THE "HALF-DEGREE" AND "DEGREE" PRINCIPLES FOR SYMMETRIC FUNCTIONS

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1. Notations

Symmetric functions

$\Sigma_d^{[n]}$: the $\mathbb{R}$-algebra of all *real symmetric polynomials of degree at most* $d \in \mathbb{N}$ *on* $\mathbb{R}^n$.

$\mathcal{R}_d(A)$: the set of all maps defined on $A \subset \mathbb{R}^n$ by *quotients of polynomials from* $\Sigma_d^{[n]}$, that is,

$$\mathcal{R}_d(A) := \left\{ q : A \to \mathbb{R} \;\middle|\; q = \tfrac{f}{g} \text{ for some } f, g \in \Sigma_d^{[n]}, \text{ with } 0 \notin g(A) \right\}.$$

Distinct components of vectors from $\mathbb{R}^n$

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, set

$$v(x) := \#(\{x_1, \ldots, x_n\}), \qquad v^*(x) := \#(\{x_1, \ldots, x_n\} \setminus \{0\}).$$

For $A \subset \mathbb{R}^n$ and $s \in \mathbb{N}^*$, set

$$A(s) := \{x \in A \mid v(x) \le s\}, \qquad A(s)^* := \{x \in A \mid v^*(x) \le s\}.$$

Stable sets

For any boolean combination $\mathcal{B}_s$ of real symmetric polynomial inequalities of degree at most s on $\mathbb{R}^n$, we may consider the sets

$$A_+ := \{x \in \mathbb{R}^n_+ \mid \mathcal{B}_s(x) \text{ is fulfilled}\}, \qquad A := \{x \in \mathbb{R}^n \mid \mathcal{B}_s(x) \text{ is fulfilled}\}.$$

Let us define

$$\mathcal{A}_s(\mathbb{R}^n_+) := \{A_+ \subset \mathbb{R}^n_+ \mid A_+ \text{ as above}\}, \qquad \mathcal{A}_s(\mathbb{R}^n) := \{A \subset \mathbb{R}^n \mid A \text{ as above}\}.$$

Stable paths

A path $\gamma : [a, b] \to \mathbb{R}^n$ is said to be (s)-*stable* (or an (s)-*path*), if and only if

$$P_1 \circ \gamma, \ldots, P_s \circ \gamma \quad \text{are constant on } [a, b]$$

(where $P_k(x) = x_1^k + \cdots + x_n^k$ is the kth symmetric power sum).

The (s)-boundary of an arbitrary set

For $A \subset \mathbb{R}^n$ and $s \in \mathbb{N}^*$, let

$$\Gamma_s(A) := \{\gamma : [0, 1] \to \mathbb{R}^n \mid \gamma \text{ is an } (s)\text{-path, with } \gamma([0, 1[) \subset A\}.$$

We define the (s)-*boundary* of A by

$$\partial_s A := \partial A \cap \{\gamma(1) \mid \gamma \in \Gamma_s(A)\}.$$

Roughly speaking, the (s)-boundary is the set of all points at which (s)-paths with initial points in A cross the topological boundary ∂A for the first time.

Minimizer of a real function

For arbitrary $f : X \to \mathbb{R}$, let

$$M(f) := \text{minimizer}(f) = \left\{ \xi \in X \;\middle|\; f(\xi) = \min_{x \in X} f(x) \right\}.$$

2. Main results (simplified setting)

The results are of the following type:

If $q : A \to \mathbb{R}$ is a rational symmetric function, then for some specific "thin" subset $A_0 \subset A$ we have

$$q > 0 \text{ on } A \iff q > 0 \text{ on } A_0.$$

Theorem 1 (enlargement and reduction). *Let a rational function $q \in \mathcal{R}_{2s+1}(A)$ defined on a set $A \in \mathcal{A}_s(\mathbb{R}^n_+)$. Assume $M(q) \neq \emptyset$ and choose an absolute minimum point $\xi \in M(q)$.*

(i): *Assume $\xi \notin A(s)^*$. Then there is an (s)-path in $M(q)$ joining ξ to some $\zeta \neq \xi$, with*

$$v^*(\zeta) = \#(\mathrm{supp}(\zeta))$$

(that is, all nonzero components of ζ are pairwise distinct).

(ii): *There is an (s)-path in $M(q)$ joining ξ to some $\zeta \in A(s)^*$. Hence $M(q|_{A(s)^*}) \neq \emptyset$ and*

$$\min_{x \in A} q(x) = \min_{x \in A(s)^*} q(x).$$

Theorem 2 (the "half-degree" principle). *Let a rational function $q \in \mathcal{R}_{2s+1}(A)$ defined on a set $A \in \mathcal{A}_s(\mathbb{R}^n_+) \cup \mathcal{A}_s(\mathbb{R}^n)$.*

(a): *If $A \in \mathcal{A}_s(\mathbb{R}^n_+)$, we have the equivalences*

$$q \geq 0 \text{ on } A \iff q \geq 0 \text{ on } A(s)^*,$$
$$q > 0 \text{ on } A \iff q > 0 \text{ on } A(s)^*.$$

In particular, we have the equality $M(q|_{A(s)^*}) = M(q) \cap A(s)^*$ and the equivalence

$$M(q) \neq \emptyset \iff M(q|_{A(s)^*}) \neq \emptyset.$$

(b): *Assume*[1] *$s \geq 2$. If $A \in \mathcal{A}_s(\mathbb{R}^n)$, we have the equivalences*

$$q \geq 0 \text{ on } A \iff q \geq 0 \text{ on } A(s),$$
$$q > 0 \text{ on } A \iff q > 0 \text{ on } A(s).$$

In particular, we have the equality $M(q|_{A(s)}) = M(q) \cap A(s)$ and the equivalence

$$M(q) \neq \emptyset \iff M(q|_{A(s)}) \neq \emptyset.$$

Corollary 3 (symmetric polynomial inequalities). *For $s \in \mathbb{N}^*$, let a symmetric polynomial inequality of degree at most $2s+1$ on $\mathbb{R}^n$.*

(a): *The inequality holds on $A \in \mathcal{A}_s(\mathbb{R}^n_+)$, if and only if it holds on $A(s)^*$.*

(b): *Assume $s \geq 2$. The inequality holds on $A \in \mathcal{A}_s(\mathbb{R}^n)$, if and only if it holds on $A(s)$.*

Corollary 4 (level sets and zeros). *Let $f \in \Sigma^{[n]}_{2s+1}$, with $s \in \mathbb{N}^*$. Then*

$$f(\mathbb{R}^n_+) = f(\mathbb{R}^n_+(s)^*),$$
$$f(\mathbb{R}^n) = f(\mathbb{R}^n(s)) \quad \text{for } s \geq 2.$$

In particular if f has a zero, then it also has a zero with at most $\frac{\deg f}{2} \vee 2$ distinct components. If f has a zero in $\mathbb{R}^n_+$, then it has such a zero with at most $\frac{\deg f}{2} \vee 1$ distinct nonzero components.

Theorem 5 (the "degree" principle for stable sets). *Let $A \in \mathcal{A}_d(\mathbb{R}^n_+) \cup \mathcal{A}_d(\mathbb{R}^n)$.*

(i): *We have $q(A) = q(A(d))$ for every rational function $q \in \mathcal{R}_d(A)$.*

(ii): *If $\mathcal{B}_d$ is a boolean combination of real symmetric polynomial inequalities of degree at most d on $\mathbb{R}^n$, then*

$$\mathcal{B}_d \text{ holds on } A \iff \mathcal{B}_d \text{ holds on } A(d).$$

Theorem 6 (the "degree" principle for arbitrary sets). *Let $A \subset \mathbb{R}^n$ and $d \in \mathbb{N}^*$.*

(i): *We have*[2] *$q(A) = q(A(d)) \cup q(\partial_d A)$ for every rational function $q \in \mathcal{R}_d(A)$.*

(ii): *If $\mathcal{B}_d$ is a boolean combination of real symmetric polynomial inequalities of degree at most d on $\mathbb{R}^n$, then*

$$\mathcal{B}_d \text{ holds on } A \iff \mathcal{B}_d \text{ holds on } A(d) \cup \partial_d A.$$

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[1]For $s = 1$ the statement is false (take $A = P_1^{-1}(\{1\}) \in \mathcal{A}_1(\mathbb{R}^n)$ and the polynomials $f = 2P_1^2 - 3P_2$ and $g = 1$).

[2]Any rational function $q \in \mathcal{R}_d(A)$ extends uniquely to $q \in \mathcal{R}_d(A \cup \partial_d A)$.