§4
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If X and Y are not disjoint, we can still look at the set of lines in $\mathbb{G}(1, n)$ joining two points $x \in X$ and $y \in Y$ with $x \neq y$. Let $\mathcal{J}(X, Y)$ be the Zariski closure of that set and, as before, let

$$J(X, Y) = \bigcup_{\ell \in \mathcal{J}(X,Y)} \ell,$$

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More generally, let $\mathcal{S}_\ell(X)$ be the closure of the set of ℓ-subspaces in $\mathbb{G}(\ell,n)$ spanned by $\ell+1$ independent points on X. This is the **variety of secant ℓ-subspaces**, and $S_\ell(X) = \bigcup_{\Lambda \in \mathcal{S}_\ell(X)} \Lambda$, is the **$\ell$th secant variety** of X

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In general, it can be quite hard to say anything substantial about the secant variety of a given projective variety. For example, what can be said about its dimension? Let $X \subset \mathbb{P}^n$ be irreducible of dimension k. It is not hard to show that

$$\dim(\mathcal{S}_1(X)) = 2k,$$

unless X is itself a linear subspace. Since lines are one-dimensional and $S_1(X)$ is a union of lines parametrized by $\mathcal{S}_1(X)$ we would therefore guess that the dimension of $S_1(X)$ is equal to $2k+1$.

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By the same argument, we would expect the dimension of $S_\ell(X)$ to be $k\ell + k + \ell$.

With a bit of dimension theory, one can establish the following, which confirms intuition.

Proposition 4.1. *If X is irreducible of dimension k, its secant variety $S_1(X)$ is of dimension at most $2k+1$, with equality if and only if there exists a point on $S_1(X)$ lying on only a finite number of secant lines to X. (In fact, if this is true for a single point, it is true for a dense set of points.)* ■

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The case of the twisted cubic is treated in the exercises.

For surfaces, things already become more complicated.

Example 4.3. The secant variety to the Veronese surface $X = v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ (the image of the map v_2 sending $[Z] \in \mathbb{P}^2$ to all quadratic monomials in Z) is only four-dimensional.

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Proposition 4.4. *Let $C \subset \mathbb{P}^n$ be a rational normal curve. The secant variety $S_\ell(C)$ has dimension $\min(2\ell + 1, n)$, for any ℓ between 1 and n.*

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Sketch of proof for the case $2\ell + 1 \leqslant n$. Let $U \subset \mathcal{S}_\ell(C)$ be an open subset consisting of secant ℓ-subspaces spanned by $\ell+1$ distinct points of C. Let $\Lambda \in U$, spanned by $p_1, \ldots, p_{\ell+1} \in C$. Since any $n+1 \geqslant 2(\ell+1)$ points on C are linearly independent, the intersection of Λ with any other secant ℓ-subspace $\Lambda' \in U$ is contained in a subspace of Λ spanned by some subset of the points $p_1, \ldots, p_{\ell+1}$. It follows that if $p \in \Lambda$ is a point not in any such subspace, then p is contained in no other secant ℓ-subspace of C in U. By Prop. 4.1, this implies that $S_1(C)$ has the expected dimension. ■

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Corollary 4.5. *Let K be an algebraically closed field of characteristic 0. For any $n \geqslant 1$ and d with $2d-1 \geqslant n$, there is a Zariski open subset U of the space $K[X,Y]_n$ such that every $F \in U$ admits a representation*

$$F = L_1^n + \cdots + L_d^n$$

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Proof. Let $V = K[X,Y]_1$ and $W = K[X,Y]_n$. If we take the monomial basis X, Y on V and $X^n, X^{n-1}Y, \ldots, Y^n$ on W, the Veronese map $v_n\colon \mathbb{P}V = \mathbb{P}^1 \to \mathbb{P}^n = \mathbb{P}W$ takes a point $[u,v]$ corresponding to a linear form $uX + vY$ to the point $[u^n, u^{n-1}v, \ldots, v^n]$. Since $\operatorname{char}(K) = 0$, the rational normal curve $v_d(\mathbb{P}V)$ is projectively equivalent to the curve

$$[u,v] \mapsto \left[u^n, nu^{n-1}v, \ldots, \binom{n}{k} u^{n-k}v^k, \ldots, v^n\right]$$

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which sends $uX + vY$ to $(uX+vY)^n$.
Hence the set of nth powers of linear forms is a rational normal curve in $\mathbb{P}W$. By the above proposition, its $(d-1)$th secant variety is all of $\mathbb{P}W$. By definition, an open dense subset of that secant variety consists of sums of d nth powers. ∎

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To obtain analogous statements for polynomials in more variables, one has to understand the secant varieties of higher-dimensional Veronese varieties. As the example of the Veronese surface in $\mathbb{P}^5$ shows, the answer becomes more complicated.

Determinantal Varieties

Let M be the projective space $\mathbb{P}(\mathrm{Mat}_{m\times n}(K)) \cong \mathbb{P}^{mn-1}$ of matrices. The **general determinantal variety of rank k** is the variety $M_k \subset M$ of matrices of rank at most k. It is closed since it is defined by the vanishing of all $(k+1)\times(k+1)$-minors.

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Example 4.6. The Segre variety $\Sigma_{m,n} \subset \mathbb{P}^{mn-1} \cong \mathbb{P}^{m-1}\times\mathbb{P}^{n-1}$ is exactly M_1. To see this, note that a matrix $Z \in \mathrm{Mat}_{m\times n}(K)$ has rank 1 if and only if it can be written as $Z = UV^T$ for $U \in \mathbb{K}^m \setminus \{0\}$, $V \in \mathbb{K}^n \setminus \{0\}$, i.e. if and only if it lies in the image of the Segre embedding.

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One can try to use this simple characterisation of secants for the general determinantal variety to study the secant varieties of other varieties defined by the vanishing of minors. We will carry this out for the rational normal curve.

Linear Determinantal Varieties

Let $\Omega = (L_{ij})_{i,j}$ be an $m \times n$-matrix with entries in $K[Z_0, \dots, Z_\ell]_1$. The variety

$$\Sigma_k(\Omega) = \big\{[Z_0, \dots, Z_\ell] : \operatorname{rank}(\Omega(Z)) \leqslant k\big\} \subset \mathbb{P}^\ell$$

is called the **linear determinantal variety** determined by Ω. It is the pullback of M_k under the linear map $\mathbb{P}^\ell \to M$ given by the linear forms L_{ij}. (In case that map is injective, $\Sigma_k(\Omega)$ can be identified with the intersection of M_k with the image of the linear map.)

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Remember from the exercises that the rational normal curve C in $\mathbb{P}^d$ is the rank-1 determinantal variety associated with the matrix

$$\Omega_k = \begin{pmatrix} Z_0 & Z_1 & Z_2 & \cdots & Z_{d-k} \\ Z_1 & Z_2 & Z_3 & \cdots & Z_{d-k+1} \\ \vdots & & & & \\ Z_k & Z_{k+1} & \cdots & \cdots & Z_d \end{pmatrix},$$

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Our goal for the rest of this lecture is

Theorem 4.7. *The secant variety $S_1(C)$ to the rational normal curve $C \subset \mathbb{P}^d$ is the rank-2 determinantal variety associated with Ω_k, for k between 2 and $d - 2$.*

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The analogous statement is true for the higher secant varieties of C.

Lemma 4.8. *Let S_d be the space of homogeneous polynomials of degree d in two variables X and Y and let $V \subsetneq S_d$ be a proper linear subspace without common zeros (i.e. $\mathcal{V}(V) = \varnothing$ in $\mathbb{P}^1$). Let $W = S_1 \cdot V$ be the subspace of S_{d+1} generated by all products of elements of V with linear forms. Then*

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Now suppose $\dim(V) = k$ and $\dim(W) < k + 2$, then

$$\mathrm{Ord}_p(W) \supset \mathrm{Ord}_p(V) \cup (\mathrm{Ord}_p(V) + 1),$$

together with the fact that the polynomials in V have no common zeros, implies

$$(i)\ \mathrm{Ord}_p(V) = \{0, 1, \ldots, k-1\} \quad \text{and} \quad (ii)\ \mathrm{Ord}_p(W) = \{0, 1, 2, \ldots, k\}.$$

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$$\mathrm{Ord}_p(W) \supset \mathrm{Ord}_p(V) \cup (\mathrm{Ord}_p(V) + 1),$$

together with the fact that the polynomials in V have no common zeros, implies

$$(i)\ \mathrm{Ord}_p(V) = \{0, 1, \ldots, k-1\} \quad \text{and} \quad (ii)\ \mathrm{Ord}_p(W) = \{0, 1, 2, \ldots, k\}.$$

By (i), we can find a basis $\{F_1, \ldots, F_k\}$ of V with $\mathrm{ord}_p(F_i) = k - i$ for all $i = 1, \ldots, k$, where we take $p = [0, 1]$, the zero of X.

Lemma 4.8. *Let S_d be the space of homogeneous polynomials of degree d in two variables X and Y and let $V \subsetneq S_d$ be a proper linear subspace without common zeros (i.e. $\mathcal{V}(V) = \varnothing$ in $\mathbb{P}^1$). Let $W = S_1 \cdot V$ be the subspace of S_{d+1} generated by all products of elements of V with linear forms. Then*

$$\dim(W) \geqslant \dim(V) + 2$$

Proof. For any point $p \in \mathbb{P}^1$ and any $U \subset S_d$, let $\mathrm{Ord}_p(U) \subset \mathbb{Z}_{\geqslant 0}$ denote the set of all vanishing orders of elements in U at the point p.
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Now the three polynomials $XF_1, YF_1, XF_2 \in W$ all vanish to order at least $k - 1$ at p, so by (ii), there must be a non-trivial linear relation between them. On the other hand, XF_1 and YF_1 are linearly independent, hence there are $a, b \in K$ such that

$$XF_2 = aXF_1 + bYF_1 = (aX + bY)F_1,$$

so F_1 and F_2 have a common divisor of degree $d - 1$.

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We proceed to show by induction that $F_1, \ldots, F_j$ have a common factor of degree at least $d - j + 1$, for $j \in \{2, \ldots, k\}$. Let $j \geqslant 3$ and assume that G is a common factor of degree $d - j + 2$ of $F_1, \ldots, F_{j-1}$, say $F_i = GF_i'$. The $2j - 1$ polynomials

$$XF_1, \ldots, XF_{j-1}, YF_1, \ldots, YF_{j-1}, XF_j$$

vanish to order order at least $k - j + 1$ at p. By (ii), they span a space of dimension at most j. On the other hand, $XF_1, \ldots, XF_{j-1}, YF_1, \ldots, YF_{j-1}$ span a space of dimension at least j, so there is an expression

$$XF_j = \sum_{i=1}^{j-1} a_i XF_i + \sum_{i=1}^{j-1} b_i YF_i = \sum_{i=1}^{j-1} (a_i X + b_i Y) F_i = G \cdot \sum_{i=1}^{j-1} (a_i X + b_i Y) F_i',$$

which shows what we want. In conclusion, $f_1, \ldots, f_j$ have at least $d - j + 1$ zeros in common. Since V has no common zeros, we conclude $d - k + 1 \leqslant 0$, hence $k = d + 1$ and $V = S_d$. ■

Proof of Thm. 4.7. First note that any point $(Z_0, \ldots, Z_d) \in K^{d+1}$ can be viewed as a linear functional φ_Z on the space S_d of polynomials of degree d in X and Y, via the rule $\varphi_Z(X^{d-i}Y^i) = Z_i$. Thus we have an identification $\mathbb{P}^d \cong \mathbb{P}(S_d^*)$.

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Otherwise, both W_1 and W_2 have codimension 2 in S_d. It follows from the lemma that each must have a common divisor P_i of degree 2 (check!) and thus

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Now there are two cases to consider:

(1) If $W_1 \neq W_2$, then $W_1 + W_2 = V$ and $W_1 \cap W_2$ has codimension 2 in V and thus codimension 3 in S_d. It follows that P_1 and P_2 must have a common zero which is then a common zero of V. So we again conclude $[Z] \in C$.

Proof (continued).

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Now there are two cases to consider:

(2) If $W_1 = W_2$, then $P_1 = P_2$ and we denote this polynomial by P. The linear functional φ_Z vanishes on PS_{d-2}. If P has distinct roots q and r in $\mathbb{P}^1$, then the point evaluations $F \mapsto F(q)$, $F \mapsto F(r)$ are distinct and thus span the two-dimensional space of functions vanishing on PS_{d-2}. Hence there exist $a, b \in K$ such that $\varphi_Z(F) = aF(q) + bF(r)$ and therefore

$$[Z] = av_d(q) + bv_d(r)$$

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∎