§3
Grassmannians

The Grassmannian

Projective space, by definition, parametrizes one-dimensional subspaces in affine space.

The **Grassmann varieties** or **Grassmannians** parametrize higher-dimensional subspaces.

The Grassmannian

Projective space, by definition, parametrizes one-dimensional subspaces in affine space.

The **Grassmann varieties** or **Grassmannians** parametrize higher-dimensional subspaces.

Let V be a finite-dimensional vector space. As a set, we define

$$G(k, V) = \{U \subset V : U \text{ is a k-dimensional subspace of } V\}$$
$$G(k, n) = \{U \subset K^n : U \text{ is a k-dimensional subspace of } K^n\}.$$

The Grassmannian

Projective space, by definition, parametrizes one-dimensional subspaces in affine space.

The **Grassmann varieties** or **Grassmannians** parametrize higher-dimensional subspaces.

Let V be a finite-dimensional vector space. As a set, we define

$$G(k,V) = \{U \subset V : U \text{ is a k-dimensional subspace of } V\}$$
$$G(k,n) = \{U \subset K^n : U \text{ is a k-dimensional subspace of } K^n\}.$$

By definition,

$$G(1,n) = \mathbb{P}^{n-1}.$$

Since a k-dimensional subspace of K^n can be identified with a $k-1$-dimensional subspace of $\mathbb{P}^{n-1}$, we will also use the notation

$$\mathbb{G}(k,n) = G(k+1, n+1)$$

when dealing with subspaces of $\mathbb{P}^n$.

The Grassmannian

Projective space, by definition, parametrizes one-dimensional subspaces in affine space.

The **Grassmann varieties** or **Grassmannians** parametrize higher-dimensional subspaces.

Let V be a finite-dimensional vector space. As a set, we define

$$G(k, V) = \{U \subset V : U \text{ is a k-dimensional subspace of } V\}$$
$$G(k, n) = \{U \subset K^n : U \text{ is a k-dimensional subspace of } K^n\}.$$

By definition,

$$G(1, n) = \mathbb{P}^{n-1}.$$

Since a k-dimensional subspace of K^n can be identified with a $k-1$-dimensional subspace of $\mathbb{P}^{n-1}$, we will also use the notation

$$\mathbb{G}(k, n) = G(k+1, n+1)$$

when dealing with subspaces of $\mathbb{P}^n$.

Grassmannians are named after **Hermann Graßmann** (1809-1877), the father of linear algebra.

The Grassmannian

Projective space, by definition, parametrizes one-dimensional subspaces in affine space.

The **Grassmann varieties** or **Grassmannians** parametrize higher-dimensional subspaces.

Let V be a finite-dimensional vector space. As a set, we define

$$G(k,V) = \{U \subset V : U \text{ is a k-dimensional subspace of } V\}$$
$$G(k,n) = \{U \subset K^n : U \text{ is a k-dimensional subspace of } K^n\}.$$

By definition,

$$G(1,n) = \mathbb{P}^{n-1}.$$

Since a k-dimensional subspace of K^n can be identified with a $k-1$-dimensional subspace of $\mathbb{P}^{n-1}$, we will also use the notation

$$\mathbb{G}(k,n) = G(k+1,n+1)$$

when dealing with subspaces of $\mathbb{P}^n$.

Grassmannians are named after **Hermann Graßmann** (1809-1877), the father of linear algebra.

The first goal is to show that the Grassmannians can be realized as projective varieties.

The Grassmannian

To turn the Grassmannian into a variety, we need a coordinate system for subspaces.

The Grassmannian

To turn the Grassmannian into a variety, we need a coordinate system for subspaces.

For projective space, a homogeneous coordinate-tuple $[Z_0, \dots, Z_n]$ represents an equivalence class of points in $\mathbb{A}^{n+1}$, namely all points on the same line through the origin.
This equivalence can be seen as coming from a **group action**. The multiplicative group K^* acts on $\mathbb{A}^{n+1} \setminus \{0\}$ by scalar multiplication and each point of $\mathbb{P}^n$ corresponds to an **orbit** of this action, in other words, $\mathbb{P}^n$ is the **quotient space** $(\mathbb{A}^{n+1} \setminus \{0\})/K^*$.

The Grassmannian

To turn the Grassmannian into a variety, we need a coordinate system for subspaces.

For projective space, a homogeneous coordinate-tuple $[Z_0, \ldots, Z_n]$ represents an equivalence class of points in $\mathbb{A}^{n+1}$, namely all points on the same line through the origin.
This equivalence can be seen as coming from a **group action**. The multiplicative group K^* acts on $\mathbb{A}^{n+1} \setminus \{0\}$ by scalar multiplication and each point of $\mathbb{P}^n$ corresponds to an **orbit** of this action, in other words, $\mathbb{P}^n$ is the **quotient space** $(\mathbb{A}^{n+1} \setminus \{0\})/K^*$.

We can try the same for the Grassmannian: A k-dimensional subspace of K^n is spanned by k vectors. So we look at the space of all k-tuples of linearly independent vectors, which we think of as the rows of $k \times n$-matrices.

The Grassmannian

To turn the Grassmannian into a variety, we need a coordinate system for subspaces.

For projective space, a homogeneous coordinate-tuple $[Z_0, \ldots, Z_n]$ represents an equivalence class of points in $\mathbb{A}^{n+1}$, namely all points on the same line through the origin.
This equivalence can be seen as coming from a **group action**. The multiplicative group K^* acts on $\mathbb{A}^{n+1} \setminus \{0\}$ by scalar multiplication and each point of $\mathbb{P}^n$ corresponds to an **orbit** of this action, in other words, $\mathbb{P}^n$ is the **quotient space** $(\mathbb{A}^{n+1} \setminus \{0\})/K^*$.

We can try the same for the Grassmannian: A k-dimensional subspace of K^n is spanned by k vectors. So we look at the space of all k-tuples of linearly independent vectors, which we think of as the rows of $k \times n$-matrices.

The group $\mathrm{GL}_k(K)$ acts on this space by multiplication from the left:

$$\begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{1,k} \\ \vdots & \ddots & \vdots \\ \lambda_{k,1} & \cdots & \lambda_{k,k} \end{pmatrix} \cdot \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,n} \end{pmatrix}$$

and two $k \times n$-matrices have the same row span if and only if they are in the same orbit under this group action. So we can identify $G(k, n)$ with the quotient space

$$\mathrm{Mat}^{(k)}_{k\times n}(K)/\mathrm{GL}_k(K).$$

where $\mathrm{Mat}^{(k)}$ is the set of matrices of rank k.

The Grassmannian

Looking further at the group action

$$\begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{1,k} \\ \vdots & \ddots & \vdots \\ \lambda_{k,1} & \cdots & \lambda_{k,k} \end{pmatrix} \cdot \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,n} \end{pmatrix},$$

we see that if the first $k \times k$-minor of the matrix on the right is non-zero, the orbit contains a unique element of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & b_{1,1} & b_{1,2} & \cdots & b_{1,n-k} \\ 0 & 1 & \cdots & 0 & b_{2,1} & b_{2,2} & \cdots & b_{2,n-k} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 & b_{k,1} & b_{k,2} & \cdots & b_{k,n-k} \end{pmatrix}.$$

The Grassmannian

Looking further at the group action

$$\begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{1,k} \\ \vdots & \ddots & \vdots \\ \lambda_{k,1} & \cdots & \lambda_{k,k} \end{pmatrix} \cdot \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,n} \end{pmatrix},$$

we see that if the first $k \times k$-minor of the matrix on the right is non-zero, the orbit contains a unique element of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & b_{1,1} & b_{1,2} & \cdots & b_{1,n-k} \\ 0 & 1 & \cdots & 0 & b_{2,1} & b_{2,2} & \cdots & b_{2,n-k} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 & b_{k,1} & b_{k,2} & \cdots & b_{k,n-k} \end{pmatrix}.$$

Conversely, we obtain a matrix of rank k for any $k \times (n-k)$-matrix B on the right. In other words, the row spans of matrices of this form are in bijection with an affine space $\mathbb{A}^{k(n-k)}$.

The Grassmannian

Looking further at the group action

$$\begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{1,k} \\ \vdots & \ddots & \vdots \\ \lambda_{k,1} & \cdots & \lambda_{k,k} \end{pmatrix} \cdot \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,n} \end{pmatrix},$$

we see that if the first $k \times k$-minor of the matrix on the right is non-zero, the orbit contains a unique element of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & b_{1,1} & b_{1,2} & \cdots & b_{1,n-k} \\ 0 & 1 & \cdots & 0 & b_{2,1} & b_{2,2} & \cdots & b_{2,n-k} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 & b_{k,1} & b_{k,2} & \cdots & b_{k,n-k} \end{pmatrix}.$$

Conversely, we obtain a matrix of rank k for any $k \times (n-k)$-matrix B on the right. In other words, the row spans of matrices of this form are in bijection with an affine space $\mathbb{A}^{k(n-k)}$.

But this involved a choice coming from the assumption that the *first* $k \times k$-minor is non-zero. In general, we have to permute columns first. So we see in this way that the Grassmannian $G(n,k)$ is covered by $\binom{n}{k}$ copies of affine spaces $\mathbb{A}^{k(n-k)}$. (Note the analogy with projective space!)

In particular, whatever the Grassmannian is as a variety, it must be of dimension $k(n-k)$.

The Grassmann algebra

While the above description of the Grassmannian in terms of matrices works fine for understanding it as a set, it is not very convenient for the goal of finding an embedding of the Grassmannian into projective space. Instead, it is better to employ some multilinear algebra.

The Grassmann algebra

While the above description of the Grassmannian in terms of matrices works fine for understanding it as a set, it is not very convenient for the goal of finding an embedding of the Grassmannian into projective space. Instead, it is better to employ some multilinear algebra.

The **Grassmann algebra** or **exterior algebra** is the algebra of antisymmetric tensors.

The Grassmann algebra

While the above description of the Grassmannian in terms of matrices works fine for understanding it as a set, it is not very convenient for the goal of finding an embedding of the Grassmannian into projective space. Instead, it is better to employ some multilinear algebra.

The **Grassmann algebra** or **exterior algebra** is the algebra of antisymmetric tensors.

Let V be a vector space of finite dimension n. The **tensor algebra** is the non-commutative algebra $T(V) = \bigoplus_{k \geqslant 0} V^{\otimes k}$, where $V^{\otimes k}$ is the k-th tensor power of V, spanned by all tensors $v_1 \otimes \cdots \otimes v_k$ with $v_1, \ldots, v_k \in V$. The product in $T(V)$ is given by the tensor product, i.e. it the map $V^{\otimes k} \times V^{\otimes \ell} \to V^{\otimes k+\ell}$, defined as the bilinear extension of $(v_1 \otimes \cdots \otimes v_k, w_1 \otimes \cdots \otimes w_\ell) \mapsto v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_\ell$.

The Grassmann algebra

While the above description of the Grassmannian in terms of matrices works fine for understanding it as a set, it is not very convenient for the goal of finding an embedding of the Grassmannian into projective space. Instead, it is better to employ some multilinear algebra.

The **Grassmann algebra** or **exterior algebra** is the algebra of antisymmetric tensors.

Let V be a vector space of finite dimension n. The **tensor algebra** is the non-commutative algebra $T(V) = \bigoplus_{k \geqslant 0} V^{\otimes k}$, where $V^{\otimes k}$ is the k-th tensor power of V, spanned by all tensors $v_1 \otimes \cdots \otimes v_k$ with $v_1, \ldots, v_k \in V$. The product in $T(V)$ is given by the tensor product, i.e. it the map $V^{\otimes k} \times V^{\otimes \ell} \to V^{\otimes k+\ell}$, defined as the bilinear extension of $(v_1 \otimes \cdots \otimes v_k, w_1 \otimes \cdots \otimes w_\ell) \mapsto v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_\ell$.

The **exterior algebra** $\bigwedge V$ is the residue class ring of $T(V)$ modulo the ideal generated by all tensors of the form $v \otimes v$ for $v \in V$. The residue class of a basis tensor $v_1 \otimes \ldots \otimes v_k$ is denoted

$$v_1 \wedge \cdots \wedge v_k.$$

The Grassmann algebra

While the above description of the Grassmannian in terms of matrices works fine for understanding it as a set, it is not very convenient for the goal of finding an embedding of the Grassmannian into projective space. Instead, it is better to employ some multilinear algebra.

The **Grassmann algebra** or **exterior algebra** is the algebra of antisymmetric tensors.

Let V be a vector space of finite dimension n. The **tensor algebra** is the non-commutative algebra $T(V) = \bigoplus_{k \geqslant 0} V^{\otimes k}$, where $V^{\otimes k}$ is the k-th tensor power of V, spanned by all tensors $v_1 \otimes \cdots \otimes v_k$ with $v_1, \ldots, v_k \in V$. The product in $T(V)$ is given by the tensor product, i.e. it the map $V^{\otimes k} \times V^{\otimes \ell} \to V^{\otimes k+\ell}$, defined as the bilinear extension of $(v_1 \otimes \cdots \otimes v_k, w_1 \otimes \cdots \otimes w_\ell) \mapsto v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_\ell$.

The **exterior algebra** $\bigwedge V$ is the residue class ring of $T(V)$ modulo the ideal generated by all tensors of the form $v \otimes v$ for $v \in V$. The residue class of a basis tensor $v_1 \otimes \ldots \otimes v_k$ is denoted

$$v_1 \wedge \cdots \wedge v_k.$$

We call the elements of $\bigwedge V$ **multivectors**. The exterior algebra inherits the grading from the tensor algebra, i.e. it has a decomposition $\bigwedge V = \bigoplus \bigwedge^k V$, where $\bigwedge^k V$ is spanned by all multivectors of the form $v_1 \wedge \cdots \wedge v_k$ for $v_1, \ldots, v_k \in V$. In particular, $\bigwedge^1 V = V$ and $\bigwedge^0 V = K$.

The Grassmann algebra

The algebra $\bigwedge V$ has the following properties for all $\omega, \eta, \vartheta \in \bigwedge V$, $\alpha \in K$.

(1) $\omega \wedge (\eta \wedge \vartheta) = (\omega \wedge \eta) \wedge \vartheta$ (Associativity)

(2) $\omega \wedge (\eta + \vartheta) = \omega \wedge \eta + \omega \wedge \vartheta$, $(\omega + \eta) \wedge \vartheta = \omega \wedge \vartheta + \eta \wedge \vartheta$ (Bilinearity)

(3) $\alpha(\omega \wedge \eta) = (\alpha\omega) \wedge \eta = \omega \wedge (\alpha\eta)$

(4) $0 \wedge \omega = \omega \wedge 0 = 0$

Futhermore, for all $v \in V = \bigwedge^1 V$, we have

(5) $v \wedge v = 0$. (Antisymmetry)

The Grassmann algebra

The algebra $\bigwedge V$ has the following properties for all $\omega, \eta, \vartheta \in \bigwedge V$, $\alpha \in K$.

(1) $\omega \wedge (\eta \wedge \vartheta) = (\omega \wedge \eta) \wedge \vartheta$ (Associativity)

(2) $\omega \wedge (\eta + \vartheta) = \omega \wedge \eta + \omega \wedge \vartheta$, $(\omega + \eta) \wedge \vartheta = \omega \wedge \vartheta + \eta \wedge \vartheta$ (Bilinearity)

(3) $\alpha(\omega \wedge \eta) = (\alpha\omega) \wedge \eta = \omega \wedge (\alpha\eta)$

(4) $0 \wedge \omega = \omega \wedge 0 = 0$

Futhermore, for all $v \in V = \bigwedge^1 V$, we have

(5) $v \wedge v = 0.$ (Antisymmetry)

From $(v+w) \wedge (v+w) = 0$, it follows that $v \wedge w = -v \wedge w$ (which is equivalent to (5) if $\mathrm{char}(K) \neq 2$) and by induction $v_1 \wedge \cdots \wedge v_k = \mathrm{sgn}(\sigma)(v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)})$ for all permutations $\sigma \in S_k$.

The Grassmann algebra

The algebra $\bigwedge V$ has the following properties for all $\omega, \eta, \vartheta \in \bigwedge V$, $\alpha \in K$.

(1) $\omega \wedge (\eta \wedge \vartheta) = (\omega \wedge \eta) \wedge \vartheta$ (Associativity)

(2) $\omega \wedge (\eta + \vartheta) = \omega \wedge \eta + \omega \wedge \vartheta$, $(\omega + \eta) \wedge \vartheta = \omega \wedge \vartheta + \eta \wedge \vartheta$ (Bilinearity)

(3) $\alpha(\omega \wedge \eta) = (\alpha\omega) \wedge \eta = \omega \wedge (\alpha\eta)$

(4) $0 \wedge \omega = \omega \wedge 0 = 0$

Futhermore, for all $v \in V = \bigwedge^1 V$, we have

(5) $v \wedge v = 0.$ (Antisymmetry)

From $(v+w)\wedge(v+w) = 0$, it follows that $v \wedge w = -v \wedge w$ (which is equivalent to (5) if $\operatorname{char}(K) \neq 2$) and by induction $v_1 \wedge \cdots \wedge v_k = \operatorname{sgn}(\sigma)(v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)})$ for all permutations $\sigma \in S_k$.

Now let $v_1, \ldots, v_n$ be a basis of V. Then we can use bilinearity to expand every multivector in $\bigwedge V$ in terms of this basis. Explicitly, we obtain

$$\left(\sum a_{i,1} v_i\right) \wedge \cdots \wedge \left(\sum a_{i,k} v_i\right) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \begin{vmatrix} a_{i_1,1} & \cdots & a_{i_1,k} \\ \vdots & & \vdots \\ a_{i_k,1} & \cdots & a_{i_k,k} \end{vmatrix} v_{i_1} \wedge \cdots \wedge v_{i_k}$$

for $k \leqslant n$. In particular, we see that every multivector in $\bigwedge^n V$ is a multiple of $v_1 \wedge \cdots \wedge v_n$, with the coefficient of a multivector of the form $w_1 \wedge \cdots \wedge w_n$ given by the determinant of the coefficient matrix of $w_1, \ldots, w_n$ in terms of the basis $v_1, \ldots, v_n$.

The Grassmann algebra

The algebra $\bigwedge V$ has the following properties for all $\omega, \eta, \vartheta \in \bigwedge V$, $\alpha \in K$.

(1) $\omega \wedge (\eta \wedge \vartheta) = (\omega \wedge \eta) \wedge \vartheta$ (Associativity)

(2) $\omega \wedge (\eta + \vartheta) = \omega \wedge \eta + \omega \wedge \vartheta$, $(\omega + \eta) \wedge \vartheta = \omega \wedge \vartheta + \eta \wedge \vartheta$ (Bilinearity)

(3) $\alpha(\omega \wedge \eta) = (\alpha\omega) \wedge \eta = \omega \wedge (\alpha\eta)$

(4) $0 \wedge \omega = \omega \wedge 0 = 0$

Futhermore, for all $v \in V = \bigwedge^1 V$, we have

(5) $v \wedge v = 0$. (Antisymmetry)

From $(v+w)\wedge(v+w) = 0$, it follows that $v \wedge w = -v \wedge w$ (which is equivalent to (5) if $\operatorname{char}(K) \neq 2$) and by induction $v_1 \wedge \cdots \wedge v_k = \operatorname{sgn}(\sigma)(v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)})$ for all permutations $\sigma \in S_k$.

Now let $v_1, \ldots, v_n$ be a basis of V. Then we can use bilinearity to expand every multivector in $\bigwedge V$ in terms of this basis. Explicitly, we obtain

$$\left(\sum a_{i,1} v_i\right) \wedge \cdots \wedge \left(\sum a_{i,k} v_i\right) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \begin{vmatrix} a_{i_1,1} & \cdots & a_{i_1,k} \\ \vdots & & \vdots \\ a_{i_k,1} & \cdots & a_{i_k,k} \end{vmatrix} v_{i_1} \wedge \cdots \wedge v_{i_k}$$

for $k \leqslant n$. In particular, we see that every multivector in $\bigwedge^n V$ is a multiple of $v_1 \wedge \cdots \wedge v_n$, with the coefficient of a multivector of the form $w_1 \wedge \cdots \wedge w_n$ given by the determinant of the coefficient matrix of $w_1, \ldots, w_n$ in terms of the basis $v_1, \ldots, v_n$.

Note that, because of (5), we never need repeated basis elements. In particular, we find

$$\dim \bigwedge^k V = \binom{n}{k}$$

for all $k \leqslant n$ and $\bigwedge^k V = 0$ for all $k > n$.

The Plücker embedding

We now use the Grassmann algebra to realize the Grassmannian as a projective variety.

The Plücker embedding

We now use the Grassmann algebra to realize the Grassmannian as a projective variety.

Let W be a k-dimensional subspace of V with basis $v_1, \dots, v_k$. The multivector $v_1 \wedge \cdots \wedge v_k \in \bigwedge^k V$ is determined by W up to a scalar, by what we just saw: If we pick a different basis, the corresponding multivector in $\bigwedge^k V$ is obtained by multiplying with the determinant of the base change. So we have a well-defined map

$$\psi \colon G(k, V) \to \mathbb{P}(\bigwedge^k V).$$

The Plücker embedding

We now use the Grassmann algebra to realize the Grassmannian as a projective variety.

Let W be a k-dimensional subspace of V with basis $v_1, \dots, v_k$. The multivector $v_1 \wedge \cdots \wedge v_k \in \bigwedge^k V$ is determined by W up to a scalar, by what we just saw: If we pick a different basis, the corresponding multivector in $\bigwedge^k V$ is obtained by multiplying with the determinant of the base change. So we have a well-defined map

$$\psi: G(k, V) \to \mathbb{P}(\bigwedge^k V).$$

The image of ψ is the set of **totally decomposable multivectors** of $\bigwedge^k V$. (While general multivectors in $\bigwedge^k V$ are sums of totally decomposable ones.)

The Plücker embedding

We now use the Grassmann algebra to realize the Grassmannian as a projective variety.

Let W be a k-dimensional subspace of V with basis $v_1, \dots, v_k$. The multivector $v_1 \wedge \dots \wedge v_k \in \bigwedge^k V$ is determined by W up to a scalar, by what we just saw: If we pick a different basis, the corresponding multivector in $\bigwedge^k V$ is obtained by multiplying with the determinant of the base change. So we have a well-defined map

$$\psi: G(k, V) \to \mathbb{P}(\bigwedge^k V).$$

The image of ψ is the set of **totally decomposable multivectors** of $\bigwedge^k V$. (While general multivectors in $\bigwedge^k V$ are sums of totally decomposable ones.)

The map ψ is injective. To see this, let

$$L_\omega = \{v \in V : \omega \wedge v = 0\}$$

for any $\omega \in \bigwedge^k V$. This is a linear subspace of V. For $\omega = v_1 \wedge \dots \wedge v_k$ as above, we find $L_\omega = W$ (see also the lemma on the next slide). So $\omega \mapsto L_\omega$ is the inverse of ψ (on its image).

The Plücker embedding

We now use the Grassmann algebra to realize the Grassmannian as a projective variety.

Let W be a k-dimensional subspace of V with basis $v_1, \dots, v_k$. The multivector $v_1 \wedge \cdots \wedge v_k \in \bigwedge^k V$ is determined by W up to a scalar, by what we just saw: If we pick a different basis, the corresponding multivector in $\bigwedge^k V$ is obtained by multiplying with the determinant of the base change. So we have a well-defined map

$$\psi: G(k, V) \to \mathbb{P}(\bigwedge^k V).$$

The image of ψ is the set of **totally decomposable multivectors** of $\bigwedge^k V$. (While general multivectors in $\bigwedge^k V$ are sums of totally decomposable ones.)

The map ψ is injective. To see this, let

$$L_\omega = \{v \in V : \omega \wedge v = 0\}$$

for any $\omega \in \bigwedge^k V$. This is a linear subspace of V. For $\omega = v_1 \wedge \cdots \wedge v_k$ as above, we find $L_\omega = W$ (see also the lemma on the next slide). So $\omega \mapsto L_\omega$ is the inverse of ψ (on its image).

In conclusion, we identified the Grassmannian $G(k, V)$ with the set of totally decomposable multivectors in $\mathbb{P}(\bigwedge^k V)$. This is called the **Plücker embedding** of $G(k, V)$.

The Plücker embedding

We now use the Grassmann algebra to realize the Grassmannian as a projective variety.

Let W be a k-dimensional subspace of V with basis $v_1, \ldots, v_k$. The multivector $v_1 \wedge \cdots \wedge v_k \in \bigwedge^k V$ is determined by W up to a scalar, by what we just saw: If we pick a different basis, the corresponding multivector in $\bigwedge^k V$ is obtained by multiplying with the determinant of the base change. So we have a well-defined map

$$\psi: G(k, V) \to \mathbb{P}(\bigwedge^k V).$$

The image of ψ is the set of **totally decomposable multivectors** of $\bigwedge^k V$. (While general multivectors in $\bigwedge^k V$ are sums of totally decomposable ones.)

The map ψ is injective. To see this, let

$$L_\omega = \{v \in V : \omega \wedge v = 0\}$$

for any $\omega \in \bigwedge^k V$. This is a linear subspace of V. For $\omega = v_1 \wedge \cdots \wedge v_k$ as above, we find $L_\omega = W$ (see also the lemma on the next slide). So $\omega \mapsto L_\omega$ is the inverse of ψ (on its image).

In conclusion, we identified the Grassmannian $G(k, V)$ with the set of totally decomposable multivectors in $\mathbb{P}(\bigwedge^k V)$. This is called the **Plücker embedding** of $G(k, V)$.

It remains to show that the totally decomposable multivectors form a closed subset of $\mathbb{P}(\bigwedge^k V)$ and to find the equations that describe it.

The Plücker embedding

Lemma 3.1. *Let* $\omega \in \bigwedge^k V$, $\omega \neq 0$. *The space* $L_\omega = \{v \in V : \omega \wedge v = 0\}$ *has dimension at most* k, *with equality occuring if and only if* ω *is totally decomposable.*

The Plücker embedding

Lemma 3.1. *Let $\omega \in \bigwedge^k V$, $\omega \neq 0$. The space $L_\omega = \{v \in V : \omega \wedge v = 0\}$ has dimension at most k, with equality occuring if and only if ω is totally decomposable.*

Proof. Pick a basis $v_1, \ldots, v_s$ of L_ω and extend to a basis $v_1, \ldots, v_n$ of V. We express ω in this basis: For any choice of indices $I = \{i_1, \ldots, i_k\}$ with $1 \leqslant i_1 < \cdots < i_k \leqslant n$ let $\omega_I = v_{i_1} \wedge \cdots \wedge v_{i_k}$. Then ω can be written as

$$\omega = \sum_{I \subset \{1,\ldots,n\}, |I|=k} c_I \omega_I$$

for some scalars $c_I \in K$. For $j \in \{1, \ldots, n\}$, we find

$$\omega \wedge v_j = \sum c_I \omega_I \wedge v_j = \sum_{I: j \notin I} c_I \omega_I \wedge v_j.$$

Now for $j \leqslant s$, we have $v_j \in L_\omega$ and the equation $\omega \wedge v_j = 0$ shows $c_I = 0$ for all I with $j \notin I$. In other words, all I with $c_I \neq 0$ must contain $\{1, \ldots, s\}$. If $s > k$, there is no such I of length k, contradicting the fact that $\omega \neq 0$. If $s = k$, then there is exactly one such I, namely $I = \{1, \ldots, k\}$, hence ω is a multiple of $v_1 \wedge \cdots \wedge v_k$. Conversely, if ω is totally decomposable, say $\omega = w_1 \wedge \cdots \wedge w_k$, then $w_1, \ldots, w_k \in L_\omega$, hence $\dim L_\omega \geqslant k$. ∎

The Plücker embedding

Lemma 3.1. *Let $\omega \in \bigwedge^k V$, $\omega \neq 0$. The space $L_\omega = \{v \in V : \omega \wedge v = 0\}$ has dimension at most k, with equality occuring if and only if ω is totally decomposable.*

This will be all we need: Fix $\omega \in \bigwedge^k V$, $\omega \neq 0$ and consider the map

$$\varphi(\omega) \colon \begin{cases} V \to \bigwedge^{k+1} V \\ v \mapsto \omega \wedge v \end{cases}.$$

By the lemma, we have $[\omega] \in G(k, V)$ if and only if the rank of $\varphi(\omega)$ is at most $n - k$.
The map $\bigwedge^k V \to \mathrm{Hom}(V, \bigwedge^{k+1} V)$ given by $\omega \mapsto \varphi(\omega)$ is linear. If we fix coordinates by fixing a basis of V, this means that the matrix $A(\omega)$ describing $\varphi(\omega)$ has linear entries, i.e. entries that are homogeneous of degree 1 in the coordinates. Therefore, $G(k, V)$ is defined by the vanishing of all $(n - k + 1) \times (n - k + 1)$-minors of this matrix. We have proved:

Theorem 3.2. *The Grassmannian $G(k, V)$ is a projective variety, embedded as a closed subset of $\mathbb{P}(\bigwedge^k V)$ under the Plücker embedding.* ■

The Plücker embedding

Theorem 3.2. *The Grassmannian $G(k, V)$ is a projective variety, embedded as a closed subset of $\mathbb{P}(\bigwedge^k V)$ under the Plücker embedding.* ∎

Fix a basis of $v_1, \dots, v_n$ of V and the corresponding basis $v_{i_1} \wedge \cdots \wedge v_{i_k}$, $1 \leqslant i_1 < \cdots < i_k \leqslant n$ of $\bigwedge^k V \cong K^{\binom{n}{k}}$. If a subspace W is represented as the row span of a $k \times n$-matrix A, the formula

$$\left(\sum a_{i,1} v_i\right) \wedge \cdots \wedge \left(\sum a_{i,k} v_i\right) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \begin{vmatrix} a_{i_1,1} & \cdots & a_{i_1,k} \\ \vdots & & \vdots \\ a_{i_k,1} & \cdots & a_{i_k,k} \end{vmatrix} v_{i_1} \wedge \cdots \wedge v_{i_k},$$

which we saw earlier, shows what the Plücker embedding does in these coordinates: It maps the matrix A to the tuple of all $k \times k$-minors of A (of which there are $\binom{n}{k} = \dim(\bigwedge^k V)$).

The Plücker embedding

Theorem 3.2. *The Grassmannian $G(k,V)$ is a projective variety, embedded as a closed subset of $\mathbb{P}(\bigwedge^k V)$ under the Plücker embedding.* ■

Fix a basis of $v_1,\dots,v_n$ of V and the corresponding basis $v_{i_1}\wedge\cdots\wedge v_{i_k}$, $1\leqslant i_1<\cdots<i_k\leqslant n$ of $\bigwedge^k V \cong K^{\binom{n}{k}}$. If a subspace W is represented as the row span of a $k\times n$-matrix A, the formula

$$\Big(\sum a_{i,1}v_i\Big)\wedge\cdots\wedge\Big(\sum a_{i,k}v_i\Big)=\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\begin{vmatrix} a_{i_1,1} & \cdots & a_{i_1,k}\\ \vdots & & \vdots\\ a_{i_k,1} & \cdots & a_{i_k,k}\end{vmatrix} v_{i_1}\wedge\cdots\wedge v_{i_k},$$

which we saw earlier, shows what the Plücker embedding does in these coordinates: It maps the matrix A to the tuple of all $k\times k$-minors of A (of which there are $\binom{n}{k}=\dim(\bigwedge^k V)$).

The Plücker embedding of $G(k,n)$ as a space of matrices is given by the $k\times k$-minors. The relations between these minors corresponding to the equations of $G(k,n)$ in $\mathbb{P}(\bigwedge^k V)$ are the **Plücker relations**.

Affine cover of the Grassmannian

We have seen how the Grassmannian $G(k, n)$ is covered by $\binom{n}{k}$ copies of $\mathbb{A}^{k(n-k)}$.
Let us see what that corresponds to under the Plücker embedding.

Affine cover of the Grassmannian

We have seen how the Grassmannian $G(k,n)$ is covered by $\binom{n}{k}$ copies of $\mathbb{A}^{k(n-k)}$.
Let us see what that corresponds to under the Plücker embedding.

First, there is an abstract description:
Let Γ be any subspace of dimension $n-k$ of V, corresponding to a multivector $\eta \in \bigwedge^{n-k} V$.
The set

$$H_\Gamma = \left\{ W \in G(k,V) : \Gamma \cap W \neq \{0\} \right\}$$

is a hyperplane in $G(k,V)$. Namely, if $W = [\omega]$ for $\omega \in \bigwedge^k V$, then $\Gamma \cap W \neq \{0\}$ is equivalent to $\omega \wedge \eta = 0$. Since $\omega \wedge \eta$ is an element of $\bigwedge^n V$, which is one-dimensional, we can identify $\bigwedge^n V$ with K and thus interpret η as a linear form on $\bigwedge^k V$ given by $\omega \mapsto \omega \wedge \eta$. (Indeed, this amounts to a natural isomorphism $\bigwedge^{n-k} V \cong \bigwedge^k V^*$, up to scaling.)

Affine cover of the Grassmannian

We have seen how the Grassmannian $G(k,n)$ is covered by $\binom{n}{k}$ copies of $\mathbb{A}^{k(n-k)}$.
Let us see what that corresponds to under the Plücker embedding.

First, there is an abstract description:
Let Γ be any subspace of dimension $n-k$ of V, corresponding to a multivector $\eta \in \bigwedge^{n-k} V$.
The set

$$H_\Gamma = \left\{ W \in G(k,V) : \Gamma \cap W \neq \{0\} \right\}$$

is a hyperplane in $G(k,V)$. Namely, if $W = [\omega]$ for $\omega \in \bigwedge^k V$, then $\Gamma \cap W \neq \{0\}$ is equivalent to $\omega \wedge \eta = 0$. Since $\omega \wedge \eta$ is an element of $\bigwedge^n V$, which is one-dimensional, we can identify $\bigwedge^n V$ with K and thus interpret η as a linear form on $\bigwedge^k V$ given by $\omega \mapsto \omega \wedge \eta$. (Indeed, this amounts to a natural isomorphism $\bigwedge^{n-k} V \cong \bigwedge^k V^*$, up to scaling.)

Thus H_Γ is the hyperplane defined by η, so that $U_\Gamma = \mathbb{P}(\bigwedge^k V) \smallsetminus H_\Gamma$ is an affine space. The intersection $G(k,V) \cap U_\Gamma$ thus corresponds to all k-dimensional subspaces of V that are complementary to Γ. Fix some k-dimensional subspace W_0 of V complementary to Γ. Then any other such subspace W can be viewed as the graph of a linear map $W_0 \to \Gamma$, and vice-versa. (Given W, the corresponding map is $w_0 \mapsto \gamma$, where $\gamma \in \Gamma$ is the unique element with $w_0 + \gamma \in W$. Conversely, given $\alpha\colon W_0 \to \Gamma$, let $W = \{w_0 + \alpha(w_0) : w_0 \in W_0\}$.) Since $W_0 \cong K^k$ and $\Gamma \cong K^{n-k}$, we find

$$G(k,V) \cap U_\Gamma \cong \mathrm{Hom}(W_0, \Gamma) \cong \mathrm{Mat}_{k\times(n-k)}(K) = \mathbb{A}^{k(n-k)}.$$

Affine cover of the Grassmannian

Now let $V = K^n$ and $\Gamma = \operatorname{span}(e_{k+1}, \dots, e_n)$. Then any subspace W complementary to Γ has a unique basis given by the rows of a $k \times n$-matrix of the form

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 & b_{1,1} & b_{1,2} & \cdots & b_{1,n-k} \\ 0 & 1 & \cdots & 0 & b_{2,1} & b_{2,2} & \cdots & b_{2,n-k} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 & b_{k,1} & b_{k,2} & \cdots & b_{k,n-k} \end{pmatrix}.$$

This yields a bijection of $G(k,n) \cap U_\Gamma$ with $\mathbb{A}^{k(n-k)}$.

Under the Plücker embedding, we know that A is mapped to the tuple of all its $k \times k$-minors. But since the left part of A is the identity, the $k \times k$-minors of A are really just the minors of the matrix B of *any* size. Hence the Plücker embedding of $G(k,n) \cap U_\Gamma$ is given by all the minors of the matrix B.

Finally, since the affine parts $G(k,n) \cap U_\Gamma$ are irreducible open subsets of dimension $k(n-k)$ and have pairwise non-empty intersection, we conclude:

Corollary 3.3. *The Grassmannian $G(k,n)$ is an irreducible variety of dimension $k(n-k)$.* ■

The Grassmannian $\mathbb{G}(1,3)$

The Grassmannian $\mathbb{G} = \mathbb{G}(1,3) = G(2,4)$ parametrizes lines in $\mathbb{P}^3$.
The Plücker embedding puts $\mathbb{G}$ into $\mathbb{P}(\bigwedge^2 K^4) \cong \mathbb{P}^5$. Writing $z_{ij} = v_i \wedge v_j$, $0 \leqslant i < j \leqslant 3$, the image is the quadratic hypersurface

$$\mathcal{V}(z_{01}z_{23} - z_{02}z_{13} + z_{03}z_{12})$$

called the **Plücker quadric**.

The Grassmannian $\mathbb{G}(1,3)$

The Grassmannian $\mathbb{G} = \mathbb{G}(1,3) = G(2,4)$ parametrizes lines in $\mathbb{P}^3$.
The Plücker embedding puts $\mathbb{G}$ into $\mathbb{P}(\bigwedge^2 K^4) \cong \mathbb{P}^5$. Writing $z_{ij} = v_i \wedge v_j$, $0 \leqslant i < j \leqslant 3$, the image is the quadratic hypersurface

$$\mathcal{V}(z_{01}z_{23} - z_{02}z_{13} + z_{03}z_{12})$$

called the **Plücker quadric**.

This and the following statements will be shown in the exercises.

Proposition 3.4. *For any point $p \in \mathbb{P}^3$ and plane $H \subset \mathbb{P}^3$ with $p \in H$, let $\Sigma_{p,H} \subset \mathbb{G}$ be the set of lines in $\mathbb{P}^3$ passing through p and lying in H. Under the Plücker embedding, $\Sigma_{p,H}$ is a line in $\mathbb{P}^5$. Conversely, every line in $\mathbb{G} \subset \mathbb{P}^5$ is of the form $\Sigma_{p,H}$ for some choice of p, H.*

Proposition 3.5. *For any point $p \in \mathbb{P}^3$, let $\Sigma_p \subset \mathbb{G}$ be the set of lines in $\mathbb{P}^3$ passing through p; for any plane $H \subset \mathbb{P}^3$, let $\Sigma_H \subset \mathbb{G}$ be the locus of lines lying in H. Under the Plücker embedding, both Σ_p and Σ_H are carried into planes in $\mathbb{P}^5$. Conversely, any plane $\Lambda \subset \mathbb{G} \subset \mathbb{P}^5$ is either of the form Σ_p for some point p or of the form Σ_H for some plane H.*

Proposition 3.6. *Let $\ell_1, \ell_2 \subset \mathbb{P}^3$ be skew lines (i.e. $\ell_1 \cap \ell_2 = \varnothing$). The set $Q \subset \mathbb{G}$ of lines in $\mathbb{P}^3$ meeting both is the intersection of $\mathbb{G}$ with a three-dimensional subspace of $\mathbb{P}^5$.*

Incidence Correspondences

Let $\mathbb{G}(k,n)$ be the Grassmannian of k-subspaces in $\mathbb{P}^n$ and put

$$\Sigma = \left\{(\Lambda, x) \in \mathbb{G}(k,n) \times \mathbb{P}^n : x \in \Lambda\right\}.$$

Incidence Correspondences

Let $\mathbb{G}(k,n)$ be the Grassmannian of k-subspaces in $\mathbb{P}^n$ and put

$$\Sigma = \left\{(\Lambda, x) \in \mathbb{G}(k,n) \times \mathbb{P}^n : x \in \Lambda\right\}.$$

So Σ is the subvariety of $\mathbb{G}(k,n) \times \mathbb{P}^n$ whose fibre over a point $\Lambda \in \mathbb{G}(k,n)$ is just Λ itself as a subset of $\mathbb{P}^n$. To see that Σ is closed, it suffices to note that

$$\Sigma = \left\{(v_1 \wedge \cdots \wedge v_k, w) : v_1 \wedge \cdots \wedge v_k \wedge w = 0\right\}.$$

Incidence Correspondences

Let $\mathbb{G}(k,n)$ be the Grassmannian of k-subspaces in $\mathbb{P}^n$ and put

$$\Sigma = \left\{(\Lambda, x) \in \mathbb{G}(k,n) \times \mathbb{P}^n : x \in \Lambda\right\}.$$

So Σ is the subvariety of $\mathbb{G}(k,n) \times \mathbb{P}^n$ whose fibre over a point $\Lambda \in \mathbb{G}(k,n)$ is just Λ itself as a subset of $\mathbb{P}^n$. To see that Σ is closed, it suffices to note that

$$\Sigma = \left\{(v_1 \wedge \cdots \wedge v_k, w) : v_1 \wedge \cdots \wedge v_k \wedge w = 0\right\}.$$

Proposition 3.7. *Let $\Phi \subset \mathbb{G}(k,n)$ be a closed subvariety. Then $\bigcup_{\Lambda \in \Phi} \Lambda$ is closed in $\mathbb{P}^n$.*

Incidence Correspondences

Let $\mathbb{G}(k,n)$ be the Grassmannian of k-subspaces in $\mathbb{P}^n$ and put

$$\Sigma = \left\{(\Lambda, x) \in \mathbb{G}(k,n) \times \mathbb{P}^n : x \in \Lambda\right\}.$$

So Σ is the subvariety of $\mathbb{G}(k,n) \times \mathbb{P}^n$ whose fibre over a point $\Lambda \in \mathbb{G}(k,n)$ is just Λ itself as a subset of $\mathbb{P}^n$. To see that Σ is closed, it suffices to note that

$$\Sigma = \left\{(v_1 \wedge \cdots \wedge v_k, w) : v_1 \wedge \cdots \wedge v_k \wedge w = 0\right\}.$$

Proposition 3.7. *Let $\Phi \subset \mathbb{G}(k,n)$ be a closed subvariety. Then $\bigcup_{\Lambda \in \Phi} \Lambda$ is closed in $\mathbb{P}^n$.*

Proof. Let π_1, π_2 be the projection maps of Σ onto $\mathbb{G}(k,n)$ and $\mathbb{P}^n$. Then

$$\bigcup_{\Lambda \in \Phi} \Lambda = \pi_2(\pi^{-1}(\Phi)).$$

■

Incidence Correspondences

Let $\mathbb{G}(k,n)$ be the Grassmannian of k-subspaces in $\mathbb{P}^n$ and put

$$\Sigma = \left\{(\Lambda, x) \in \mathbb{G}(k,n) \times \mathbb{P}^n : x \in \Lambda\right\}.$$

So Σ is the subvariety of $\mathbb{G}(k,n) \times \mathbb{P}^n$ whose fibre over a point $\Lambda \in \mathbb{G}(k,n)$ is just Λ itself as a subset of $\mathbb{P}^n$. To see that Σ is closed, it suffices to note that

$$\Sigma = \left\{(v_1 \wedge \cdots \wedge v_k, w) : v_1 \wedge \cdots \wedge v_k \wedge w = 0\right\}.$$

Proposition 3.7. *Let $\Phi \subset \mathbb{G}(k,n)$ be a closed subvariety. Then $\bigcup_{\Lambda \in \Phi} \Lambda$ is closed in $\mathbb{P}^n$.*

Proof. Let π_1, π_2 be the projection maps of Σ onto $\mathbb{G}(k,n)$ and $\mathbb{P}^n$. Then

$$\bigcup_{\Lambda \in \Phi} \Lambda = \pi_2(\pi^{-1}(\Phi)).$$ ∎

Proposition 3.8. *Let $X \subset \mathbb{P}^n$ be a projective variety. Then $\mathcal{C}_k(X) = \left\{\Lambda \in \mathbb{G}(k,n) : \Lambda \cap X \neq \varnothing\right\}$ is closed in $\mathbb{G}(k,n)$.*

Incidence Correspondences

Let $\mathbb{G}(k,n)$ be the Grassmannian of k-subspaces in $\mathbb{P}^n$ and put

$$\Sigma = \{(\Lambda, x) \in \mathbb{G}(k,n) \times \mathbb{P}^n : x \in \Lambda\}.$$

So Σ is the subvariety of $\mathbb{G}(k,n) \times \mathbb{P}^n$ whose fibre over a point $\Lambda \in \mathbb{G}(k,n)$ is just Λ itself as a subset of $\mathbb{P}^n$. To see that Σ is closed, it suffices to note that

$$\Sigma = \{(v_1 \wedge \cdots \wedge v_k, w) : v_1 \wedge \cdots \wedge v_k \wedge w = 0\}.$$

Proposition 3.7. *Let $\Phi \subset \mathbb{G}(k,n)$ be a closed subvariety. Then $\bigcup_{\Lambda \in \Phi} \Lambda$ is closed in $\mathbb{P}^n$.*

Proof. Let π_1, π_2 be the projection maps of Σ onto $\mathbb{G}(k,n)$ and $\mathbb{P}^n$. Then

$$\bigcup_{\Lambda \in \Phi} \Lambda = \pi_2(\pi^{-1}(\Phi)). \qquad \blacksquare$$

Proposition 3.8. *Let $X \subset \mathbb{P}^n$ be a projective variety. Then $\mathcal{C}_k(X) = \{\Lambda \in \mathbb{G}(k,n) : \Lambda \cap X \neq \varnothing\}$ is closed in $\mathbb{G}(k,n)$.*

Proof. We have

$$\mathcal{C}_k(X) = \pi_1(\pi_2^{-1}(X)). \qquad \blacksquare$$

The variety $\mathcal{C}_k(X)$ is called the **variety of incident subspaces**.

Incidence Correspondences

Let $\mathbb{G}(k,n)$ be the Grassmannian of k-subspaces in $\mathbb{P}^n$ and put

$$\Sigma = \left\{(\Lambda, x) \in \mathbb{G}(k,n) \times \mathbb{P}^n : x \in \Lambda\right\}.$$

So Σ is the subvariety of $\mathbb{G}(k,n) \times \mathbb{P}^n$ whose fibre over a point $\Lambda \in \mathbb{G}(k,n)$ is just Λ itself as a subset of $\mathbb{P}^n$. To see that Σ is closed, it suffices to note that

$$\Sigma = \left\{(v_1 \wedge \cdots \wedge v_k, w) : v_1 \wedge \cdots \wedge v_k \wedge w = 0\right\}.$$

Proposition 3.7. *Let $\Phi \subset \mathbb{G}(k,n)$ be a closed subvariety. Then $\bigcup_{\Lambda\in\Phi} \Lambda$ is closed in $\mathbb{P}^n$.*

Proof. Let π_1, π_2 be the projection maps of Σ onto $\mathbb{G}(k,n)$ and $\mathbb{P}^n$. Then

$$\bigcup_{\Lambda\in\Phi} \Lambda = \pi_2(\pi^{-1}(\Phi)). \qquad \blacksquare$$

Proposition 3.8. *Let $X \subset \mathbb{P}^n$ be a projective variety. Then $\mathcal{C}_k(X) = \left\{\Lambda \in \mathbb{G}(k,n) : \Lambda \cap X \neq \varnothing\right\}$ is closed in $\mathbb{G}(k,n)$.*

Proof. We have

$$\mathcal{C}_k(X) = \pi_1(\pi_2^{-1}(X)). \qquad \blacksquare$$

The variety $\mathcal{C}_k(X)$ is called the **variety of incident subspaces**.

Proposition 3.9. *Let $X, Y \subset \mathbb{P}^n$ be two disjoint projective varieties. Let $J(X,Y)$ be the union of all lines $\overline{pq}$ with $p \in X$, $q \in Y$, called the* ***join of X and Y****. Then $J(X,Y)$ is closed in $\mathbb{P}^n$.*

Incidence Correspondences

Let $\mathbb{G}(k,n)$ be the Grassmannian of k-subspaces in $\mathbb{P}^n$ and put

$$\Sigma = \left\{(\Lambda, x) \in \mathbb{G}(k,n) \times \mathbb{P}^n : x \in \Lambda\right\}.$$

So Σ is the subvariety of $\mathbb{G}(k,n) \times \mathbb{P}^n$ whose fibre over a point $\Lambda \in \mathbb{G}(k,n)$ is just Λ itself as a subset of $\mathbb{P}^n$. To see that Σ is closed, it suffices to note that

$$\Sigma = \left\{(v_1 \wedge \cdots \wedge v_k, w) : v_1 \wedge \cdots \wedge v_k \wedge w = 0\right\}.$$

Proposition 3.7. *Let $\Phi \subset \mathbb{G}(k,n)$ be a closed subvariety. Then $\bigcup_{\Lambda\in\Phi} \Lambda$ is closed in $\mathbb{P}^n$.*

Proof. Let π_1, π_2 be the projection maps of Σ onto $\mathbb{G}(k,n)$ and $\mathbb{P}^n$. Then

$$\bigcup_{\Lambda\in\Phi} \Lambda = \pi_2(\pi^{-1}(\Phi)).$$ ■

Proposition 3.8. *Let $X \subset \mathbb{P}^n$ be a projective variety. Then $\mathcal{C}_k(X) = \left\{\Lambda \in \mathbb{G}(k,n) : \Lambda \cap X \neq \varnothing\right\}$ is closed in $\mathbb{G}(k,n)$.*

Proof. We have

$$\mathcal{C}_k(X) = \pi_1(\pi_2^{-1}(X)).$$ ■

The variety $\mathcal{C}_k(X)$ is called the **variety of incident subspaces**.

Proposition 3.9. *Let $X, Y \subset \mathbb{P}^n$ be two disjoint projective varieties. Let $J(X,Y)$ be the union of all lines $\overline{pq}$ with $p \in X$, $q \in Y$, called the **join of X and Y**. Then $J(X,Y)$ is closed in $\mathbb{P}^n$.*

Proof. The set $\mathcal{J}(X,Y) = \mathcal{C}_1(X) \cap \mathcal{C}_1(Y)$ is closed in the Grassmannian, hence $J(X,Y) = \bigcup_{\ell\in\mathcal{J}} \ell$ is closed in $\mathbb{P}^n$. ■

Fano varieties

Let $X \subset \mathbb{P}^n$ be a projective variety. Then $F_k(X) = \left\{\Lambda \in \mathbb{G}(k,n) : \Lambda \subset X\right\}$ is the variety of k-subspaces contained in X, called the **kth Fano variety of X**.

Fano varieties

Let $X \subset \mathbb{P}^n$ be a projective variety. Then $F_k(X) = \{\Lambda \in \mathbb{G}(k,n) : \Lambda \subset X\}$ is the variety of k-subspaces contained in X, called the **kth Fano variety of X**.

Proposition 3.9. *The Fano variety $F_k(X)$ is closed in $\mathbb{G}(k,n)$.*

Fano varieties

Let $X \subset \mathbb{P}^n$ be a projective variety. Then $F_k(X) = \left\{\Lambda \in \mathbb{G}(k,n) : \Lambda \subset X\right\}$ is the variety of k-subspaces contained in X, called the **kth Fano variety of X**.

Proposition 3.9. *The Fano variety $F_k(X)$ is closed in $\mathbb{G}(k,n)$.*

Proof. Let $X = \mathcal{V}(H_1, \ldots, H_r)$. We fix an $(n-k)$-subspace Γ of K^{n+1} and consider the affine open subset U_Γ of $\mathbb{G}(k,n) = G(k+1, n+1)$ of $(k+1)$-subspaces complementary to Γ. We determine explicit equations for $U_\Gamma \cap F_k(X)$. After changing coordinates, we may assume as before that Γ is spanned by $e_{k+1}, \ldots, e_n$. We have seen that any subspace in $G(k+1, n+1) \cap U_\Gamma$ is uniquely represented as the row span of a matrix

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 & b_{0,1} & b_{0,2} & \cdots & b_{0,n-k} \\ 0 & 1 & \cdots & 0 & b_{1,1} & b_{1,2} & \cdots & b_{1,n-k} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 & b_{k,1} & b_{k,2} & \cdots & b_{k,n-k} \end{pmatrix}.$$

Fano varieties

Let $X \subset \mathbb{P}^n$ be a projective variety. Then $F_k(X) = \{\Lambda \in \mathbb{G}(k,n) : \Lambda \subset X\}$ is the variety of k-subspaces contained in X, called the **kth Fano variety of X**.

Proposition 3.9. *The Fano variety $F_k(X)$ is closed in $\mathbb{G}(k,n)$.*

Proof. Let $X = \mathcal{V}(H_1, \ldots, H_r)$. We fix an $(n-k)$-subspace Γ of K^{n+1} and consider the affine open subset U_Γ of $\mathbb{G}(k,n) = G(k+1, n+1)$ of $(k+1)$-subspaces complementary to Γ. We determine explicit equations for $U_\Gamma \cap F_k(X)$. After changing coordinates, we may assume as before that Γ is spanned by $e_{k+1}, \ldots, e_n$. We have seen that any subspace in $G(k+1, n+1) \cap U_\Gamma$ is uniquely represented as the row span of a matrix

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 & b_{0,1} & b_{0,2} & \cdots & b_{0,n-k} \\ 0 & 1 & \cdots & 0 & b_{1,1} & b_{1,2} & \cdots & b_{1,n-k} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 & b_{k,1} & b_{k,2} & \cdots & b_{k,n-k} \end{pmatrix}.$$

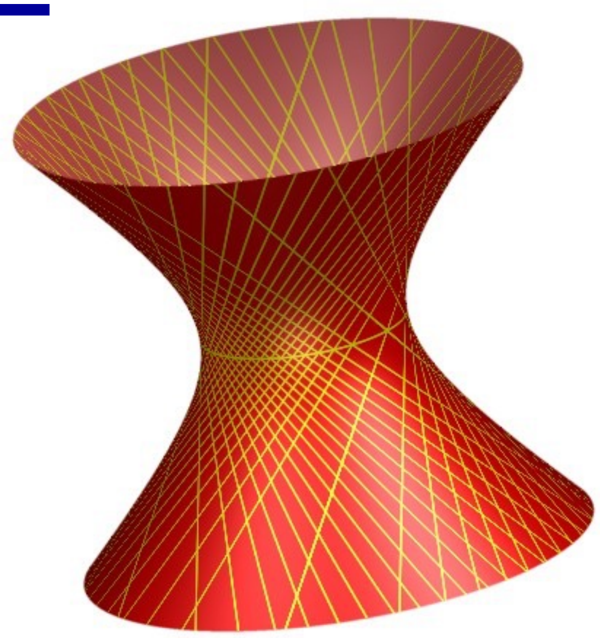
The entries $b_{i,j}$ are regular functions (even coordinates) on U_Γ via the Plücker embedding. For $\lambda \in K^{k+1}$, let $a(\lambda) = \sum_{i=0}^{k} \lambda_i a_{i\bullet}$, where $a_{i\bullet}$ is the ith row vector of A. Then the subspace spanned by the rows of A is contained in X if and only if

$$H_i(a(\lambda)_0, \ldots, a(\lambda)_n) = 0$$

for all $\lambda \in K^{k+1}$ and $i = 1, \ldots, r$. Taking coefficients in λ, this amounts to a set of polynomial equations in the cordinates $b_{i,j}$, which defines $F_k(X)$ in U_Γ. ■

Example of a Fano variety

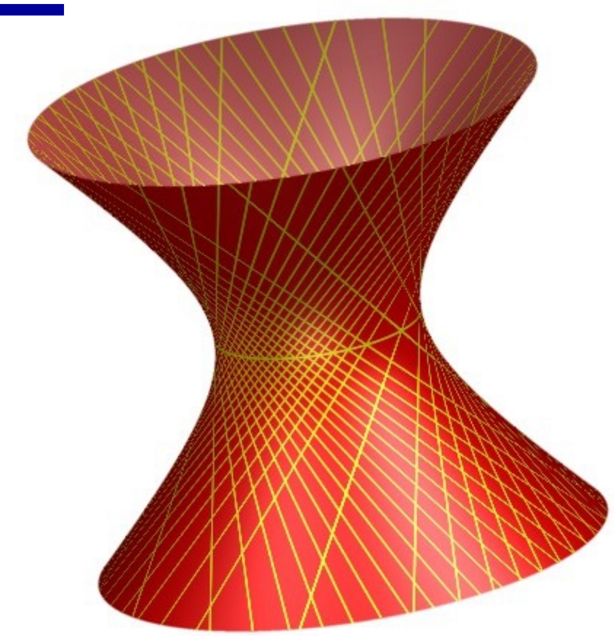
Let $Q = \mathcal{V}(Z_0Z_3 - Z_1Z_2)$ be a quadratic surface in $\mathbb{P}^3$. The surface Q contains two families of linear subspaces, which can be seen in the real affine picture on the right. This corresponds to the fact that Q is exactly the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$, so the two families of lines are $\{p\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{q\}$, for $p, q \in \mathbb{P}^1$.



[taken from R. Vakil's homepage

Example of a Fano variety

Let $Q = \mathcal{V}(Z_0Z_3 - Z_1Z_2)$ be a quadratic surface in $\mathbb{P}^3$. The surface Q contains two families of linear subspaces, which can be seen in the real affine picture on the right. This corresponds to the fact that Q is exactly the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$, so the two families of lines are $\{p\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{q\}$, for $p, q \in \mathbb{P}^1$.

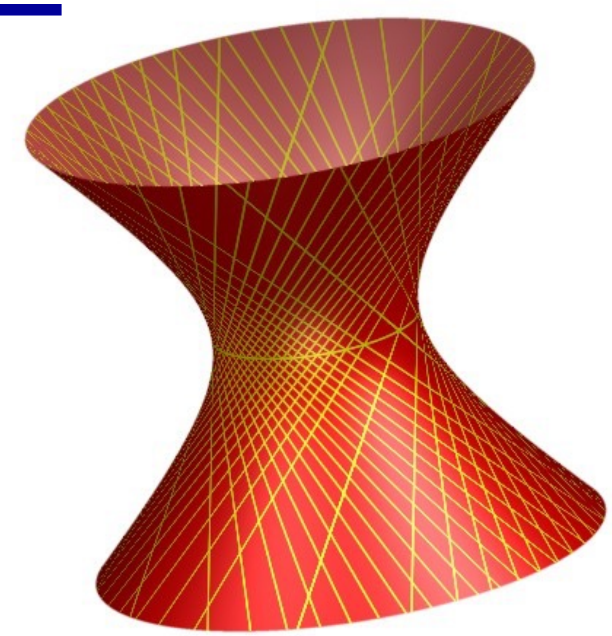


[taken from R. Vakil's homepage

How does this translate into the Fano variety $F_1(Q)$?

Example of a Fano variety

Let $Q = \mathcal{V}(Z_0 Z_3 - Z_1 Z_2)$ be a quadratic surface in $\mathbb{P}^3$. The surface Q contains two families of linear subspaces, which can be seen in the real affine picture on the right. This corresponds to the fact that Q is exactly the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$, so the two families of lines are $\{p\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{q\}$, for $p, q \in \mathbb{P}^1$.



[taken from R. Vakil's homepage

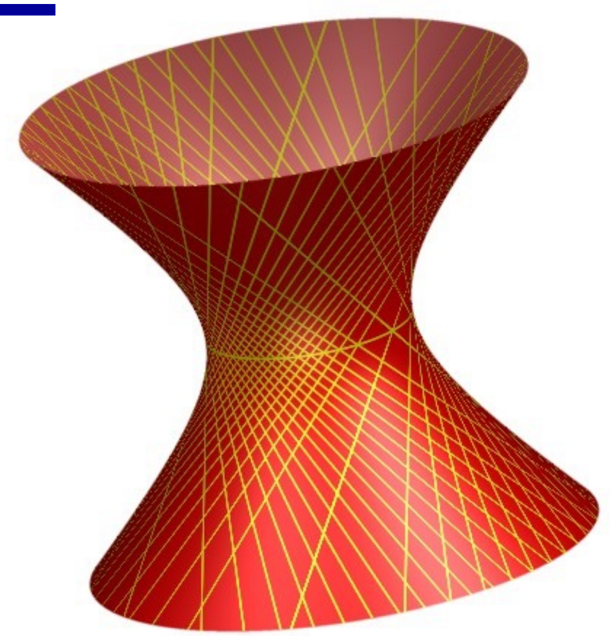
How does this translate into the Fano variety $F_1(Q)$?

Instead of doing the computation by hand, we are lazy and ask Macaulay2.

```
i1 : R = QQ[Z0,Z1,Z2,Z3];

i2 : F = Fano(1,ideal(Z0*Z3-Z1*Z2))

                                                       2
o2 = ideal (p p , p p  + p p , p p , p p  - p p , p p , p  - p p  - p p , p p  -
             4 5   2 5    3 5   1 5   2 4    3 4   0 4   3    1 4    0 5   2 3
     ----------------------------------------------------------------------------
                 2
     p p  + p p , p  - p p  - p p , p p  - p p , p p  + p p , p p )
      1 4    0 5   2    1 4    0 5   1 2    1 3   0 2    0 3   0 1

o2 : Ideal of QQ[p , p , p , p , p , p ]
                  0   1   2   3   4   5

i3 : decompose F

                           2                                  2
o3 = {ideal (p  - p , p , p , - p  + p p ), ideal (p  + p , p , p , - p  + p p )}
              2    3   5   0     3    1 4           2    3   4   1     3    0 5
```

Example of a Fano variety

Let $Q = \mathcal{V}(Z_0Z_3 - Z_1Z_2)$ be a quadratic surface in $\mathbb{P}^3$. The surface Q contains two families of linear subspaces, which can be seen in the real affine picture on the right. This corresponds to the fact that Q is exactly the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$, so the two families of lines are $\{p\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{q\}$, for $p, q \in \mathbb{P}^1$.



[taken from R. Vakil's homepage

How does this translate into the Fano variety $F_1(Q)$?

Instead of doing the computation by hand, we are lazy and ask Macaulay2.

```
i1 : R = QQ[Z0,Z1,Z2,Z3];

i2 : F = Fano(1,ideal(Z0*Z3-Z1*Z2))

                                                      2
o2 = ideal (p p , p p  + p p , p p , p p  - p p , p p , p  - p p  - p p , p p  -
             4 5   2 5    3 5   1 5   2 4    3 4   0 4   3    1 4    0 5   2 3
     ----------------------------------------------------------------------------
                 2
     p p  + p p , p  - p p  - p p , p p  - p p , p p  + p p , p p )
      1 4    0 5   2    1 4    0 5   1 2    1 3   0 2    0 3   0 1

o2 : Ideal of QQ[p , p , p , p , p , p ]
                  0   1   2   3   4   5

i3 : decompose F

                                  2                                     2
o3 = {ideal (p  - p , p , p , - p  + p p ), ideal (p  + p , p , p , - p  + p p )}
              2    3   5   0     3    1 4           2    3   4   1     3    0 5
```

Conclusion.
$F_1(Q)$ is the union of two plane quadrics in $\mathbb{G}(1,3) \subset \mathbb{P}^5$, one for each of to the two families of lines in Q.