CLASSICAL ALGEBRAIC GEOMETRY

Daniel Plaumann
Universität Konstanz

Summer 2015

A brief inaccurate history of algebraic geometry

1800 - 1880	**Projective geometry.** Emergence of 'analytic' geometry with cartesian coordinates, as opposed to 'synthetic' (axiomatic) geometry in the style of Euclid. (*Celebrities:* Plücker, Hesse, Cayley)
1820 - 1920	**Complex analytic geometry.** Powerful new tools for the study of geometric problems over $\mathbb{C}$. (*Celebrities:* Abel, Jacobi, Riemann)
1880 - 1940	**Classical school.** Perfected the use of existing tools without any 'dogmatic' approach. (*Celebrities:* Castelnuovo, Segre, Severi, M. Noether)
1920 - 1950	**Algebraization.** Development of modern algebraic foundations ('commutative ring theory') for algebraic geometry. (*Celebrities:* Hilbert, E. Noether, Zariski)
from 1950	**Modern algebraic geometry.** All-encompassing abstract frameworks (schemes, stacks), greatly widening the scope of algebraic geometry. (*Celebrities:* Weil, Serre, Grothendieck, Deligne, Mumford)
from 1990	**Computational algebraic geometry** Symbolic computation and discrete methods, many new applications. (*Celebrities:* Buchberger)

Literature

Primary source

[Ha] J. Harris, *Algebraic Geometry: A first course*. Springer GTM 133 (1992)

Classical algebraic geometry

[BCGB] M. C. Beltrametti, E. Carletti, D. Gallarati, G. Monti Bragadin. *Lectures on Curves, Surfaces and Projective Varieties. A classical view of algebraic geometry*. EMS Textbooks (translated from Italian) (2003)

[Do] I. Dolgachev. *Classical Algebraic Geometry. A modern view*. Cambridge UP (2012)

Algorithmic algebraic geometry

[CLO] D. Cox, J. Little, D. O'Shea. *Ideals, Varieties, and Algorithms*. Springer UTM (1992)

[EGSS] D. Eisenbud, D. R. Grayson, M. Stillman, B. Sturmfels. *Computations in Algebraic Geometry with Macaulay* 2. Springer (2002).

'Big books'

[GH] P. Griffiths, J. Harris. *Principles of Algebraic Geometry*. John Wiley & Sons (1978)

[Hs] R. Hartshorne. *Algebraic Geometry*. Springer GTM 52 (1977)

[Sh] I. Shafarevich. *Basic Algebraic Geometry I: Varieties in projective space*. Springer (translated from Russian) (1974)

§1
Projective Varieties

Affine varieties

K algebraically closed field
$\mathbb{A}^n = K^n$ affine space

$V \subset \mathbb{A}^n$ is an **affine variety** if there is a set of polynomials $M \subset K[x_1, \ldots, x_n]$ such that

$$V = \mathcal{V}(M) = \{p \in \mathbb{A}^n : f(p) = 0 \text{ for all } f \in M\}.$$

Affine varieties

K algebraically closed field
$\mathbb{A}^n = K^n$ affine space

$V \subset \mathbb{A}^n$ is an **affine variety** if there is a set of polynomials $M \subset K[x_1, \ldots, x_n]$ such that

$$V = \mathcal{V}(M) = \{p \in \mathbb{A}^n : f(p) = 0 \text{ for all } f \in M\}.$$

If I is the **ideal** generated by M, then $\mathcal{V}(M) = \mathcal{V}(I)$. By the **Hilbert Basis Theorem**, there is a **finite** subset $M' \subset M$ that also generates I, so in particular $\mathcal{V}(M') = \mathcal{V}(M)$.

Affine varieties

K algebraically closed field
$\mathbb{A}^n = K^n$ affine space

$V \subset \mathbb{A}^n$ is an **affine variety** if there is a set of polynomials $M \subset K[x_1, \ldots, x_n]$ such that

$$V = \mathcal{V}(M) = \{p \in \mathbb{A}^n : f(p) = 0 \text{ for all } f \in M\}.$$

If I is the **ideal** generated by M, then $\mathcal{V}(M) = \mathcal{V}(I)$. By the **Hilbert Basis Theorem**, there is a **finite** subset $M' \subset M$ that also generates I, so in particular $\mathcal{V}(M') = \mathcal{V}(M)$.

If I and J are two ideals in $K[x_1, \ldots, x_n]$, then

$$V(I) \cup V(J) = V(IJ) = \mathcal{V}(I \cap J)$$
$$V(I) \cap V(J) = V(I + J)$$

where IJ is the ideal generated by all products fg, $f \in I$, $g \in J$.

Projective space

Let V be a K-vector space.

$$\mathbb{P}(V) = \{\text{one-dimensional subspaces of } V\}$$, the **projective space of V**

$$\mathbb{P}^n = \mathbb{P}K^{n+1} = \left(K^{n+1} \smallsetminus \{0\}\right) / \sim$$

where $v \sim w \iff \exists \lambda \in K^\times : v = \lambda w$.

Projective space

Let V be a K-vector space.

$$\mathbb{P}(V) = \{\text{one-dimensional subspaces of } V\}, \text{ the } \textbf{projective space of } V$$

$$\mathbb{P}^n = \mathbb{P}K^{n+1} = \left(K^{n+1} \setminus \{0\}\right)/\sim$$

where $v \sim w \iff \exists \lambda \in K^\times : v = \lambda w$.

Points of $\mathbb{P}^n$ are denoted in **homogeneous coordinates** $[Z_0, \ldots, Z_n]$ where

$$[Z_0, \ldots, Z_n] = [\lambda Z_0, \ldots, \lambda Z_n] \text{ for } \lambda \in K^\times.$$

Projective space

Let V be a K-vector space.

$$\mathbb{P}(V) = \{\text{one-dimensional subspaces of } V\}, \text{ the } \textbf{projective space of } V$$

$$\mathbb{P}^n = \mathbb{P}K^{n+1} = \left(K^{n+1} \setminus \{0\}\right)/\sim$$

where $v \sim w \iff \exists \lambda \in K^\times : v = \lambda w$.

Points of $\mathbb{P}^n$ are denoted in **homogeneous coordinates** $[Z_0, \ldots, Z_n]$ where

$$[Z_0, \ldots, Z_n] = [\lambda Z_0, \ldots, \lambda Z_n] \text{ for } \lambda \in K^\times.$$

Projective varieties

A polynomial $F \in K[Z_0, \ldots, Z_n]$ is **not** a function on $\mathbb{P}^n$, since in general

$$F(Z_0, \ldots, Z_n) \neq F(\lambda Z_0, \ldots, \lambda Z_n).$$

If F is **homogeneous** of degree d, then

$$F(\lambda Z_0, \ldots, \lambda Z_n) = \lambda^d F(Z_0, \ldots, Z_n).$$

So given a set M of homogeneous polynomials in $K[Z_0, \ldots, Z_n]$, it makes sense to define

$$\mathcal{V}(M) = \left\{p \in \mathbb{P}^n : f(p) = 0 \text{ for all } f \in M\right\}, \text{ a } \textbf{projective variety}.$$

The Zariski topology

The projective (resp. affine) varieties in $\mathbb{P}^n$ (resp. $\mathbb{A}^n$) form the closed sets of a topology, the **Zariski topology**. Projective space is covered by the open subsets

$$U_i = \{[Z_0, \ldots, Z_n] \in \mathbb{P}^n : Z_i \neq 0\} = \{[Y_0, \ldots, Y_{i-1}, 1, Y_{i+1}, \ldots, Y_n] \in \mathbb{P}^n\}.$$

The map

$$U_i \to \mathbb{A}^n, [Z_0, \ldots, Z_n] \mapsto (Z_0/Z_i, \ldots, Z_{i-1}/Z_i, Z_{i+1}/Z_i, \ldots, Z_n)$$

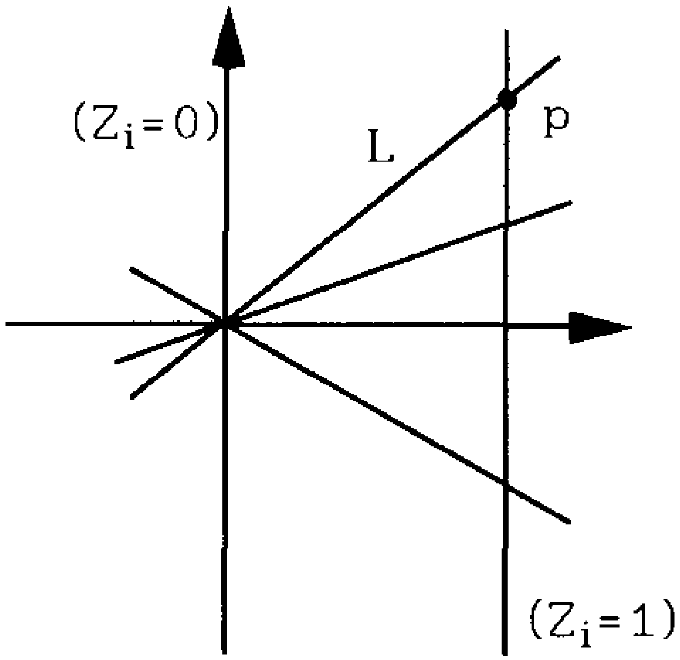
is a homeomorphism. The inverse map is

$$\mathbb{A}^n \to U_i, (z_0, \ldots, z_{i-1}, z_{i+1}, z_n) \mapsto [z_1, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n].$$

Thus $\mathbb{P}^n$ is covered by $n+1$ copies of $\mathbb{A}^n$.

How to think about projective space

Projective line

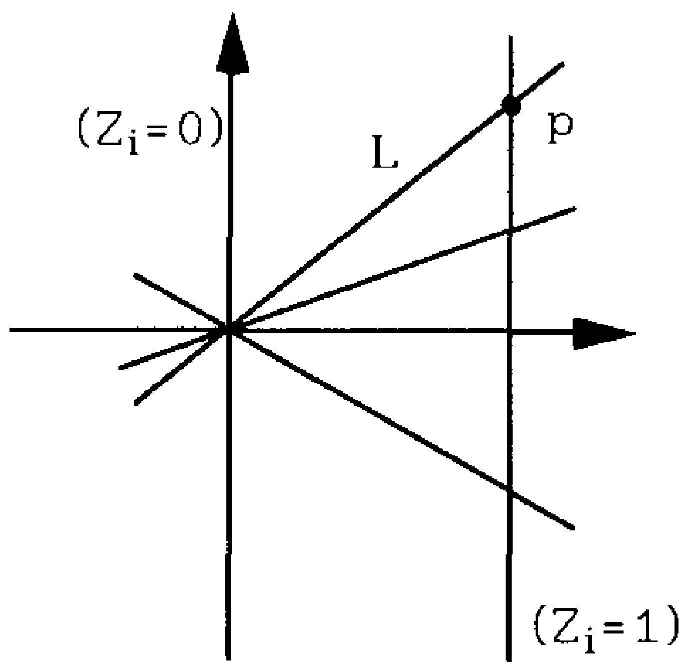


Exact picture

[Ha, p.4]

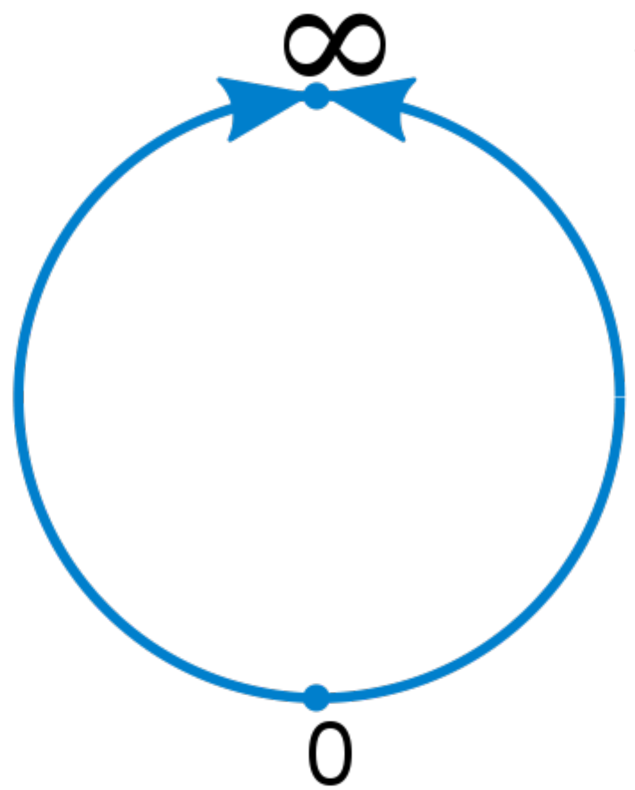
How to think about projective space

Projective line



Exact picture

[Ha, p.4]

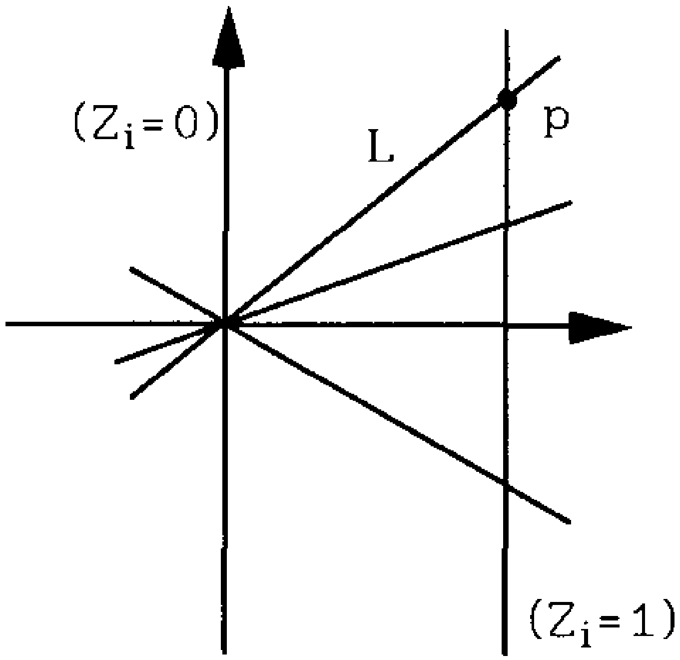


Real topological picture

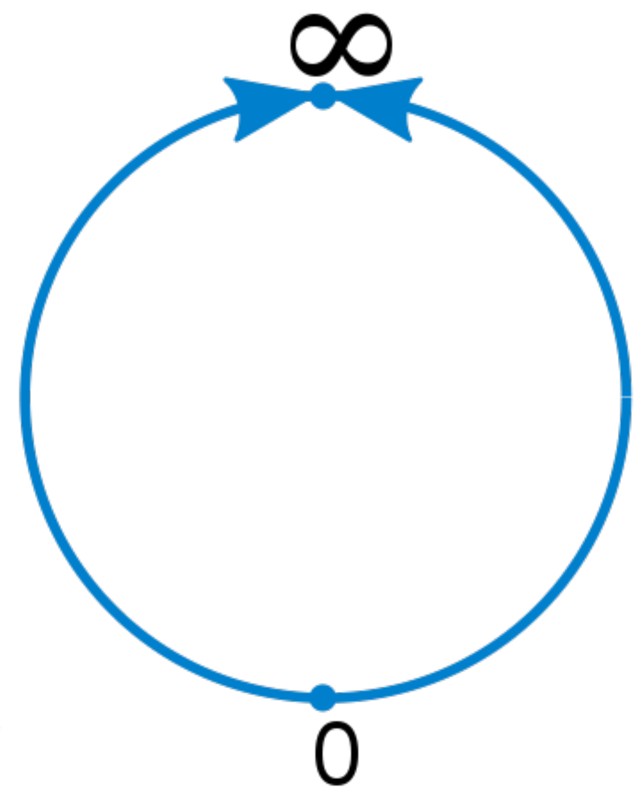
(O. Alexandrov - Wikimedia Commons)

How to think about projective space

Projective line



Exact picture
[Ha, p.4]

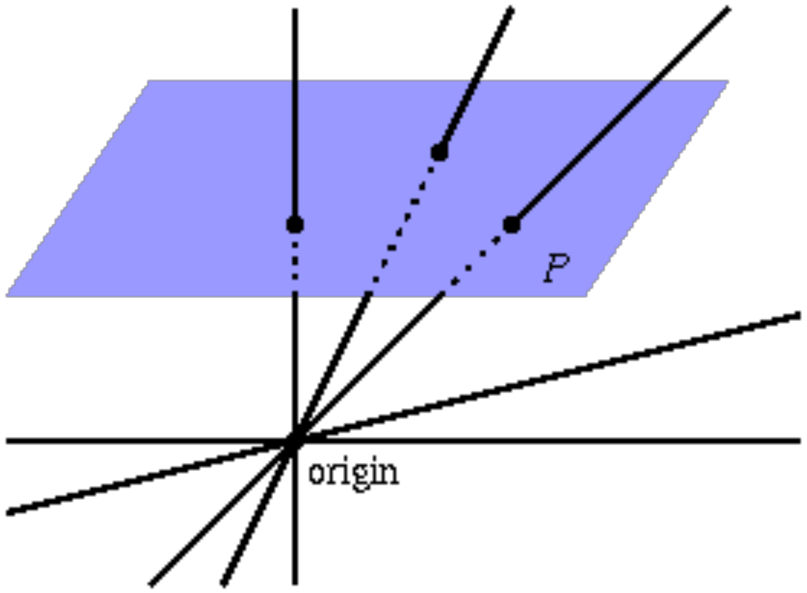


Real topological picture
(O. Alexandrov - Wikimedia Commons)

Intuitive picture

How to think about projective space

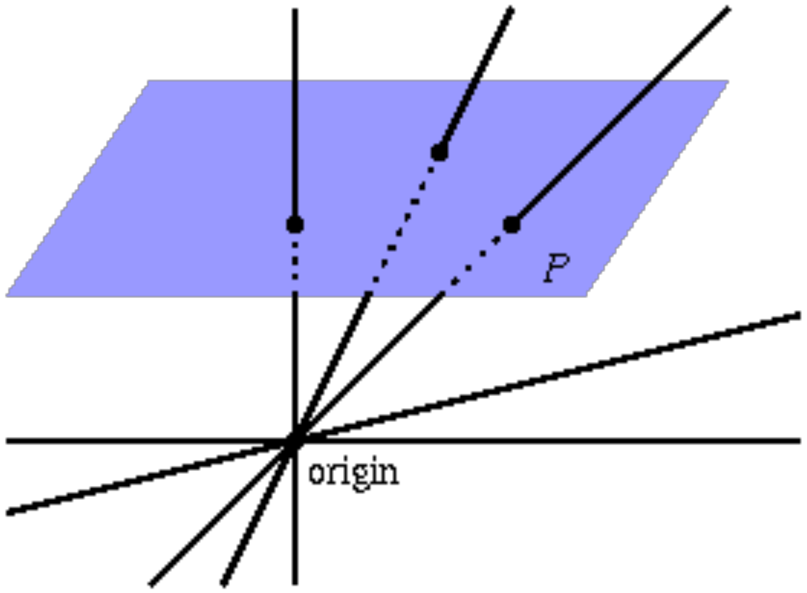
Projective plane



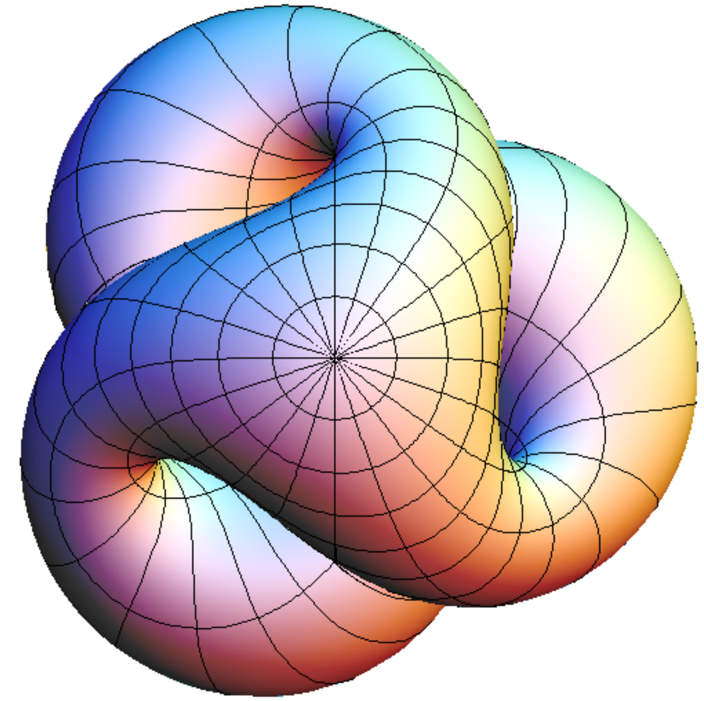
Exact picture
[Univ. of Toronto Math Network]

How to think about projective space

Projective plane



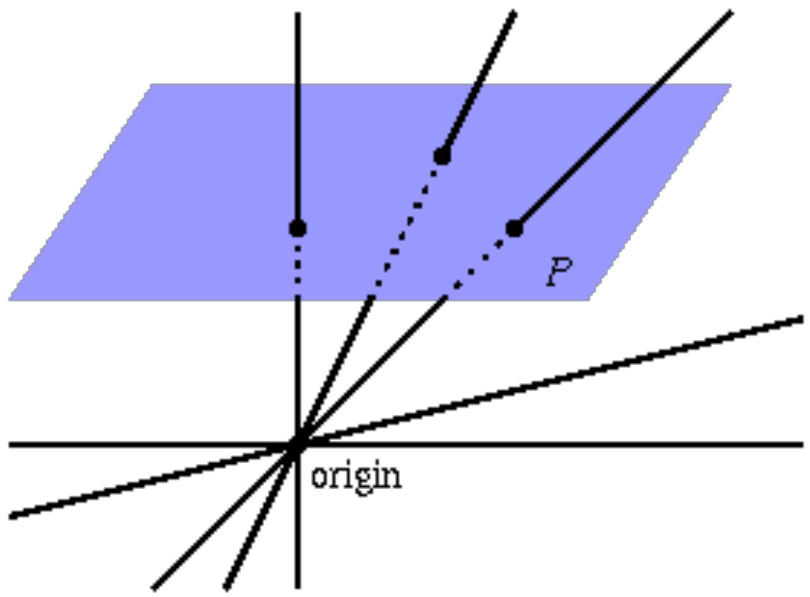
Exact picture
[Univ. of Toronto Math Network]



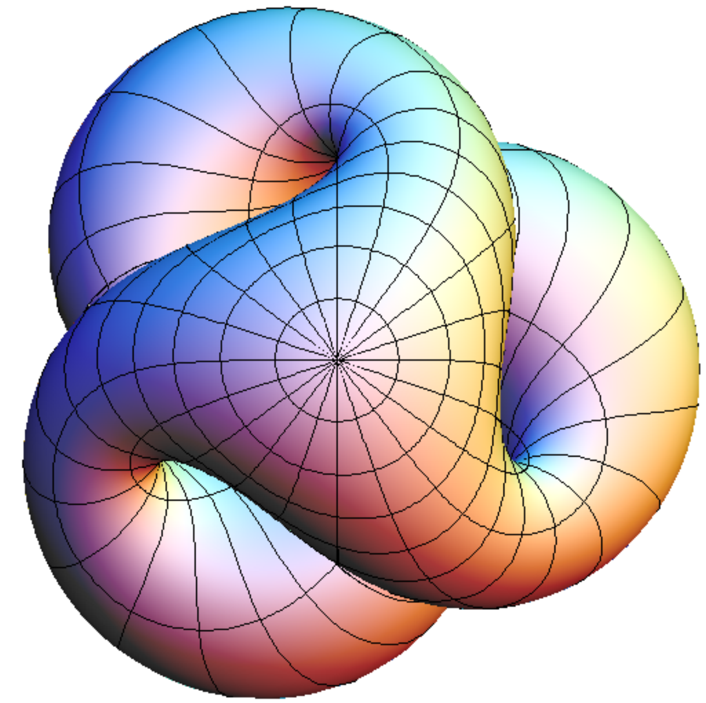
Real topological picture - Boy's surface
(virtualmathmuseum.org)

How to think about projective space

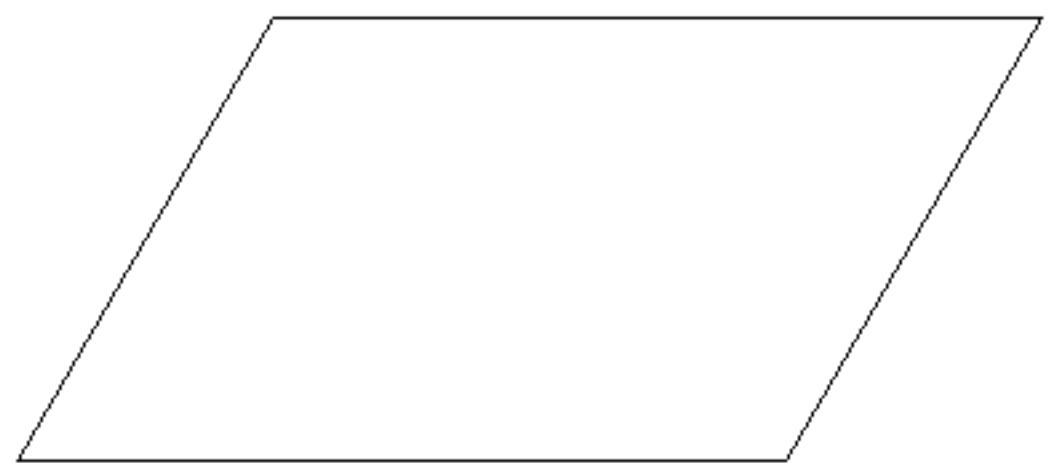
Projective plane



Exact picture
[Univ. of Toronto Math Network]



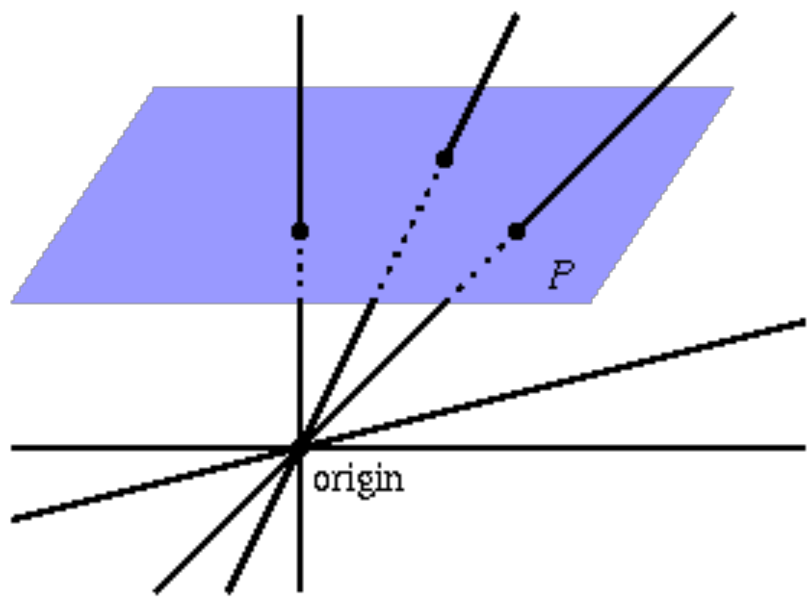
Real topological picture - Boy's surface
(virtualmathmuseum.org)



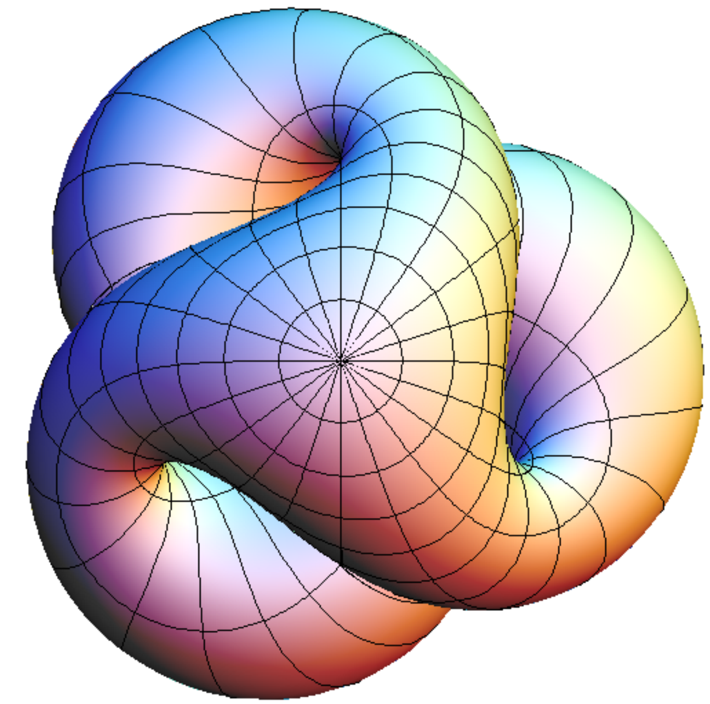
Intuitive picture
[freehomeworkmathhelp.com]

How to think about projective space

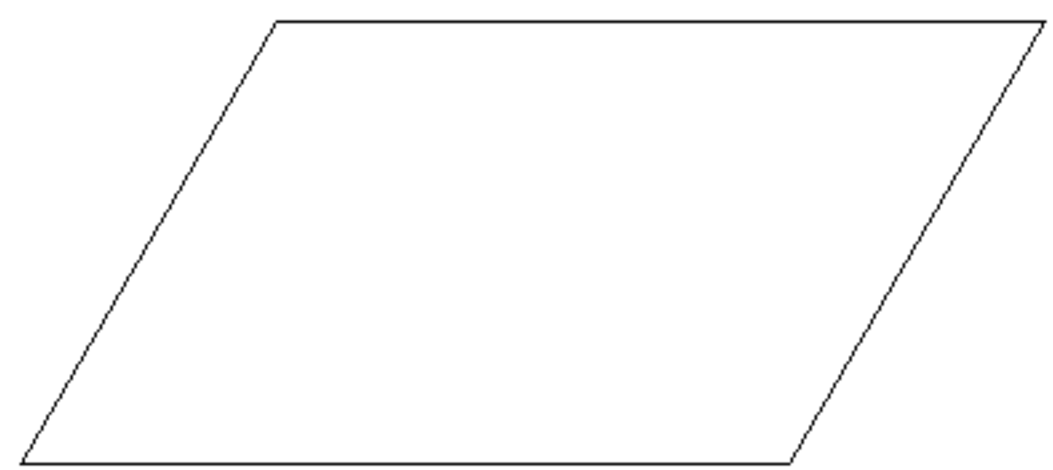
Projective plane



Exact picture
[Univ. of Toronto Math Network]



Real topological picture - Boy's surface
(virtualmathmuseum.org)



Intuitive picture
[freehomeworkmathhelp.com]

We think of projective space over an algebraically closed field just as real affine space, together with the *idea* that taking intersections always works perfectly.

Linear Spaces

If $W \subset V$ is a linear subspace, then $\mathbb{P}W \subset \mathbb{P}V$ is a projective subspace, a **linear space** of dimension $\dim \mathbb{P}W = \dim W - 1$ in $\mathbb{P}V$.

Linear Spaces

If $W \subset V$ is a linear subspace, then $\mathbb{P}W \subset \mathbb{P}V$ is a projective subspace, a **linear space** of dimension $\dim \mathbb{P}W = \dim W - 1$ in $\mathbb{P}V$.

$\dim \mathbb{P}W = 0$	**point**
$\dim \mathbb{P}W = 1$	**line**
$\dim \mathbb{P}W = 2$	**plane**
$\dim \mathbb{P}W = \dim \mathbb{P}V - 1$	**hyperplane**

If $L = \mathbb{P}W$, $L' = \mathbb{P}W'$, write

$$\overline{LL'} = \mathbb{P}(W + W').$$

We have

$$\dim \overline{LL'} = \dim L + \dim L' - \dim L \cap L'.$$

Dimension

Let X be a variety. X is **reducible** if it is the union of two proper, non-empty closed subvarieties; otherwise it is called **irreducible.**

Dimension

Let X be a variety. X is **reducible** if it is the union of two proper, non-empty closed subvarieties; otherwise it is called **irreducible.**

The **Dimension of X** is the largest integer k such that there exists a chain

$$\varnothing \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_{k-1} \subsetneq X$$

of irreducible closed subvarieties.

In particular, $\dim \mathbb{P}^n = \dim \mathbb{A}^n = n$.

Dimension

Let X be a variety. X is **reducible** if it is the union of two proper, non-empty closed subvarieties; otherwise it is called **irreducible.**

The **Dimension of X** is the largest integer k such that there exists a chain

$$\emptyset \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_{k-1} \subsetneq X$$

of irreducible closed subvarieties.

In particular, $\dim \mathbb{P}^n = \dim \mathbb{A}^n = n$.

If $X \subset \mathbb{P}^n$ or $X \subset \mathbb{A}^n$ is irreducible:

$\dim X = 0$	**point**
$\dim X = 1$	**curve**
$\dim X = 2$	**surface**
$\dim X = 3$	**threefold**
$\dim X = n-1$	**hypersurface**

Dimension

Let X be a variety. X is **reducible** if it is the union of two proper, non-empty closed subvarieties; otherwise it is called **irreducible.**

The **Dimension of X** is the largest integer k such that there exists a chain

$$\varnothing \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_{k-1} \subsetneq X$$

of irreducible closed subvarieties.

In particular, $\dim \mathbb{P}^n = \dim \mathbb{A}^n = n$.

If $X \subset \mathbb{P}^n$ or $X \subset \mathbb{A}^n$ is irreducible:

$\dim X = 0$	**point**
$\dim X = 1$	**curve**
$\dim X = 2$	**surface**
$\dim X = 3$	**threefold**
$\dim X = n-1$	**hypersurface**

Theorem. *The hypersurfaces in $\mathbb{P}^n$ are exactly the varieties defined by a single equation.*

The hypersurfaces in $\mathbb{P}^2$ are the **plane projective curves**.

Points

Proposition 1.1. *Any finite set of d points in $\mathbb{P}^n$ is described by polynomials of degree at most d.*

Points

Proposition 1.1. *Any finite set of d points in $\mathbb{P}^n$ is described by polynomials of degree at most d.*

Proof. Let $\Gamma = \{p_1, \ldots, p_d\}$. For $q \notin \Gamma$, let $L_{q,i}$ be a linear form with $L_{q,i}(p_i) = 0$ and $L_{q,i}(q) \neq 0$. Put

$$F_q = L_{q,1} \cdots L_{q,d}.$$

Then $\Gamma = \mathcal{V}(F_q : q \notin \Gamma)$. ∎

Points

Proposition 1.1. *Any finite set of d points in $\mathbb{P}^n$ is described by polynomials of degree at most d.*

Proof. Let $\Gamma = \{p_1, \ldots, p_d\}$. For $q \notin \Gamma$, let $L_{q,i}$ be a linear form with $L_{q,i}(p_i) = 0$ and $L_{q,i}(q) \neq 0$. Put

$$F_q = L_{q,1} \cdots L_{q,d}.$$

Then $\Gamma = \mathcal{V}(F_q : q \notin \Gamma)$. ∎

Definition. Let $p_1, \ldots, p_d \in \mathbb{P}^n$. If $d \leqslant n + 1$, the points p_i are **independent** if

$$\dim(\overline{p_1 \cdots p_d}) = d - 1,$$

otherwise **dependent**.

If $d > n + 1$, the p_i are in (linearly) **general position** if no $n + 1$ of them are dependent (i.e. lie in a hyperplane).

Theorem 1.2. *Any collection of at most $2n$ points in general position in $\mathbb{P}^n$ can be described by quadratic forms.*

Proof. Let $\Gamma \subset \mathbb{P}^n$ be such a collection. We may assume that Γ contains exactly $2n$ points.

Theorem 1.2. *Any collection of at most $2n$ points in general position in $\mathbb{P}^n$ can be described by quadratic forms.*

Proof. Let $\Gamma \subset \mathbb{P}^n$ be such a collection. We may assume that Γ contains exactly $2n$ points. Let $q \in \mathbb{P}^n$ be such that

$$F|_\Gamma = 0 \quad \Longrightarrow \quad F(q) = 0$$

holds for all quadratic forms F. We must show $q \in \Gamma$.

Theorem 1.2. *Any collection of at most $2n$ points in general position in $\mathbb{P}^n$ can be described by quadratic forms.*

Proof. Let $\Gamma \subset \mathbb{P}^n$ be such a collection. We may assume that Γ contains exactly $2n$ points.
Let $q \in \mathbb{P}^n$ be such that

$$F|_{\Gamma} = 0 \quad \Longrightarrow \quad F(q) = 0$$

holds for all quadratic forms F. We must show $q \in \Gamma$.

(1) If $\Gamma = \Gamma_1 \cup \Gamma_2$ with $|\Gamma_1| = |\Gamma_2| = n$, then Γ_i spans a hyperplane $H_i = \mathcal{V}(L_i)$, defined by a linear form L_i, and $H_1 \cup H_2 = \mathcal{V}(L_1 L_2)$. So $L_1 L_2(q) = 0$ by hypothesis. Hence $q \in H_1 \cup H_2$.

Theorem 1.2. *Any collection of at most $2n$ points in general position in $\mathbb{P}^n$ can be described by quadratic forms.*

Proof. Let $\Gamma \subset \mathbb{P}^n$ be such a collection. We may assume that Γ contains exactly $2n$ points.
Let $q \in \mathbb{P}^n$ be such that

$$F|_\Gamma = 0 \quad \Longrightarrow \quad F(q) = 0$$

holds for all quadratic forms F. We must show $q \in \Gamma$.

(1) If $\Gamma = \Gamma_1 \cup \Gamma_2$ with $|\Gamma_1| = |\Gamma_2| = n$, then Γ_i spans a hyperplane $H_i = \mathcal{V}(L_i)$, defined by a linear form L_i, and $H_1 \cup H_2 = \mathcal{V}(L_1 L_2)$. So $L_1 L_2(q) = 0$ by hypothesis. Hence $q \in H_1 \cup H_2$.

(2) Let $\{p_1, \ldots, p_k\} \in \Gamma$ be a minimal subset of Γ with the property $q \in \overline{p_1 \cdots p_k}$.
By (1), we can find such a subset with $k \leqslant n$.

Claim: $k = 1$ ($\iff q = p_1$)

Theorem 1.2. *Any collection of at most $2n$ points in general position in $\mathbb{P}^n$ can be described by quadratic forms.*

Proof. Let $\Gamma \subset \mathbb{P}^n$ be such a collection. We may assume that Γ contains exactly $2n$ points. Let $q \in \mathbb{P}^n$ be such that

$$F|_\Gamma = 0 \quad \Longrightarrow \quad F(q) = 0$$

holds for all quadratic forms F. We must show $q \in \Gamma$.

(1) If $\Gamma = \Gamma_1 \cup \Gamma_2$ with $|\Gamma_1| = |\Gamma_2| = n$, then Γ_i spans a hyperplane $H_i = \mathcal{V}(L_i)$, defined by a linear form L_i, and $H_1 \cup H_2 = \mathcal{V}(L_1 L_2)$. So $L_1 L_2(q) = 0$ by hypothesis. Hence $q \in H_1 \cup H_2$.

(2) Let $\{p_1, \ldots, p_k\} \in \Gamma$ be a minimal subset of Γ with the property $q \in \overline{p_1 \cdots p_k}$.
By (1), we can find such a subset with $k \leqslant n$.

Claim: $k = 1$ ($\iff q = p_1$)

Take $\Sigma \subset \Gamma \setminus \{p_1, \ldots, p_k\}$ with $|\Sigma| = n - k + 1$. By hypothesis, the n points $\{p_2, \ldots, p_k\} \cup \Sigma$ span a hyperplane H that does not contain p_1. Since $p_1 \in \overline{q p_2 \cdots p_k}$, it follows that $q \notin H$.

Theorem 1.2. *Any collection of at most $2n$ points in general position in $\mathbb{P}^n$ can be described by quadratic forms.*

Proof. Let $\Gamma \subset \mathbb{P}^n$ be such a collection. We may assume that Γ contains exactly $2n$ points.
Let $q \in \mathbb{P}^n$ be such that

$$F|_\Gamma = 0 \quad \Longrightarrow \quad F(q) = 0$$

holds for all quadratic forms F. We must show $q \in \Gamma$.

(1) If $\Gamma = \Gamma_1 \cup \Gamma_2$ with $|\Gamma_1| = |\Gamma_2| = n$, then Γ_i spans a hyperplane $H_i = \mathcal{V}(L_i)$, defined by a linear form L_i, and $H_1 \cup H_2 = \mathcal{V}(L_1 L_2)$. So $L_1 L_2(q) = 0$ by hypothesis. Hence $q \in H_1 \cup H_2$.

(2) Let $\{p_1, \ldots, p_k\} \in \Gamma$ be a minimal subset of Γ with the property $q \in \overline{p_1 \cdots p_k}$.
By (1), we can find such a subset with $k \leqslant n$.
Claim: $k = 1$ ($\iff q = p_1$)

Take $\Sigma \subset \Gamma \smallsetminus \{p_1, \ldots, p_k\}$ with $|\Sigma| = n - k + 1$. By hypothesis, the n points $\{p_2, \ldots, p_k\} \cup \Sigma$ span a hyperplane H that does not contain p_1. Since $p_1 \in \overline{q p_2 \cdots p_k}$, it follows that $q \notin H$.
By (1), q lies in the hyperplane spanned by the remaining n points. It follows that q lies on the hyperplane spanned by p_1 and any $n-1$ of the points $p_{k+1}, \ldots, p_{2n}$. The intersection of all such hyperplanes is just p_1, hence $q = p_1$. ∎

Projective equivalence

The group $\mathrm{PGL}_{n+1}K = (\mathrm{GL}_{n+1}K)\big/K^\times I$ acts on $\mathbb{P}^n$. Two varieties $X, Y \subset \mathbb{P}^n$ are **projectively equivalent** if there exists $A \in \mathrm{PGL}_{n+1}K$ such that $A \cdot X = Y$.
Any two ordered sets of $n+2$ points in general position in $\mathbb{P}^n$ are projectively equivalent.

Projective equivalence

The group $\mathrm{PGL}_{n+1}K = (\mathrm{GL}_{n+1}K)\big/K^\times I$ acts on $\mathbb{P}^n$. Two varieties $X, Y \subset \mathbb{P}^n$ are **projectively equivalent** if there exists $A \in \mathrm{PGL}_{n+1}K$ such that $A \cdot X = Y$.
Any two ordered sets of $n+2$ points in general position in $\mathbb{P}^n$ are projectively equivalent.

The group $\mathrm{PGL}_2 K$ acts on $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$ through **Möbius transformations:**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}_2 K \text{ induces } z \mapsto \frac{az+b}{cz+d}.$$

Projective equivalence

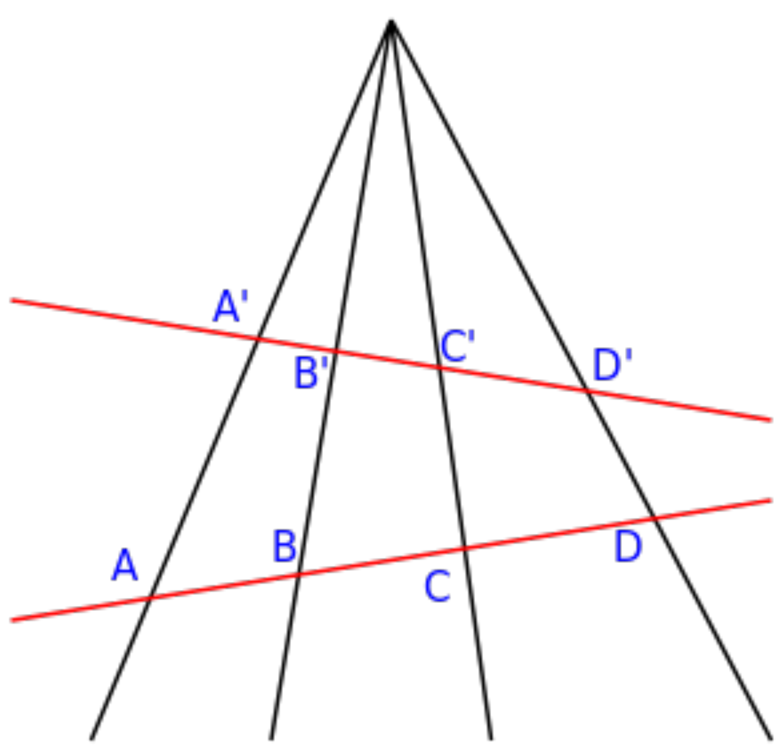
The group $\mathrm{PGL}_{n+1}K = (\mathrm{GL}_{n+1}K)/K^{\times}I$ acts on $\mathbb{P}^n$. Two varieties $X, Y \subset \mathbb{P}^n$ are **projectively equivalent** if there exists $A \in \mathrm{PGL}_{n+1}K$ such that $A \cdot X = Y$.
Any two ordered sets of $n+2$ points in general position in $\mathbb{P}^n$ are projectively equivalent.

The group PGL_2K acts on $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$ through **Möbius transformations:**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}_2K \text{ induces } z \mapsto \frac{az+b}{cz+d}.$$

Two sets of *four* points in $\mathbb{P}^1$ are projectively equivalent if and only if they have the same **cross-ratio**, defined by

$$\lambda(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}.$$



Cross-ratio
[Krishnavedala - Wikimedia Commons]

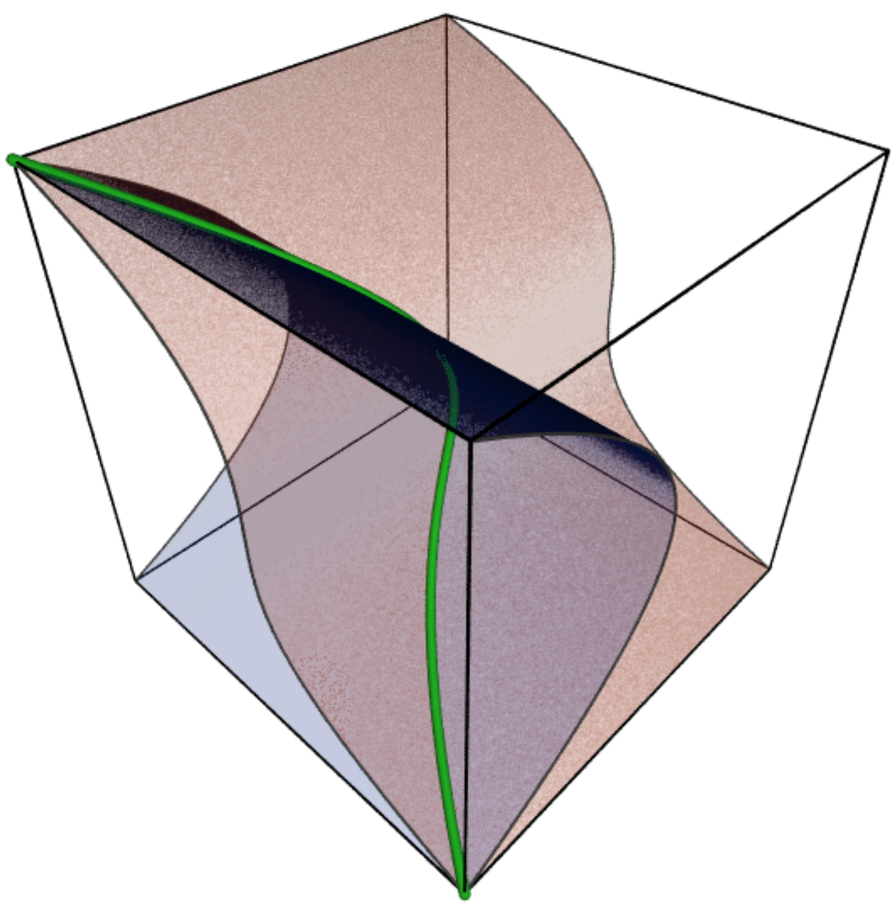
The twisted cubic

Let $\nu: \mathbb{P}^1 \to \mathbb{P}^3$, $[X_0, X_1] \mapsto [X_0^3, X_0^2 X_1, X_0 X_1^2, X_1^3]$.
The image $C = \nu(\mathbb{P}^1)$ is the **twisted cubic** in $\mathbb{P}^3$. It is defined by

$$C = \mathcal{V}(F_0, F_1, F_2)$$

where

$$F_0 = Z_0 Z_2 - Z_1^2$$
$$F_1 = Z_0 Z_3 - Z_1 Z_2$$
$$F_2 = Z_1 Z_3 - Z_2^2.$$



Claudio Rocchini - Wikimedia Commons

The twisted cubic

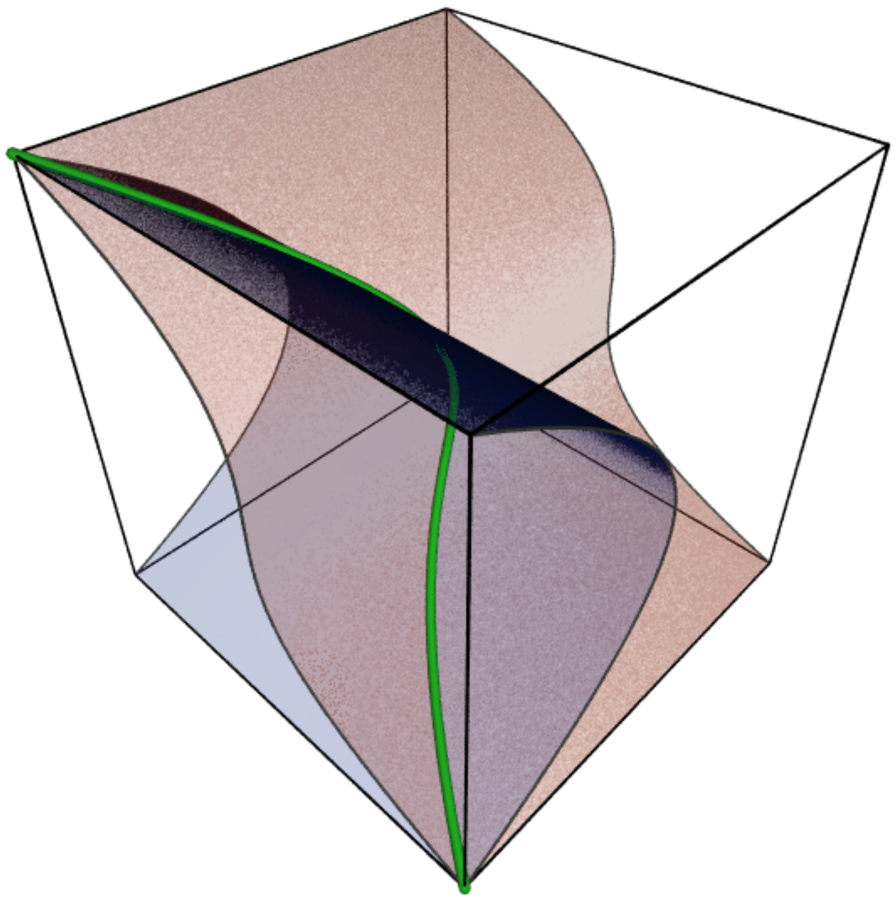
Let $\nu \colon \mathbb{P}^1 \to \mathbb{P}^3$, $[X_0, X_1] \mapsto [X_0^3, X_0^2 X_1, X_0 X_1^2, X_1^3]$.
The image $C = \nu(\mathbb{P}^1)$ is the **twisted cubic** in $\mathbb{P}^3$. It is defined by

$$C = \mathcal{V}(F_0, F_1, F_2)$$

where

$$F_0 = Z_0 Z_2 - Z_1^2$$
$$F_1 = Z_0 Z_3 - Z_1 Z_2$$
$$F_2 = Z_1 Z_3 - Z_2^2.$$

It is *not* defined by any two of these.
For example, F_0 and F_1 define the union of C and the line $\{Z_0 = Z_1 = 0\}$.



Claudio Rocchini - Wikimedia Commons

The twisted cubic

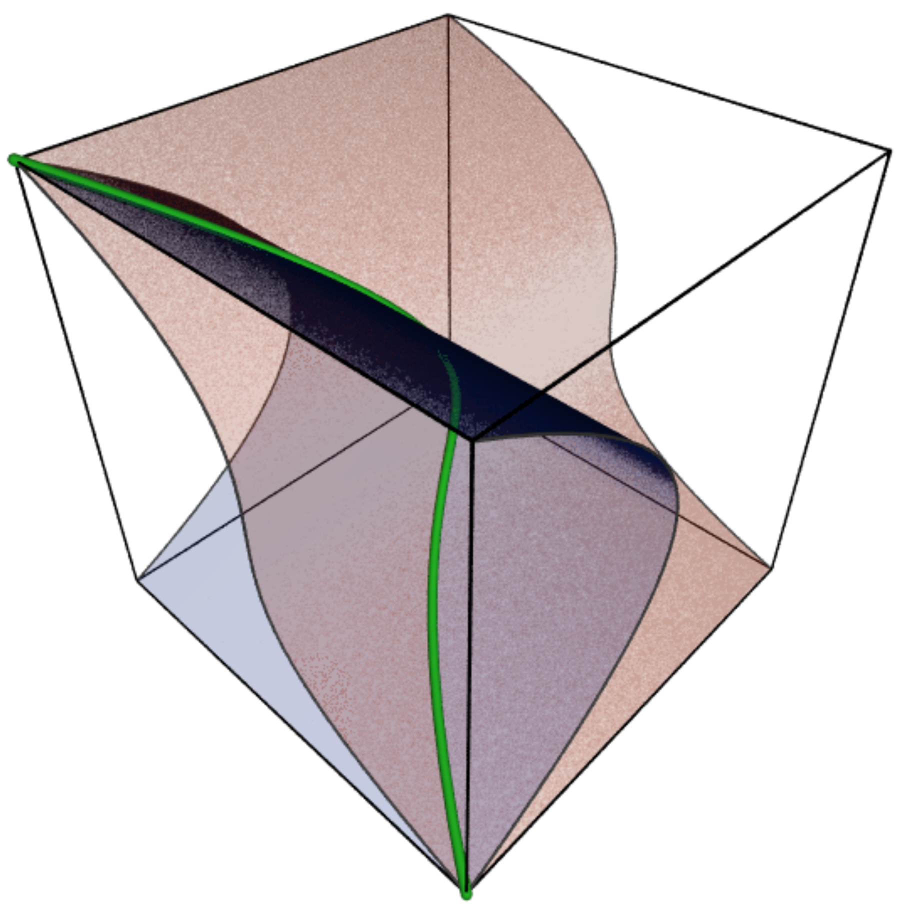
Let $\nu\colon \mathbb{P}^1 \to \mathbb{P}^3$, $[X_0, X_1] \mapsto [X_0^3, X_0^2 X_1, X_0 X_1^2, X_1^3]$.
The image $C = \nu(\mathbb{P}^1)$ is the **twisted cubic** in $\mathbb{P}^3$. It is defined by

$$C = \mathcal{V}(F_0, F_1, F_2)$$

where

$$F_0 = Z_0 Z_2 - Z_1^2$$
$$F_1 = Z_0 Z_3 - Z_1 Z_2$$
$$F_2 = Z_1 Z_3 - Z_2^2.$$

It is *not* defined by any two of these.
For example, F_0 and F_1 define the union of C and the line $\{Z_0 = Z_1 = 0\}$.



Claudio Rocchini - Wikimedia Commons

Exercise (1.11. in [Ha])

b. More generally, for any $\lambda = [\lambda_0, \lambda_1, \lambda_2]$, let

$$F_\lambda = \lambda_0 \cdot F_0 + \lambda_1 \cdot F_1 + \lambda_2 \cdot F_2$$

and let Q_λ be the surface defined by F_λ. Show that for $\mu \neq \nu$, the quadrics Q_ν and Q_μ intersect in the union of C and a line $L_{\mu,\nu}$. (A slick way of doing this problem is described after Exercise 9.16; it is intended here to be done naively, though the computation is apt to get messy.)

Corollary 1.3. *If L is any secant line of C (i.e. $L = \overline{pq}$ with $p, q \in C$), there exist μ, ν with*

$$Q_\mu \cap Q_\nu = C \cup L.$$

Corollary 1.3. *If L is any secant line of C (i.e. $L = \overline{pq}$ with $p, q \in C$), there exist μ, ν with*

$$Q_\mu \cap Q_\nu = C \cup L.$$

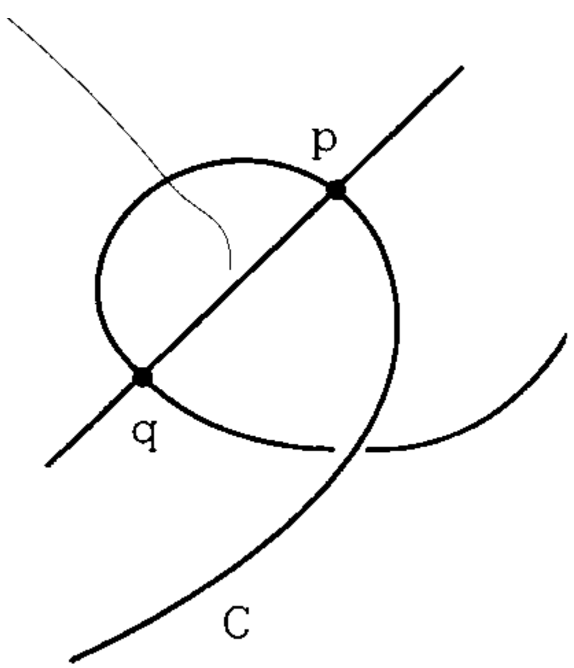
Proof. For any $r \in \overline{pq}$, $r \neq p$, $r \neq q$, the space

$$\{F_\lambda : F_\lambda(r) = 0\}$$

is 2-dimensional.
Let F_μ, F_ν be a basis. Since F_μ, F_ν vanish at the three points p, q, r on L, we have $L \subset \mathcal{V}(F_\mu, F_\nu)$, hence

$$Q_\mu \cap Q_\nu = \mathcal{V}(F_\mu, F_\nu) = C \cup L$$

by the exercise above. ■



The rational normal curve

The **rational normal curve** in $\mathbb{P}^d$ is the Veronese embedding of $\mathbb{P}^1$ of degree d. It is the image of the map

$$\nu_d: \mathbb{P}^1 \to \mathbb{P}^d, \ [X_0, X_1] \mapsto [X_0^d, X_0^{d-1}X_1, \ldots, X_0X_1^{d-1}, X_1^d].$$

Any $d+1$ points on a rational normal curve are linearly independent. For given distinct points $p_0, \ldots, p_d \in \mathbb{P}^1$, we may assume $p_i \neq [1, 0]$ for all i, say $p_i = [Y_i, 1]$. The matrix

$$\begin{bmatrix} Y_0^d & Y_0^{d-1} & Y_0^{d-2} & \cdot & Y_0 & 1 \\ Y_1^d & Y_1^{d-1} & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ Y_d^d & \cdot & \cdot & \cdot & Y_d & 1 \end{bmatrix}$$

is a Vandermonde matrix with determinant $\prod_{i<j}(Y_i - Y_j) \neq 0$, showing that $\nu_d(p_0), \ldots, \nu_d(p_d)$ are independent.

The rational normal curve

The **rational normal curve** in $\mathbb{P}^d$ is the Veronese embedding of $\mathbb{P}^1$ of degree d. It is the image of the map

$$\nu_d \colon \mathbb{P}^1 \to \mathbb{P}^d, \ [X_0, X_1] \mapsto [X_0^d, X_0^{d-1}X_1, \ldots, X_0 X_1^{d-1}, X_1^d].$$

Any $d+1$ points on a rational normal curve are linearly independent. For given distinct points $p_0, \ldots, p_d \in \mathbb{P}^1$, we may assume $p_i \neq [1,0]$ for all i, say $p_i = [Y_i, 1]$. The matrix

$$\begin{bmatrix} Y_0^d & Y_0^{d-1} & Y_0^{d-2} & \cdot & Y_0 & 1 \\ Y_1^d & Y_1^{d-1} & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ Y_d^d & \cdot & \cdot & \cdot & Y_d & 1 \end{bmatrix}$$

is a Vandermonde matrix with determinant $\prod_{i<j}(Y_i - Y_j) \neq 0$, showing that $\nu_d(p_0), \ldots, \nu_d(p_d)$ are independent.

Any curve projectively equivalent to *the* rational normal curve is also *a* rational normal curve. In particular, if $H_0, \ldots, H_d$ is any basis of $K[X_0, X_1]_d$, then

$$\nu_d \colon [X_0, X_1] \mapsto [H_0(X_0, X_1), \ldots, H_d(X_0, X_1)]$$

is a rational normal curve.

Theorem 1.4. *Through any* $d + 3$ *points in general position in* $\mathbb{P}^d$*, there passes a unique rational normal curve.*

Theorem 1.4. *Through any $d+3$ points in general position in $\mathbb{P}^d$, there passes a unique rational normal curve.*

Construction. Let $\mu_0, \ldots, \mu_d, \nu_0, \ldots, \nu_d \in K^\times$ with $[\mu_i, \nu_i] \neq [\mu_j, \nu_j]$ for all $i \neq j$ and consider

$$
\begin{aligned}
G &= \prod_{i=0}^{d} (\mu_i X_0 - \nu_i X_1) \in K[X_0, X_1]_{d+1} \\
H_i &= \frac{G}{\mu_i X_0 - \nu_i X_1}, \qquad i = 0, \ldots, d.
\end{aligned}
$$

Then $H_0, \ldots, H_d$ are a basis of $K[X_0, X_1]_d$. For if $\sum_{i=0}^{d} a_i H_i = 0$ is any linear relation, evaluation at $[\mu_i, \nu_i]$ gives $a_i = 0$.

Theorem 1.4. *Through any $d+3$ points in general position in $\mathbb{P}^d$, there passes a unique rational normal curve.*

Construction. Let $\mu_0, \ldots, \mu_d, \nu_0, \ldots, \nu_d \in K^\times$ with $[\mu_i, \nu_i] \neq [\mu_j, \nu_j]$ for all $i \neq j$ and consider

$$\begin{aligned} G &= \prod_{i=0}^{d} (\mu_i X_0 - \nu_i X_1) \in K[X_0, X_1]_{d+1} \\ H_i &= \frac{G}{\mu_i X_0 - \nu_i X_1}, \qquad i = 0, \ldots, d. \end{aligned}$$

Then $H_0, \ldots, H_d$ are a basis of $K[X_0, X_1]_d$. For if $\sum_{i=0}^{d} a_i H_i = 0$ is any linear relation, evaluation at $[\mu_i, \nu_i]$ gives $a_i = 0$.

Thus

$$\nu_d \colon [X_0, X_1] \mapsto [H_0(X_0, X_1), \ldots, H_d(X_0, X_1)]$$

is a rational normal curve. We find

$$\begin{aligned} &\nu_d([\mu_0, \nu_0]) = [1, 0, \ldots, 0], \ldots, \nu_d([\mu_d, \nu_d]) = [0, \ldots, 0, 1] \\ &\nu_d([1, 0]) = \left[\frac{\mu_0 \cdots \mu_d}{\mu_0}, \ldots, \frac{\mu_0 \cdots \mu_d}{\mu_d}\right] = [\mu_0^{-1}, \ldots, \mu_d^{-1}] \quad \text{and} \quad \nu_d([0, 1]) = [\nu_0^{-1}, \ldots, \nu_d^{-1}]. \end{aligned}$$

Theorem 1.4. *Through any $d+3$ points in general position in $\mathbb{P}^d$, there passes a unique rational normal curve.*

Construction. Let $\mu_0, \dots, \mu_d, \nu_0, \dots, \nu_d \in K^\times$ with $[\mu_i, \nu_i] \neq [\mu_j, \nu_j]$ for all $i \neq j$ and consider

$$G = \prod_{i=0}^{d} (\mu_i X_0 - \nu_i X_1) \in K[X_0, X_1]_{d+1}$$
$$H_i = \frac{G}{\mu_i X_0 - \nu_i X_1}, \qquad i = 0, \dots, d.$$

Then $H_0, \dots, H_d$ are a basis of $K[X_0, X_1]_d$. For if $\sum_{i=0}^d a_i H_i = 0$ is any linear relation, evaluation at $[\mu_i, \nu_i]$ gives $a_i = 0$.

Thus

$$\nu_d \colon [X_0, X_1] \mapsto [H_0(X_0, X_1), \dots, H_d(X_0, X_1)]$$

is a rational normal curve. We find

$$\nu_d([\mu_0, \nu_0]) = [1, 0, \dots, 0], \dots, \nu_d([\mu_d, \nu_d]) = [0, \dots, 0, 1]$$
$$\nu_d([1,0]) = \left[\frac{\mu_0 \cdots \mu_d}{\mu_0}, \dots, \frac{\mu_0 \cdots \mu_d}{\mu_d}\right] = [\mu_0^{-1}, \dots, \mu_d^{-1}] \quad \text{and} \quad \nu_d([0,1]) = [\nu_0^{-1}, \dots, \nu_d^{-1}].$$

So let $p_0, \dots, p_{d+2}$ be any $d+3$ points in general position in $\mathbb{P}^d$.
We can assume $p_i = [e_i]$ for $i = 0, \dots, d$ by projective equivalence. Then p_{d+1} and p_{d+2} have non-zero coordinates. We can choose $[\mu_0, \nu_0], \dots, [\mu_d, \nu_d] \in \mathbb{P}^1$ such that $\nu_d[1,0] = p_{d+1}$ and $\nu_d[0,1] = p_{d+2}$. Uniqueness is left as an exercise. ■