslant m_0$, so that $q = \widetilde{q} + c_{k+1}$ satisfies the claim. ∎

We are now ready for the proof of Thm. 7.3.

Theorem. *Let $X \subset \mathbb{P}^n$ be a projective variety with Hilbert function h_X. Then there exists a polynomial p_X in one variable with rational coefficients such that $h_X(m) = p_X(m)$ for all sufficiently large $m \in \mathbb{N}$. The degree of p_X is the dimension of X.*

Proof. Let $k = \dim(X)$ and let W be a linear subspace of dimension $n - k$ that meets X in finitely many points and satisfies $\mathrm{Sat}(I(X), I(L)) = I(X \cap W)$. Such W exists by Thm. 7.9 (Note that k is the highest dimension of any irreducible component of X). Let $L_1, \dots, L_k$ be linear forms that generate $I(W)$. Consider the chain of ideals

$$I(X) = I^{(0)} \subset I^{(1)} \subset I^{(k)} \subset K[Z_0, \dots, Z_n]$$

given by $I^{(j)} = (I(X), L_1, \dots, L_j)$. Let $S^{(j)} = K[Z_0, \dots, Z_n]/I^{(j)}$ and

$$h^{(j)}(m) = \dim(S^{(j)})_m,$$

so that $h^{(0)}(m) = h_X(m)$. Since the saturation of $I^{(k)}$ is $I(X \cap W)$, the function $h^{(k)}$ agrees with the Hilbert function $h_{X \cap W}$ for all sufficiently large arguments. Also, since $X \cap W$ consists of finitely many points, we know that $h_{X \cap W}(m) = d$ for large values of m, where $d = \#(X \cap W)$ (c.f. Exercise 8.1).

Now since the dimension of $X \cap W$ is 0, it follows that the variety $\mathcal{V}(I^{(j)})$ must have dimension exactly $k - j$. (Reason: The intersection of a variety of dimension ℓ with a hyperplane has dimension at least $\ell - 1$. So if the dimension between $\mathcal{V}(I^{(0)}) = X$ and $\mathcal{V}(I^{(k)}) = X \cap W$ drops from k to 0, it must drop by one in each step.) In particular, this means that the image of L_{j+1} in $S^{(j)}$ is not a zero-divisor, since otherwise L_{j+1} would generate a zero-dimensional ideal in $S^{(j)}$. Thus multiplication by L_{j+1} defines an inclusion $(S^{(j)})_{m-1} \hookrightarrow (S^{(j)})_m$ for all m. The quotient $(S^{(j)})_m/(S^{(j)})_{m-1}$ with respect to this inclusion is exactly $(S^{(j+1)})_m$. This tells us that

$$h^{(j+1)}(m) = h^{(j)}(m) - h^{(j)}(m-1).$$

Since $h^{(k)}$ is constant for large arguments, we can apply Lemma 7.10 inductively and conclude that $h^{(0)}$ for large arguments agrees with a polynomial of degree k. ■

Remark. While the Hilbert function of a projective variety eventually agrees with a polynomial, it is in general very hard to predict at what degree the equality occurs. When trying to analyze the above proof to extract that degree, the problem lies in the fact that we have to take the saturation of the radical ideal of X with the added linear forms. It is a non-trivial fact that given any polynomial $p \in \mathbb{Q}[z]$, there exists a constant m_0 such that $h_X(m) = p(m)$ holds for all $m \geqslant m_0$ and all varieties X with $p_X = p$. Very little is known in general about the size of this m_0, even though this is a highly relevant question for modern constructions like the so-called Hilbert-scheme.

7.5. The degree of a projective variety

Together with the dimension, the degree is an important invariant of projective varieties. Unlike the dimension, however, it is not invariant under isomorphism but rather depends on the embedding into projective space.

A hypersurface in $\mathbb{P}^n$ is defined by a square-free[2] homogeneous polynomial $F \in K[Z_0, \dots, Z_n]$, which is unique up to scaling. The degree of the hypersurface is then the degree of X. We would like to extend this definition to general subvarieties of $\mathbb{P}^n$. We say that $X \subset \mathbb{P}^n$ has **pure dimension** k if every irreducible component of X has dimension k.

Proposition 7.11. *Let $X \subset \mathbb{P}^n$ be a projective variety of pure dimension k. Then there exists a number $d \geqslant 0$ such that the general subspace of dimension $n-k$ intersects X in exactly d points. That number is equal to $k!$ times the leading coefficient of the Hilbert polynomial of X.*

The number d is called the **degree** of X.

Proof. We know that the general subspace of dimension $n-k$ meets X in finitely many points. We need to show that this number does not depend on the choice of subspace, i.e. there is some open subset U of $\mathbb{G}(n, n-k)$ such that the number of points in $X \cap W$ is the same for all $W \in U$. We could argue using Thm. 5.3. Alternatively, the proof of Thm. 7.3 shows that if W is any subspace meetings X transversally (i.e. satisfying the conclusion of Thm. 7.9), then the number of points in $X \cap W$ is equal to the kth difference function of the Hilbert function. This is exactly $k!$ times the leading coeffcient of the Hilbert polynomial. ■

It is not hard to see that this definition of degree agress with the old one for hypersurfaces: If X is a hypersurface defined by a square-free homogeneous polynomial F, then for a general line $W \subset \mathbb{P}^n$, the restriction $F|_W$ will have $\deg(F)$-many distinct roots. (We can argue with the discriminant as we did in the case of curves in Chapter 6.) Thus $\deg(F)$ is the degree of X.

Examples 7.12. (1) If $X \subset \mathbb{P}^n$ is set of d points, the degree is d by definition. Indeed, we determined in Example 7.4(1) that the Hilbert polynomial of X is the constant d.

(2) If X is a plane curve defined by a square-free polynomial F of degree d, we determined in Example 7.4(2) that the Hilbert polynomial is $p_X(t) = d \cdot t - d(d-3)/2$. This confirms that the degree of X is d. (It would not be hard to do a similar computation in the case of hypersurfaces in $\mathbb{P}^n$.)

(3) Likewise, it follows from 7.4(3) that the degree of the rational normal curve in $\mathbb{P}^d$ is d. Again, it is not hard to verify directly that the general hyperplane in $\mathbb{P}^d$ meets the rational normal curve in d distinct points.

(4) In Exercise 9.3, we showed that the Hilbert polynomial of the Segre variety $\Sigma_{m,n} = \sigma(\mathbb{P}^m + \mathbb{P}^n)$ is a polynomial of degree $m+n$ with leading coefficient $1/m!n!$. Thus the degree of $\Sigma_{m,n}$ is $(m+n)!/m!n! = \binom{m+n}{m}$.

[2]This means that F has distinct irreducible factors, so that the ideal (F) is radical.

7.6. Bézout theorems

The simplest form of Bézout's theorem asserts that two plane projective curves X and Y of degree d and e that intersect transversally have exactly $d \cdot e$ intersection points. In the case of curves, transversal intersection just means[3] that $I(X \cap Y) = I(X) + I(Y)$. Here is the precise statement:

Theorem 7.13. *(Bézout) Let X and Y be curves in $\mathbb{P}^2$ without common components. Then $X \cap Y$ consists of at most $\deg(X) \cdot \deg(Y)$ points, with equality if and only if X and Y intersect transversely.*

Proof. Let $d = \deg(X)$, $e = \deg(Y)$, $I(X) = (F)$, $I(Y) = (G)$ with $F, G \in K[Z] = K[Z_0, Z_1, Z_2]$ and put $I = (F, G) \subset K[Z]$. We wish to compute the Hilbert polynomial of $X \cap Y = \mathcal{V}(I)$. To do this, let $m \geqslant d + e$ and consider the linear map

$$\alpha\colon \begin{cases} K[Z]_{m-d} \times K[Z]_{m-e} & \to \quad K[Z]_m \\ \qquad (A, B) & \mapsto \quad AF + BG \end{cases}.$$

The image of α is exactly I_m. We have $\alpha(A, B) = 0$ if and only if $AF = -BG$. Since F and G are coprime by hypothesis, this implies $G|A$ and $F|B$. Putting $A = A'G$ and $B = B'F$, we find $A' = B'$. So the kernel of α consists exactly of all elements of the form $(RG, -RF)$ for $R \in K[Z]_{m-d-e}$. Therefore, we have

$$\begin{aligned} \dim(I_m) &= \dim(K[Z]_{m-d} \times K[Z]_{m-e}) - \dim K[Z]_{m-d-e} \\ &= \binom{m-d+2}{2} + \binom{m-e+2}{2} - \binom{m-d-e+2}{2} = \binom{m+2}{2} - de \\ &= \dim K[Z]_m - de. \end{aligned}$$

Thus $\operatorname{codim}(I_m) = de$ for all $m \geqslant d + e$. Now if X and Y intersect transversely, then I is the vanishing ideal of $X \cap Y$. We then have $\operatorname{codim}(I_m) = h_{X \cap Y}(m)$, hence $p_{X \cap Y} = de$. This shows that $X \cap Y$ is 0-dimensional of degree de, as claimed.

If X and Y do not intersect transvesely, then $I(X \cap Y) = \sqrt{(F, G)}$ and strictly includes (F, G). Therefore, $h_{X \cap Y}(m)$ must be strictly less than de for large m. ∎

The statement of Bézout's theorem can be made more precise and can also be generalized to higher dimensions. We indicate some such statements.

(1) A more precise version for curves takes **multiplicities** into account. This generalizes the fact that a polynomial of degree d in one variable has exactly d zeros, *counted with mulitplicities*. If X and Y are two curves without common components and $p \in X \cap Y$, the **intersection multiplicity** of X and Y at p is denoted $m_p(X, Y)$.

Theorem. *Let X and Y be curves in $\mathbb{P}^2$ without common components. Then*

$$\deg(X) \cdot \deg(Y) = \sum_{p \in X \cap Y} m_p(X, Y).$$

The intersection multiplicity $m_p(X, Y)$ is defined locally: If $p \in U_i \cong \mathbb{A}^2$ and f resp. g are polynomials defining $X \cap U_i$ resp. $Y \cap U_i$, then $m_p(X, Y)$ is defined as the dimension of the K-vector space $\mathcal{O}_p/(f, g)$, where $\mathcal{O}_p$ is the local ring $k[z_1, z_2]_{m_p}$. With a little bit of local algebra, this definition is quite simple to handle.

[3]Alternatively, X and Y intersect transversely if and only if every point $P \in X \cap Y$ is a smooth point of X and Y and the tangent lines $\mathbb{T}_p X$ and $\mathbb{T}_p Y$ are distinct.

(2) The most straightforward generalization of Thm. 7.13 to higher dimensions is the following.

Theorem. *Let $X_1, \dots, X_n$ be hypersurfaces in $\mathbb{P}^n$ and assume that $Z = X_1 \cap \cdots \cap X_n$ is finite with $I(Z) = \sum_{i=1}^n I(X_i)$. Then Z consists of* $\deg(X_1) \cdots \deg(X_n)$ *points.*

This becomes harder to prove than in the case of curves because if $I(X_i) = (F_i)$ and we look at the map $(A_1, \dots, A_n) \mapsto \sum_{i=1}^n A_i F_i$ as in the proof of Thm. 7.13, it is more difficult to compute the dimension of the kernel in degree m. The appropriate technical tool is the **Koszul complex**, sketched at the end of Chapter 13 in [Ha].

(3) If the intersection is allowed to be higher-dimensional, things become more complicated. For example, we know that if X and Y are two distinct quadric surfaces in $\mathbb{P}^3$, then $X \cap Y$ consists of a twisted cubic C and a line L. So we find $\deg(X) \cdot \deg(Y) = 4 = \deg(C) + \deg(L) = \deg(X \cap Y)$, as expected. On the other hand, the fact that the intersection with another quadric Z containing C defines C, which is of degree 3, while $\deg(X) \cdot \deg(Y) \cdot \deg(Z) = 8$, is less easy to explain in this way.

(4) In general, the condition that $I(X)$ and $I(Y)$ generate $I(X \cap Y)$ needs to be replaced by the following: Let $X, Y \subset \mathbb{P}^n$ and let $Z_1, \dots, Z_k$ be the irreducible components of $X \cap Y$. We say that X and Y intersect **generically transversely** if for each i the general point p on Z_i is a smooth point of X and Y and the tangent spaces $\mathbb{T}_p(X)$ and $\mathbb{T}_p(Y)$ span $\mathbb{T}_p(\mathbb{P}^n)$.

Theorem. *Let X and Y be subvarieties of pure dimensions k and ℓ in $\mathbb{P}^n$ with $k + \ell \geqslant n$ and suppose they intersect generically transversely. Then*

$$\deg(X \cap Y) = \deg(X) \cdot \deg(Y).$$

In particular, if $k + \ell = n$, this implies that $X \cap Y$ consists of $\deg(X) \cdot \deg(Y)$ *points.*

(5) Again, if we are content with an inequality, we can get away with less. We say that X and Y **intersect properly** if $\dim(X) + \dim(Y) \geqslant n$ and every irreducible component of $X \cap Y$ has the expected dimension, i.e. $\dim(X) + \dim(Y) - n$.

Theorem. *Let X, Y be subvarieties of $\mathbb{P}^n$ of pure dimension intersecting properly, then*

$$\deg(X \cap Y) \leqslant \deg(X) \deg(Y).$$

(6) There is a corresponding version with intersection multiplicities that looks analogous to the case of curves.

Theorem. *Let X, Y be subvarieties of $\mathbb{P}^n$ of pure dimension intersecting properly, then*

$$\deg(X) \cdot \deg(Y) = \sum_{Z \subset X \cap Y} m_Z(X, Y) \cdot \deg(Z)$$

where the sum is taken over all irreducible components Z of $X \cap Y$ and $m_Z(X, Y)$ is the intersection multiplicity of X and Y along Z.

However, it becomes *much* more complicated in general to correctly define the intersection multiplicity, and, naturally, also to prove the statement.