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9. CUBIC SURFACES

9.1. Twenty-seven lines on a cubic surface

The goal of this chapter is to prove the famous 'theorem of the 27 lines'. Let K be a field of characteristic different from 2 or 3.

Theorem 9.1 (Cayley-Salmon). *A smooth cubic surface in $\mathbb{P}^3$ contains exactly 27 lines.*

An equivalent way of saying this is that the Fano variety of a smooth cubic surface is 0-dimensional and of degree 27.

There is a wide variety of different proofs. The classical proof is very explicit, relying essentially on the clever use of resultants. We will take a more modern approach using the Grassmannian and incidence correspondences. Our presentation is based on the lecture notes of A. Gathmann (Algebraic Geometry, Kaiserslautern, 2014). The idea is simple: We start with a particular smooth cubic for which the 27 lines are easy to compute. We then use topological arguments to show that the number of lines contained in a smooth cubic cannot change as we move about in the set of smooth cubics. The only downside is that this argument only works over the complex numbers. First, here is the particular example we will start from.

Theorem 9.2. *The Fermat cubic in $\mathbb{P}^3$ defined by the equation $Z_0^3 + Z_1^3 + Z_2^3 + Z_3^3 = 0$ is smooth and contains exactly 27 lines.*

Proof. Let X be the Fermat cubic. Smoothness is immediate by examining the gradient. (Note that $\mathrm{char}(K) \neq 3$.) Up to permutation of the coordinates, a line in $\mathbb{P}^3$ is given by two linear equations of the form

$$\begin{aligned} Z_0 &= a_2Z_2 + a_3Z_3 \\ Z_1 &= b_2Z_2 + b_3Z_3 \end{aligned}$$

for $a_2, a_3, b_2, b_3 \in K$. Such a line is contained in X if and only if

$$(a_2Z_2 + a_3Z_3)^3 + (b_2Z_2 + b_3Z_3)^3 + Z_2^3 + Z_3^3 = 0$$

for all Z_2, Z_3. Comparing coefficients shows this to be the case if and only if

$$\begin{aligned} a_2^3 + b_2^3 &= -1 \\ a_3^3 + b_3^3 &= -1 \\ a_2^2a_3 &= -b_2^2b_3 \\ a_2a_3^2 &= -b_2b_3^2. \end{aligned}$$

If a_2, a_3, b_2, b_3 are all non-zero, squaring the third equation and using the fourth shows $a_2^3 = -b_2^3$, which contradicts the first equation. So suppose that $a_2 = 0$. Then the equations become equivalent to

$$b_2^3 = -1, b_3 = 0 \text{ and } a_3^3 = -1.$$

Hence we obtain 9 lines in X by choosing a primitive third root of unity ω and putting

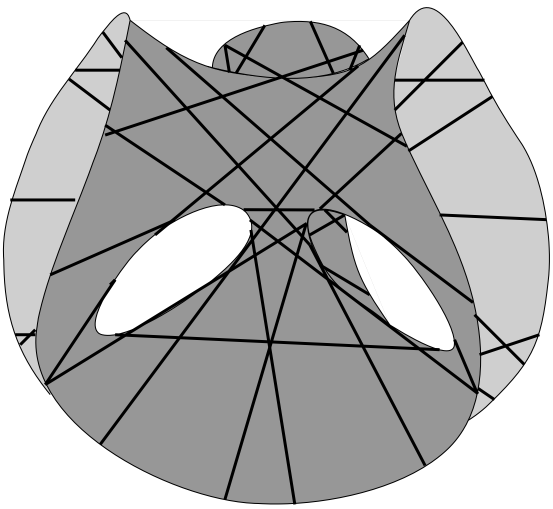
$$b_2 = -\omega^j, a_3 = -\omega^k \quad \text{for } 0 \leqslant j, k \leqslant 2.$$

Taking all possible permutations of the coordinates, we find that there are exactly 27 lines on X, namely

$$\begin{aligned} Z_0 + \omega^k Z_3 = Z_1 + \omega^j Z_2 = 0, &\quad 0 \leqslant j, k \leqslant 2, \\ Z_0 + \omega^k Z_2 = Z_3 + \omega^j Z_1 = 0, &\quad 0 \leqslant j, k \leqslant 2, \\ Z_0 + \omega^k Z_1 = Z_3 + \omega^j Z_2 = 0, &\quad 0 \leqslant j, k \leqslant 2. \end{aligned}$$ ■

Example 9.3. The Fermat cubic does not make for a very exciting picture, since most of the lines it contains are not real. But it is possible for the 27 lines on a real smooth cubic to be all real. An example of such a cubic was constructed by Clebsch in 1871, the **Clebsch diagonal surface**, given by the equation

$$Z_0^3 + Z_1^3 + Z_2^3 + Z_3^3 - (Z_0 + Z_1 + Z_2 + Z_3)^3 = 0.$$



Source: K. Hulek, Elementare Algebraische Geometrie, S. 118.

It is also possible to find cubics defined over $\mathbb{Q}$ for which all 27 lines are rational. A simple description of such cubics has been given by P. Swinnerton-Dyer in an unpublished note ('Cubic surfaces with 27 rational lines').

To proceed with the proof of Thm. 9.1, we introduce some notation. A cubic surface in $\mathbb{P}^3$ is given by a homogeneous polynomial $F_c = \sum_\alpha c_\alpha Z_0^{\alpha_0} Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3}$, where the sum is taken over all $\alpha = (\alpha_0, \ldots, \alpha_3) \in \mathbb{Z}^4$ with $0 \leqslant \alpha_i \leqslant 3$ and $\sum_{i=0}^3 \alpha_i = 3$. The space of cubics in four variables has dimension $\binom{3+3}{3} = 20$, so that $\mathbb{P}K[Z_0, \ldots, Z_3]_3$ is a projective space of dimension 19. Let U be the open-dense subset of smooth cubics, the complement of the discriminant hypersurface in $\mathbb{P}^{19}$.

Consider the following incidence correspondence:

$$\Sigma = \left\{(L, X): L \text{ is a line contained in the smooth cubic X}\right\} \subset \mathbb{G}(1,3) \times U.$$

Let $\pi: \Sigma \to U$ be the projection onto the second factor. We want to show that every fibre of π contains exactly 27 elements.

Lemma 9.4.

(1) *The incidence correspondence Σ is a closed subvariety of $\mathbb{G}(1,3) \times U$.*

(2) *Assume $K = \mathbb{C}$. For every pair $(L, X) \in \Sigma$, there exists an open neighbourhood $V \times W$ of (L, X) in $\mathbb{G}(1,3) \times U$ in the Euclidean topology and a holomorphic function $\Psi: W \to V$ whose graph in $\mathbb{G}(1,3) \times W$ is exactly $\Sigma \cap (V \times W)$.*

Proof. The proof of (1) is similar to that of Prop. 3.10, in which we showed that the Fano variety of a projective variety is closed. Let $(L, X) \in \Sigma$. After a linear change of coordinates, we may assume that $L = \mathcal{V}(Z_2, Z_3)$. Therefore, L lies in the open subset $\Omega \subset \mathbb{G}(1,3)$ of lines complementary to (i.e. not coplanar with) the line $\mathcal{V}(Z_0, Z_1)$. Such a line is given as the row span of a matrix of the form

$$\begin{bmatrix} 1 & 0 & a_2 & a_3 \\ 0 & 1 & b_2 & b_3 \end{bmatrix}$$

with $a_2, a_3, b_2, b_3 \in K$, with the origin corresponding to L. On U we use coordinates $(c_\alpha) \in \mathbb{P}^{19}$, as above. Write $(a, b, c) = (a_2, a_3, b_2, b_3, c_\alpha) \in \Omega \times U$, then $(a, b, c) \in \Sigma$ if and only if $F_c(r(1, 0, a_2, a_3) + s(0, 1, b_2, b_3)) = 0$ for all $r, s \in K$. Expanding, we find

$$\textstyle\sum_\alpha c_\alpha r^{\alpha_0} s^{\alpha_1} (ra_2 + sb_2)^{\alpha_2} (ra_3 + sb_3)^{\alpha_3} = 0 \text{ for all } r, s \in K.$$

This is a homogeneous polynomial in r, s with coefficients in a, b, c. Let $G_i(a, b, c)$ be the coefficient of $r^i s^{3-i}$, then $(a, b, c) \in \Sigma$ if and only if the polynomial

$$\textstyle\sum_{i=0}^{3} r^i s^{3-i} G_i(a, b, c) = 0$$

in r, s is zero, i.e. if and only if $G_i(a, b, c) = 0$ for $i = 0, \ldots, 3$. This proves (1).

To prove (2), we find Ψ by applying the implicit function theorem, as cited below, to the map $G = (G_0, G_1, G_2, G_3): \mathbb{C}^4 \times \mathbb{C}^{20} \to \mathbb{C}^4$ at some point $(0, 0, c) \in \Sigma$. We must show that the Jacobian $J = \partial G / \partial(a, b)$ of G at $(0, 0, c)$ is invertible. We compute

$$\begin{aligned} \frac{\partial}{\partial a_2} \left(\textstyle\sum_i r^i s^{3-i} G_i\right)(0, 0, c) &= \frac{\partial}{\partial a_2} F_c(r, s, ra_2 + sb_2, ra_3 + sb_3)(0, 0, c) \\ &= r \frac{\partial F_c}{\partial Z_2}(r, s, 0, 0). \end{aligned}$$

The coefficients of this polynomial in r, s form the first column of J. Similarly, we find the other columns and obtain

$$J = \left[r \tfrac{\partial F_c}{\partial Z_2}(r, s, 0, 0) \quad r \tfrac{\partial F_c}{\partial Z_3}(r, s, 0, 0) \quad s \tfrac{\partial F_c}{\partial Z_2}(r, s, 0, 0) \quad s \tfrac{\partial F_c}{\partial Z_3}(r, s, 0, 0)\right].$$

Thus if J had rank less than 4, we would have a linear relation among the four columns, i.e. a relation

$$(\lambda_1 r + \lambda_3 s) \frac{\partial F_c}{\partial Z_2}(r, s, 0, 0) + (\lambda_2 r + \lambda_4 s) \frac{\partial F_c}{\partial Z_3}(r, s, 0, 0) = 0$$

with $\lambda_1, \dots, \lambda_4 \in \mathbb{C}$, not all zero. Factoring both summands into linear factors in r, s, we see that $\frac{\partial F_c}{\partial Z_2}(r, s, 0, 0)$ and $\frac{\partial F_c}{\partial Z_3}(r, s, 0, 0)$ must have a common linear factor. In other words, there is a point $p = [p_0, p_1, 0, 0] \in L$ such that

$$\frac{\partial F_c}{\partial Z_2}(p) = \frac{\partial F_c}{\partial Z_3}(p) = 0.$$

But since $L = \mathcal{V}(Z_2, Z_3)$ is contained in the surface $\mathcal{V}(F_c)$, we have $F_c(Z_0, Z_1, 0, 0) = 0$ and $F_c(Z_0, Z_1, 0, 0) = 0$, thus also $\frac{\partial F_c}{\partial Z_0}(p) = 0$ and $\frac{\partial F_c}{\partial Z_1}(p) = 0$. Hence $\nabla(F_c)(p) = 0$, which is a contradiction, since $\mathcal{V}(F_c)$ is assumed to be smooth. This shows that J is invertible, so that (2) is proved. ∎

In topological terms, Lemma 9.4 says that the projection $\pi\colon \Sigma \to U$ is a covering map.

The proof made use of the implicit function theorem for holomorphic maps, which we state below. A reference is the book of Kaup and Kaup [Ka, Thm. 8.6].

Theorem 9.5 (Implicit mapping theorem). *Let $k, n \geqslant 1$ and let $\Phi\colon \mathbb{C}^{n+k} \to \mathbb{C}^n$ be a map. If $p \in \mathbb{C}^{n+k}$ is a point such that Φ is holomorphic in an open neighbourhood of p, $\Phi(p) = 0$ and the matrix*

$$\left[\frac{\partial \Phi_j}{\partial z_i}(p)\right]_{i,j=1,\dots,n}$$

is invertible, then there is an open neighbourhood $V \times W$ of $p \in \mathbb{C}^n \times \mathbb{C}^k$ and a holomorphic map $\Psi\colon W \to V$ with $\Psi(p_{n+1}, \dots, p_{n+k}) = (p_1, \dots, p_n)$ such that

$$\left\{(v, w) \in V \times W\colon \Phi(v, w) = 0\right\} = \Gamma_\Psi = \left\{(\Psi(w), w)\colon w \in W\right\}.$$

Before we can complete the proof of Thm. 9.1, we need a lemma from topology.

Lemma 9.6. *If $Y \subsetneq \mathbb{P}^n$ is a projective variety over $\mathbb{C}$, then $\mathbb{P}^n \setminus Y$ is connected in the Euclidean topology.*

Sketch of proof. Since Y is a proper closed subvariety, it has codimension at least 1 in $\mathbb{P}^n$. But then the codimension in the Euclidean topology is at least 2. This implies that the complement of Y is (path-)connected. [1] ∎

Proof of Thm. 9.1. We assume $K = \mathbb{C}$ and work in the Euclidean topology. Let X be a smooth cubic surface and L any line. There are two possibilities.

(1) Either $L \subset X$, then Lemma 9.4 shows that there exists an open neighbourhood $V_L \times W_L$ of (L, X) in $\mathbb{G}(1,3) \times U$ such that $\Sigma \cap (V_L \times W_L)$ is the graph of a holomorphic function $V_L \to W_L$. In particular, every cubic in W_L contains exactly one line from V_L.

(2) Or $L \not\subset X$, then since Σ is closed in $\mathbb{G}(1,3) \times U$, there is an open neighbourhood $V_L \times W_L$ of (L, X) such that no cubic in W_L contains any line from V_L.

Now keep X fixed and let L vary. Since $\mathbb{G}(1,3)$ is a projective variety, it is compact in the Euclidean topology. Hence there is a finite set $\mathcal{L} \subset \mathbb{G}(1,3)$ such that the open sets $\{V_L\colon L \in \mathcal{L}\}$ cover $\mathbb{G}(1,3)$. Let $W = \bigcap_{L \in \mathcal{L}} W_L$, an open neighbourhood of X. By

[1] To make this whole line of reasoning at least plausible, just think of the case of finitely many points on the projective line. In the Euclidean topology, we are dealing with points on the Riemann sphere, so that the complement remains connected.

construction, all cubics in W contain the same number of lines, namely the number of $L \in \mathcal{L}$ such that the first case above occurs.

Thus we have shown that the function $U \to \mathbb{Z} \cup \{\infty\}$ that assigns to a cubic X the number of lines contained in X is locally constant. Since U is connected by the preceding lemma, the number is indeed constant on U. The example of the Fermat cubic shows it to be equal to 27. ■

Remark 9.7. The proof of Thm. 9.1 given here only works for $K = \mathbb{C}$. In fact, a proof over $\mathbb{C}$ implies that the statement holds over any field of characteristic 0, by the so-called Lefschetz principle. It is also possible to make an analogous argument in positive characteristic, but more theory is needed to rephrase the statements about the covering map in an algebraic way (keyword: étale morphisms).

9.2. Configuration of the 27 lines - rationality and blow-ups

The 27 lines on a cubic surface are arranged in a very particular configuration. Let us first take another look at the lines on the Fermat cubic.

Corollary 9.8. *Let X be the Fermat cubic from Thm. 9.2.*

(1) *Given any line L in X, there are exactly 10 other lines in X that intersect L.*
(2) *Given any two disjoint lines L_1, L_2 in X, there are exactly 5 other lines in X meeting both L_1 and L_2.*

Proof. (1) After permuting the variables and multiplying with a primitive third root of unity ω, we may assume that L is the line given by $Z_0+Z_3 = Z_1+Z_2 = 0$. Inspection of the remaining lines shows that the 10 meeting L are exactly the ones given by

$$\begin{array}{lll} Z_0 + \omega^k Z_3 = Z_1 + \omega^j Z_2 = 0, & (j,k) = (1,0),(2,0),(0,1),(0,2) & \text{(4 lines)}, \\ Z_0 + \omega^j Z_2 = Z_3 + \omega^j Z_1 = 0, & 0 \leqslant j \leqslant 2 & \text{(3 lines)}, \\ Z_0 + \omega^j Z_1 = Z_3 + \omega^j Z_2 = 0, & 0 \leqslant j \leqslant 2 & \text{(3 lines)}. \end{array}$$

(2) We may again assume that $L_1 = Z_0 + Z_3 = Z_1 + Z_2 = 0$. Using (1), we can compare the lists of 10 lines meeting one of L_1, L_2 and find the ones that meet both. For instance, for $L_2 = Z_0 + \omega Z_3 = Z_1 + \omega Z_2 = 0$, we find the five lines

$$Z_0 + Z_3 = Z_1 + \omega Z_2 = 0$$
$$Z_0 + \omega Z_3 = Z_1 + Z_2 = 0$$
$$Z_0 + Z_1 = Z_3 + Z_2 = 0$$
$$Z_0 + \omega Z_1 = Z_3 + \omega Z_2 = 0$$
$$Z_0 + \omega^2 Z_1 = Z_3 + \omega^2 Z_2 = 0.$$ ■

Theorem 9.9. *The statements (1) and (2) in Cor. 9.8 hold for any smooth cubic surface.*

This can be proved via a refinement of the argument we used to prove the Theorem of the 27 lines, by considering an incidence correspondence of a cubic and up to three lines. Alternatively, it can be shown directly, as part of a more of explicit proof of the theorem. To give the spirit, let us prove the first statement. We follow the exposition in the book of Hulek [Hu].

Sketch of Proof of Thm. 9.9(1). Let X be a smooth cubic surface in $\mathbb{P}^3$ and let L be a line on X. Let H be any plane containing L. Then $X \cap H$ will decompose into L and a conic Q. We wish to show that there are exactly five planes H for which Q factors into two distinct lines. (This will in fact prove a slightly stronger statement than (1), namely that the lines meeting L come in 5 coplanar pairs.)

First, it is not hard to show that Q can never be a double line, since X is smooth. So we only need to worry about whether or not Q is irreducible. After a change of coordinates, we may assume that L is the line given by $Z_2 = Z_3 = 0$. If $X = \mathcal{V}(G)$ with $G \in K[X, Y, Z]_3$, then $L \subset X$ implies that G is of the form

$$G = AZ_0^2 + BZ_0Z_1 + CZ_1^2 + DZ_0 + EZ_1 + F$$

with $A, B, C, D, E, F \in K[Z_2, Z_3]$ homogeneous, where A, B and C are of degree 1, D and E of degree 2 and F of degree 3. The planes containing L are exactly the planes $H_{\lambda,\mu}$ defined by $\lambda Z_3 - \mu Z_2 = 0$, $[\lambda, \mu] \in \mathbb{P}^1$. Suppose $\mu \neq 0$, then we may assume $\mu = 1$, so that $H_{\lambda,1}$ is given by $Z_2 = \lambda Z_3$. Substituting into G gives

$$G|_{H_{\lambda,1}} = Z_3 Q(Z_0, Z_1, Z_3)$$

where

$$\begin{aligned}
&Q = \widetilde{A}Z_0^2 + \widetilde{B}Z_0Z_1 + \widetilde{C}Z_1^2 + \widetilde{D}Z_0Z_3 + \widetilde{E}Z_1Z_3 + \widetilde{F}Z_3^2,\\
&\widetilde{A} = A(\lambda,1), \widetilde{B} = B(\lambda,1), \widetilde{C} = C(\lambda,1), \widetilde{D} = D(\lambda,1), \widetilde{E} = E(\lambda,1), \widetilde{F} = F(\lambda,1).
\end{aligned}$$

Hence Q defines a singular conic if and only if the determinant

$$\det \begin{bmatrix} \widetilde{A} & \frac{1}{2}\widetilde{B} & \frac{1}{2}\widetilde{D} \\ \frac{1}{2}\widetilde{B} & \widetilde{C} & \frac{1}{2}\widetilde{E} \\ \frac{1}{2}\widetilde{D} & \frac{1}{2}\widetilde{E} & \widetilde{F} \end{bmatrix} = \widetilde{A}\widetilde{C}\widetilde{F} + \frac{1}{4}\big(\widetilde{B}\widetilde{D}\widetilde{E} - \widetilde{C}\widetilde{D}^2 - \widetilde{A}\widetilde{E}^2 - \widetilde{B}^2\widetilde{F}\big)$$

is zero. Allowing for the case $\mu = 0$, $\lambda = 1$, which is analogoues, we see that the values of the parameter $[\lambda, \mu] \in \mathbb{P}^1$ for which $X \cap H_{\lambda,\mu}$ factors into three lines are exactly the zeros of the homogeneous polynomial

$$\Delta(Z_2, Z_3) = ACF + \frac{1}{4}\big(BDE - CD^2 - AE^2 - B^2F\big).$$

Since Δ has degree 5, the claim is almost proved. Note that Δ cannot be the zero polynomial, since that would mean that X contains infinitely many lines. What remains show is that the roots of Δ are distinct. This can be checked directly. We refer to Hulek [Hu] for this last part of the argument. ∎

Remark 9.10. Further statements about the configuration are possible. The full truth consists of information about all the pairwise intersections and about which triples of lines are coplanar. Again, all this information is independent of the chosen cubic.

The intersection of the lines can also be understood as a graph with 27 vertices with an edge if and only if the two corresponding lines intersect. The complementary graph, in which there is an edge if and only if the corresponding lines are skew, is called the **Schläfli graph** (see Fig. 1), named after the Swiss mathematician Ludwig Schläfli. It has very particular combinatorial properties studied in graph theory.

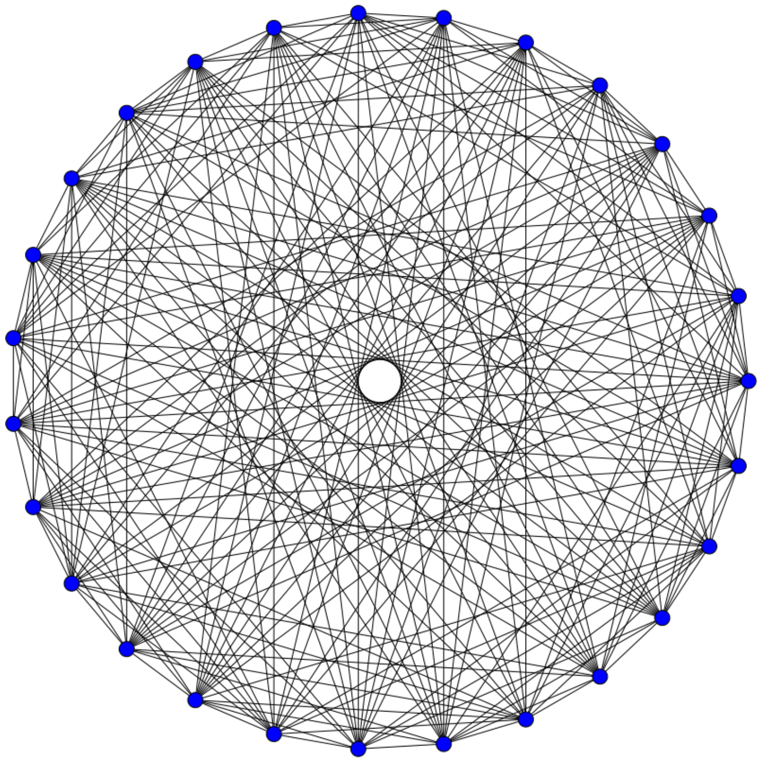


FIGURE 1. The Schläfli graph
Source: Wikimedia Commons (Claudio Rocchini)

We now apply some of our knowledge about the configuration of lines to improve our understanding of the geometry of the smooth cubic surface.

Proposition 9.11. *Any smooth cubic surface is rational.*

Proof. Let X be a smooth cubic surface in $\mathbb{P}^3$. By Thm. 9.9, there are two disjoint lines $L_1, L_2 \subset X$. We show that X is birational to $L_1 \times L_2 \cong \mathbb{P}_1 \times \mathbb{P}_1$ (which in turn is rational because it contains an open dense copy of $\mathbb{A}^2$).

Given any point $p \in \mathbb{P}^3$, $p \notin L_1 \cup L_2$, there is a unique line through p intersecting both L_1 and L_2 (see Exercise 13.1). Denote this line by L_p and let $\varphi\colon X \dashrightarrow L_1 \times L_2$ be the map that sends p to the pair $(L_1 \cap L_p, L_2 \cap L_p)$. It is easy to check that φ is a morphism[2] on $X \smallsetminus (L_1 \cup L_2)$. The inverse of φ is the rational map taking $(q_1, q_2) \in L_1 \times L_2$ to the intersection point of the line $\overline{q_1 q_2}$ with $X \smallsetminus (L_1 \cup L_2)$. Thus this is defined whenever $\overline{q_1, q_2}$ is not one of the 27 lines contained in X. ∎

Remark 9.12. This result is in contrast with the case of curves: A smooth cubic in $\mathbb{P}^2$ is *not* rational. By a famous result of Clemens and Griffiths already mentioned earlier, a general cubic threefold in $\mathbb{P}^4$ is not rational. Whether a general cubic hypersurface in $\mathbb{P}^5$ is rational is an open question. On the other hand, any cubic hypersurface $X \subset \mathbb{P}^n$ with $n \geqslant 3$ is unirational, i.e. there exists a dominant rational map $\mathbb{P}^{n-1} \to X$ (see [Ha], Prop. 18.20 for the case $n = 4$).

The construction in the proof of Prop. 9.11 shows much more than just rationality.

[2]For example, we may change coordinates on $\mathbb{P}^3$ in such a way that $L_1 = \mathcal{V}(Z_0, Z_1)$, $L_2 = \mathcal{V}(Z_2, Z_3)$. Then φ is just the map $[Z] \mapsto \big([Z_2, Z_3], [Z_0, Z_1]\big)$.

Theorem 9.13. *A smooth cubic surface in $\mathbb{P}^3$ is isomorphic to the blow-up of $\mathbb{P}^2$ in six suitably chosen points.*

Sketch of proof. Let X be a smooth cubic surface in $\mathbb{P}^3$ and L_1, L_2 two disjoint lines in X; let $\varphi\colon X \dashrightarrow L_1 \times L_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ be the birational map constructed in the proof of the previous proposition. The first claim is that φ is in fact a morphism. In other words, we claim that the morphism φ defined on $X \smallsetminus (L_1 \cup L_2)$ extends to all of X. To see this, we use a different description of φ. Given $p \in X$, $p \notin L_1$, let H_p be the plane in $\mathbb{P}^3$ spanned by L_1 and p and put $\varphi_2(p) = H_p \cap L_2$. (Note that L_2 cannot be contained in H since it does not intersect L_1). Define $\varphi_1(p)$ for $p \in X \smallsetminus L_2$ in the analogous fashion. Then clearly $\varphi(p) = (\varphi_1(p), \varphi(p))$ for all $p \in X \smallsetminus (L_1 \cup L_2)$. Now if $p \in L_1$, let H_p be the tangent plane $\mathbb{T}_p X$ and again set $\varphi_2(p) = H_p \cap L_2$. Extending φ_1 in the same way, we see that φ is indeed a morphism.

As noted in the proof of Prop. 9.11, the inverse map φ^{-1} is undefined in any point $(q_1, q_2) \in L_1 \times L_2$ for which the line $\overline{q_1 q_2}$ is contained in X. For in this case, the whole line $\overline{q_1, q_2}$ is mapped to (q_1, q_2) under φ. By Thm. 9.9, there are exactly five such lines. It can now be checked that $\varphi\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ is indeed the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ in the five points to which the five lines are contracted under φ.

As shown in §5, the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ in a point is isomorphic to the blow-up of $\mathbb{P}^2$ in two points. It follows that the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ in five points is isomorphic to the blow-up of $\mathbb{P}^2$ in six points. ∎

Remark 9.14. There is a different, perhaps more suggestive way to look at the isomorphism between a smooth cubic surface and the blow-up of six points in $\mathbb{P}^2$. Let $\Gamma = \{p_1, \ldots, p_6\}$ be six points in $\mathbb{P}^2$ in general position, by which we mean that no three are collinear and not all six lie on a conic. The space of cubics $I(\Gamma)_3$ vanishing on Γ has dimension $10 - 6 = 4$ (equivalently $h_\Gamma(3) = 6$). Let $F_0, \ldots, F_3 \in I(\Gamma)_3$ be a basis and consider the rational map

$$\Phi\colon \begin{cases} \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^3 \\ [Z] & \mapsto & [F_0(Z), F_1(Z), F_2(Z), F_3(Z)] \end{cases}.$$

It is defined exactly on $\mathbb{P}^2 \smallsetminus \Gamma$. In fact, $F_0, \ldots, F_3$ generate the ideal $I(\Gamma)$, so that the graph of Φ is preciselyy the blow-up of $\mathbb{P}^2$ in Γ. The projection of this graph onto $\mathbb{P}^3$ is an isomorphism and the image is a smooth cubic hypersurface. This is somewhat analogous to the example of the projection of the quadric surface in $\mathbb{P}^3$ studied in §5, but it is a bit harder to proof. One can show further that every smooth cubic surface arises in this way, which also shows again that all smooth cubic surfaces are rational.

The realization of a smooth cubic surface as the blow-up of six points in $\mathbb{P}^2$ also gives an alternative description of the 27 lines and their configuration. These are

- The six exceptional lines of the blow-up.
- The strict transforms of the $15 = \binom{6}{2}$ lines passing through two of the six points.
- The strict transforms of the $6 = \binom{6}{5}$ conics passing through five of the six points.

It can be checked explicitly that these lines and conics indeed correspond to lines in the blow-up.