§6
Smoothness and Tangent Spaces

Smooth points on a hypersurface

Let $f \in K[z_1, \ldots, z_n]$ be an irreducible polynomial and let $X = \mathcal{V}(f) \subset \mathbb{A}^n$ be the hypersurface defined by f. A point $p \in X$ is called **smooth** if the gradient $(\nabla f)(p)$ is non-zero.

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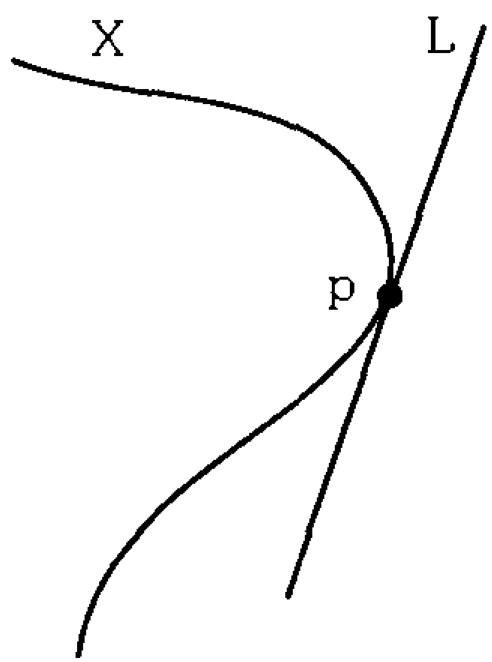
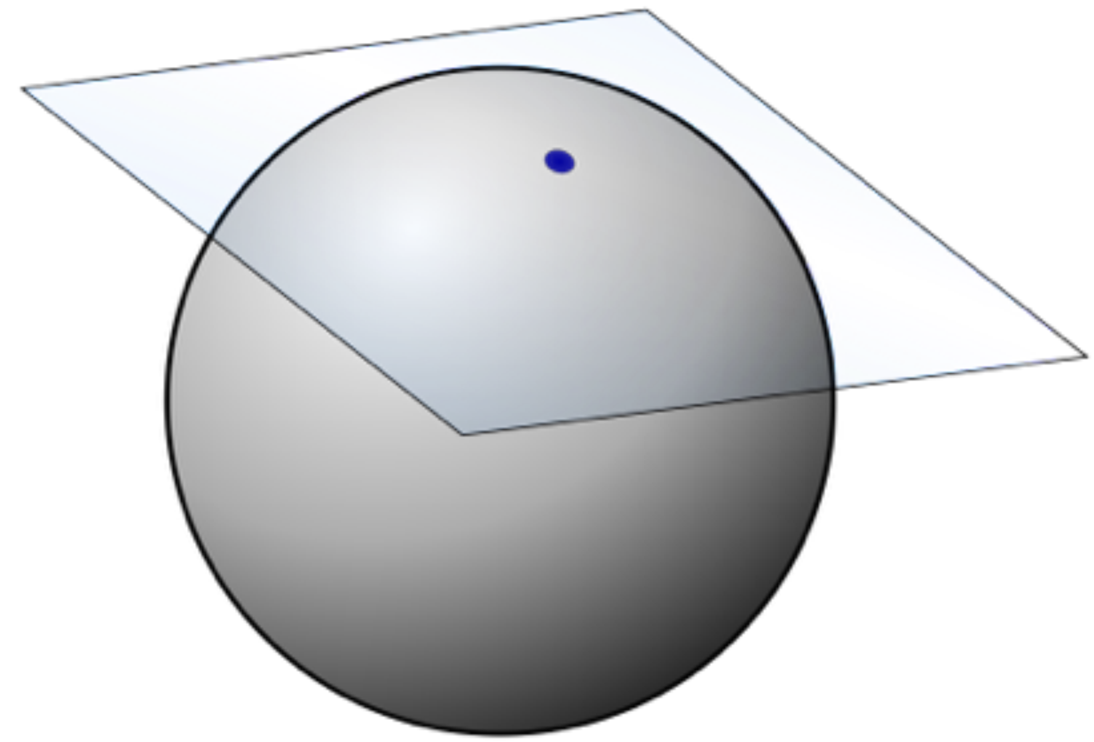


Tangent line to a curve [Ha, p. 175]



Tangent plane to a sphere [Wikimedia Commons]

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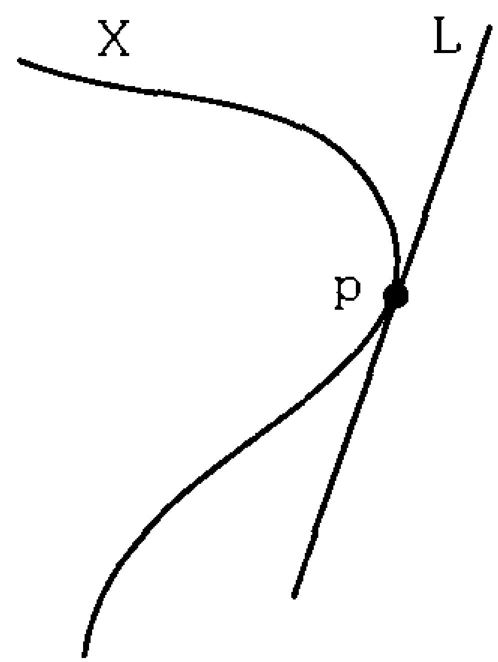

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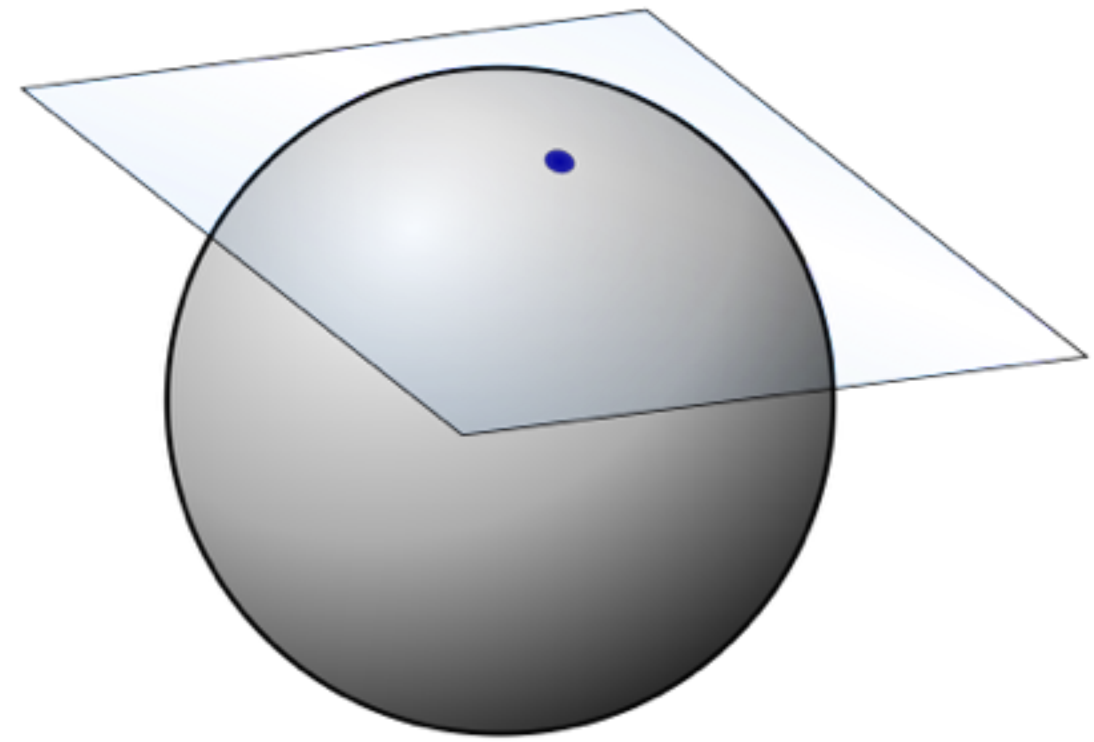

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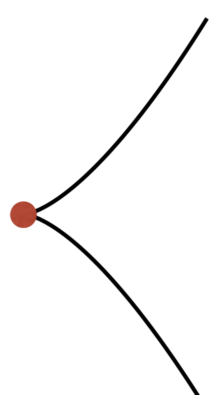
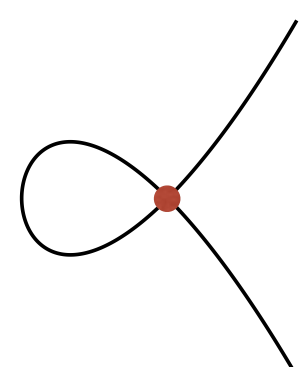
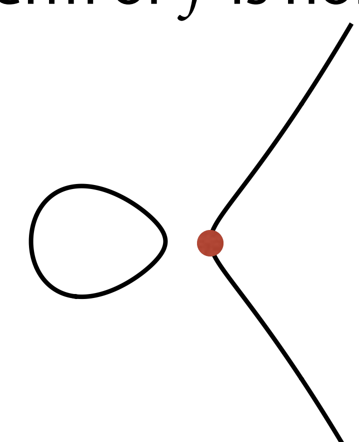
In particular, the origin $p = 0$ is a smooth point if and only if the linear term of f is non-zero.



$x^3 - y^2 = 0$



$x^3 + x^2 - y^2 = 0$



$4x^3 + 5x^2 - 4y^2 + x = 0$

The tangent space to an affine variety

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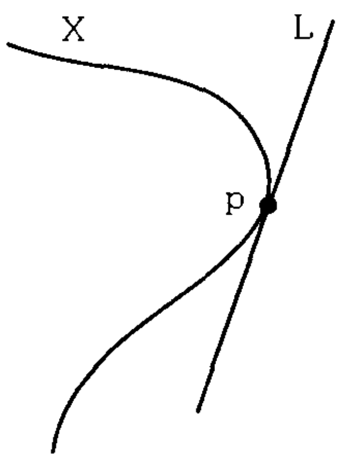
By definition, the tangent space is a linear subspace of $T_p(\mathbb{A}^n)$ and thus passing through the origin. However, it is often more in accordance with geometric intuition to picture the tangent space as an affine space through the point p. Thus the shifted tangent space $p + T_p(X)$ is called the **affine tangent space**.



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For the hypersurface $X = \mathcal{V}(f)$, this means

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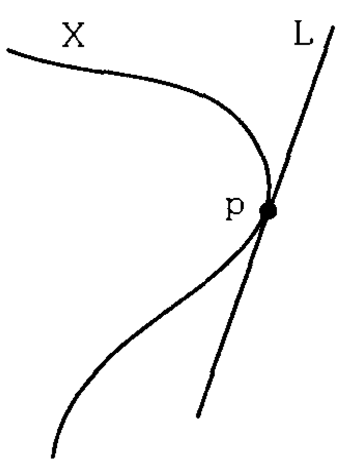

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Proposition 6.1. *Let $f_1, \ldots, f_\ell$ be generators of the ideal $I(X)$ and let J be the $\ell \times n$-matrix*

$$J_{ij} = (\partial f_i/\partial z_j)_{i,j}.$$

Then $T_p(X)$ is the kernel of $J(p)$.

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Thus if v is in the kernel of $J(p)$, we see that $D_v(f)(p) = \sum_{j=1}^{\ell} g_j(p)\left(\sum_{i=1}^n v_i(\partial f_j/\partial z_i)(p)\right) = 0$. Hence $T_p(X)$ contains the kernel of $J(p)$. Conversely, if $v \in T_p(X)$, then $D_v(f_j) = 0$ for all $j = 1, \ldots, \ell$, which implies $v \in \ker J(p)$. ∎

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First, $\overline{\alpha}$ is well-defined: For $v \in T_p(X)$ and $f \in I(X)$, we have $D_v(f)(p) = 0$, by the definition of $T_p(X)$. So if f and g in M are such that $f - g \in I(X) + M^2$ and $v \in T_p(X)$, then $D_v(f)(p) = D_v(g)(p)$, so that $\overline{\alpha}$ is well-defined.

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Now let $X \subset \mathbb{A}^n$ and $I(X) \subset K[z_1, \dots, z_n]$. Let $M = (z_1 - p_1, \dots, z_n - p_n)$ be the maximal ideal of p in $K[z_1, \dots, z_n]$, so that $m = M/I(X)$.

$$\alpha: \begin{cases} T_p(\mathbb{A}^n) \times M/M^2 & \to & K \\ (v, f) & \mapsto & D_v(f)(p) \end{cases}.$$

We claim that when we work modulo $I(X)$, α induces a perfect pairing $\overline{\alpha}: T_p(X) \times (m/m^2) \to K$.

First, $\overline{\alpha}$ is well-defined: For $v \in T_p(X)$ and $f \in I(X)$, we have $D_v(f)(p) = 0$, by the definition of $T_p(X)$. So if f and g in M are such that $f - g \in I(X) + M^2$ and $v \in T_p(X)$, then $D_v(f)(p) = D_v(g)(p)$, so that $\overline{\alpha}$ is well-defined.

To see that $\overline{\alpha}$ is perfect, we need to check that the kernels on both sides are zero:

Proposition 6.2. *Let X be an affine variety with coordinate ring $A(X)$. Let $p \in X$ be a point and let m be the maximal ideal of p in $A(X)$. Then there is a natural isomorphism*

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On the left side, this is clear, since we only restricted from $T_p(\mathbb{A}^n)$ to $T_p(X)$.

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On the right side, we work directly in the finite-dimensional vector space M/M^2. Let $f_1, \ldots, f_\ell$ be generators of $I(X)$ and let U be the subspace spanned by $\overline{f_1}, \ldots, \overline{f_\ell}$ in M/M^2. Let U' be a complement of U in M/M^2, i.e. $M/M^2 = U \oplus U'$ and pick $g_1, \ldots, g_r \in M$ such that $\overline{g_1}, \ldots, \overline{g_r}$ form a basis of U'. Since α is a perfect pairing, we can pick a dual basis given by elements $v_1, \ldots, v_r \in T_p(\mathbb{A}^n)$ satisfying $D_{v_i}(g_j)(p) = \delta_{ij}$, and the subspace V spanned by $v_1, \ldots, v_r$ in $T_p(\mathbb{A}^n)$ satisfies $T_p(\mathbb{A}^n) = T_p(X) \oplus V$. Now V is exactly the kernel of the map $T_p(\mathbb{A}^n) \to (U')^*$, $v \mapsto \alpha(v, -) \in (U')^*$. Since $m/m^2 = (M/M^2)/U \cong U'$, this shows that $T_p(X) \to (m/m^2)^*$, $v \mapsto \alpha(v, -)$ is injective, as claimed. ■

The tangent space

Proposition 6.2. *Let X be an affine variety with coordinate ring $A(X)$. Let $p \in X$ be a point and let m be the maximal ideal of p in $A(X)$. Then there is a natural isomorphism*

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The description of the tangent space furnished by Prop. 6.2 is **local**, since it involves only the maximal ideal of the point in question.

To make this precise, if X is a quasi-projective variety, $p \in X$ a point and U an open-affine subvariety with $p \in X$, we define the **tangent space to X at p** to be $(m/m^2)^*$, where m is the maximal ideal of p in $K[U]$. (Of course, it has to be checked that this is independent of the choice of U. This is easy to see using the language of local rings, but we omit it here.)

Smooth points and singular points

A point $p \in X$ is called **smooth** (or X is called smooth at p) if p is contained in a single irreducible component of X and

$$\dim T_p(X) = \dim_p(X),$$

where $\dim_p(X)$ is the local dimension of X at p, i.e. the dimension of the irreducible component containing p. A point at which X is not smooth is called a **singular point** or a **singularity**.

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Remark. In modern algebraic geometry, what we call 'smooth' is often called 'non-singular', while the word smooth is reserved for a stronger property. The difference is only relevant in characteristic p. In many texts, the terms 'smooth' and 'non-singular' are used interchangeably, but in characteristic p, one has to exercise caution.

Smooth points and singular points

Proposition 6.3. *The set of smooth points of a variety X is open and dense in X.*

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Proof. First note that the set of points in X that are contained in more than one irreducible component of X are a closed subset with dense complement. Since all those points are singular by definition, we may assume that X is irreducible. Furthermore, since X is covered by open affine subvarieties, we may also assume that X is affine.

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If X is a hypersurface defined by a single irreducible polynomial $f \in K[z_1, \dots, z_n]$, the singular points of X are the points $p \in X$ in which the gradient $(\nabla f)(p)$ vanishes. So the singular points form the subvariety $\mathcal{V}(\partial f/\partial z_1, \dots, \partial f/\partial z_n) \cap X$ of X. This is a proper subvariety, unless the derivatives $\partial f/\partial z_1, \dots, \partial f/\partial z_n$ are all zero in $A(X)$. Since $\partial f/\partial z_i$ has lower degree in z_i than f, it cannot be divisible by f unless it is already 0 in $K[z_1, \dots, z_n]$.

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In characteristic 0, this is only possible if the variable z_i does not occur in f. Since some variable has to occur, ∇f cannot vanish identically on X.

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In characteristic 0, this is only possible if the variable z_i does not occur in f. Since some variable has to occur, ∇f cannot vanish identically on X.

If $\operatorname{char}(K) = p$, then $\partial f/\partial z_i = 0$ if and only if f is a polynomial in z_i^p. If this were to happen for all $i = 1, \ldots, n$, we could take the pth root of each coefficient, since K is algebraically closed, and conclude that $f = g^p$ for some $g \in K[z_1, \ldots, z_n]$. This would contradict the fact that f is irreducible. Hence the claim is proved if X is a hypersurface.

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Proposition 6.3. *The set of smooth points of a variety X is open and dense in X.*

Proof (continued).

If X is not a hypersurface, we can apply Thm. 5.2: Since X is birational to a hypersurface, there is an open dense subset U of X that is isomorphic to an open dense subset of a hypersurface. Therefore, U contains an open dense subset U' consisting of smooth points.

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Finally, we show that the set of singular points is closed in X. Let $f_1, \dots, f_\ell$ be generators of $I(X)$ and let J be the matrix with entries $(\partial f_i / \partial z_j)$ as before. By Prop. 6.1, the smooth points of X are exactly the points $p \in X$ in which $J(p)$ has rank $n - \dim(X)$.

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Since the smooth points are dense in X, the matrix $J(p)$ can never have rank bigger than $n - \dim(X)$. For if the rank of $J(p)$ were bigger for some $p \in X$, the same would happen on some non-empty open subset of X, which is impossible. (It would mean that some minor of size $r \times r$, with $r > n - \dim(X)$ would not vanish at p. But then that same minor would be non-vanishing on some non-empty open subset of X.)

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Hence the singular points of X are precisely the points at which $J(p)$ has rank less than $n - \dim(X)$. This is the closed subset given by the vanishing of all minors of size $n - \dim(X)$. ■

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The proof has shown the following.

Corollary 6.4. *If X is an irreducible variety of dimension k, then*

$$\dim T_p(X) \geqslant k$$

holds for all $p \in X$, with equality on the open-dense subset of smooth points of X. ■

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$$p + T_p(X \cap U_i) \subset U_i \subset \mathbb{P}^n$$

in $\mathbb{P}^n$. It not too hard to show that this definition does not depend on the choice of U_i. But it is not a very convenient description.

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We first consider again the case of a hypersurface X defined by a homogeneous polynomial $F(Z) \in K[Z_0, \dots, Z_n]$. Consider the open affine subset U_0 with affine coordinates $z_i = Z_i/Z_0$.

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Then $X \cap U_0$ is defined by $f(z_1, \ldots, z_n) = F(1, z_1, \ldots, z_n)$.
For a point $p = (w_1, \ldots, w_n) \in X \cap U_0$, the affine tangent space is given by

$$p + T_p(X) = \left\{ (z_1, \ldots, z_n) : \sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(p) \cdot (z_i - w_i) = 0 \right\}.$$

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By definition, the projective tangent space is defined by the homogenized equation:

$$\mathbb{T}_p(X) = \left\{ [Z_0, Z_1, \ldots, Z_n] : \sum_{i=1}^{n} \frac{\partial F}{\partial Z_i}(1, w_1, \ldots, w_n) \cdot (Z_i - w_i Z_0) = 0 \right\}.$$

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We conclude

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The point p is singular if and only if all the partial derivatives of F vanish at p, i.e. if and only if $\mathbb{T}_p(X) = \mathbb{P}^n$. In view of the Euler relation, the vanishing of all partial derivatives also implies the vanishing of F (unless $\mathrm{char} K$ divides d), so that the singular locus of $\mathcal{V}(F)$ is defined by all the partial derivatives.

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$$\sum_{i=1}^{n} \frac{\partial F}{\partial Z_i}(1, w_1, \ldots, w_n) \cdot (-w_i \cdot Z_0) = \frac{\partial F}{\partial Z_0}(1, w_1, \ldots, w_n) \cdot Z_0.$$

We conclude

$$\mathbb{T}_p(X) = \left\{[Z_0, \ldots, Z_n] \in \mathbb{P}^n : \sum_{i=0}^{n} \frac{\partial F}{\partial Z_i}(P) \cdot Z_i = 0\right\}.$$

The point p is singular if and only if all the partial derivatives of F vanish at p, i.e. if and only if $\mathbb{T}_p(X) = \mathbb{P}^n$. In view of the Euler relation, the vanishing of all partial derivatives also implies the vanishing of F (unless $\mathrm{char} K$ divides d), so that the singular locus of $\mathcal{V}(F)$ is defined by all the partial derivatives.

Remark. Yes another way to define the projective tangent space: If $\widehat{X}$ is the cone defined by F in $\mathbb{A}^{n+1}$ and $p \neq 0$ any point on $\widehat{X}$, then $\mathbb{T}_p(X) = \mathbb{P}T_p(\widehat{X})$, by the description above.

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In particular, if the homogeneous ideal $I(X)$ is generated by $F_1, \ldots, F_\ell$, then

$$\begin{aligned}\mathbb{T}_p(X) &= \bigcap_{i=1}^{\ell} \mathbb{T}_p(\mathcal{V}(F_i)) = \left\{[Z_0, \ldots, Z_n] \in \mathbb{P}^n : \sum_{i=0}^{n} \frac{\partial F_j}{\partial Z_i}(P) \cdot Z_i = 0, j = 1, \ldots, \ell\right\} \\ &= \mathbb{P}(\ker J)\end{aligned}$$

where J is the $\ell \times n$-matrix with entries $J_{ij} = (\partial F_i / \partial Z_j)(P)$.

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The existence of a resolution of singularities as above for varieties of any dimension over fields of prime characteristic remains unknown.

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We will discuss each of these examples in turn, starting from the following definition.

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Example. We showed that the set of smooth points of any variety is open and dense. Thus 'the general point of a variety is smooth'.

Why this seemingly complicated terminology? The main reason is that the *existence* of an open subset of points satisfying some property is often much more significant than being able to describe the subset U explicitly.

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An n-tuple of points $(v_1, \ldots, v_n) \in V^n$ forms a basis if and only if $v_1, \ldots, v_n$ are linearly independent. This means that the $n \times n$-matrix with row vectors v_i has non-zero determinant. We can view the determinant as a polynomial D on $V^n = K^{n\times n}$. Thus the statement '$(v_1, \ldots, v_n)$ form a basis of V' holds on the open set $\mathbb{A}^{n\times n} \smallsetminus \mathcal{V}(D)$ and therefore generically.

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We proved in Thm. 5.3:

Theorem. *Let $\varphi\colon X \to Y$ be a rational map between irreducible varieties. If $\mathrm{char}(K) = 0$ and the degree of the field extension $K(X)/K(Y)$ is d, then there exists a non-empty Zariski-open subset U of Y such that the fibre $\varphi^{-1}(y)$ consists of exactly d points.*

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In the proof, we also determined *in principle* equations that define the complement of U in Y. However, in general, these equations are quite complicated and we do not usually care much what they look like. It is often enough to know that the subset U exists.

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To see this, consider the correspondence

$$\Theta = \{([p], [F]) \in \mathbb{P}^n \times \mathbb{P}V : [p] \in \mathcal{V}(F)_{\mathrm{sing}}\}.$$

The set Θ is closed in $\mathbb{P}^n \times \mathbb{P}V$, since it is defined by the equations $F(p) = 0$ and $(\nabla F)(p) = 0$, interpreted as equations in p and the coefficients of F. It follows from elimination theory that $\Delta = \mathrm{pr}_2(\Theta)$ is also closed.

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It turns out that Δ is a hypersurface, called the **discriminant**. It is defined by a homogeneous polynomial in the coefficients of F of degree $(n+1)(d-1)^n$. In general, no one has much of an idea as to what this polynomial looks like (c.f. the book of Gelfand, Kapranov and Zelevinsky).

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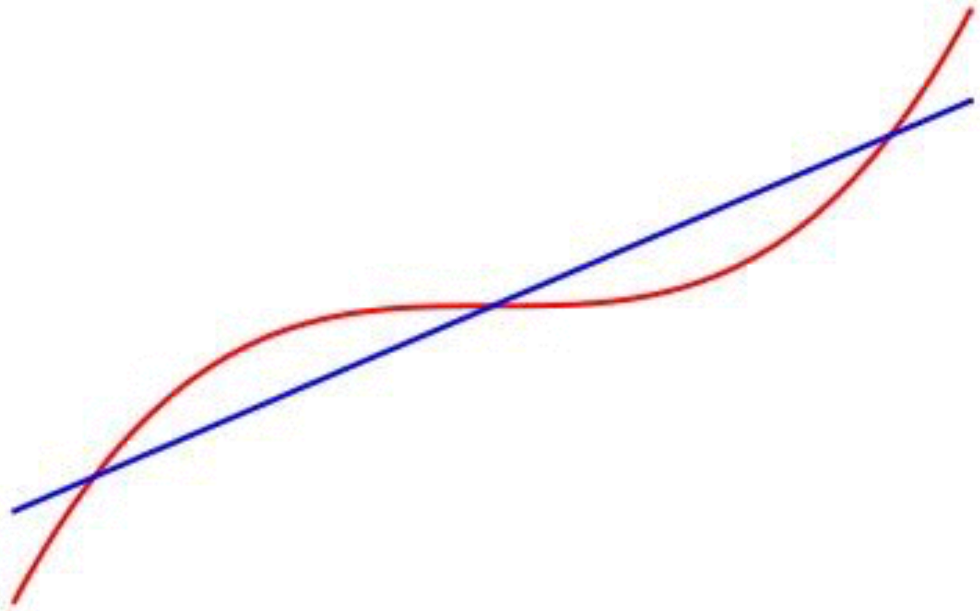
For $d = 1$, we clearly have $\Delta = \varnothing$. The case $d = 2$ is also easy: A quadratic form $F \in K[Z]_d$ can be uniquely expressed as $F = Z^T A Z$, where $Z = (Z_0, \dots, Z_n)^T$ and A is a symmetric $(n+1)\times(n+1)$-matrix. Since F is singular if and only if A has rank less than $n+1$, the discriminant hypersurface is defined by the determinant, a polynomial of degree $n+1$ in the $\binom{n+2}{2}$ entries of A.

Examples

(4) Let $X \subset \mathbb{P}^2$ be an irreducible plane curve of degree d. A **general line** in $\mathbb{P}^2$ meets X in d distinct points.



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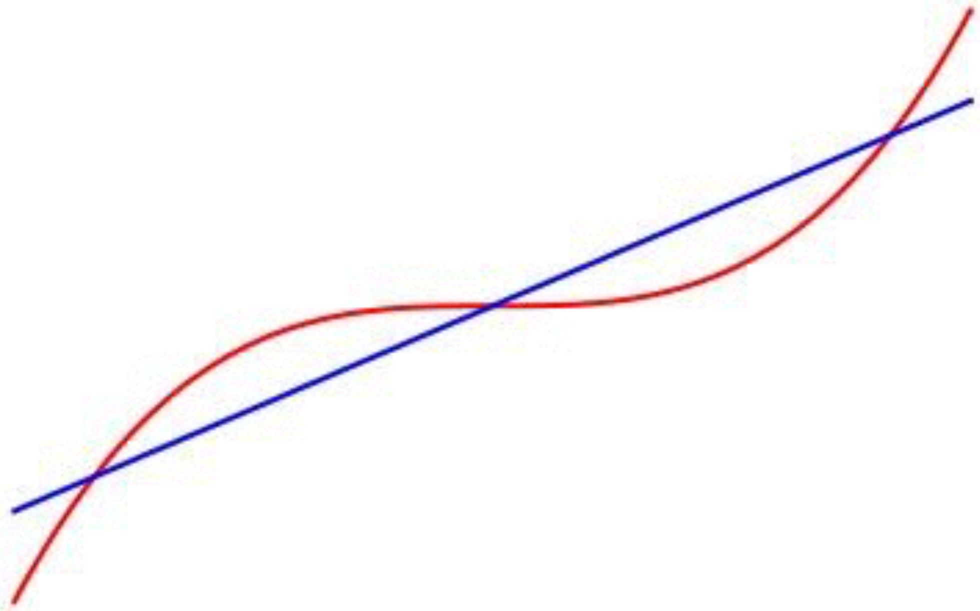

Here the genericity refers to the space of lines in $\mathbb{P}^2$. This is the Grassmannian $\mathbb{G}(2,1)$, which is identified with the dual space $(\mathbb{P}^2)^*$. Explicitly, a line in $\mathbb{P}^2$ is defined by a linear form $aX+bY+cZ$ corresponding to the point $[a,b,c] \in \mathbb{P}^2$.

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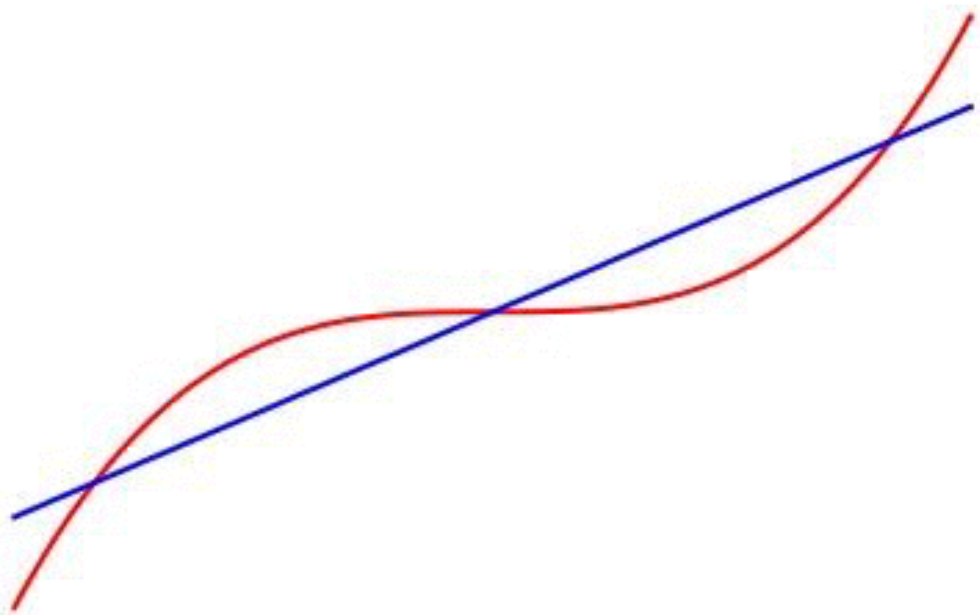

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Assume $\mathrm{char}(K) = 0$. Let X be given by an irreducible polynomial $F \in K[X,Y,Z]_d$. We may restrict to the open set of lines as above with $a \neq 0$. Then we put $a = 1$ and substitute $X = -bY - cZ$ into F. This results in a homogeneous polynomial

$$G_{b,c}(Y,Z) = F(-bY - cZ, Y, Z)$$

of degree d. We are interested in the set of parameters b, c for which $G_{b,c}$ has no multiple roots, and thus d distinct roots in K. This corresponds to the set of b, c for which the discriminant $R(G_{b,c}, \partial G_{b,c}/\partial Y)$ is non-zero. This is a polynomial condition in b, c.

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In algebraic geometry, the intuition behind statements that hold generically is very similar.
The exceptional set is contained in some proper subvariety, but it may be quite hard to describe that subvariety explicitly.

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A possible solution (in some applications) is to choose the polynomial 'at random' in some way.

An algebraic definition of genericity

Let $V \subset \mathbb{A}^n$ be an irreducible variety over $\mathbb{C}$ and $k \subset \mathbb{C}$ a subfield.
A **k-generic point** of V is a point $x \in V$ with the property that every polynomial $f \in k[x_1, \ldots, x_n]$ with $f(x) = 0$ vanishes at every point of V.

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Proof. Let $f_1, \ldots, f_m$ be generators of $I(V)$. Let $\widetilde{k}$ be the field extension of k obtained by adjoining all the coefficients of $f_1, \ldots, f_m$ to k. Since there are only finitely many coefficients, $\mathbb{C}/\widetilde{k}$ still has infinite transcendence degree.

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Since $f_i \in I_0$, we have $f_i(\overline{x_1}, \ldots, \overline{x_n}) = 0$ in L, hence $f_i(a_1, \ldots, a_n) = \varphi(f_i(\overline{x_1}, \ldots, \overline{x_n})) = 0$, so $a \in V$. Now if $f \in k[x_1, \ldots, x_n]$ and $f \notin I(V)$, then $f \notin I_0$, hence $f(\overline{x_1}, \ldots, \overline{x_n}) \neq 0$ in L. Therefore, $f(a_1, \ldots, a_n) = \varphi(f(\overline{x_1}, \ldots, \overline{x_n})) \neq 0$. ■

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For example, the variety V itself might be defined by polynomials with rational coefficients. Then we might take $k = \overline{\mathbb{Q}}$. A k-generic point of V would then be a general point with respect to any property defined (in a suitable sense) over any number field.

Bertini's theorem

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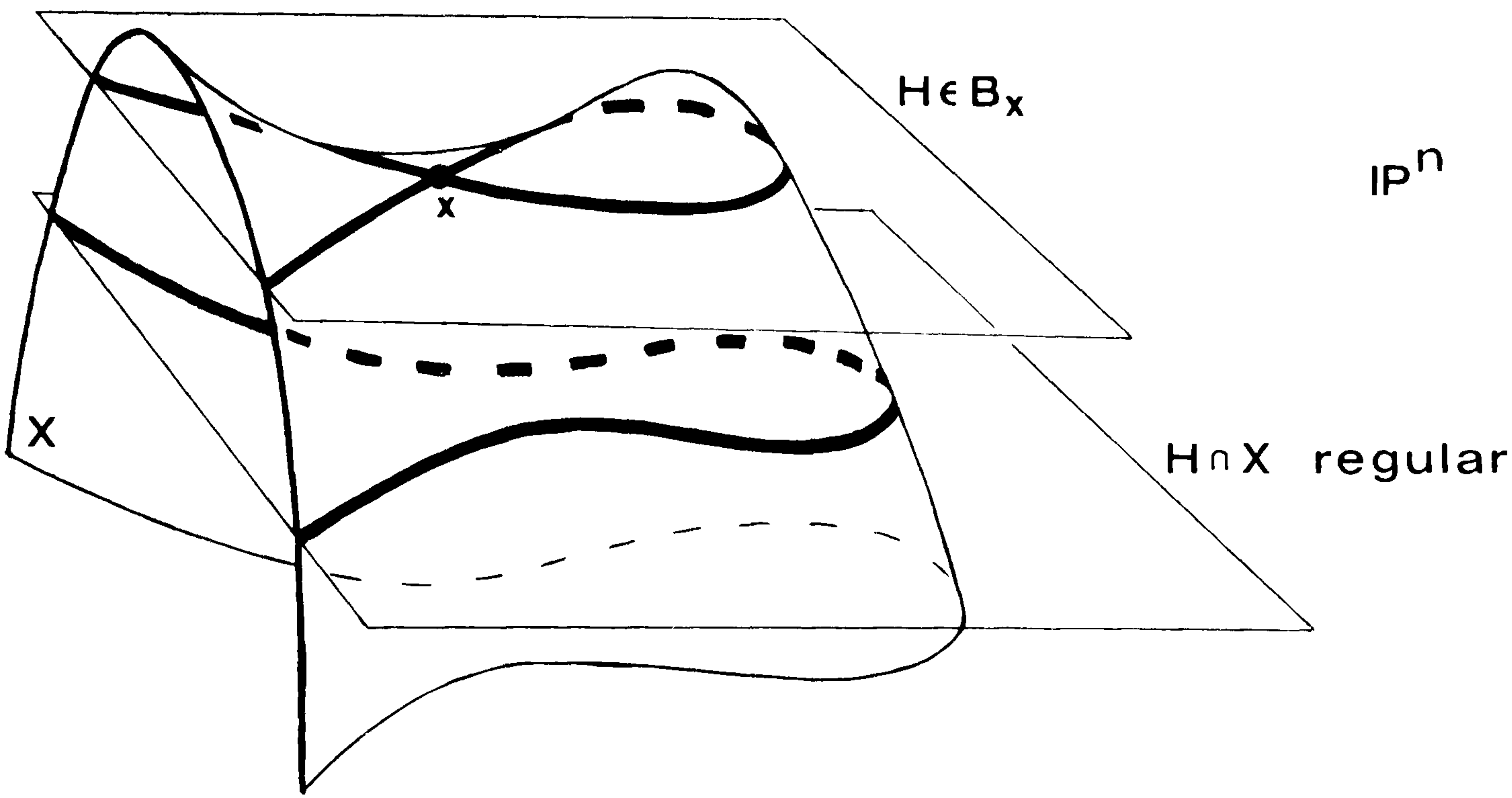

From Hartshorne [Hs], p.179

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There exist many stronger and refined versions of Bertini's theorem, like the following

Theorem 6.7. *Assume $\mathrm{char}(K) = 0$. Let X be a quasi-projective variety over K and $f\colon X \to \mathbb{P}^n$ a morphism. Then the general linear subspace $L \subset \mathbb{P}^n$ satisfies*

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Applications of Bertini's theorem will follow later.

Dimension of fibres

The second important genericity statement is the following.

Theorem 6.8 (Fibre-dimension theorem). *Let X and Y be irreducible varieties and $\varphi\colon X \to Y$ a dominant morphism. Then the fibre of φ over a general point of Y has dimension $\dim(X) - \dim(Y)$.*

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(1) For every $y \in \varphi(X)$, we have $\dim(X_y) \geqslant \dim X - \dim Y$.

(2) For every $k \geqslant 0$, the set

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Examples 6.9.

(1) Consider the map $\varphi\colon \mathbb{A}^2 \to \mathbb{A}^2$, $(x, y) \mapsto (x, xy)$.
We find $\varphi(\mathbb{A}^2) = \{(u, v) \in \mathbb{A}^2 : u \neq 0\} \cup \{(0, 0)\}$. If $(u, v) \in \mathbb{A}^2$ with $u \neq 0$, then $\varphi^{-1}(u, v) = \{(u, v/u)\}$ has dimension 0.
The exceptional fibre $\varphi^{-1}(0, 0)$ is the line $x = 0$ in $\mathbb{A}^2$ and has dimension 1.

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(2) Let $X \subset \mathbb{A}^n$ be an irreducible affine variety of dimension k and let

$$\Theta = \{(p, v) \in X \times \mathbb{A}^n : v \in T_p(X)\}.$$

It is not hard to verify that Θ is closed in $X \times \mathbb{A}^n$. Let $\pi_1\colon \Theta \to X$ be the first projection. For $p \in X$, the fibre $\pi_1^{-1}(p)$ is exactly $T_p(X)$. From what we know about tangent spaces, it follows that the general fibre of π_1 has dimension k, so that $\dim \Theta = k + \dim X = 2k$. Furthermore, statement (2) in the fibre dimension theorem is exactly what we showed in Problem 7.2.

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(3) In the chapter about secant varieties, we saw the following statement.

Proposition 4.1. *If $X \subset \mathbb{P}^n$ is irreducible of dimension k, its secant variety $S_1(X)$ is of dimension at most $2k+1$, with equality if and only if X is not a line and there exists a point on $S_1(X)$ lying on only a finite number of secant lines to X. (If this is true for a single point, it is true for a dense set of points.)*

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Now we apply the fibre-dimension theorem and conclude $2\dim(\widehat{X}) - \dim S_1(X) = 1$ where $\dim(\widehat{X}) = k+1$, hence $\dim S_1(X) = 2k+1$, as claimed.

Dimension of fibres

Corollary 6.10. *Let* $X \subsetneq \mathbb{P}^n$ *be a projective variety. For* $p \in \mathbb{P}^n \smallsetminus X$*, let* π_p *be the projection from* p *onto a hyperplane* $H \cong \mathbb{P}^{n-1}$*. Then* $\dim(\pi_p(X)) = \dim(X)$.

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Proof. Apply the fibre-dimension theorem to the morphism $\pi_p\colon X \to \pi_p(X)$. For every $q \in \pi_p(X)$, the fibre $\pi_p^{-1}(q)$ consists of the intersection points of the line $\overline{pq}$ with X. Since $p \notin X$ but $q \in \pi_p(X)$, these are finitely many points. Thus every fibre is 0-dimensional, which implies $\dim(X) = \dim(\pi_p(X))$. ∎

Linear spaces of complementary dimension

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Proof. We first show that $k = \dim(X)$ has this property. We do induction on $\operatorname{codim}(X) = n - \dim(X)$. If $\dim(X) = n$, the claim is clear.

If $\dim(X) \leqslant n-1$, choose any point $p \in \mathbb{P}^n \setminus X$ and consider the projection $X' = \pi_p(X)$ onto $H \cong \mathbb{P}^{n-1}$. Since $\dim(X) = \dim(X')$, we have $\operatorname{codim}(X') = \operatorname{codim}(X) - 1$. By the induction hypothesis, the general subspace of H of dimension $n-k-1$ meets X' in finitely many points. If L is any such subspace, then $\overline{\pi_p^{-1}(L)}$ is a subspace of dimension $n-k$ in $\mathbb{P}^n$ (spanned by L and p) still meeting X in finitely many points. Thus the general subspace of dimension $n-k$ through p meets X in finitely many points. Since p is any point not on X, this shows the claim.

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Proof (continued).

To show that $\dim(X)$ is the only number with this property, suppose the general subspace of dimension $n-k$ meets X in finitely many points. If $n = k$, then this implies $X = \mathbb{P}^n$, so $k = \dim X$. So we may assume $k < n$.

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Let L be a subspace of dimension $n-k$ that meets X in only finitely many points. Then L contains a subspace $L^{(0)}$ of dimension $n-k-1$ which does not meet X at all. Let $p_0 \in L^{(0)}$ and let π_0 be the projection from p_0 onto $H \cong \mathbb{P}^{n-1}$. If $n-k-1 > 1$, then the image $L^{(1)} = \pi_0(L^{(0)})$ is a subspace of H of dimension $n-k-2$ which does not meet $\pi_0(X)$. Repeating this step $n-k-1$ times, we arrive at a subspace $L^{(n-k-1)} \subset \mathbb{P}^{n-(n-k-1)} = \mathbb{P}^{k+1}$ of dimension 0 which is disjoint from the image $(\pi_{n-k-1} \circ \cdots \circ \pi_0)(X)$. We can then project from this point one more time. Since the dimension of X stays the same under all these projections by Cor. 6.10 and the image of X is a subvariety of $\mathbb{P}^k$, we must have $\dim(X) \leqslant k$.

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Essentially the same argument shows the converse: Let $r = \operatorname{codim}(X)$. By Cor. 6.10, we can successively project X from points $p_1, \ldots, p_r$ outside X. Then $p_1, \ldots, p_r$ span an $r-1$-dimensional subspace disjoint from X. Since each p_i can be chosen from an open subset, this shows that the general subspace of dimension $r-1$ does not meet X. So we must have $n-k > r-1$, hence $\dim(X) \geqslant k$. ∎