Bounded Polynomials, Sums of Squares, and the Moment Problem

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CONTENTS

INTRODUCTION

Let $h_1, \ldots, h_r$ be polynomials in n variables $x = (x_1, \ldots, x_n)$ with real coefficients, and let

$$S = \left\{a \in \mathbf{R}^n \,\middle|\, h_1(a) \geqslant 0, \ldots, h_r(a) \geqslant 0\right\}$$

be the basic closed semialgebraic subset determined by them. We study the cone

$$\mathcal{P}(S) = \left\{f \in \mathbf{R}[x] \,\middle|\, \forall a \in S\colon f(a) \geqslant 0\right\}$$

of all polynomials that are non-negative on S, in particular how it relates to the cone (more accurately the preordering) generated by $h_1, \ldots, h_r$:

$$T = \mathrm{PO}(h_1, \ldots, h_r) = \left\{ \sum_{i \in \{0,1\}^r} s_i h_1^{i_1} \cdots h_r^{i_r} \,\middle|\, s_i \in \sum \mathbf{R}[x]^2 \right\},$$

where $\sum \mathbf{R}[x]^2 = \{\sum_{j=1}^k g_j^2 \mid g_1, \ldots, g_k \in \mathbf{R}[x], k \geqslant 0\}$ is the set of all sums of squares in $\mathbf{R}[x]$. The inclusion $T \subset \mathcal{P}(S)$ is obvious, and the natural question is whether equality holds or, more generally, to what extent something close to equality can be established. Questions of this kind have been extensively studied within the last 15 years. A classical result of Hilbert says that not every non-negative polynomial on $\mathbf{R}^n$ is a sum of squares in $\mathbf{R}[x]$ if $n \geqslant 2$. More generally, Scheiderer has shown that $\mathcal{P}(S)$ is not finitely generated, i.e. cannot coincide with a cone of the form $\mathrm{PO}(h_1, \ldots, h_r)$, if S has dimension at least 3. Equality between T and $\mathcal{P}(S)$ is a phenomenon of lower dimensional cases only. On the other hand, weaker statements can be proved in arbitrary dimensions: In 1991, Schmüdgen proved that if S is compact, then every polynomial that is strictly positive on S is contained in T. More generally, he defined the *moment property* for T based on the moment problem of functional analysis, which he was studying. It requires that T be dense in $\mathcal{P}(S)$ in the sense that T and $\mathcal{P}(S)$ cannot be separated by any linear functional. It can also be thought of as an approximation property for elements in $\mathcal{P}(S)$ by elements in T (see section 2.1 for precise statements and references for the above results).

Concerning the moment property, Schmüdgen went on to prove a much stronger version of his theorem, namely

Theorem (Schmüdgen 2003) — *Let $\varphi\colon \mathbf{R}^n \to \mathbf{R}^m$ be a polynomial map such that the image $\varphi(S)$ is bounded in $\mathbf{R}^m$. Then T has the moment property if and only if the restriction of T to each fibre $\varphi^{-1}(a)$, $a \in \mathbf{R}^m$, has the moment property.*

(The restriction of a preordering is defined in section 2.1). In order to apply Schmüdgen's theorem in a given situation, one must first know if any map φ as in the statement exists. We therefore consider the subring of $\mathbf{R}[x]$ of polynomials that are bounded on S:

$$B(S) = \left\{ f \in \mathbf{R}[x] \,\middle|\, \exists \lambda \in \mathbf{R} \forall a \in S\colon |f(a)| \leqslant \lambda \right\}.$$

Clearly, $B(S) = \mathbf{R}[x]$ holds if and only if S is compact. More generally, the size of the ring $B(S)$ can be regarded as a measure for the "compactness" of the set S. Schmüdgen's theorem indicates that the bigger the ring $B(S)$ is, the closer T should be to $\mathcal{P}(S)$. This point of view has been suggested to me for this thesis. Apart from Schmüdgen's theorems, it is also motivated by results in the one-dimensional case, i.e. when S is contained in an algebraic curve. Kuhlmann, Marshall, and Scheiderer have almost completely answered all of the above questions in this situation and found a dichotomy between the cases $B(S) = \mathbf{R}$ and $B(S) \neq \mathbf{R}$.

We give a brief overview of the questions and results in this work:

(1) One task is to determine $B(S)$, in particular, to decide whether $B(S) = \mathbf{R}$ or $B(S) \neq \mathbf{R}$. This is achieved to some degree by interpreting $B(S)$ as the ring of polynomial functions on a suitable algebraic compactification of $\mathbf{R}^n$ (Prop. 1.12). This construction is carried out in chapter 1, but we can only prove the strongest result if S has dimension at most 2 (Thm. 1.22). We also address the question whether $B(S)$ is a finitely generated $\mathbf{R}$-algebra. We prove that this is true if S is sufficiently regular and of dimension at most 2, while in higher dimensions there exist otherwise well-behaved counterexamples (section 1.6).

(2) If $B(S) = \mathbf{R}$, the statement of Schmüdgen's theorem becomes void, so some other means of deciding the moment problem for T have to be found. Here we use (and somewhat extend) a result due to Powers and

Scheiderer showing that if $B(S) = \mathbf{R}$, then T can often be shown to be stable, which roughly means the existence of degree bounds for representations in T. This prevents T from having the moment property. These notions and results are explained in section 2.2.

(3) If $B(S) \neq \mathbf{R}$, then Schmüdgen's theorem can be applied, and by restricting to each fibre one can in principle hope to decide the moment problem by an inductive process. We try to make this as explicit as possible in the case when S is of dimension 2. The fibres are then typically curves, and one can make use of Scheiderer's extensive results for irreducible curves. But even in good cases, the curves that come up as fibres need not be irreducible. Therefore, we try to extend Scheiderer's results to the reducible case. This is completely achieved for the moment problem for sums of squares (section 3.2) while for general preorderings we only treat some special cases (section 3.3). We apply these results in chapter 4 to obtain new examples and criteria in dimension 2.

(4) Beyond the moment problem, we also prove a number of results about saturatedness of finitely generated preorderings, i.e. conditions that imply $\mathcal{P}(S) = \mathrm{PO}(h_1, \dots, h_r)$ when S is of dimension 1 or 2. Namely, for curves the above mentioned results in the reducible case often concern this stronger property. In dimension 2, we reprove a theorem by Roggero concerning the divisor class group of a real variety (Thm. 4.9) and combine this with Scheiderer's results about saturated preorderings in the 2-dimensional compact case. This leads to a systematic way of producing new non-compact examples in which the correspoding finitely generated preordering is saturated (Thm. 4.12).

(5) In some one-dimensional cases, the solvability of the moment problem for $T = \mathrm{PO}(h_1, \dots, h_r)$ is known to depend on the choice of generators $h_1, \dots, h_r$. Now if suitable generators can be found for the restriction of T to the fibres of φ, this poses the problem of lifting these generators to T. It comes down to the following general extension problem: Given a closed subvariety Z in $\mathbf{R}^n$ (or any ambient real variety) and a polynomial $f \in \mathbf{R}[x]$ such that f is non-negative on $S \cap Z$, does there exist $g \in \mathbf{R}[x]$ such that g is non-negative on S and such that f and g agree on Z? We extend a result of Scheiderer for the case when Z is a non-singular curve from global positivity to positivity on a semialgebraic set (Thm. 3.35).

Two technical notes are in place: In this introduction, we have put the initial situation into affine space, i.e. $S \subset \mathbf{R}^n$, $h_1, \dots, h_r \in \mathbf{R}[x]$. But it is much more natural and technically convenient to replace affine space by the Zariski closure V of S, an affine $\mathbf{R}$-variety. The principal advantage is that S always has non-empty interior within $V(\mathbf{R})$, a hypothesis that is needed most of the time. The other remark concerns the ground field: Some of the above results use the topology and the archimedean property of the classical real numbers while others work over any real closed field. This distinction is important for connections to model theory, and while we do not make use of it at all, we will require the classical real numbers only when necessary.

CHAPTER 1

BOUNDED FUNCTIONS ON REAL VARIETIES

1.1. REAL VARIETIES

In this chapter, R will always denote a real closed field (e.g. the field of real numbers, which will be denoted by the boldface letter $\mathbf{R}$). By an R-variety we mean a reduced, separated scheme of finite type over $\operatorname{Spec} R$, not necessarily irreducible. The use of schemes is convenient in some places but not essential. We now briefly establish notation and terminology for affine R-varieties in the simplest way and refer the reader to the appendix for a complete list of notations and some generalities on real varieties and semialgebraic sets.

Given polynomials $f_1, \ldots, f_r \in R[x_1, \ldots, x_n]$, let $I = \sqrt{(f_1, \ldots, f_r)}$ be the radical ideal in $R[x_1, \ldots, x_n]$ generated by $f_1, \ldots, f_r$. Put $V = \operatorname{Spec}\big(R[x_1, \ldots, x_n]/I\big)$ and $R[V] = R[x_1, \ldots, x_n]/I$. Then V is called an affine R-variety and $R[V]$ its ring of regular functions. We write $V(R)$ for $\{a \in R^n \mid f_1(a) = \cdots = f_r(a) = 0\}$, the set of real points of V and consider $R[V]$ as a ring of functions on $V(R)$. We will usually identify $V(R)$ with the subset of points of V that have residue field R. Write $K = R(\sqrt{-1})$, then the ideal I (and hence the ring $R[V]$) can be recovered from $V(K) = \{a \in K^n \mid f_1(a) = \cdots = f_r(a) = 0\}$, the set of complex points of V, namely $I = \mathcal{I}(V(K)) = \{f \in R[x_1, \ldots, x_n] \mid \forall a \in V(K)\colon\ f(a) = 0\}$ by Hilbert's Nullstellensatz. Thus we could work with just $V(R)$ and $V(K)$ and forget about the scheme V. It should be pointed out that our notion of R-variety is different from the one often used in real geometry where only the set of real points is considered (for example in Bochnak, Coste, and Roy [**6**]). We say that an R-variety is real if it has a non-singular R-rational point (see also Prop. A.1).

If V and W are two affine R-varieties and $\varphi\colon V \to W$ is a morphism of varieties (i.e. a polynomial map), then φ induces a map $V(R) \to W(R)$ which we denote by φ_R. Furthermore, φ induces a homomorphism $\varphi^\#\colon R[W] \to R[V]$

of R-algebras by the rule $f \mapsto f \circ \varphi$. The map $\varphi \mapsto \varphi^{\#}$ yields an equivalence of categories between finitely generated reduced R-algebras and affine R-varieties.

The set $V(R)$ comes equipped with two topologies, the R-Zariski-topology, where the closed sets are given by the vanishing of polynomials with coefficients in R, and the strong topology induced by the ordering of R (which is the euclidean topology for $R = \mathbf{R}$). We will often consider the closure of subsets of $V(R)$ with respect to either of these topologies. To avoid confusion, the closure of $S \subset V(R)$ with respect to the Zariski topology will be denoted by $\overline{S}$ (also for subsets of V) while the closure with respect to the strong topology will be denoted by $\operatorname{clos}_{V(R)}(S)$ or just $\operatorname{clos}(S)$.

A subset S of $V(R)$ is called basic closed if there exist $h_1, \dots, h_r \in R[V]$ such that $S = \{a \in V(R) \mid h_1(a) \geqslant 0, \dots, h_r(a) \geqslant 0\}$. The set S is called semialgebraic if it can be written as a finite boolean combination of basic closed sets.

1.2. THE RING OF BOUNDED FUNCTIONS

Let V be an affine R-variety and S a semialgebraic subset of $V(R)$. A function $f \in R[V]$ is *bounded on S (over R)* if there exists $\lambda \in R$ such that $-\lambda \leqslant f(a) \leqslant \lambda$ holds in all points $a \in S$. The functions in $R[V]$ that are bounded on S clearly form a subring of $R[V]$. In this section, we will gather some fundamental properties of this ring.

Notation — We write

$$B_V(S) = \{f \in R[V] \mid \exists \lambda \in R \forall a \in S: |f(a)| \leqslant \lambda\}$$

for the *ring of bounded functions on S (over R)*. In the case $S = V(R)$, we will also write $B(V)$ instead of $B_V(V(R))$.

Proposition 1.1 — *Let V be an affine R-variety, Z a closed subvariety of V with vanishing ideal $\mathcal{I}_V(Z) \subset R[V]$, and let S, S' be two semialgebraic subsets of $V(R)$.*

(1) $B_V(S \cup S') = B_V(S) \cap B_V(S')$;
(2) $B_V(S \cap Z(R)) / \mathcal{I}_V(Z) = B_Z(S \cap Z(R))$;
(3) $B_V(S) = B_V(\operatorname{clos}(S))$;
(4) $B_V(S) = R[V]$ *if and only if* $\operatorname{clos}(S)$ *is semialgebraically compact.*

Proof — (1) and (2) are immediate. (3) comes from the fact that we only consider functions that are regular on all of $V(R)$ and hence extend continuously to $\operatorname{clos}(S)$. (4) If $\operatorname{clos}(S)$ is semialgebraically compact, then its image under any continuous semialgebraic map is again semialgebraically compact, and in particular, any regular function on V is bounded on S. Conversely, if $B_V(S) = R[V]$, this implies that $x_1^2 + \cdots + x_n^2$ is bounded on S for any embedding of V into affine space with coordinates $x_1, \ldots, x_n$, thus $\operatorname{clos}(S)$ is semialgebraically compact. □

Proposition 1.2 — *The ring $B_V(S)$ is integrally closed in $R[V]$.*

Proof — The proof is well-known in the theory of real holomorphy rings. Suppose that $f \in R[V]$ satisfies an equation $f^n + b_{n-1}f^{n-1} + \cdots + b_0 = 0$ with $b_i \in B_V(S)$, $i = 0, \ldots, n-1$. Now we apply the following basic

Lemma — *Let $f = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in R[x]$. Every real root $\xi \in R$ of f satisfies*

$$|\xi| \leqslant \max\{1, |a_0| + \cdots + |a_{n-1}|\}.$$

Proof — If $\xi \neq 0$, then $\xi = -(a_{n-1}\xi^{-1} + \cdots + a_0\xi^{1-n})$. Apply the triangle inequality to complete the proof. □

In our situation, this gives

$$|f(a)| \leqslant \max\{1, |b_0(a)| + \cdots + |b_{n-1}(a)|\}$$

for all $a \in V(R)$. Since $|\lambda| \leqslant 1 + \lambda^2$ for any $\lambda \in R$, it follows that

$$|f(a)| \leqslant \big(n + 1 + b_0^2(a) + \cdots + b_{n-1}^2(a)\big)$$

for all $a \in V(R)$, so $|f|$ is bounded on S by an element of R since the right hand side is. □

Definition — Let V be an R-variety and S a semialgebraic subset of $V(R)$. We call the morphism $V \to \operatorname{Spec}\big(B_V(S)\big)$ induced by the inclusion $B_V(S) \subset R[V]$ the *canonical bounded morphism over S*.

Let $\varphi\colon V \to \operatorname{Spec}\big(B_V(S)\big)$ denote the canonical bounded morphism over S. Any morphism $\psi\colon V \to \mathbf{A}_R^n$ such that $\operatorname{clos}\psi_R(S)$ is semialgebraically compact factors uniquely through φ:

$$\begin{array}{ccc} V & \xrightarrow{\ \varphi\ } & \operatorname{Spec}\big(B_V(S)\big) \\ & \underset{\psi}{\searrow} & \big\downarrow \exists! \\ & & \mathbf{A}_R^n. \end{array}$$

For if clos $\psi_R(S)$ is semialgebraically compact, then the image of the homomorphism $\psi^\#: R[x_1, \dots, x_n] \to R[V]$ is contained in $B_V(S)$, so the decomposition $\psi^\#: R[x_1, \dots, x_n] \to B_V(S) \to R[V]$ yields the desired decomposition $\psi: V \to \operatorname{Spec}\big(B_V(S)\big) \to \mathbf{A}^n_R$.

It is important to point out that the ring $B_V(S)$ is not in general a finitely generated R-algebra, hence $\operatorname{Spec}\big(B_V(S)\big)$ is a not a variety but only an affine scheme. This is a major obstacle in the study of rings of bounded functions. It can be overcome by putting everything in the context of schemes (or real spectra) and working with the more general notion of rings of bounded elements or real holomorphy rings (see section 2.3). The question of when the ring of bounded functions is in fact finitely generated will be addressed in section 1.6.

1.3. FIBRES OF BOUNDED MORPHISMS

Let V be an irreducible affine R-variety, S a semialgebraic subset of $V(R)$, and $W = \operatorname{Spec}\big(B_V(S)\big)$. Assume that $B_V(S)$ is a finitely generated R-algebra (c.f. section 1.6), so that W is a variety. The goal of this section is to study the fibres of the canonical bounded morphism $V \to W$ which will always be denoted by φ. This will gain us some understanding of the fibres of any regular map that is bounded on S since the fibres of such a map are unions of fibres of φ.

We write φ_R for the induced map $V(R) \to W(R)$ on R-rational points. Then $\varphi_R^{-1}(w) = \varphi^{-1}(w) \cap V(R)$ is the real fibre for every $w \in W(R)$. We begin by showing that a function that is bounded outside a Zariski-closed subset consisting of fibres of φ_R can be expressed as a quotient of bounded functions: For $f \in R[V]$, let

$$\Omega(f) = \big\{ w \in \varphi_R(S) \,\big|\, f \notin B_V(\varphi_R^{-1}(w) \cap S) \big\}.$$

Note that $\Omega(f)$ is a semialgebraic subset of $W(R)$ (since it spells out to $\Omega(f) = \big\{ w \in W(R) \mid \forall \lambda \in R\ \exists x \in S: \varphi_R(x) = w \wedge |f(x)| > \lambda \big\}$).

Proposition 1.3 — *Assume that $B_V(S)$ is a finitely generated R-algebra. For $f \in R[V]$, the following statements are equivalent:*

(1) $f \in \operatorname{qf}\big(B_V(S)\big)$;
(2) $\varphi_R^{-1}(\Omega(f))$ *is not Zariski-dense in* V.
(3) $\Omega(f)$ *is not Zariski-dense in* W.

Proof — Write $A = R[V]$ and $B = B_V(S)$.

(1) ⇒ (2): Let $h \in B$, $h \neq 0$ and $hf \in B$. Then h is constant along all fibres $\varphi^{-1}(w)$, $w \in W(R)$. Now if $w \in \Omega(f)$, i.e. if f is unbounded on $\varphi_R^{-1}(w) \cap S$, then we must have $h = 0$ on $\varphi^{-1}(w)$, since hf is bounded on $\varphi_R^{-1}(w) \cap S$. Hence $\varphi^{-1}(\Omega(f)) \subset \mathcal{V}(h)$, so (2) holds.

(2) ⇒ (3): Note first that if W is a real variety, then so is V. For if V is non-real, then there exists an ideal $0 \subsetneq I \subsetneq R[V]$ such that all elements of I vanish on $V(R)$. Clearly, $I \subset B_V(S)$ which implies $\mathrm{qf}(B_V(S)) = R(V) = R(W)$. Hence W cannot be real either (by Prop. A.1).

Now suppose that $\Omega(f)$ is Zariski-dense in W, so that W and V are real. Since $\Omega(f)$ is semialgebraic and Zariski-dense in W and φ is a dominant morphism, $\Omega(f) \cap \varphi_R(V_{\mathrm{reg}}(R))$ contains a non-empty open subset U. This implies that $\varphi_R^{-1}(U)$ contains a non-empty open subset of real regular points of V and is therefore Zariski-dense in V, hence so is $\varphi_R^{-1}(\Omega(f))$.

(3) ⇒ (1): To show that $f \in \mathrm{qf}(B)$, we define a map $\widetilde{f}\colon \mathrm{clos}(\varphi_R(S)) \to R \cup \{\infty\}$ by

$$\widetilde{f}(w) = \frac{1}{\sup\left\{|f(v)| \,\middle|\, v \in \varphi_R^{-1}(w) \cap S\right\}}$$

for $w \in \varphi_R(S)$ (where $1/\infty := 0$ and $1/0 := \infty$) and put $\widetilde{f}(w) = 1$ if $w \in \mathrm{clos}(\varphi_R(S)) \setminus \varphi_R(S)$. The map $\widetilde{f}$ is semialgebraic. Let N be the set of points where $\widetilde{f}$ is not continuous. Then N is not Zariski-dense in W since $\widetilde{f}$ is semialgebraic. Furthermore, $\Omega(f) = \left\{w \in \varphi_R(S) \,\middle|\, \widetilde{f}(w) = 0\right\}$ is not Zariski-dense, by hypothesis. By the subsequent lemma, there exists a continuous semialgebraic function $\widetilde{h}\colon \mathrm{clos}(\varphi_R(S)) \to R$ such that $|\widetilde{h}| \leqslant \widetilde{f}$ and such that the set $Z = \left\{w \in \mathrm{clos}(\varphi_R(S)) \,\middle|\, \widetilde{h}(w) = 0\right\}$ is contained in $\mathrm{clos}(N) \cup \Omega(f)$ and is therefore not Zariski-dense.

Let $h \in B$, $h \neq 0$, be such that $Z \subset \mathcal{V}_R(h)$. Since $\mathrm{clos}(\varphi_R(S))$ is compact, there exist $c > 0$ and $N \in \mathbf{N}$ such that $|h^N| \leqslant c\widetilde{h}$ holds on $\mathrm{clos}(\varphi_R(S))$, by the Łojasiewicz inequality (see e.g. Bochnak, Coste, and Roy [**6**], Cor. 2.6.7). Now let $v \in S$ and $w = \varphi(v)$. We have

$$|(c^{-1}h^N f)(v)| = c^{-1}|h^N(w)f(v)| \leqslant |\widetilde{h}(w)f(v)| \leqslant \widetilde{f}(w)|f(v)| \leqslant 1.$$

It follows that $c^{-1}h^N f \in B$, hence $f \in \mathrm{qf}(B)$. □

Lemma 1.4 — *Let M be a compact semialgebraic subset of R^n, and let $f\colon M \to R \cup \{\infty\}$ be a semialgebraic map with $f \geqslant 0$ on M. Let N be the set of points where f is not continuous. Then there exists a continuous semialgebraic map $g\colon M \to R$*

such that $|g| \leqslant f$ *on* M *and such that*

$$g(x) = 0 \quad \Longrightarrow \quad \big(f(x) = 0 \quad \vee \quad x \in \mathrm{clos}(N)\big)$$

holds for all $x \in M$.

Proof — Upon replacing f by $\min\{f, 1\}$, we may assume that f is bounded on M. The function $d_N: M \to R$, $x \mapsto \inf\{\|x - y\| \mid y \in N\}$ is continuous, semialgebraic and vanishes precisely on $\mathrm{clos}(N)$. Let m be its maximum on M. Then $g(x) = f(x) \cdot d_N(x) \cdot m^{-1}$, $x \in M$, has the desired properties. □

Example — Let $V = \mathbf{A}_R^2$ with coordinates x, y, let $S = \mathcal{S}(x, y, 1-x, 1-xy)$. We have $B = R[x, xy]$, $\varphi_R: R^2 \to R^2$, $(x, y) \mapsto (x, xy)$; hence $y \notin B$, but $y = \frac{xy}{x} \in \mathrm{qf}(B)$. This is because the function x vanishes on $\varphi_R^{-1}(\Omega(y)) = \{(0, a) \,|\, a \in R\}$.

Now let $\Omega = \bigcup_{f \in R[V]} \Omega(f)$, then

$$\Omega = \big\{w \in \varphi_R(S) \,\big|\, \mathrm{clos}(\varphi_R^{-1}(w) \cap S) \text{ is not semialgebraically compact}\big\}.$$

by Prop. 1.1 (4). If $x_1, \ldots, x_n$ are generators for the R-algebra $R[V]$, then $\Omega = \Omega(x_1) \cup \cdots \cup \Omega(x_n)$. Thus Ω is again a semialgebraic subset of $W(R)$.

Corollary 1.5 — *Assume that* $B_V(S)$ *is a finitely generated* R*-algebra, and let* W, $\varphi: V \to W$ *be as before. The following are equivalent:*

(1) φ *is birational;*
(2) $\dim W = \dim V$*;*
(3) $\varphi_R^{-1}(\Omega)$ *is not Zariski-dense in* V.
(4) Ω *is not Zariski-dense in* W*;*

Proof — (1) ⇒ (2): is clear, since birationally equivalent varieties have the same dimension.

(2) ⇒ (3): If $\dim W = \dim V$, then there is a Zariski-open subset U of W such that $\varphi^{-1}(w)$ is a finite set for all $w \in U$ (see Hartshorne [**9**], chap. 2, ex. 3.22). Clearly, $U(R) \cap \Omega = \varnothing$, so Ω is not Zariski-dense in W.

(3) ⇒ (4): with the same reasoning as in the proof of the proposition.

(4) ⇒ (1): For every $f \in R[V]$, $\Omega(f) \subset \Omega$ is not Zariski-dense, so $f \in \mathrm{qf}(B)$ by the proposition. Therefore, $\mathrm{qf}(B) = \mathrm{qf}(R[V]) = R(V)$, so φ is birational. □

Corollary 1.6 — *If all fibres* $\varphi_R^{-1}(w) \cap S$, $w \in W(R)$, *are semialgebraically compact, then* φ *is birational.*

Proof — In this case, we have $\Omega = \varnothing$. □

Remark 1.7 — The implication (2) ⇒ (1) in Cor. 1.5 says that if the canonical bounded morphism has generically finite fibres, then it must be birational, i.e. generically one-to-one. In algebraic terms this is equivalent to saying that if the quotient fields of $R[V]$ and $B_V(S)$ have the same transcendence degree over R, then they conincide. If V is normal, then $R[V]$ is integrally closed in $R(V)$ and this claim can be directly deduced from the fact that $B_V(S)$ is integrally closed in $R[V]$ (Prop. 1.2). Namely, it amounts to the following, purely algebraic, statement:

Lemma — *Let B be a domain, $K = \mathrm{qf}(B)$ and L/K an algebraic field extension. If B is integrally closed in L, then $L = K$.*

Proof — Take $a \in L$, and let $f \in K[x]$ be the minimal polynomial of a over K. After clearing denominators in $f(a) = 0$, we find $b \in B$ such that ab is integral over B; hence $ab \in B$ and thus $a \in K$. □

1.4. COMPLETIONS OF QUASI-PROJECTIVE VARIETIES

In this section, we will show how the ring of bounded functions can often be described in terms of a prescribed distribution of poles on a suitable compactification. Let X be an irreducible R-variety, and let Z be an irreducible subvariety of codimension 1 in X. If X is normal along Z, then the local ring $\mathcal{O}_{X,Z}$ of Z in X is a discrete valuation ring with valuation v_Z (see section A.3). For any rational function $f \in R(X)$, the value $-v_Z(f)$ is called the *pole order of f along Z*; we say that *f has a pole along Z* if the pole order of f along Z is strictly positive.

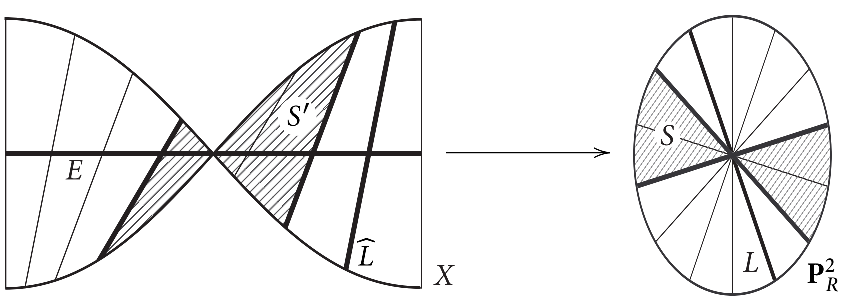
Lemma 1.8 — *Let X be an irreducible R-variety, and let Z be an irreducible subvariety of codimension 1 in X such that X is normal along Z. Let $V = X \setminus Z$, and let S be a semialgebraic subset of $V(R)$. Assume that $\mathrm{clos}_{X(R)}(S) \cap Z(R)$ is Zariski-dense in Z. If a rational function $f \in R(X)$ has a pole along Z, then f is unbounded on $S \cap \mathrm{dom}(f)$.*

Proof — Put $S' = \mathrm{clos}_{X(R)}(S)$, and let $t \in \mathcal{O}_{X,Z}$ be a regular parameter, i.e. an element satisfying $v_z(t) = 1$. Let U be an open affine subset of X with $U \cap Z \neq \emptyset$ and $t \in \mathcal{O}_X(U)$. Since Z is a pole of f, we can write $f = g/t^i$ where $i \geqslant 1$ and $g \in \mathcal{O}_{X,Z}^{\times}$ does not vanish identically on $Z(R)$. For any $\lambda \in R$, set $U_\lambda = \{x \in U(R) \cap \mathrm{dom}(g) \mid |g(x)| > \lambda\}$. Then $\bigcup_{\lambda \in R} U_\lambda$ is dense in $Z(R)$, so there exists $\mu \in R$ such that $S' \cap U_\mu \cap Z(R)$ is non-empty. By the curve selection lemma (see Thm. 2.5.5. in Bochnak, Coste, and Roy [**6**]), there exists a continuous semialgebraic map $\varphi\colon [0,1] \to X(R)$ such that $\varphi((0,1]) \subset S \cap U_\mu$ and $\varphi(0) \in S' \cap U_\mu \cap Z(R)$. Since $t(\varphi(0)) = 0$ and $g(\varphi(a)) > \mu$ for all $a \in (0,1]$, we see that $f = g/t^i$ is unbounded on $\varphi((0,1]) \subset S$. □

Examples 1.9 — (1) Let V be the affine plane $\mathbf{A}^2_R$ with coordinates u, v. A non-constant polynomial $f \in R[u,v]$ is never bounded on R^2. Of course, this is easy to see directly, but it can also be deduced from the lemma: Choose homogeneous coordinates x, y, z on $\mathbf{P}^2_R$ and put $u = \frac{x}{z}$, $v = \frac{y}{z}$, so that V is identified with $\mathbf{P}^2 \smallsetminus \{z = 0\}$. Then $\operatorname{clos}_{X(R)}(V(R)) = \mathbf{P}^2(R)$ so that any function with a pole along $\{z = 0\}$ must be unbounded on $V(R)$. We have $R(V) \cong R(u,v) = \left\{ \frac{F}{G} \,\middle|\, F, G \in R[x,y,z] \text{ homogeneous and } \deg F = \deg G \right\}$. If f has total degree n, we write $f = \sum a_{ij} u^i v^j = \frac{\sum a_{ij} x^i y^j z^{n-i-j}}{z^n}$, then the enumerator is a polynomial not divisible by z, so that n is the pole order of f along the line $\{z = 0\}$. We find

$$\begin{aligned} B_V(R^2) &= \left\{ f \in R[u,v] \,\middle|\, f \text{ has no pole along } \{z = 0\} \right\} \\ &= \left\{ f \in R[u,v] \,\middle|\, \deg(f) = 0 \right\} = R. \end{aligned}$$

(2) With V as before, let $S = \left\{ (u,v) \in R^2 \,\middle|\, -1 \leqslant u \leqslant 1 \right\}$. We use the lemma to show that $B_V(S) = R[u]$. Let $f \in R[V] = R[u,v]$, $\deg f = n$, and write $f = \sum a_{ij} u^i v^j = \frac{\sum a_{ij} x^i y^j z^{n-i-j}}{z^n}$. The set $\operatorname{clos}_{\mathbf{P}^2(R)}(S)$ meets the line $L = \{z = 0\}$ in the single point $P = (0:1:0)$. Therefore, the lemma does not apply, and that f has a pole along L, i.e. $n > 0$, does not imply that f is unbounded on S. However, there is a simple remedy: Let $\varphi\colon X \to \mathbf{P}^2_R$ be the blowing-up of $\mathbf{P}^2_R$ in $(0:1:0)$ with exceptional divisor E, and let $\widehat{L}$ be the strict transform of L in $\mathbf{P}^2_R$. In affine coordinates this means that the local chart $\left(\frac{x}{y}, \frac{z}{y}\right)$ centered around $(0:1:0)$ on $\mathbf{P}^2_R$ is replaced by a new chart U with coordinates $(r,s) = \left(\frac{x}{y}, \frac{z}{x}\right)$. Let $S' = \operatorname{clos}_X(S)$, then the situation looks as follows:



Indeed, we have $E \cap U = \{r = 0\}$ and $\widehat{L} \cap U = \{s = 0\}$. Furthermore, $S' \cap U = \{(r,s) \mid (s \leqslant -1) \vee (s \geqslant 1)\}$. This shows that $S' \cap \widehat{L}(R) = \varnothing$ while $S' \cap E(R)$ is an interval and hence Zariski-dense in $E(R)$. Since $f = \sum a_{ij} u^i v^j = \sum a_{ij} r^{-j} s^{-i-j}$, we see that f has a pole along E if and only if $j > 0$, hence

$$B_V(S) = \{f \in R[u,v] \mid f \text{ does not have a pole along } E\} = R[u].$$

We will now carry out this procedure in greater generality. The example we have just seen suggests that we should pass from the natural compactification $\mathbf{A}^2_R \hookrightarrow \mathbf{P}^2_R$ to other compactifications, obtained by blowing up. But it is best to first deal with this from a more general point of view. We will be looking for compactifications with the following properties:

Definition 1.10 — Let V be a quasi-projective R-variety.

(1) An open dense embedding $V \hookrightarrow X$ of V into a projective variety X is called a *completion of* V. We will say that a completion $V \hookrightarrow X$ is *good* if
   - $X \smallsetminus V$ has pure codimension 1 in X;
   - X is normal along $X \smallsetminus V$ (c.f. section A.3).

(2) Let S be a semialgebraic subset of $V(R)$. We say that a completion $V \hookrightarrow X$ has *dense boundaries for* S or that S *has dense boundaries in* X if every irreducible component Z of $X \smallsetminus V$ satisfies the following condition:
   - $\mathrm{clos}_{X(R)}(S) \cap Z(R)$ is either empty or Zariski-dense in Z.

Remarks — (1) Note that if $\mathrm{clos}_{X(R)}(S) \cap Z(R)$ is Zariski-dense in Z for some irreducible component Z of $X \smallsetminus Y$, then, in particular, $Z(R)$ itself must be Zariski-dense in Z so that Z is a real variety (see Prop. A.1).

(2) Since $\mathrm{clos}_{X(R)}(S) = \mathrm{clos}_{X(R)}(\mathrm{clos}_{V(R)}(S))$, we see that S has dense boundaries in a completion X if and only if $\mathrm{clos}_{V(R)}(S)$ has dense boundaries in X.

Examples 1.11 — (1) In Examples 1.9 (1) and (2), $\mathbf{P}^2_R$ resp. X were good completions of $V = \mathbf{A}^2_R$ with dense boundaries for R^2 resp. S.

(2) Let $V = \mathbf{A}^n_R$ be affine space and consider the standard embedding $\mathbf{A}^n_R \hookrightarrow \mathbf{P}^n_R$ into projective space, which identifies $\mathbf{A}^n_R$ with the complement of a hyperplane in $\mathbf{P}^n_R$. A semialgebraic subset $S \subset R^n$ has dense boundaries in $\mathbf{P}^n_R$ if and only if it contains an open convex cone (i.e. if there exist $x \in R^n$ and an open subset U of R^n such that $x + \lambda U \subset S$ holds for all $\lambda > 0$ in R).

We will prove the existence of good completions with dense boundaries for several particular cases in the next section. Let us first prove that such a completion yields a representation of the ring of bounded functions:

Proposition 1.12 — *Let V be an affine R-variety and S a semialgebraic subset of $V(R)$. Assume that V possesses a good completion $V \hookrightarrow X$ with dense boundaries for S. Let Z be the union of those irreducible components of $X \setminus V$ that are disjoint from* $\operatorname{clos}_{X(R)}(S)$. *Then*

$$B_V(S) = \mathcal{O}_X(X \setminus Z).$$

Thus a function is bounded on S if and only if it has poles only in components of Z.

Proof — If $V_1, \dots, V_r$ are the irreducible components of V, let X_i denote the Zariski-closure of V_i in X, and put $S_i = S \cap V_i(R)$, and Z_i the union of all irreducible components of $X_i \setminus V_i$ that are disjoint from $\operatorname{clos}_{X_i(R)}(S_i)$. Then $V_i \hookrightarrow X_i$ is a good completion with dense boundaries for S_i, and for every $f \in R[V]$, we have $f \in B_V(S)$ (resp. $f \in \mathcal{O}_X(X \setminus Z)$) if and only if $f|_{V_i} \in B_{V_i}(S_i)$ (resp. $f|_{V_i} \in \mathcal{O}_{X_i}(X_i \setminus Z_i)$) for each $i \in \{1, \dots, r\}$. We may therefore assume that V is irreducible.

Put $W = X \setminus Z$ and $S' = \operatorname{clos}_{X(R)}(S)$. Since X is projective, S' is semialgebraically compact (see Prop. A.4) and $S' \subset W(R)$, so every element of $\mathcal{O}_X(W)$ is bounded on S', hence $\mathcal{O}_X(W) \subset B_V(S)$. For the converse, let $f \in R[V] = \mathcal{O}_X(V)$ and assume $f \notin \mathcal{O}_X(W)$. Let $Z_1, \dots, Z_r$ be those irreducible components of $X \setminus Z$ not contained in Z, then, by hypothesis, $S' \cap Z_i(R)$ is Zariski-dense in Z_i for $i = 1, \dots, r$. We have

$$\mathcal{O}_X(W) = \mathcal{O}_X(V) \cap \mathcal{O}_{X,Z_1} \cap \cdots \cap \mathcal{O}_{X,Z_r}$$

by Prop. A.5, because X is normal along $X \setminus V$. Therefore, $f \in \mathcal{O}_X(V) \setminus \mathcal{O}_X(W)$ implies $f \notin \mathcal{O}_{X,Z_i}$ for some i, in other words f has a pole along Z_i. Therefore, f is unbounded on S by Lemma 1.8 which means $f \notin B_V(S)$. □

1.5. EXISTENCE OF GOOD COMPLETIONS WITH DENSE BOUNDARIES

If V is a quasi-projective R-variety, then for any embedding $V \hookrightarrow \mathbf{P}^n_R$ the Zariski-closure $\overline{V}$ of V in $\mathbf{P}^n_R$ is a completion of V. In constructing good completions and completions with dense boundaries, we always start from such an embedding which is then gradually improved. The case of curves is quite simple:

Proposition 1.13 — *Let C be a curve over R.*

(1) *C has a good completion, unique up to isomorphism.*
(2) *If C is non-singular, then so is the good completion of C.*
(3) *Any completion of C has dense boundaries for all semialgebraic subsets of $C(R)$.*

Proof — Let $C \hookrightarrow X_0$ be any completion. Since C is dense in X_o, the complement $X_0 \smallsetminus C$ consists of finitely many points. That any semialgebraic subset of $C(R)$ has dense boundaries in $X(R)$ is therefore obvious. We can remove any singularities in $X_0 \smallsetminus C$ by successive blow-ups or by taking the normalization of $X_0 \smallsetminus C_{\text{sing}}$ and then glueing with C (see for example Serre [**35**], ch. IV, §1). Thus we obtain a completion $C \hookrightarrow X$ such that all points of $X \smallsetminus C$ are non-singular. Clearly, this completion is good, and if C is non-singular, then so is X. If $C \hookrightarrow X$ and $C \hookrightarrow X'$ are two good completions, the birational map $X \dashrightarrow X'$ given by the identity $C \to C$ extends to an isomorphism, since $X \smallsetminus C$ and $X' \smallsetminus C$ consist of non-singular points (see Hartshorne [**9**], ch. I, Prop. 6.8.). □

Definition — Let C be a curve over R, and let $C \hookrightarrow X$ be a good completion of C. The finitely many points in $X \smallsetminus C$ are called the *points at infinity of C*.

We have just seen that the points at infinity of C are uniquely determined up to isomorphism as a zero-dimensional variety, in other words their number and their residue fields are uniquely determined.

Corollary 1.14 — *Let C be a curve over R, with good completion X, and let S be a semialgebraic subset of $C(R)$. Then $B_C(S) = R$ if and only if all points at infinity of C are real and contained in* $\text{clos}_{X(R)}(S)$.

Proof — Let $T = \{P \in X \smallsetminus C \mid P \notin X(R) \text{ or } P \in X(R) \smallsetminus \text{clos}_{X(R)}(S)\}$. Then $B_C(S) = \mathcal{O}_X(X \smallsetminus T)$ by Prop. 1.12. We have $\mathcal{O}_X(X \smallsetminus T) = R$ if and only if $T = \varnothing$, as a consequence of the Riemann-Roch theorem. □

Remark 1.15 — On a side note, the statement of the corollary does not generalise to higher dimensions in any reasonable way because if X has dimension at least 2, it may happen that $\mathcal{O}_X(X \smallsetminus Z) = R$ for some non-empty divisor Z on X. For concrete examples where this occurs in the construction of a good completion, see Examples 4.5.

In higher dimensions, it is more difficult to prove the existence of good completions with dense boundaries. It is not hard to see that every normal variety has a good completion: If V is normal and $V \hookrightarrow X$ is any completion, then the normalization $X' \to X$ is an isomorphism over V, and should any component of $X' \setminus V$ have codimension greater than 1, this can be resolved by blowing up. It is the condition of dense boundaries that is critical. We begin with the case $S = V(R)$:

Proposition 1.16 — *Let V be a quasi-projective variety over R, and assume that V has a good completion $V \hookrightarrow X$ such that every irreducible component of $X \setminus V$ is non-singular. Then $V(R)$ has dense boundaries in X.*

Proof — Let $d = \dim V$. The local dimension of $X(R)$ in any regular point is d while $\dim(X(R) \setminus V(R)) \leqslant d-1$. Therefore, the complement of $\mathrm{clos}_{X(R)}(V(R))$ in $X(R)$ must be contained in the singular locus of X and is therefore of dimension at most $d-2$, since X is normal along $X \setminus V$. Now if Z is any irreducible component of $X \setminus V$, then either $Z(R)$ is empty or $\dim Z(R) = d-1$, since Z is non-singular, so $\mathrm{clos}_{X(R)}(V(R)) \cap Z(R)$ is Zariski-dense in Z. □

Proposition 1.17 — *Let V be a non-singular, quasi-projective R-variety. There is a good completion $V \hookrightarrow X$ such that X is non-singular and all irreducible components of $X \setminus V$ are also non-singular.*

Proof — Starting from any open embedding of V into a projective variety X_0, we may first apply resolution of singularities (as stated in Thm. A.7) to X_0 and obtain an open embedding $V \hookrightarrow X_1$ such that X_1 is again non-singular. Then apply embedded resolution of singularities (Thm. A.8) to $X_1 \setminus V$ to get a non-singular completion $V \hookrightarrow X$ of V where all irreducible components of $X \setminus V$ are non-singular and of codimension 1 in X. □

Corollary 1.18 — *Let V be a non-singular, quasi-projective variety over R. Then V possesses a good completion with dense boundaries for $V(R)$.* □

Remark 1.19 — Good completions in dimensions $\geqslant 2$ are not unique, but if $V \hookrightarrow X_1$ and $V \hookrightarrow X_2$ are two good completions, one may ask whether there exists a common refinement, i.e. a good completion $V \hookrightarrow X'$ together with birational morphisms $X' \to X_1$ and $X' \to X_2$ that are isomorphisms over V. This is always true in the case of non-singular surfaces, since any birational map $\varphi\colon X_1 \dashrightarrow X_2$ of non-singular surfaces can be factored into a series of blow-downs $X_1 \leftarrow X'$ followed by a series of blow-ups $X' \to X_2$, both with centers away from $\mathrm{dom}(\varphi)$ (see for example Beauville [3], Cor. II.12; the analogous statement in dimensions $\geqslant 3$ goes by the name "strong factorization conjecture", and no proof is known so far; see Abramovich, Karu, Matsuki, and Włodarczyk [1]). Applying this to the map φ induced by the identity on V gives the desired refinement X'. It is not hard to see that if some semialgebraic subset of $V(R)$ has dense boundaries in $X_1(R)$ or $X_2(R)$, then this can be arranged also for $X'(R)$.

As an example in dimension 2, consider $\mathbf{P}^2_R$ and $\mathbf{P}^1_R \times \mathbf{P}^1_R$ as completions of $\mathbf{A}^2_R$ in the natural way. The two are not directly comparable, since the first has only one component at infinity while the second has two, and there is no birational morphism (and hence no non-constant morphism that preserves the embedding of $\mathbf{A}^2_R$) from one to the other. But the birational map $\varphi\colon \mathbf{P}^1_R \times \mathbf{P}^1_R \dashrightarrow \mathbf{P}^2_R$, obtained by embedding $\mathbf{P}^1_R \times \mathbf{P}^1_R$ into $\mathbf{P}^3_R$ as a non-singular quadric and projecting from a point, fits into a diagram

$$\begin{array}{ccc} & \widehat{Q} & \\ \swarrow & & \searrow \\ \mathbf{P}^1_R \times \mathbf{P}^1_R & \overset{\varphi}{\dashrightarrow} & \mathbf{P}^2_R \end{array}$$

where $\widehat{Q} \to \mathbf{P}^1_R \times \mathbf{P}^1_R$ is a blow-up in one point and $\widehat{Q} \to \mathbf{P}^2_R$ a blow-up in two points.

We now turn to the case of general semialgebraic subsets of $V(R)$.

Definition 1.20 — Let V be an R-variety and S a semialgebraic subset of $V(R)$.

(1) S is called *regular* if there exists an open subset U of $V(R)$ such that $\operatorname{clos}(S) = \operatorname{clos}(U)$.
(2) S is called *regular at infinity* if $\operatorname{clos}(S)$ is the union of a regular and a semialgebraically compact subset of $V(R)$.

Examples 1.21 — (1) Put $V = \mathbf{A}^2_R$. The set $S = \{(x,y) \in R^2 \mid (x^2+y^2 \leqslant 1) \vee (x = 0 \wedge -2 \leqslant y \leqslant 2)\}$ is not regular, but it is regular at infinity. The set $S = \{(x,y) \in R^2 \mid (x^2+y^2 \leqslant 1) \vee (x = 0)\}$ is not even regular at infinity.
(2) For every semialgebraic subset S of $V(R)$, both $\operatorname{int}(S)$ and $\operatorname{clos}(\operatorname{int}(S))$ are always regular. If V is non-singular and of dimension d, then $S \neq \varnothing$ is regular if and only if S is of pure dimension d, i.e. $\dim_x(S) = d$ for all $x \in S$ (where $\dim_x$ denotes the local semialgebraic dimension in x). Furthermore, if V is regular at infinity and $V \hookrightarrow X$ is any completion of V, then $\dim_x(\operatorname{clos}_{X(R)}(S)) = d$ holds for every $x \in \operatorname{clos}_{X(R)}(S) \cap$

$(X \smallsetminus Z)(R)$. This can be seen by taking a suitable semialgebraic cellular decomposition of $\operatorname{clos}_{X(R)}(S)$.

(3) Let $V = \mathbf{A}^2_R$, $S_1 = \{(x,y) \in R^2 \mid -1 \leqslant x \leqslant 1\}$, $S_2 = \{(x,0) \mid x \in R\}$, and $S = S_1 \cup S_2$. Then V has a good completion $V \hookrightarrow X$ with dense boundaries for S_1, namely the blow-up of $\mathbf{P}^2_R$ in a point; $X \smallsetminus V$ consists of two lines L and E, and $\operatorname{clos}_{X(R)}(S_1) \cap E(R)$ is an interval while $\operatorname{clos}_{X(R)}(S_1) \cap L(R) = \varnothing$ (see Example 1.9 (2)). But S does not have dense boundaries in X because $\operatorname{clos}_{X(R)}(S) \cap L(R)$ is a point and therefore not Zariski-dense in L. In fact, there is no way to fix this: V does not have a good completion with dense boundaries for S.

Theorem 1.22 — *Let V be a quasi-projective surface over R with at most isolated singularities. Let S be a semialgebraic subset of $V(R)$ that is regular at infinity. Then V possesses a good completion with dense boundaries for S. If, in addition, V is non-singular (resp. normal), there exists such a completion that is again non-singular (resp. normal).*

Proof — Assume first that V is real and non-singular and that S is basic closed, say $S = \mathcal{S}(h_1, \ldots, h_r)$. By Prop. 1.17, there exists a non-singular good completion X_0 of V such that all irreducible components of $E = X_0 \smallsetminus V$ are also non-singular. Let $C = \bigcup_i \overline{\mathcal{V}_V(h_i)}$ (where $\overline{\mathcal{V}_V(h_i)}$ is the Zariski-closure of $\mathcal{V}_V(h_i)$ in X_0). Now apply embedded resolution of curves in surfaces as stated in Thm. A.9 to the surface X_0, the curve $C \cup E$ and the finite set of intersection points of $C \cup E$. We obtain a birational morphism $\pi\colon X \to X_0$ with the following properties:

(1) X is non-singular and projective;
(2) $\pi|_{\pi^{-1}(V)}\colon \pi^{-1}(V) \to V$ is an isomorphism;
(3) all irreducible components of $X \smallsetminus V = \pi^{-1}(E)$ are non-singular and $X \smallsetminus V$ has only normal crossings;
(4) let $\widehat{C}$ denote the strict transform of C in X. Then all points in $\widehat{C} \cap \pi^{-1}(E)$ are also normal crossings.

Let us now convince ourselves that $V \hookrightarrow X$ has the desired properties: It is clearly a good completion of V, because X is non-singular and $X \smallsetminus V = \pi^{-1}(E)$ is a curve. To see that S has dense boundaries in X, let Z be an irreducible component of $X \smallsetminus V$ and assume that $\operatorname{clos}_{X(R)}(S) \cap Z(R)$ is non-empty. We have to show that it is Zariski-dense in Z. Since Z is non-singular, $Z(R)$ must be Zariski-dense in Z. Hence it suffices to show that $\operatorname{clos}_{X(R)}(S) \cap Z(R)$ is Zariski-dense in $Z(R)$.

Let $x \in \operatorname{clos}_{X(R)}(S) \cap Z(R)$, and let $u \in \mathcal{O}_{X,x}$ be a local equation for Z. There can be at most one other irreducible component $Z' \neq Z$ of $\widehat{C} \cup (X \smallsetminus V)$ with $x \in Z'(R)$ because x must be a normal crossing point. If such Z' exists, let $v \in \mathcal{O}_{X,x}$ be a local equation for Z'. Then u, v is a regular system of parameters for $\mathcal{O}_{X,x}$. If no such Z' exist, let $v \in \mathcal{O}_{X,x}$ be some element of $\mathcal{O}_{X,x}$ such that u, v is a regular system of parameters for $\mathcal{O}_{X,x}$.

Now choose an open neighbourhood U of x in $X(R)$ such that $U \cap Z''(R) = \varnothing$ for all irreducible components Z'' of $\widehat{C} \cup (X \smallsetminus V)$ other than Z and possibly Z'. Then the sign of h_k, $k = 1, \ldots, r$, on $V(R) \cap U$ depends only on the sign of u and v since all poles and zeros of the h_i are contained in $\widehat{C} \cup (X \smallsetminus V)$.

After intersecting U with a suitable Zariski-open subset of X, we can also assume that u and v are regular on U. Since S is regular at infinity, $\operatorname{clos}_{X(R)}(S) \cap U$ has dimension 2 (see Example 1.21 (2)). From all this we see that $\operatorname{clos}_{X(R)}(S) \cap U$ must contain a quadrant, i.e. a set of the form

$$\left\{x \in U \,\middle|\, \varepsilon u(x) \geqslant 0 \wedge \delta v(x) \geqslant 0\right\}$$

where $\varepsilon, \delta \in \{\pm 1\}$. Since u, v is a regular system of parameters and $Z(R) \cap U = \{u = 0\} \cap U$, both $Z(R) \cap \{v \geqslant 0\}$ and $Z(R) \cap \{v \leqslant 0\}$ are Zariski-dense in $Z(R)$. This shows that $\operatorname{clos}_{X(R)}(S) \cap U \cap Z(R)$ is Zariski-dense in $Z(R)$.

This completes the proof if V is real and non-singular and S is basic closed. If S is not basic closed, we may first replace S by $\operatorname{clos}_{V(R)}(S)$ (see remark 1.4 (2)), so we can assume that S is closed. By the finiteness theorem (see e.g. Bochnak, Coste, and Roy [**6**], Thm. 2.7.2), S can then be written as a union of finitely many basic closed sets. The construction used above can now be applied consecutively to each of these.

If V has isolated singularities, let $V \hookrightarrow X$ be a completion, and let $\varphi\colon \widetilde{X} \to X$ be a resolution of singularities for X (Thm. A.7). Put $\widetilde{V} = \varphi^{-1}(V)$ and $\widetilde{S} = \varphi_R^{-1}(S)$. Consider the commutative square

$$\begin{array}{ccccccc}
\widetilde{S} \subset \widetilde{V}(R) & \subset & \widetilde{V} & \longrightarrow & \widetilde{X} \\
\downarrow & & \downarrow & & \downarrow \\
S \subset V(R) & \subset & V & \longrightarrow & X.
\end{array}$$

By what we have already proved, we can construct a good completion of $\widetilde{V}$ with dense boundaries for $\widetilde{S}$ by blowing up in points of $\widetilde{X} \smallsetminus \widetilde{V}$. This completion will fit into the same commutative square in place of $\widetilde{X}$, so we may assume without loss of generality that $\widetilde{X}$ is a good completion of $\widetilde{V}$ with dense boundaries for

$\widetilde{S}$. Since V_{sing} is finite (and in particular $\overline{V}_{\text{sing}} \cap (X \setminus V) = \varnothing$), the divisor $\varphi^{-1}(y)$ of $\widetilde{X}$ is contractible for every $y \in V_{\text{sing}}$, so there exists $\psi\colon \widetilde{X} \to X'$, X' projective, such that $\psi(\widetilde{V}) \cong V$ and $\psi(\widetilde{X} \setminus \varphi^{-1}(V_{\text{sing}})) \xrightarrow{\sim} (X \setminus V_{\text{sing}})$ so that X' is a good completion of V with dense boundaries for S.

Finally, if V is not real, then $V(R)$ and hence S are contained in V_{sing}, so S is finite. Therefore, any good completion of V (which exists by the construction in the preceding paragraph) has dense boundaries for S. The additional claim about non-singularity resp. normality of X is clear from the construction. □

We have shown that under certain conditions a ring of bounded functions can be represented as the ring of regular functions of a quasi-projective variety. The converse also holds. To show this, we need the following

Lemma 1.23 — *Let V be a non-singular, irreducible, affine R-variety, $V \hookrightarrow X$ a good completion of V, and $Z_1, \dots, Z_r$ the irreducible components of $X \setminus V$. Let $s \geqslant 0$, and assume that $Z_1, \dots, Z_s$ are real. Then there exists a regular, basic closed subset S of $V(R)$ such that* $\operatorname{clos}_{X(R)}(S) \cap Z_i(R)$ *is Zariski-dense in Z_i for all $i \leqslant s$ and* $\operatorname{clos}_{X(R)}(S) \cap Z_i(R) = \varnothing$ *for all $i > s$.*

Proof — Let U be an open affine subvariety of X with $U(R) = X(R)$ (see Lemma A.2), and fix an embedding of U into affine space with coordinates $x_1, \dots, x_n$. If $s = 0$, there is nothing to show. Otherwise, let $1 \leqslant i \leqslant s$ and write $E_i = \bigcup_{j \neq i} Z_j$. We will first show that there exists $h_i \in R[V]$ such that $\operatorname{clos}_{X(R)}(\mathcal{S}_V(h_i)) \cap Z_i(R)$ is Zariski-dense and $\operatorname{clos}_{X(R)}(\mathcal{S}_V(h_i)) \cap E_i(R) = \varnothing$. Let $(a_1, \dots, a_n) \in Z_i(R) \setminus E_i(R)$ be a non-singular point of Z_i, and for every $\varepsilon > 0$ in R, let $g_{i,\varepsilon}$ be the class of $(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 - \varepsilon$ in $R[U]$, so that $\mathcal{S}(g_{i,\varepsilon})$ is the intersection of $U(R)$ with a closed ball of radius ε around $(a_1, \dots, a_n)$. Choose $\varepsilon > 0$ small enough, such that $\mathcal{S}_U(g_i) \cap E_i(R) = \varnothing$ for $g_i = g_{i,\varepsilon}$. On the other hand, $\mathcal{S}_U(g_i) \cap Z_i(R)$ is Zariski-dense in Z_i. Since $U(R) = X(R)$, there exists $t_i \in R[V]$ such that $t_i^2 g_i \in R[V]$ and t_i has no zeros on $V(R)$. Put $h_i = t_i^2 g_i$, then $\mathcal{S}_V(h_i)$ behaves as desired.

Now choose h_i as above for each $1 \leqslant i \leqslant s$. We may assume that the $\mathcal{S}(h_i)$ are pairwise disjoint, after choosing balls of smaller radius if necessary. This implies $\bigcup_i \mathcal{S}(h_i) = \mathcal{S}((-1)^{r+1} h_1 \cdots h_r)$, so $S = \bigcup_i \mathcal{S}(h_i)$ is basic closed and has the desired properties. □

Theorem 1.24 — *Let W be a non-singular, irreducible, real, quasi-projective R-variety. Then there exists an affine open subvariety V of W and a regular, basic* *closed subset S of $V(R)$ such that*

$$B_V(S) \cong \mathcal{O}_W(W).$$

Proof — Let $W \hookrightarrow X$ be a good completion of W such that X is again non-singular (see Prop. 1.17). Since W is real, there exists a real divisor Z of X such that $V = W \smallsetminus (Z \cap W)$ is affine. Let $Z_1, \ldots, Z_s$ be the irreducible components of Z and $Z_{s+1}, \ldots, Z_r$ those of $X \smallsetminus W$. By the preceding lemma, there exists a basic closed subset S of $V(R)$ such that $\operatorname{clos}_{X(R)}(S) \cap Z_i$ is Zariski-dense in Z_i for $1 \leqslant i \leqslant s$ and empty otherwise. Therefore, we have $B_V(S) = \mathcal{O}_W(W)$ by Prop. 1.12. □

1.6. FINITE GENERATION OF THE RING OF BOUNDED FUNCTIONS

In this section, we ask when the ring of bounded functions is a finitely generated R-algebra. The case of curves is very simple:

Proposition 1.25 — *Let C be an irreducible curve over R and S a semialgebraic subset of $C(R)$. Then $B_C(S)$ is a finitely generated R-algebra.*

Proof — By Prop. 1.13, there exists a quasi-projective curve C' such that $B_C(S) = \mathcal{O}_{C'}(C')$. But every quasi-projective curve is either affine or projective (an easy consequence of the Riemann-Roch theorem; see Hartshorne [9], ch. IV, ex. 1.1), so $\mathcal{O}_{C'}(C')$ is a finitely generated R-algebra. □

But in higher dimensions, the situation is again much more complicated:

Example 1.26 — Let V be the affine plane over R and S a semialgebraic subset of R^2. If S is not regular at infinity, $B_V(S)$ need not be a finitely generated R-algebra. For example, let $S = \mathcal{S}(x^2(1 - x^2 - y^2))$. Clearly, $B_V(S)$ consists of all polynomials that are bounded and hence constant on the y-axis. Therefore, it is the subalgebra of $R[x, y]$ generated by all monomials xy^i, $i \geqslant 0$. This cannot be finitely generated over R, for the corresponding lattice $\{(i, j) \mid i \geqslant 1, j \geqslant 0\}$ in $\mathbf{N}_0^2$ is not finitely generated over $\mathbf{N}_0$.

However, this cannot happen if V is a normal surface and S is regular at infinity:

Theorem 1.27 (Zariski) — *Let W be a normal, quasi-projective surface over a field k of characteristic zero. Then the ring of regular functions $\mathcal{O}_W(W)$ is a finitely generated k-algebra.*

Proof — See Zariski [40], remarks on Thm. 1 at the end of the article. See Krug [13] for an alternative proof in modern language, based on methods by Nagata. □

Corollary 1.28 — *Let V be a normal, quasi-projective surface over R, and let S a semialgebraic subset of $V(R)$ that is regular at infinity. Then the ring of bounded functions $B_V(S)$ is a finitely generated R-algebra.*

Proof — Indeed, under the hypotheses, V possesses a normal good completion with dense boundaries for S by Thm. 1.22, hence $B_V(S)$ is the ring of regular functions of a normal quasi-projective surface by Prop. 1.12 which is finitely generated over R by the above theorem. □

Zariski's theorem and our corollary do not hold in higher dimensions: For example, there exist non-singular quasi-projective threefolds over R (or indeed over any field that is not algebraic over a finite field) whose ring of regular functions is not a finitely generated R-algebra. By Thm. 1.24, such a ring is also a ring of bounded functions of a suitable regular, basic closed subset of some non-singular affine R-variety. Examples of such quasi-projective varieties can be constructed in various ways but are not entirely elementary; see for example Krug [13]; in an unpublished note [37], Vakil has given an example that is the total space of a vector bundle of rank 2 over an elliptic curve.

If the semialgebraic set S is not regular at infinity, then at least the interior $\text{int}(S)$ of S is regular, and one can hope to understand the ring $B_V(S)$ by first considering $B_V(\text{int}(S))$ and then proceed inductively. We need the following lemma (see also [4], Prop. 2.3):

Lemma 1.29 — *Let A be a domain, and B a subring of A that is integrally closed in A. If I is a non-trivial ideal of A contained in B, then* $\text{qf}(A) = \text{qf}(B)$ *and either $B = A$ or B is not noetherian.*

Proof — Let $c \in I$, $c \neq 0$. Then for any $a/b \in \text{qf}(A)$, we have $a/b = ca/cb \in \text{qf}(B)$; hence $\text{qf}(A) = \text{qf}(B)$. Now suppose that B is noetherian. Since A is a domain and $I \neq (0)$, we have an embedding $\lambda\colon A \hookrightarrow \text{End}_B(I)$ given by multiplication from the left. Let $a \in A$. Since B is noetherian, I is a finitely generated B-module. By the Cayley-Hamilton theorem, there exists a monic polynomial f with coefficients in B such that $f(\lambda(a)) = 0$ in $\text{End}_B(I)$. Since $f(\lambda(a)) = \lambda(f(a))$, we have $f(a) = 0$ in A. Therefore, a is integral over B, hence $a \in B$. □

Proposition 1.30 — *Let V be an irreducible affine R-variety, S a semialgebraic subset of $V(R)$ and $S_0 = \operatorname{int}(S)$. Assume that* $\dim(S \setminus S_0) < \operatorname{trdeg} B_V(S_0)$. *Then $B_V(S) = B_V(S_0)$ or $B_V(S)$ is not noetherian.*

Proof — Let $S' = S \setminus S_0$, and let $I = \{f \in B_V(S_0) \mid \forall x \in S' : f(x) = 0\}$. If $S' = \varnothing$, the claim is trivial. So we may assume that $S' \neq \varnothing$, hence $I \neq B_V(S_0)$. We claim that $I \neq (0)$. Indeed, let $W = \operatorname{Spec}(B_V(S_0))$, and let $\varphi\colon V \to W$ be the canonical bounded morphism. Then I is the vanishing ideal of $\varphi_R(S')$. But $\dim(S') < \operatorname{trdeg} B_V(S_0)$ by hypothesis, hence $\varphi_R(S')$ cannot be Zariski-dense in W which shows $I \neq (0)$. Now the preceding lemma applies (with $A = B_V(S_0)$, $B = B_V(S)$), since obviously $I \subset B_V(S)$ and $B_V(S)$ is integrally closed in $R[V]$, hence also in $B_V(S_0)$. □

Corollary 1.31 — *Let V be an irreducible affine R-variety, S a semialgebraic subset of $V(R)$. If* $\dim S < \dim V$, *then $B_V(S) = R[V]$ or $B_V(S)$ is not noetherian.*

Proof — Either V is real, then $\dim V(R) = \dim V$. In this case, $\operatorname{int}(S) = \varnothing$, so $\dim S < \dim V = \operatorname{trdeg} R[V] = \operatorname{trdeg} B_V(\operatorname{int}(S))$, and the proposition applies. Or V is non-real, then there is a non-zero ideal $I \subset R[V]$ such that $V(R) \subset \mathcal{V}(I)$ by Prop. A.1 which implies $I \subset B_V(S)$. Thus $B_V(S) = R[V]$ or $B_V(S)$ is not noetherian by the lemma above. □

Corollary 1.32 — *Let V be an irreducible, normal, real, affine surface over R, S a semialgebraic subset of $V(R)$ and $S_0 = \operatorname{int}(S)$. Then $B_V(S)$ is a finitely generated R-algebra if and only if* $\operatorname{trdeg}(B_V(S_0)) \leqslant 1$ *or $B_V(S) = B_V(S_0)$.*

Proof — We do sufficiency first: By Cor. 1.28 above, $B_V(S_0)$ is finitely generated. If $\operatorname{trdeg}(B_V(S_0)) = 0$, then $B_V(S) = B_V(S_0) = R$, so there is nothing to show. If $\operatorname{trdeg}(B_V(S_0)) = 1$, then $W = \operatorname{Spec}(B_V(S_0))$ is a curve over R and $B_V(S) = B_W(\varphi_R(S \setminus S_0))$, where $\varphi\colon V \to W$ is the canonical bounded morphism. Therefore, $B_V(S)$ is finitely generated by 1.25. For necessity, assume that $\operatorname{trdeg}(B_V(S_0)) = 2$ and $B_V(S)$ is finitely generated. Then $B_V(S) = B_V(S_0)$ by the proposition. □

Remark 1.33 — If S is regular at infinity, then clearly $B_V(S) = B_V(S_0)$. The converse, however, is not true, even if $\operatorname{trdeg}(B_V(S)) = 2$. For example, let $V = \mathbf{A}^2_R$ with coordinates u, v, and put $S = \mathcal{S}(u, 1-u, u^2(uv-1), u^2(1-u^2v))$, then $B_V(S) = B_V(S_0) = R[u, uv]$ but S is not regular at infinity.

CHAPTER 2

POSITIVITY AND SUMS OF SQUARES

2.1. PREORDERINGS, POSITIVITY, AND THE MOMENT PROBLEM

In this section, we set up the basic terminology concerning preorderings of a ring and give precise formulations of the problems and known results described in the introduction. Let A be a ring. A subset T of A is called a *preordering*[1] if it has the following properties:

(1) $T + T \subset T$ and $T \cdot T \subset T$;
(2) $a^2 \in T$ for all $a \in A$.

The set $\sum A^2$ of all sums of squares of elements of A is obviously a preordering which is contained in every preordering of A. If A contains $\frac{1}{2}$ and T is a preordering of A, then $A = T - T$, and the set $T \cap (-T)$ is an ideal of A, called the *support of* T and denoted by $\mathrm{supp}(T)$. To prove both of these facts, note that every $a \in A$ can be written as $a = \left(\frac{a+1}{2}\right)^2 - \left(\frac{a-1}{2}\right)^2$ which proves $A = T - T$. For every $t \in T \cap (-T)$ we have $at = \left(\frac{a+1}{2}\right)^2 t - \left(\frac{a-1}{2}\right)^2 t \in T \cap (-T)$, so $T \cap (-T)$ is an ideal.

Definition — Let A be a ring and $\mathcal{H}$ a subset of A. We denote by $\mathrm{PO}_A(\mathcal{H})$ the intersection of all preorderings of A containing $\mathcal{H}$, the *preordering generated by* $\mathcal{H}$. We say that a preordering T of A is *finitely generated* if there exists a finite subset $\mathcal{H}$ of A such that $T = \mathrm{PO}_A(\mathcal{H})$. We will drop A from the notation when no confusion is likely to arise. For $h_1, \ldots, h_r \in A$, we write $\mathrm{PO}_A(h_1, \ldots, h_r)$ instead of $\mathrm{PO}_A(\{h_1, \ldots, h_r\})$, and using the notation $\underline{h}^i = h_1^{i_1} \cdots h_r^{i_r}$, $i \in \{0,1\}^r$,

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[1] The letter T is commonly used for preorderings in the literature. The origin of this notation is not known to me.

it is easy to see that

$$\mathrm{PO}_A(h_1,\ldots,h_r) = \left\{ \sum_{i\in\{0,1\}^r} s_i \underline{h}^i;\ s_i \in \Sigma A^2 \right\}.$$

Consider now the following geometric situation: Let V be an affine R-variety, let $h_1,\ldots,h_r \in R[V]$, and put $S = \mathcal{S}_V(h_1,\ldots,h_r)$, $T = \mathrm{PO}(h_1,\ldots,h_r)$. The first question is whether the equivalence

$$\forall x \in S\colon f(x) \geqslant 0 \quad \Longleftrightarrow \quad f \in T$$

holds for every $f \in R[V]$. If this is the case, then T is called *saturated*. Or, starting from S, one may ask whether the saturated preordering

$$\mathcal{P}_V(S) = \{f \in R[V] \mid f(x) \geqslant 0 \text{ for all } x \in S\}$$

is finitely generated and how to find the generators if it is. Unfortunately, $\mathcal{P}_V(S)$ is rarely finitely generated: Using the existence of non-negative homogeneous polynomials in three variables that are not sums of squares of forms, Scheiderer has constructed a local obstruction to membership in the preordering $\mathcal{P}_V(S)$ if S has dimension at least 3, giving the following result:

Theorem 2.1 (Scheiderer [27], Cor. 1.3) — *Let V be an affine R-variety and S a basic closed subset of $V(R)$ of dimension at least 3. Then the preordering $\mathcal{P}_V(S)$ is not finitely generated.*

The case when S is of dimension at most 2 remains, however, and has been extensively studied by Scheiderer ([27], [28], [30]), and by Kuhlmann, Marshall, and Schwartz ([14],[15]). We will also be concerned with saturatedness of preorderings on curves and surfaces and shall extend a few of these results later on. On the other hand, one can study weaker conditions than saturatedness. The following result by Schmüdgen was the starting point for many new variations of Positivstellensätze:

Theorem 2.2 (Schmüdgen's Positivstellensatz [31]) — *Let V be an affine $\mathbf{R}$-variety, and T a finitely generated preordering of $\mathbf{R}[V]$ such that the set $S = \mathcal{S}(T)$ is compact. Then T contains every $f \in \mathbf{R}[V]$ that is strictly positive on S.*

Schmüdgen's original proof is of an analytical nature, but meanwhile there exist several different proofs that are purely algebraic and various refined versions of Schmüdgen's theorem; see for example Berr and Wörmann [5], Prestel and Delzell [24] (Thm. 6.3.6, Thm. 8.3.3), Schweighofer [34] and [33].

The context in which Schmüdgen proved his theorem was the moment problem in functional analysis: Given a closed subset S of $\mathbf{R}^n$ and a linear functional $L: \mathbf{R}[x_1, \ldots, x_n] \to \mathbf{R}$ on the polynomial ring, the existence part of the moment problem asks whether there exists a positive Borel measure μ supported on K such that $L(f) = \int_S f d\mu$ holds for all $f \in \mathbf{R}[x_1, \ldots, x_n]$. However, we will work exclusively in an algebraic setup and refer the reader to Schmüdgen [**31**], to Kuhlmann and Marshall [**14**], and to Powers and Scheiderer [**23**] for the connection between the analytical and the algebraic world. For our purposes, the moment problem shall be the question whether a finitely generated preordering T of $R[V]$ has the (strong) moment property, defined as follows: Consider the *natural linear topology on* $R[V]$ in which a subset M of $R[V]$ is called closed if and only if $M \cap U$ is closed in U for every finite dimensional subspace U of $R[V]$ (where on U we use the strong topology induced by the ordering of R, the unique vector space topology).

Definition — Let V be an affine R-variety, let T be a preordering of $R[V]$, and put $S = \mathcal{S}(T)$. Then T is said to have the *strong moment property* (SMP for short) if T is dense in $\mathcal{P}_V(S)$ with respect to the natural linear topology on $R[V]$. It is said to have the *moment property* (MP) if the closure of T contains all $f \in R[V]$ that are non-negative on $V(R)$.

If T has the (strong) moment property, one says that T *solves the (strong) moment problem for* S. Starting from S, one says that the *(strong) moment problem for S is finitely solvable* if there exists a finitely generated preordering T of $R[V]$ such that $S = \mathcal{S}(T)$ and such that T has the (strong) moment property. The question whether any of these conditions is satisfied is collectively referred to as the *moment problem*. Note that the moment problem in this form makes sense over any real closed field while the connection to analysis exists only for $R = \mathbf{R}$. Another distinct feature of the case $R = \mathbf{R}$ is that one can use the Hahn-Banach separation theorem for convex cones to characterise the closure of T, namely it consists of all elements that cannot be separated from T by any linear form:

$$\overline{T} = \left\{ f \in \mathbf{R}[V] \,\middle|\, \forall L \in \mathbf{R}[V]^\vee : (L|_T \geqslant 0 \Longrightarrow L(f) \geqslant 0) \right\},$$

where $\mathbf{R}[V]^\vee$ is the vector space of all linear forms $\mathbf{R}[V] \to \mathbf{R}$. Now Schmüdgen's Positivstellensatz implies that a finitely generated preordering T has the strong moment property whenever $\mathcal{S}(T)$ is compact. For given a linear form $L: \mathbf{R}[V] \to \mathbf{R}$ such that $L|_T \geqslant 0$, and given $f \in \mathcal{P}_V(S)$, the Positivstellensatz says that $f + \varepsilon \in T$ and therefore $L(f + \varepsilon) \geqslant 0$ hold for every $\varepsilon > 0$. From $L(f) = L(f + \varepsilon) - \varepsilon L(1)$ it follows that $L(f) \geqslant 0$, as desired.

We cite some basic facts from Scheiderer [**29**]:

Lemma 2.3 — *Let V be an affine R-variety, I an ideal of $R[V]$, and T a finitely generated preordering of $R[V]$.*

(1) *If (SMP) holds for T, then it also holds for $T + I$.*
(2) *The following are equivalent:*
  (a) *(SMP) holds for $T + I$.*
  (b) *(SMP) holds for $T + \sqrt{I}$.*
  (c) *(SMP) holds for $T + \sqrt[re]{I}$.*
  (d) *(SMP) holds for $(T + I)/I \subset A/I$.*

Proof — see Scheiderer [**29**]: The statement in (1) is Prop. 4.6 *ibid.*. In (2), (a) and (d) are equivalent by Lemma 4.5 *ibid.*, since $I \subset \operatorname{supp}(T + I)$. For the equivalence of (a), (b) and (c), note that $\mathcal{S}(T + I) = \mathcal{S}(T + \sqrt{I}) = \mathcal{S}(T + \sqrt[re]{I}) = \mathcal{S}(T) \cap \mathcal{V}(I)(R)$ which shows (a)⇒(b)⇒(c). Conversely, if (SMP) holds for $T + \sqrt{I}$ or $T + \sqrt[re]{I}$, then the same is true for $T + \sqrt{\operatorname{supp}(T + I)} = T + \sqrt[re]{\operatorname{supp}(T + I)}$, thus (SMP) holds for $T + I$ by Cor. 4.7 *ibid.*. □

In the situation of the lemma, write $Z = \mathcal{V}(I)$ for the closed subscheme of V determined by I. We denote the preordering $(T + I)/I$ of $R[Z] = R[V]/I$ by $T|_Z$ and call it the *restriction of T to Z*. Part (1) of the lemma says that the strong moment property is preserved under restriction to a closed subscheme. Part (2) says that we may replace Z by the subvariety (reduced subscheme) $\mathcal{V}(\sqrt{I})$ and even the real reduced subvariety $\mathcal{V}(\sqrt[re]{I})$. In particular, a necessary condition for T to have the strong moment property is that $T|_Z$ should have the strong moment property for all closed subvarieties $Z \subsetneq V$. This condition is not sufficient, but the following recent theorem of Schmüdgen says that it is sufficient if V can be fibred by a bounded polynomial map:

Theorem 2.4 (Schmüdgen's fibre theorem [**32**]**)** — *Let V be an affine $\mathbf{R}$-variety, T a finitely generated preordering of $\mathbf{R}[V]$ and $S = \mathcal{S}(T)$. Assume that $\varphi\colon V \to \mathbf{A}^m$ is a polynomial map such that the closure of $\varphi(S)$ in $\mathbf{R}^m$ is compact. If (SMP) holds for $T|_{\varphi^{-1}(a)}$ for all $a \in \mathbf{R}^m$, then (SMP) holds for T itself.*

Here, $\varphi^{-1}(a)$ denotes the reduced fibre, i.e. if we write $a = (a_1, \dots, a_m)$, $\varphi = (\varphi_1, \dots, \varphi_m)$ with $\varphi_i \in B_V(\mathcal{S}(T))$, and $I_a = (\varphi_i - a_i \mid i \in \{1, \dots, m\})$, then $\varphi^{-1}(a) = \mathcal{V}(\sqrt{I_a})$ so that $T|_{\varphi^{-1}(a)} = (T + \sqrt{I_a})/\sqrt{I_a}$. Schmüdgen states his theorem for $T + I_a$, and one has to use Lemma 2.3 to pass from his version to the one above. (By the preceding discussion, we could in fact also take $\varphi^{-1}(a)$ to be the real reduced fibre $\mathcal{V}(\sqrt[re]{I_a})$.)

We do not prove the theorem here. Schmüdgen's proof relies heavily on results from operator theory. In contrast to his Positivstellensatz, no purely algebraic proof is known for the fibre theorem so far. However, Marshall and Netzer have (independently) developed much more elementary proofs, using only the Radon-Nikodym theorem; see Netzer [**21**].

Example 2.5 — Let W be the plane circle $\mathcal{V}(x^2+y^2-1) \subset \mathbf{A}^2_R$ and $V = W \times \mathbf{A}^1_R$ a cylinder. Schmüdgen's fibre theorem tells us that the preordering $T = \sum \mathbf{R}[V]^2$ has the strong moment property: The image of $V(\mathbf{R})$ under the projection $\pi: V \to W$ onto the first factor is compact, and all fibres of π are copies of $\mathbf{A}^1_R$, so $\mathbf{R}[\pi^{-1}(a)] = \mathbf{R}[t_a]$ is a polynomial ring in one variable for each $a \in W(\mathbf{R})$ and $T|_{\pi^{-1}(a)} = \sum \mathbf{R}[t_a]^2$. Since every non-negative polynomial in one variable is a sum of (two) squares, we see that $T|_{\pi^{-1}(a)}$ has the strong moment property for all $a \in W(\mathbf{R})$.

In fact, one can show more in this kind of example with more explicit methods: Kuhlmann and Marshall have proved that if $S = \mathcal{S}(T)$ is a cylinder with compact base, then for every $f \in \mathcal{P}_V(S)$ there exists $q \in \mathbf{R}[V]$ such that $f + \varepsilon q \in T$ holds for every $\varepsilon > 0$ (Thm. 5.1 in [**14**]). In particular, it follows easily that T has the strong moment property. We sketch the idea of their proof in the above example as an illustration to the general fibre theorem:

Let $f = \sum^n f_i t^i \in \mathbf{R}[V] = \mathbf{R}[W][t]$, $f_n \neq 0$, and assume $f \geqslant 0$ on $V(\mathbf{R})$. Note first that this implies that $f_n \geqslant 0$ on $W(\mathbf{R})$ and $n = 2k$ must be even. For every $\varepsilon > 0$, put $g_\varepsilon = f + \varepsilon t^n + \varepsilon$, then g_ε is strictly positive on $V(\mathbf{R})$, and the leading coefficient of g_ε is strictly positive on $W(\mathbf{R})$. Now for every $w \in W(\mathbf{R})$, $g_\varepsilon(w,t) \in \mathbf{R}[t]$ factorizes over $\mathbf{C}$ as

$$g_\varepsilon(w,t) = c(w)^2 \prod_{i=1}^{k} (t - \alpha_i(w))(t - \overline{\alpha_i(w)})$$

where $\alpha_1, \ldots, \alpha_k$ are continuous semialgebraic functions $W(\mathbf{R}) \to \mathbf{C}$ and c is a continuous semialgebraic function $W(\mathbf{R}) \to \mathbf{R}$. Put $\widetilde{g}_\varepsilon = c \prod_{i=1}^k (t - \alpha_i)$, then $g_\varepsilon = (\operatorname{Re} \widetilde{g}_\varepsilon)^2 + (\operatorname{Im} \widetilde{g}_\varepsilon)^2$. Now use Weierstraß approximation for the semialgebraic function $\widetilde{g}_\varepsilon$ on the compact set $W(\mathbf{R})$ to get an approximate sums of squares representation of f. The only remaining difficulty is to control the dependence on ε to get something of the form $f + \varepsilon q$ for fixed q. Kuhlmann and Marshall achieve this with a neat application of Schmüdgen's Positivstellensatz.

Whether $\sum \mathbf{R}[V]^2$ is in fact saturated, i.e. whether every non-negative function on $V(\mathbf{R})$ is a sum of squares, remains unknown, even for strict positivity in place of non-negativity. An analogous open question is whether the preordering $T = \mathrm{PO}(1-x^2)$ in $\mathbf{R}[x,y]$ is saturated, or contains at least every polynomial that is strictly positive on $[-1,1] \times \mathbf{R}$. (*Added in proof*: This problem has just been resolved by Marshall who has shown that T is saturated.)

2.2. STABLE PREORDERINGS

A finitely generated preordering $T = \mathrm{PO}(h_1, \dots, h_r)$ of the polynomial ring $R[x_1, \dots, x_n]$ is called stable if all elements $f \in T$ have representations $f = \sum s_i \underline{h}^i$ in T such that the degree of the sums of squares s_i can be bound in terms of the degree of f. The question for such degree bounds was first considered by Stengle (to the knowledge of the author) in a concrete example, namely the preordering $\mathrm{PO}((1-x^2)^3)$ of the polynomial ring in one variable for which he proved (among other things) that such bounds cannot exist ([**36**]). The notion of stability in the context of finitely generated algebras was first defined in Powers and Scheiderer [**23**], though similar conditions have been used before in the study of operator algebras. It is used in connection with the moment problem to show that a finitely generated preordering is closed in the natural linear topology and hence can have the moment property only if it is saturated. This relation will be explained below.

Let k be a field, and let A be a finitely generated k-algebra. By a subspace of A we will always mean a k-linear subspace of the k-vector space A. If U is a subspace of A, we write ΣU^2 for the set of all sums of squares $\sum_i u_i^2$ of elements $u_i \in U$. Let $(\Gamma, +, \leqslant)$ be an ordered abelian semigroup. A Γ-*filtration* of A is a collection $\mathfrak{U} = \{U_c\}_{c\in\Gamma}$ of subspaces of A such that $U_c \subset U_d$ whenever $c \leqslant d$, $A = \bigcup_{c\in\Gamma} U_c$ and $U_c \cdot U_d \subset U_{c+d}$ for all $c, d \in \Gamma$.

Examples — (1) Any Γ-graduation $A = \bigoplus_{c\in\Gamma} A_c$ of A induces a natural Γ-filtration by putting $U_c = \sum_{d\leqslant c} A_d$.

(2) If A is integral, Γ a group, and $v\colon \mathrm{qf}(A)^\times \to \Gamma$ a valuation on the quotient field of A with value group Γ, then $U_c = \{f \in A \mid v(f) \geqslant -c\}$, $c \in \Gamma$, is a Γ-filtration of A.

Lemma and Definition — Let A be a finitely generated k-algebra, $\mathfrak{U} = \{U_c\}_{c\in\Gamma}$ a Γ-filtration of A, and let $T = \mathrm{PO}(h_1, \dots, h_r)$ be a finitely generated preordering of A.

(1) We say that T is *quasi-stable with respect to* $\mathfrak{U}$ if there exists a function $\delta\colon \Gamma \to \Gamma$ such that for every $c \in \Gamma$ and any element $f \in T \cap U_c$, there exists an expression

$$f = \sum_{i \in \{0,1\}^r} s_i \underline{h}^i$$

with $s_i \in \Sigma U^2_{\delta(c)}$ for all i. The function δ is called a *degree bound for T with respect to* $\mathfrak{U}$.

(2) We say that T is *stable* if all subspaces U_c in $\mathfrak{U}$ are finite-dimensional over k and T is quasi-stable with respect to $\mathfrak{U}$.

These notions are well-defined as stated: Quasi-stability depends only on T and $\mathfrak{U}$, but not on the choice of generators $h_1, \ldots, h_r$ of T. Stability depends only on T, not on Γ and the filtration $\mathfrak{U}$ by finite-dimensional subspaces.

Proof — For the independence from the choice of generators, it suffices to show the following: For every $g \in T$, the preordering T satisfies the quasi-stability condition with respect to $\mathfrak{U}$ for the generators $h_1, \ldots, h_r$ if and only if the same is true for the generators $h_1, \ldots, h_r, g$. Clearly, if T is quasi-stable w.r.t. $\mathfrak{U}$ for $h_1, \ldots, h_r$, then it is also quasi-stable w.r.t. $\mathfrak{U}$ for $h_1, \ldots, h_r, g$. For the converse, assume that T is quasi-stable for $h_1, \ldots, h_r, g$. Let $\delta\colon \Gamma \to \Gamma$ be such that every $f \in T \cap U_c$ has a representation $f = \sum s_i \underline{h}^i + \sum t_i \underline{h}^i g$ with $s_i, t_i \in \Sigma U^2_{\delta(c)}$. Write $g = \sum_i u_i \underline{h}^i$ and choose $d \in \Gamma$ with $u_i \in \Sigma U^2_d$ and $h_i \in U_d$ for all $i \in \{0,1\}^r$. Then $f = \sum_i (s_i + \sum_j t_i u_j \underline{h}^j) \underline{h}^i$ which, after extracting all squares contained in the $\underline{h}^i \underline{h}^j$, can be rewritten as $\sum v_i \underline{h}^i$ with $v_i \in \Sigma U^2_{\delta(c)+2d}$.

If T is stable for some Γ-filtration $\mathfrak{U} = \{U_c\}_{c \in \Gamma}$, suppose that Γ' is another ordered abelian semigroup and that $\mathfrak{V} = \{V_c\}_{c \in \Gamma'}$ is a Γ'-filtration of A with all V_c finite-dimensional over k. Then T is stable for $\mathfrak{V}$ because every subspace U_c, $c \in \Gamma$, is contained in some V_d, $d \in \Gamma'$. □

Example — Let k be a real field, and let $A = k[x_1, \ldots, x_n]$ be the polynomial ring. Then the preordering $\sum A^2$ is stable. For if $f = \sum s_i^2$, then the total degree of f is even and $\deg s_i \leqslant \frac{1}{2}\deg(f)$ because the leading coefficients of the s_i cannot all cancel out since k is real. Thus $\sum A^2$ is quasi-stable with respect to the natural filtration of A by degrees.

This example has a powerful generalisation due to Powers and Scheiderer where the degree of a polynomial is replaced by suitable valuations. In a somewhat different direction, Netzer has given many new examples of stable preorderings in the polynomial ring (see [20]) by refining the degree to more general graduations adapt to the situation. Recall that a valuation $v\colon K^\times \to \Gamma$ of a field K is called compatible with an ordering P of K if $a \leqslant_P b$ implies $v(a) \geqslant v(b)$ for all $a, b \in K^\times$. This is equivalent to saying that P induces an ordering on the residue field of v. The fact that leading terms cannot cancel in the above example in the polynomial ring corresponds to the following simple fact concerning compatible valuations:

Lemma 2.6 — *Let K be an ordered field equipped with a valuation v that is compatible with the ordering of K. If $a_1, \dots, a_r \in K$ have the same sign, then*

$$v(a_1 + \cdots + a_r) = \min\{v(a_1), \dots, v(a_r)\}.$$

Proof — Since $a_1, \dots, a_r$ have the same sign in K, we have $|a_i| \leqslant |\sum a_j|$ and hence $v(\sum a_j) \leqslant v(a_i)$ for all i. The opposite inequality holds trivially. □

Corollary 2.7 — *Let k be a real field equipped with a real valuation v, and let $a_1, \dots, a_r \in k$. Then $v(a_1^2 + \cdots + a_r^2) = 2\min\{v(a_1), \dots, v(a_r)\}$.*

Proof — By the Baer-Krull theorem (see e.g. Bochnak, Coste, and Roy [**6**], Thm. 10.1.10), there exists an ordering of k that is compatible with the valuation v. Now the lemma applies. □

Proposition 2.8 — *Let V be an irreducible, affine R-variety, and let v be a discrete valuation of $R(V)$ with real residue field. For all $n \geqslant 0$, write*

$$U_n = \left\{ f \in R[V] \,\middle|\, v(f) \geqslant -n \right\}.$$

Let P be an ordering of $R(V)$ that is compatible with v. Then for all $h_1, \dots, h_r \in P \cap R[V]$, the preordering $\mathrm{PO}(h_1, \dots, h_r)$ is quasi-stable with respect to the filtration $\{U_n\}_{n\geqslant 0}$ of $R[V]$.

Proof — Let $n \geqslant 0$ and assume that $f = \sum_{i=1}^r (s_{i,1}^2 + \cdots + s_{i,p_i}^2)\underline{h}^i \in T \cap U_n$. Then $v(f) = \min_{i,j}\{2v(s_{i,j}) + v(h_i)\}$ by Lemma 2.6, so for all i, j we get

$$v(s_{i,j}) \geqslant \frac{1}{2}\big(v(f) - m\big),$$

where $m = \max_{i,l}\big\{v(h_i)\big\}$, so that $\delta\colon n \mapsto \left\lceil \frac{1}{2}(n-m) \right\rceil$ is a degree bound for T with respect to $\{U_n\}_{n\geqslant 0}$. □

This reduces the task of showing the stability of a given preordering of $R[V]$ to finding suitable valuations and compatible orderings and ensuring that the resulting spaces U_n are finite-dimensional. From a geometric point of view, the total degree of a polynomial in several variables corresponds to a discrete valuation of the rational function field supported on the hyperplane at infinity. This generalises to the following

Theorem 2.9 — *Let V be an irreducible, affine R-variety, let $h_1, \dots, h_r \in R[V]$, and put $S = \mathcal{S}_V(h_1, \dots, h_r)$ and $T = PO(h_1, \dots, h_r)$. Assume that V possesses a good completion $V \hookrightarrow X$ with dense boundaries for S. Let $Z_1, \dots, Z_r$ be all irreducible components of $X \setminus V$ such that* $\mathrm{clos}_{X(R)}(S) \cap Z_i(R) \neq \varnothing$*, and let v_i be the valuation of $R(V)$ corresponding to Z_i. Then T is quasi-stable with respect to the filtration*

$$U_n = \{ f \in R[V] \mid v_i(f) \geqslant -n \text{ for all } i \in \{1, \dots, r\} \}.$$

of $R[V]$.

Proof — In view of the preceding proposition, it will suffice to show the following: Let Z be an irreducible component of $X \setminus V$ such that $\mathrm{clos}_{X(R)}(S) \cap Z(R)$ is Zariski-dense in Z. Then there exists an ordering P of $R(V)$ that is compatible with the valuation v_Z and such that $h_i \in P$ holds for all $i \in \{1, \dots, r\}$.

To show this, choose an affine open subvariety U of X such that $Z' = Z \cap U$ is non-empty, and write U_r for the real spectrum $\mathrm{Sper}(R[U])$ corresponding to U. We refer to Bochnak, Coste, and Roy [**6**] for a number of results about the real spectrum. Let $\widetilde{S}$ denote the constructible subset of U_r associated with S. We have $\mathrm{clos}_{U_r}(\widetilde{S}) = \widetilde{\mathrm{clos}_{U(R)}(S)}$ (see *ibid.* Thm. 7.2.3). Now $\mathrm{clos}_{U(R)}(S) \cap Z'(R)$ is Zariski-dense in Z' which implies the existence of a point $\beta \in Z'_r \cap \mathrm{clos}_{U_r}(\widetilde{S})$ that is supported on the generic point of Z'. Since $\beta \in \mathrm{clos}_{U_r}(\widetilde{S})$, it follows that there exists $\alpha \in \widetilde{S}$ with $\beta \in \mathrm{clos}_{U_r}(\{\alpha\})$, since $\widetilde{S}$ is construcible (see *ibid.* Prop. 7.1.21). Note that $\alpha \neq \beta$ since $\alpha \in \widetilde{S}$ but $\beta \notin \widetilde{S}$. This implies a proper inclusion of prime ideals $\mathrm{supp}(\alpha) \subsetneq \mathrm{supp}(\beta)$ in $R[U]$, but $\mathrm{supp}(\beta)$ is an ideal of height 1 in $R[U]$ since Z' is a divisor, so $\mathrm{supp}(\alpha) = (0)$. Let P_α be the ordering of $R(V)$ corresponding to α, then $h_i \in P_\alpha$ for all $i \in \{1, \dots, r\}$ because $\alpha \in \widetilde{S}$, and P_α is compatible with the valuation v_Z since α specialises to β (see *ibid.* Prop. 10.2.3). □

In the situation of the theorem, note that T is shown to be stable if all the vector spaces U_n are finite-dimensional over R. These spaces can be realized as spaces of sections of coherent sheaves, namely if Z is the union of the divisors $Z_1, \dots, Z_r$ and Z' is the union of the remaining irreducible components of $X \setminus V$, then

$$U_n = \Gamma(X - Z', (\mathcal{I}_Z^n)^\vee),$$

(see section A.3 for the definition of $(\mathcal{I}_Z^n)^\vee$ and a proof of this equality). The most important case is when Z is a Cartier divisor on X (for example if X is non-singular along Z), then $(\mathcal{I}_Z^n)^\vee$ is just the line bundle $\mathcal{O}_X(nZ)$.

Corollary 2.10 (Powers-Scheiderer) — *Let V be an irreducible affine R-variety, let $h_1, \dots, h_r \in R[V]$, and put $S = \mathcal{S}_V(h_1, \dots, h_r)$ and $T = \mathrm{PO}(h_1, \dots, h_r) \subset R[V]$. Assume that V possesses a good completion $V \hookrightarrow X$ such that $\mathrm{clos}_{X(R)}(S) \cap Z(R)$ is Zariski-dense in Z for all irreducible components Z of $X \setminus V$. Then T is stable.*

Proof — In this case, Z' as above is empty, so $U_n = \Gamma(X, (\mathcal{I}_Z^n)^\vee)$ is of finite dimension over R since X is projective (see Hartshorne [9], Thm. 5.19). □

This corollary is clearly the most important instance of the theorem, but we will prove a somewhat stronger version for surfaces in Chapter 4. We note the following corollary for curves:

Corollary 2.11 — *Let C be an affine curve over R, let $h_1, \dots, h_r \in R[C]$, and put $S = \mathcal{S}(h_1, \dots, h_r)$ and $T = \mathrm{PO}(h_1, \dots, h_r)$. If $B_C(S) = R$, then T is stable. Additionally, if $f_1, \dots, f_r \in R[C]$ are such that $\sum f_i^2 \in R$, then $f_i \in R$ holds for all $i \in \{1, \dots, r\}$.*

Proof — The hypothesis $B_C(S) = R$ just means that all points at infinity of C are real and contained in the closure of S by Cor. 1.14. Thus the Powers-Scheiderer theorem applies to T. To prove the supplement, the argument in the proof of Prop. 2.8 (with all $h_i = 1$) shows that if $\sum f_i^2 \in R$, then the f_i have no poles in any point at infinity and hence must be constant. □

The connection between stability and the moment problem is the following: If a finitely generated preordering T of $R[V]$ is stable, it can be shown that T is closed in the natural linear topology on $R[V]$ (possibly only after passing to the residue ring $R[V]/\sqrt[\mathrm{re}]{T}$; see Powers and Scheiderer [23], Cor. 2.11). Thus if T also had the strong moment property, it would follow that T is in fact saturated. But this is impossible if $\dim(\mathcal{S}(T)) \geqslant 3$ by Thm. 2.1. Thus stability and the moment property are mutually exclusive in dimension $\geqslant 3$. This result extends, in fact, to dimension 2:

Theorem 2.12 (Scheiderer [29], Thm. 5.3) — *Let V be an affine R-variety, and let T be a finitely generated preordering of $R[V]$. If T is stable and $\mathcal{S}(T)$ has dimension at least 2, then T does not have the strong moment property.*

However, the simple proof sketched above does not work for $\dim(\mathcal{S}(T)) = 2$, and the argument in [**29**] is considerably more involved. This recent result by Scheiderer can also be used to simplify the proofs of some other statements in dimension 2 where more complicated arguments via restrictions to curves had been necessary before. As an example, we deduce the following theorem by Kuhlmann and Marshall:

Theorem 2.13 ([14], Cor. 3.10) — *Let $h_1 \ldots, h_r \in R[x_1, \ldots, x_n]$ be polynomials, $T = \mathrm{PO}(h_1, \ldots, h_r)$, and assume that $\mathcal{S}(h_1, \ldots, h_r)$ contains a two-dimensional convex cone. Then T does not have the strong moment property.*

Proof — Let K be a two-dimensional convex cone contained in $\mathcal{S}(h_1, \ldots, h_r)$. The Zariski-closure V of K in $\mathbf{A}^n_R$ is isomorphic to an affine plane $\mathbf{A}^2_R$. Write $S = \mathcal{S}(h_1, \ldots, h_r) \cap V(R)$, and let $V \hookrightarrow \mathbf{P}^2_R$ be the standard embedding that identifies V with the complement of a line L in $\mathbf{P}^2_R$. Now $K \subset S$ implies that $\mathrm{clos}(S) \cap L(R)$ has non-empty interior in $L(R) \cong \mathbf{P}^1(R)$ and is thus Zariski-dense in L (see also Example 1.11 (2)). So the Powers-Scheiderer theorem together with Thm. 2.12 apply to V, showing that (SMP) cannot hold for $T|_V$, hence neither for T by Lemma 2.3 (1). □

2.3. POSITIVSTELLENSÄTZE FOR BOUNDED FUNCTIONS

We begin by briefly explaining the relationship between the ring of bounded functions and the more general concept of the real holomorphy ring. Let A be an R-algebra, and let X be a subset of $\mathrm{Sper}(A)$, the real spectrum of A. Then

$$H_A(X) = \left\{ a \in A \,\middle|\, \exists r \in R \,\forall x \in X \colon |a(x)| \leqslant r \right\}$$

is called the *ring of (geometrically) bounded elements* or *real holomorphy ring* on X (over R). The definition originates from field theory. Rings of bounded elements in the context of rings and algebras over a real field have been studied by Becker and Powers [**4**], by Monnier [**18**], and by Schweighofer [**33**].
For any finitely generated preordering T of A, we write

$$\mathcal{X}_A(T) = \{x \in \mathrm{Sper}\, A \mid \forall f \in T \colon f(x) \geqslant 0\}.$$

If V is an affine R-variety, $T \subset R[V]$, and $S = \mathcal{S}(T)$, then

$$B_V(S) = H_{R[V]}(\mathcal{X}_{R[V]}(T)).$$

One consequence of the main result of Schweighofer's thesis [**33**] is the following generalisation of Schmüdgen's Positivstellensatz:

Theorem 2.14 ([33], Thm. 4.15) — *Let A be an $\mathbf{R}$-algebra of finite transcendence degree over $\mathbf{R}$, and let T be a finitely generated preordering of A. If $H_A(\mathcal{X}_A(T)) = A$, then T contains every element of A that is strictly positive on $\mathcal{X}_A(T)$.*

(Note that the transcendence degree of an $\mathbf{R}$-algebra is defined as in the case of fields and coincides with the trancendence degree of the quotient field in the case of a domain.)

Proposition 2.15 — *Let A be an $\mathbf{R}$-algebra of finite transcendence degree over $\mathbf{R}$. Let T be a finitely generated preordering of A, $X = \mathcal{X}_A(T)$, $B = H_A(X)$, and let $\varphi\colon \operatorname{Sper} A \to \operatorname{Sper} B$ be the canonical bounded morphism. Put $T' = T \cap B$ and $X' = \mathcal{X}_B(T')$. Consider the following statements:*

(1) *Every element $h \in B$ that satisfies $h \geqslant 1$ on X is contained in T.*
(2) $X' = \operatorname{clos}_{\operatorname{Sper} B}(\varphi(X))$.
(3) $H_B(X') = B$.
(4) *Every element of B that is strictly positive on X' is contained in T.*

The implications (1)⇔(2)⇒(3)⇒(4) *hold.*

Proof — Suppose that (1) holds and (2) does not. Note that $\operatorname{clos}_{\operatorname{Sper} B}(\varphi(X)) \subset X'$ always holds, thus there exists $x \in X'$ such that $\operatorname{clos}(\{x\}) \cap \operatorname{clos}(\varphi(X)) = \varnothing$, since $\operatorname{clos}(\varphi(X))$ and X' are both constructible. This implies the existence of an element $h \in B$ such that $h(x) < 0$ but $h \geqslant 1$ on $\varphi(X)$ (see lemma below) hence $h \in T$, a contradiction. To prove (2)⇒(3), use the equality $H_B(X') = H_B(\operatorname{clos}(\varphi(X)) = H_B(\varphi(X)) = \bigcap_{x \in X} H_B(\{\varphi(x)\})$ (see Knebusch and Scheiderer [**12**], Kap. III, §11; in particular Satz 6 and Kor. 6). But clearly $H_B(\{\varphi(x)\}) = B$ for all $x \in X$ by the definition of $B = H_A(X)$. The implication (3)⇒(4) is just Schweighofer's theorem applied to B and T'. To get back to (1), assume that (2) is satisfied, and let $h \in B$ be such that $h \geqslant 1$ holds on X. Then $h \geqslant 1$ also holds on X' by continuity, hence $h \in T$ by (3) and Schweighofer's theorem. □

Lemma 2.16 — *Let A be a ring, X a closed subset of* $\operatorname{Sper} A$, *and $y \in \operatorname{Sper} A$ a point that does not specialise to any point of X. Then there exists $f \in A$ such that $f(y) = 0$ and $f(x) \geqslant 1$ for all $x \in X$.*

Proof — That y does not specialise to any point of X just says that $\operatorname{clos}(\{y\}) \cap X = \varnothing$. Equivalently, the ideal $\operatorname{supp}(y)$ is not contained in any P_x-convex ideal of A for any $x \in X$ (see Knebusch and Scheiderer [**12**], Kap. III, §7, Thm. 2), which means that the convex hull of $\operatorname{supp}(y)$ in A with respect to P_x contains 1. Thus for every $x \in X$ there exists $f_x \in \operatorname{supp}(y)$ such that $f_x(x) > 1$. Since X is compact in the Harrison topology, there exist finitely many elements $f_1, \dots, f_n \in \operatorname{supp}(y)$ such that the sets $\mathcal{Y}_A(f_i - 1)$ cover X, hence $f = f_1^2 + \dots + f_n^2$ will do what we want. □

The following example is based on Example 8.2 in Marshall [**16**].

Let $q = x^2(1 - x^2 - y^2)$, let $A = R[x, y]_q$ be the localisation of the polynomial ring in q, and put $V = \operatorname{Spec}(A)$. Let T be the preordering $\mathrm{PO}(\frac{1}{q} - 1)$ of A. Then $V(R)$ is canonically identified with $R^2 \setminus \mathcal{V}_R(q)$ and $S = \mathcal{S}(T)$ with $\mathcal{U}(q)$. Consider the good completion $V \hookrightarrow \mathbf{A}^2_R \hookrightarrow \mathbf{P}^2_R$. The complement of V in $\mathbf{P}^2_R$ consists of l_∞, the line at infinity, and the curves $C_1 = \overline{\mathcal{V}(x)}$ and $C_2 = \overline{\mathcal{V}(1 - x^2 - y^2)}$. Furthermore, S has dense boundaries in $\mathbf{P}^2_R$, since $\operatorname{clos}_{\mathbf{P}^2(R)}(S) \cap C_1(R)$ resp. $\operatorname{clos}_{\mathbf{P}^2(R)}(S) \cap C_2(R)$ are Zariski dense in C_1 resp. C_2, and $\operatorname{clos}_{\mathbf{P}^2(R)}(S) \cap l_\infty(R) = \varnothing$, so that $B_V(S) = \Gamma(\mathbf{P}^2_R - l_\infty, \mathcal{O}_{\mathbf{P}^2_R}) = R[x, y]$.

Write B for $B_V(S)$. We claim that $T \cap B = \mathrm{PO}_B(q)$. Since $q = q^2(\frac{1}{q} - 1) + q^2$, we have $\mathrm{PO}_B(q) \subset T \cap B$. For the converse, note first that $1 - q = (x^4 - x^2 + 1) + x^2 y^2$ is a sum of squares in B. Now let

$$f = s + t\left(\frac{1}{q} - 1\right)$$

with s and t sums of squares in A and $f \in B$. Multiplying by q^{2i} for any i, we have $fq^{2i} = sq^{2i} + tq^{2i}(\frac{1}{q} - 1) = sq^{2i} + tq^{2i-1}(1 - q)q$. Now if $i \geqslant 1$, then, using the fact that $1 - q$ is a sum of squares in B, we get an expression

$$fq^{2i} = s' + t'q$$

such that s' and t' are sums of squares in B. Furthermore, we see that each term in s' must be divisible by q, by Lemma 2.17 below, hence $s' = s''q^2$ with s'' still a sum of squares in B. Dividing by q gives $fq^{2i-1} = s''q + t'$ and applying Lemma 2.17 again, we see that $t' = t''q^2$ with t'' a sum of squares in B. Thus we get an equation

$$fq^{2(i-1)} = s'' + t''q.$$

This argument can be repeated inductively until $i = 1$, which shows that $f \in \mathrm{PO}_B(q)$.

The whole point of this example lies in the fact that $\mathcal{S}(q)$ is not compact so that $B(T \cap B) = R[x] \neq B$. The compact set $\operatorname{clos}(\varphi(S)) = \mathcal{S}(1 - x^2 - y^2)$ is strictly contained in $\mathcal{S}(q)$ in accordance with the proposition. The proposition also says that there must exist a bounded polynomial that is greater or equal to 1 on S but not an element of T.

Now consider the following modification: Put $p = 2 - x^2 - y^2$ and $T = \mathrm{PO}(p, \frac{1}{q} - 1)$. Using the same arguments as before, one can again show that $B = R[x, y]$ and that $T \cap B = \mathrm{PO}(p, q)$. This time, $\mathcal{S}(p, q)$ is compact so that $B(T \cap B) = B$. But still $\mathrm{clos}(\varphi(S)) = \mathcal{S}(1 - x^2 - y^2) \subsetneq \mathcal{S}(q)$ which shows that the implication (3)⇒(2) in the proposition does not hold.

The following lemma was used in the example above:

Lemma 2.17 — *Let R be a real closed field, and let $q \in R[x_1, \dots, x_n]$ be a square-free polynomial such that the variety $\mathcal{V}(q)$ is real. Let $f = g_1^2 + \cdots + g_r^2$ be a sum of squares in $R[x_1, \dots, x_n]$. If q divides f, then q divides g_i for every i. In particular, if f is divisible by q, it is divisible by q^2.*

The condition that $\mathcal{V}(q)$ be real can be restated as follows: For any irreducible factor r of q there exists $x \in R^n$ such that $r(x) = 0$ and $\big(\partial r/\partial x_1, \dots, \partial r/\partial x_n\big)(x) \neq 0$ (see Prop. A.1).

Proof — Write $q = q_1 \cdots q_s$ with q_i irreducible. By hypothesis, each q_i defines a real valuation v_i of the rational function field $R(x_1, \dots, x_n)$ (see Prop. A.1) and $v_i(g_1^2 + \cdots + g_r^2) \geqslant 1$. Now $v_i(g_j^2) \geqslant 1$ by Cor. 2.7, so q_i divides g_j for every j. This we may do for every i which completes the proof. □

CHAPTER 3

CURVES

3.1. SOME FACTS ABOUT REDUCIBLE CURVES

In this section, we gather some facts about curves and their irreducible components that will be needed later. Let C be a curve over a field k (like all our varieties reduced, in particular without multiple components, but not necessarily irreducible). To avoid confusion, note first that we will be looking at three kinds of components of C: *Irreducible* components of C (in the Zariski-topology); *connected* components of C (again in the Zariski-topology); and occasionally, if $k = R$ is a real closed field, connected components of $C(R)$ (in the strong topology). Any irreducible component is connected, and any connected component is a union of irreducible components. But even if C is irreducible, $C(R)$ need not be connected. An *intersection point* of C is any point contained in at least two irreducible components of C.

Definition — Let C be a curve. We say that a closed point $P \in C$ with residue field K is an *ordinary multiple point with independent tangents* if the completion of the local ring $\mathcal{O}_{C,P}$ is isomorphic to $K[[x_1, \dots, x_n]]/(x_i x_j \mid 1 \leqslant i < j \leqslant n)$ for some n.

Examples — (1) Every non-singular point on a curve is an ordinary multiple point with independent tangents.

(2) The point $(0,0)$ on the plane curve $\{xy = 0\}$ is an ordinary multiple point with independent tangents.

(3) More generally, if C_1 and C_2 are two curves contained in a non-singular surface X, then an intersection point $P \in C_1 \cap C_2$ is an ordinary multiple point with independent tangents if and only if the intersection is

transversal, i.e. the maximal ideal of the local ring $\mathcal{O}_{X,P}$ is generated by local equations for C_1 and C_2.

Lemma 3.1 — *For any field K and any $n \geqslant 1$, there is an isomorphism*

$$\begin{aligned} &K[[x_1,\dots,x_n]]/(x_ix_j \mid 1 \leqslant i < j \leqslant n) \\ &\cong \big\{(f_1,\dots,f_n) \in K[[t_1]] \times \cdots \times K[[t_n]] \,\big|\, f_1(0) = \cdots = f_n(0)\big\}. \end{aligned}$$

given by the the map $\varphi\colon K[[x_1,\dots,x_n]]/(x_ix_j \mid i,j) \longrightarrow \prod_i K[[t_i]]$ that sends the class of $f \in K[[x_1,\dots,x_n]]$ to $\big(f(t_1,0,\dots,0),\dots,f(0,\dots,0,t_n)\big)$,

Proof — One checks that φ is well-defined and injective. To see that the image is as claimed, let $(f_1,\dots,f_n)$ be in the right-hand side such that $f_1(0) = \cdots = f_n(0)$. Then $\varphi\big(f_1(x_1) + \cdots + f_n(x_n) - (n-1)f_1(0)\big) = (f_1,\dots,f_n)$. □

Corollary 3.2 — *Let C be a curve with irreducible components $C_1,\dots,C_m$ over a field k. If every intersection point of C is an ordinary multiple point with independent tangents, then the diagonal map*

$$k[C] \longrightarrow \Big\{(f_i)_i \in \prod_i k[C_i] \,\Big|\, f_i(P) = f_j(P) \text{ for all } P \in C_i \cap C_j,\ i,j = 1,\dots,m\Big\}$$

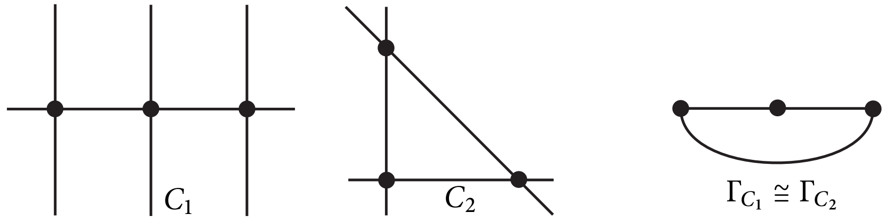
is an isomorphism.

Proof — By the preceding lemma, the diagonal map becomes an isomorphism after completing with respect to any maximal ideal of $k[C]$. Thus it is an isomorphism (see for example Bourbaki [7], §3, no. 5, Cor. 5). □

We associate with the curve C a finite graph Γ_C as follows: The vertices of Γ_C are the intersection points of C, and we put an edge between two vertices $P \neq Q$ for every irreducible component of C that contains both P and Q. Recall that a *cycle* in a graph is a subgraph that is homeomorphic to S^1.

Definition — We say that a cycle in Γ_C is *mixed* if it consists of edges that correspond to at least two different irreducible components of C.

This is not actually a property of the graph Γ_C itself but rather of the relationship between C and Γ_C. For example, the plane curve $C_1 = \{x(x-1)(x-2)y = 0\}$ has four irreducible components, and Γ_{C_1} is a basic cycle with three vertices and three edges. This cycle is not mixed, since all edges of Γ_{C_1} correspond to the same component $\{y = 0\}$. But if we take C_2 to be the plane curve $\{xy(x+y-1) = 0\}$, the resulting graph Γ_{C_2} is the same, yet the cycle is mixed.



The notion of a mixed cycle could be made more formally rigid, for example by equipping Γ_C with an edge colouring reflecting the origin of each edge, but that does not seem necessary for our modest purposes.

Lemma 3.3 — *Let C be a curve with irreducible components $C_1, \dots, C_m$. Then the graph Γ_C does not contain any mixed cycles if and only if $C_1, \dots, C_m$ can be relabelled in such a way that $C_i \cap (C_1 \cup \dots \cup C_{i-1})$ consists of at most one point for every $1 < i \leqslant m$.*

Proof — Clearly, we may assume that C is connected, otherwise we can treat all connected components separately. The case $m = 1$ is trivial, so we assume $m \geqslant 2$. Put $C'_i = \bigcup_{j \neq i} C_j$ for every $1 \leqslant i \leqslant m$. Note first that since C is connected, there exists a component C_i such that C'_i remains connected. (Proof: Start with any component C_j of C, and let A be a non-empty connected component of C'_j. If $A \subsetneq C'_j$, then there exists an irreducible component C_k of C such that $C_k \cap A = \varnothing$. On the other hand, $C_j \cap A \neq \varnothing$ since C is connected, so we see that the connected component of A in C'_k is strictly bigger than A. Thus we can repeat this argument until $A = C'_j$.)

Now assume that Γ_C does not contain any mixed cycles. We do induction on m. We may relabel and assume that C'_m is connected. Then $C_m \cap C'_m$ consists of exactly one point. For if we had $P \neq Q \in C_m \cap C'_m$, then there would be two distinct paths in Γ joining P and Q, namely one along C_m and one along C'_m. This would constitute a mixed cycle in Γ_C. Now apply the induction hypothesis to C'_m.

For the converse, we again use induction on m; Clearly, $\Gamma_{C'_m}$ is a subgraph of Γ_C and by the induction hypothesis, it does not contain any mixed cycles. Let $C_m \cap C'_m = \{P\}$. If P is already an intersection point of C'_m, then $\Gamma_C = \Gamma_{C'_m}$, and we are done. If P is not an intersection point of C'_m, it follows that $P \in C_j$ for exactly one $j < m$. The point P gives a vertex of Γ_C that is not a vertex of $\Gamma_{C'_m}$, and it is joint by edges to all points $Q \in C_j \cap C'_j$, $Q \neq P$. But all edges in $\Gamma_{C'_m}$ between such vertices must correspond to C_j, since $\Gamma_{C'_m}$ contains no mixed cycles. This shows that P cannot be contained in a mixed cycle of Γ_C, so Γ_C contains no mixed cycles. □

If C is a real curve with only real intersection points, the condition that Γ_C contain no mixed cycles can sometimes be expressed in terms of the semialgebraic topology of $C(R)$:

Proposition 3.4 — *Let R be a real closed field, and let C be a curve over R with irreducible components $C_1, \dots, C_m$. Assume that all intersection points of C are real and that $C_i(R)$ is simply connected for all $1 \leqslant i \leqslant m$. Then Γ_C contains no mixed cycles if and only if every connected component of $C(R)$ is simply-connected.*

Proof — It suffices to note that, under the hypotheses, mixed cycles in Γ_C correspond exactly to non-trivial 1-cycles in $C(R)$. □

Example 3.5 — Note that if the $C_i(R)$ are not simply connected, then it does not suffice to take only the real picture into account. For example, let C be the plane curve $\{(xy-1)(x-y) = 0\}$, a hyperbola intersecting a line. Clearly, $C(R)$ is simply connected, yet Γ_C is the cycle consisting of two vertices and two edges, and that cycle is mixed.

3.2. SUMS OF SQUARES ON CURVES

Let us first state the general problem of psd vs. sos for reducible varieties: Let always R be a real closed field, and let V be an affine variety over R. We say that *psd=sos holds in $R[V]$* if every $f \in R[V]$ that is non-negative on $V(R)$ is a sum of squares in $R[V]$. Now let $V_1, \dots, V_m$ be the irreducible components of V, and assume that psd=sos holds in $R[V_1], \dots, R[V_m]$. When does psd=sos hold in $R[V]$? Note first that if V has connected components $W_1, \dots, W_l$, then $R[V] \cong R[W_1] \times \cdots \times R[W_l]$ and psd=sos holds in $R[V]$ if and only if psd=sos holds in all $R[W_1], \dots, R[W_l]$. Thus we can always assume that V is connected, and the interesting data is how the irreducible components $V_1, \dots, V_m$ intersect. We begin with the simplest conceivable case:

Proposition 3.6 — *Let V_1 and V_2 be affine varieties over R, $P \in V_1(R)$ and $Q \in V_2(R)$. Assume that psd=sos holds in $R[V_1]$ and $R[V_2]$. Then psd=sos also holds in*

$$A = \big\{(f,g) \in R[V_1] \times R[V_2] \,\big|\, f(P) = g(Q)\big\}.$$

Proof — Let $(f,g) \in A$ be a psd element, i.e. f is psd on V_1 and g is psd on V_2. Since psd=sos holds in $R[V_1]$ and $R[V_2]$, we can write $f = \sum_{i=1}^n f_i^2$, $g = \sum_{i=1}^n g_i^2$ with $f_1,\dots,f_n \in R[V_1]$ and $g_1,\dots,g_n \in R[V_2]$ (not necessarily $\neq 0$). The vectors $v = \big(f_1(P),\dots,f_n(P)\big)^t$ and $w = \big(g_1(Q),\dots,g_n(Q)\big)^t$ in R^n have the same euclidean length since $f(P) = g(Q)$. Therefore, there exists an orthgonal matrix $B \in O_n(R)$ such that $Bv = w$. Put $(\widetilde{f_1},\dots,\widetilde{f_n})^t = B\cdot(f_1,\dots,f_n)^t$, then $f = \sum \widetilde{f_i}^2$ and $(\widetilde{f_i}, g_i) \in A$ for all i, hence $(f,g) = \sum_{i=1}^n (\widetilde{f_i}, g_i)^2$. □

Example 3.7 — (1) Note that $\operatorname{Spec} A$, with A as in the proposition, is the affine variety obtained from V_1 and V_2 by glueing along P and Q. For example, let V_1 and V_2 be two copies of $\mathbf{A}_R^1$, then $A \cong R[x,y]/(xy)$ and $C = \operatorname{Spec} A$ is the plane curve $\{xy = 0\}$, a pair of crossing lines. Now psd=sos holds in $R[C]$ by the proposition.

(2) More generally, let C be a curve that consists of two irreducible components C_1 and C_2 intersecting in only one point P that is a real ordinary multiple point with independent tangents. Then $R[C] = \big\{(f,g) \in R[C_1]\times R[C_2] \,\big|\, f(P) = g(P)\big\}$ by Cor. 3.2, and if psd=sos holds in $R[C_1]$ and $R[C_2]$, it also holds in $R[C]$.

To obtain more general results, we from now on restrict our attention to the case of curves. Let C be an affine curve over R with irreducible components $C_1,\dots,C_m$. The first case we consider is when every irreducible component C_i of C admits a bounded function, i.e. $B(C_i) \neq R$ for all i. By Cor. 1.14, this is equivalent to saying that all irreducible components of C have a non-real point at infinity. In Scheiderer's terminology from [**28**], this means that $C(R)$ is *virtually compact*. For $R = \mathbf{R}$, he has proved the following:

Theorem 3.8 (Scheiderer) — *Let C be an affine curve over $\mathbf{R}$, and assume that $B(C_i) \neq \mathbf{R}$ holds for every irreducible component C_i of C. If C has no other real singularities than ordinary multiple points with independent tangents, then every psd function in $\mathbf{R}[C]$ with only finitely many zeros is a sum of squares in $\mathbf{R}[C]$.*

Proof — see Cor. 4.15 in Scheiderer [**28**]. □

Corollary 3.9 — *Let C be an affine curve over $\mathbf{R}$, and assume that $B(C_i) \neq \mathbf{R}$ holds for every irreducible component C_i of C. If all intersection points of C are real and C has no other real singularities than ordinary multiple points with independent tangents, then psd=sos holds in $\mathbf{R}[C]$.*

Proof — Let $C_1, \dots, C_m$ be the irreducible components of C. By Cor. 3.2, we have

$$\mathbf{R}[C] \cong \Big\{(f_i)_i \in \prod_i \mathbf{R}[C_i] \,\Big|\, f_i(P) = f_j(P) \text{ für alle } P \in C_i \cap C_j, 1 \leqslant i,j \leqslant m\Big\}.$$

Let $f = (f_1, \dots, f_m) \in \mathbf{R}[C]$ be psd. Upon relabelling, we may assume that f has only finitely many zeros on $C_1, \dots, C_l$ and vanishes identically on $C_{l+1}, \dots, C_m$ for some $0 \leqslant l \leqslant m$. Put $C' = \bigcup_{i=1}^{l} C_i$. Then $g = f|_{C'}$ is a sum of squares in $\mathbf{R}[C']$ by the preceding theorem, say $g = \sum g_i^2$, $g_i \in \mathbf{R}[C']$. For every $P \in C' \cap C_j$, $j > l$, we have $g(P) = f(P) = 0$, and since P is real by hypothesis, it follows that $g_i(P) = 0$ for all i. This implies that $f_i = \big(g_i|_{C_1}, \dots, g_i|_{C_l}, 0, \dots, 0\big)$ is a function on C, by the above description of $\mathbf{R}[C]$, and that $f = \sum f_i^2$. □

On the other hand, we have the following negative results that hold over any real closed field R and regardless of the existence of bounded functions:

Proposition 3.10 (Scheiderer) — *Let C be an affine curve over R. If C has a real singular point that is not an ordinary multiple point with independent tangents, then psd=sos does not hold in $R[C]$.*

see Cor. 4.22 in Scheiderer [**28**] for the proof.

Lemma 3.11 — *Let C be an affine curve over R all of whose real intersection points are ordinary multiple points with independent tangents. Let $C' \subset C$ be a closed subcurve of C. Then the restriction map $R[C]_+ \to R[C']_+$ is surjective, i.e. every psd function on C' can be extended to a psd function on C.*

Proof — Let f be a psd function on C', and let D be the union of all irreducible components of C not contained in C'. Let $Z = C' \cap D$ be the scheme-theoretic intersection. We have to find a psd function on D that agrees with f on the closed subscheme Z. If Z is supported on r real points and s non-real points, then the ring of regular functions $R[Z]$ of Z is a direct product

$$R[Z] \cong \underbrace{R \times \cdots \times R}_{r} \times A_1 \times \cdots \times A_s$$

with $R(\sqrt{-1}) \subset A_i$ for all $i \in \{1, \dots, s\}$. This implies $\sum A_i^2 = A_i$ for all i, and since f takes only non-negative values in the real points of Z, it follows that the class of f in $R[Z]$ is a sum of squares in $R[Z]$. Therefore, it can be lifted to a sum of squares in $R[D]$ which, in particular, is psd on D. □

Example — The condition on intersection points cannot be dropped in general: If $C = \{y(y^2 - x^3) = 0\}$, the function x is psd on $C' = \{y^2 - x^3 = 0\}$ but cannot be extended to a psd function on C.

Proposition 3.12 — *Let C be an affine curve over R. If psd=sos holds in $R[C]$, then psd=sos also holds in $R[C']$ for every closed subcurve C' of C.*

Proof — If all real singularities of C are ordinary multiple points with independent tangents, then every psd function f on C' can be extended to a psd function g on C by Lemma 3.11. Then g is sos in $R[C]$ by hypothesis, hence f is sos in $R[C']$. If C has a real singular point that is not an ordinary multiple point with independent tangents, then psd=sos does not hold in $R[C]$ by the preceding proposition, so the statement is empty. □

Proposition 3.13 — *Let C be a real curve over R, and assume that C has a non-real intersection point. Then psd=sos does not hold in $R[C]$.*

Proof — Let C_1 and C_2 be two irreducible components of C that intersect in a non-real point. By Prop. 3.12, it suffices to show that psd=sos fails in $R[C_1 \cup C_2]$. Let $I = \mathcal{I}_{R[C_1 \cup C_2]}(C_2)$ be the vanishing ideal of C_2 in $R[C_1 \cup C_2]$, and let $J = \mathcal{I}_{R[C_1]}(C_1 \cap C_2)$ be the vanishing of the scheme-theoretic intersection $C_1 \cap C_2$ (considered as a subscheme of C_1) in $R[C_1]$. The restriction map $\pi\colon R[C_1 \cup C_2] \to R[C_1]$ induces an isomorphism $\pi\colon I \xrightarrow{\sim} J$ (of $R[C_1 \cup C_2]$-modules). Now if a function $f \in I$ is a sum of squares $f = \sum_{i=1}^r f_i^2$ in $R[C_1 \cup C_2]$, it follows that $f_i \in I$ for all $i \in \{1, \ldots, r\}$, since C_2 is real. Therefore, to prove the claim, it suffices to produce an element $f \in J \subset R[C_1]$ that is non-negative on $C_1(R)$ and such that f cannot be written as a sum of squares of elements in J.

Write $C_1 \cap C_2 = S \cup T$ with closed subschemes S and T of C_1 such that S is supported on real points of C_1 and T is supported on non-real points. Let $I_S = \mathcal{I}_{R[C_1]}(S)$ and $I_T = \mathcal{I}_{R[C_1]}(T)$ be the vanishing ideals of S resp. T in $R[C_1]$, so that $J = I_S \cap I_T$. Since T is non-empty by hypothesis, we have $(0) \subsetneq I_T \subsetneq R[C_1]$. Put $B = R[C_1]/I_T^2$, and choose $h \in I_T \setminus I_T^2$, i.e. h vanishes on T but not on the closed subscheme $\operatorname{Spec} B$ of C_1 (note that $I_T \neq I_T^2$ by the Krull intersection theorem; see Bourbaki [7], §3.2). Since T has non-real support, we have $\sqrt{-1} \in B$, so every element of B is sos, in particular $h + I_T^2$, the class of h in B, is sos in B. Therefore, the element $(0 + I_S, h + I_T^2) \in (R[C_1]/I_S) \times B$ is sos in $(R[C_1]/I_S) \times B \cong R[C_1]/(I_S \cap I_T^2)$. Thus there exists a sum of squares $f \in \sum R[C_1]^2$ that restricts to $(0 + I_S, h + I_T^2)$ modulo $I_S \cap I_T^2$. Since $h \in I_T$, we have $f \in J$, but f cannot be sos in J. For if it were, say $f = \sum f_i^2$ with $f_i \in J$, we could conclude $f \in J^2 \subset (I_S \cap I_T^2)$, contradicting the choice of h. □

Putting together Proposition 3.13, Corollary 3.9, and Proposition 3.10, we immediately get the following

Corollary 3.14 — *Let C be a real affine curve over $\mathbf{R}$ such that $B(C_i) \neq \mathbf{R}$ for every irreducible component C_i of C. Then psd=sos holds in $\mathbf{R}[C]$ if and only if the following conditions are satisfied:*

(1) *all real points of C are ordinary multiple points with independent tangents.*
(2) *all intersection points of C are real.*

□

Example 3.15 — Let $C = \{(x^2+y^2-1)((x-1)^2+y^2-1) = 0\}$, two intersecting circles in the plane. Then psd=sos holds in $R[C]$. But psd=sos does not hold in $R[C]$ for $C = \{(x^2+y^2-1)((x-3)^2+y^2-1) = 0\}$, since the two circles intersect in a non-real point.

We now turn to the case $B(C) = R$. Again, the complete answer is known in the irreducible case:

Theorem 3.16 (Scheiderer) — *Let C be an irreducible affine curve over R. Assume that $B(C) = R$. Then psd=sos holds in $R[C]$ if and only if C is an open subcurve of $\mathbf{A}_R^1$.*

Proof — By Cor. 1.14, $B(C) = R$ if and only if all points of C at infinity are real. This is the way the hypotheses are stated in Scheiderer [**28**], Thm. 4.17. The proof for psd=sos on an open subcurve of $\mathbf{A}_R^1$ can be reduced to the case of the polynomial ring in one variable by elementary arguments; see also Scheiderer [**27**], Prop. 2.17. □

The main result of this section gives a complete answer to the question of psd vs. sos for affine curves over R that admit no bounded functions:

Theorem 3.17 — *Let C be an affine curve over R, and assume that $B(C) = R$. Then psd=sos holds in $R[C]$ if and only if the following conditions are satisfied:*

(1) *all irreducible components of C are isomorphic to open subcurves of $\mathbf{A}_R^1$;*
(2) *all intersection points of C are real ordinary multiple points with independent tangents;*
(3) *the graph Γ_C does not contain any mixed cycles.*

Proof — Let $C_1, \ldots, C_m$ be the irreducible components of C.

Assume that C satisfies all the conditions listed in the theorem. Condition (1) implies that psd=sos holds on $C_1, \ldots, C_m$ by Thm. 3.16. Condition (3) implies by Lemma 3.3 that we may rearrange the C_i in such a way that $C_i \cap (C_1 \cup \cdots \cup C_{i-1})$ consists of at most one point for all $1 \leqslant i \leqslant m$. Put $E_i = C_1 \cup \cdots \cup C_i$ for all $1 \leqslant i \leqslant m$ and use induction on i: We already know that psd=sos holds in $R[E_1]$. For $i \geqslant 2$, psd=sos holds in $R[C_i]$ and in $R[E_{i-1}]$ by the induction hypothesis. Now C_i and E_{i-1} intersect in at most one point which must then be a real ordinary multiple point with independent tangents. Therefore,

$$R[E_i] = \begin{cases} R[C_i] \times R[E_{i-1}], & C_i \cap E_i = \varnothing \\ \{(f,g) \in R[C_i] \times R[E_{i-1}] \mid f(P) = g(P)\}, & C_i \cap E_i = \{P\} \end{cases}$$

by Cor. 3.2. Thus psd=sos holds in $R[E_i]$ by Prop. 3.6. Eventually, we reach $i = m$, and we see that psd=sos holds in $R[E_m] = R[C]$.

For the converse, assume that psd=sos holds in $R[C]$. Then psd=sos holds in $R[C_1], \ldots, R[C_m]$ by Prop. 3.12, so Thm. 3.16 implies condition (1). Furthermore, Prop. 3.10 and Prop. 3.13 imply condition (2). Thus we are left with the case when conditions (1) and (2) are satisfied, but (3) is not, i.e. the graph Γ_C contains a mixed cycle. Let C_i be an irreducible component of C corresponding to an edge in a mixed cycle of Γ_C. Let $C_i' = \bigcup_{j \neq i} C_j$ as before. Then there is a connected component E of C_i' such that $E \cap C_i$ contains at least two distinct points. It suffices to show that psd=sos fails in $R[C_i \cup E]$ by Lemma 3.12, so we may assume right away that C_i' is connected. Write $C_i \cap C_i' = \{P_1, \ldots, P_r\}$, $r \geqslant 2$, and let

$$A = \{f \in R[C_i] \mid f(P_1) = \cdots = f(P_r)\}.$$

From Cor. 3.2 and the fact that C_i' is connected, we see that the restriction map $R[C] \to R[C_i]$ induces an isomorphism

$$\{f \in R[C] \mid f|_{C_i'} \text{ is constant}\} \xrightarrow{\sim} A.$$

Now if an element in A is a sum of squares in $R[C]$, then it is a sum of squares in A by Cor. 2.11, since $B(C) = R$.

We will make a similar argument as in the proof of Prop. 3.13 and construct an element of A that is not sos in A as follows: Let $C_i \hookrightarrow \mathbf{P}^1_R$ be the good completion of C_i, and let $\mathbf{P}^1_R \setminus C_i = \{Q_1, \ldots, Q_s\}$ be the points at infinity of C_i. Since $B(C_i) = R$, all Q_j are real by Cor. 1.14. Furthermore, since $\mathbf{P}^1(R)$ is topologically a circle, we can relabel $P_1, \ldots, P_r$ and assume that P_j is next to P_{j+1}, i.e. if U_1 and U_2 are the two connected components of $\mathbf{P}^1(R) \setminus \{P_j, P_{j+1}\}$, either U_1 or

U_2 contains none of $P_1,\dots,P_r$, for $1 \leqslant j \leqslant r-1$. We denote that connected component by (P_j, P_{j+1}). Assume further that $Q_1 \in (P_r, P_1)$.

Fix $j \in \{1,\dots,r\}$. Since C_i is rational, there exists $h_j \in R[C_i]$ such that

$$\operatorname{div}_{\mathbf{P}^1_R}(h_j) = P_1 + \cdots + \widehat{P_j} + \cdots + P_r - (r-1)Q_1.$$

Then $h_j(P_j) \neq 0$, and after multiplying with $(-1)^j \cdot h(P_j)^{-1}$ we can assume that $h_j(P_j) = (-1)^j$. Put $f = \sum_j h_j$; then $f(P_j) = (-1)^j$. We have $v_{Q_1}(f) \geqslant -(r-1)$ and $v_P(f) \geqslant 0$ for all $P \in \mathbf{P}^1_R$, $P \neq Q_1$, so f has poles only in Q_1, and f can have at most $r-1$ distinct zeros. But f changes sign on (P_j, P_{j+1}), so it must have a zero $R_j \in (P_j, P_{j+1})$ for every $j = 1,\dots,r-1$. We conclude that

$$\operatorname{div}_{\mathbf{P}^1_R}(f) = R_1 + \cdots + R_{r-1} - (r-1)Q_1.$$

Now since $f^2(P_j) = 1$ for all $1 \leqslant j \leqslant r$, we have $f^2 \in A$, but $f \notin A$, since $f(P_j) = (-1)^j$ for $j = 1,\dots,r$ and $r \geqslant 2$. We will show that f^2 cannot be a sum of squares in A: For if $f^2 = \sum f_j^2$ with $f_j \in R[C_i]$, then $f_j(R_l) = 0$ for every $1 \leqslant l \leqslant r-1$ and every j, so each f_j has at least $r-1$ distinct zeros. On the other hand, $v_{Q_l}(f_j) \geqslant v_{Q_l}(f)$ for $l = 1,\dots,s$ by Cor. 2.7, so $v_{Q_1}(f_j) \geqslant -(r-1)$ and $v_P(f_j) \geqslant 0$ for all $P \in \mathbf{P}^1_R$, $P \neq Q_1$. It follows that $\operatorname{div}_X(f_j) = R_1 + \cdots + R_{r-1} - (r-1)Q_1 = \operatorname{div}_X(f)$, so there exists $c_j \in R$ such that $f_j = c_j f$, hence $f_j \notin A$. □

Corollary 3.18 — *Let C be an affine curve over R all of whose irreducible components are isomorphic to $\mathbf{A}^1_R$. Then psd=sos holds in $R[C]$ if any only if the following conditions are satisfied:*

(1) *all intersection points of C are real ordinary multiple points with independent tangents;*
(2) *all connected components of $C(R)$ are simply connected.*

Proof — Combine the theorem with Prop. 3.4. □

Examples 3.19 — (1) Let $C = \{y(y - x^2 + 1) = 0\}$, the union of a parabola and a line with two real intersection points. Then Γ_C contains a mixed cycle, so psd=sos fails in $R[C]$. Explicitly, write $R[C] \cong \{(f,g) \in R[t] \times R[u] \mid f(0) = g(0), f(1) = g(1)\}$, and let $f = 2t - 1$, $g = 1$. Then (f^2, g) is an element of $R[C]$ which is clearly psd but not sos. As in the proof of the theorem, if $(f^2, g) = \sum (f_i, g_i)^2$ in $R[t] \times R[u]$, then $g_i \in R$ and $f_i = c_i f$ with $c_i \in R$, so $(f_i, g_i) \notin R[C]$, since f changes its sign between 0 and 1.

(2) Let $C = \{y(y - x^2) = 0\}$, the union of a parabola with a tangent line. Then psd=sos does not hold by Prop. 3.10, because the origin is not an ordinary

multiple point of C with independent tangents. Explicitly, write $R[C] \cong \{(f,g) \in R[t] \times R[u] \mid f(0) = g(0), f'(0) = g'(0)\}$. Then $(t^2, 0) \in R[C]$ is not a sum of squares in $R[C]$, since t^2 cannot be written as $t^2 = \sum f_i^2$ with $f_i'(0) = 0$.

(3) Let $C = \{y(y-x^2-1) = 0\}$ the union of a parabola with a line intersecting in a pair of complex conjugate points. Again, psd=sos does not hold in $R[C]$ by Prop. 3.13. Explicitly, write $R[C] \cong \{(f,g) \in R[t] \times R[u] \mid f(i) = g(i)\}$, and put $f(t) = t^2 + 1$ and $g = 0$. Then (f,g) is a psd function on C but it is not a sum squares because a linear function cannot vanish in the point i.

(4) For another example, let $C = \{xy(1-x-y) = 0\}$, three lines forming a triangle. Once more, psd=sos does not hold in $R[C]$ since Γ_C contains a mixed cycle. For a funny example of a psd function in $R[C]$ that is not sos, write $R[C] \cong \{(f,g,h) \in R[t] \times R[u] \times R[v] \mid f(0) = g(1), f(1) = h(0), g(0) = h(1)\}$, and put $f = 2t-1$, $g = 2u-1$, $h = 2v-1$. Then (f^2, g^2, h^2) is an element of $R[C]$ which is clearly psd. But if $f^2 = \sum f_i^2$, with all $f_i \neq 0$, then $f_i\left(\frac{1}{2}\right) = 0$ and $\deg f_i = 1$, so $f_i = a_i f$ with constants $a_i \in R$ for all i. The same is true for g and h. Therefore, if we had $(f^2, g^2, h^2) = \sum s_i^2$ in $R[C]$, then each s_i would have to be given as $s_i = (af, bg, ch)$ with constants $a, b, c \neq 0$. Such a triple can never define a function on C, for if $af(0)$ and $bg(1)$ have the same sign and $af(1)$ and $ch(0)$ have the same sign, then $bg(0)$ and $ch(1)$ will have different signs.

Note that the curve $C' = \{x(x-1)(x-2)y\}$ has the same graph as C, but the cycle in Γ_C is unmixed (see illustration in section 3.1), and psd=sos holds in $R[C]$ by the above corollary.

There remains the "mixed case" when $B(C_i) = R$ holds for some components of C while $B(C_i) \neq R$ holds for others. Does psd=sos hold in this case if it holds in all $R[C_i]$ and all intersection points of C are real ordinary multiple points with independent tangents? If there is only one transversal intersection point, the answer is again given by Prop. 3.6. For example, if C is the space curve $\{x^2 + y^2 = 1, z = 0\} \cup \{x = 0, y = 1\}$, a line intersecting a circle transversally in a single real point, then psd=sos holds in $R[C]$. But what if C has more than one intersection point between two components? For example, does psd=sos hold for the plane curve $\{(x^2 + y^2 - 1)x = 0\}$? We do not know how to answer this question. However, Thm. 3.17 suffices to settle the moment problem for $\sum \mathbf{R}[C]^2$ on any curve C over $\mathbf{R}$ using Schmüdgen's fibre theorem.

Theorem 3.20 — *Let C be an affine curve over $\mathbf{R}$, and let C' be the union of all irreducible components C_i of C for which $B(C_i) = \mathbf{R}$. The following are equivalent:*

(1) $\sum \mathbf{R}[C]^2$ *has the moment property;*
(2) $\sum \mathbf{R}[C']^2$ *has the moment property;*
(3) *psd=sos holds in* $\mathbf{R}[C']$.

Proof — We prove this more generally for preorderings in the next section (see Thm. 3.28). □

3.3. PREORDERINGS ON CURVES

The case of general preorderings in place of just sums of squares is substantially harder already in the irreducible case and is not completely solved for curves with singularities. As far as the irreducible case is concerned, we content ourselves here with citing the results of Scheiderer for non-singular curves and the results of Kuhlmann and Marshall for subsets of the line:

Theorem 3.21 (Scheiderer [28], Thm. 5.17 and [27], Thm. 3.5)
Let C be a non-singular affine curve over $\mathbf{R}$, $\mathcal{H}$ a finite subset of $\mathbf{R}[C]$, $T = \mathrm{PO}(\mathcal{H})$, and $S = \mathcal{S}(T)$. Assume that $B_C(S) \neq \mathbf{R}$ holds. Then T is saturated if and only if the following conditions are satisfied:

(1) *For every boundary point P of S in $C(\mathbf{R})$ there exists an element $h \in \mathcal{H}$ such that* $\mathrm{ord}_P(h) = 1$.
(2) *For every isolated point P of S there exist $h_1, h_2 \in \mathcal{H}$ such that* $\mathrm{ord}_P(h_1) = \mathrm{ord}_P(h_2) = 1$ *and $h_1 h_2 \leqslant 0$ holds in a neighbourhood of P in $C(\mathbf{R})$.*

The results in the singular case are more complicated to state, and we refer the reader to Scheiderer [28]. The complementary case $B_C(S) = \mathbf{R}$ is covered by the following result:

Theorem 3.22 (Scheiderer [27], Thm. 3.5) — *Let C be a non-singular, affine curve over $\mathbf{R}$, and let S be a basic closed subset of $C(\mathbf{R})$ such that $B_C(S) = \mathbf{R}$. Then the preordering $\mathcal{P}_C(S)$ is finitely generated if and only if C is an open subcurve of $\mathbf{A}^1_{\mathbf{R}}$.*

This statement is conjectured to be true for singular curves, as well. We note that Thm. 2.7 in [27], which is essential to the proof of the above theorem, has been extended by Monnier ([19], Thm. B) from non-singular curves to those that have only real ordinary double points. This can be used to extend Scheiderer's result to such curves.

Finally, in the case of the affine line, it is possible to say precisely what the generators of the saturated preordering $\mathcal{P}_C(S)$ must look like:

Theorem 3.23 (Kuhlmann-Marshall [14], Thm 2.2) — *Let $\mathcal{H}$ be a finite subset of $\mathbf{R}[t]$, $T = \mathrm{PO}(\mathcal{H})$, and $S = \mathcal{S}(T)$. Assume that $B(S) = \mathbf{R}$ (i.e. S is not compact). Then T is saturated if and only if the following hold:*

(1) *If $a = \min(S)$ exists, then $\lambda(t-a) \in \mathcal{H}$ for some $\lambda > 0$ in $\mathbf{R}$.*
(2) *If $a = \max(S)$ exists, then $\lambda(a-t) \in \mathcal{H}$ for some $\lambda > 0$ in $\mathbf{R}$.*
(3) *If $a < b \in S$ are such that $(a,b) \cap S = \varnothing$, then $\lambda(t-a)(t-b) \in \mathcal{H}$ for some $\lambda > 0$ in $\mathbf{R}$.*

Properties (1)–(3) in the theorem specify a minimal set of generators for T (unique up to positive scalars) that Kuhlmann and Marshall call the *natural generators*.

We now turn our attention to reducible curves, but will only treat the simplest case and some examples. We need the following

Lemma 3.24 — *Let C be a connected affine curve over R with irreducible components $C_1, \dots, C_m$. Assume that all intersection points of C are ordinary multiple points with independent tangents and that the graph Γ_C contains no mixed cycles. Then for all $i \in \{1, \dots, m\}$ and all $f \in R[C]$, there exists a unique function $f_{(i)} \in R[C]$ such that $f_{(i)}|_{C_i} = f|_{C_i}$ and $f_{(i)}|_{C_j}$ is constant for all $j \neq i$.*

Proof — The claim is trivial for $m = 1$, so assume $m \geqslant 2$ and put $C' = C_1 \cup \dots \cup C_{m-1}$. By Lemma 3.3, we may relabel and assume that $C_m \cap C'$ consists of a single point P. Now $R[C] = \{(f, f') \in R[C_m] \times R[C'] \mid f(P) = f'(P)\}$ by Cor. 3.2, so for every $(f, f') \in R[C]$ we can form $(f'(P), f') \in R[C]$ and $(f, f(P)) \in R[C]$. From this the claim follows easily by induction. □

Proposition 3.25 — *Let C be a connected affine curve over R with irreducible components $C_1, \dots, C_m$, let $\mathcal{H} \subset R[C]$ be a finite subset, and let $S = \mathcal{S}(\mathcal{H})$ and $T = \mathrm{PO}(\mathcal{H})$. Assume that the following conditions are satisfied:*

(1) *The induced preordering $T|_{C_i}$ is saturated for all $i \in \{1, \dots, m\}$;*
(2) *All intersection points of C are real ordinary multiple points with independent tangents and are contained in S;*
(3) *The graph Γ_C contains no mixed cycles.*

(4) *If* $h \in \mathcal{H}$, *then* $h_{(i)} \in \mathcal{H}$ *for all* $i \in \{1, \dots, m\}$, *where* $h_{(i)}$ *is defined as in the preceding lemma.*

Then T is saturated.

Proof — The claim is trivial for $m = 1$ because of condition (1), so assume $m \geqslant 2$. Write $C = C_m \cup C'$, $C_m \cap C' = \{P\}$, as in the proof of the preceding lemma using hypotheses (2) and (3), and write $\mathcal{H} = \{H_1, \dots, H_r\}$, $H_i = (h_i, h'_i)$. We first show that for every $f \in R[C_m]$ with $f \geqslant 0$ on $S \cap C_m(R)$, the function $(f, f(P)) \in R[C]$ is contained in T. By hypothesis (1), f has a representation $f = \sum_i s_i \underline{h}^i$ with $s_i \in \sum R[C_m]^2$. Thus we can write $(f, f(P)) = \sum_i (s_i, s_i(P))(\underline{h}^i, \underline{h}^i(P)) = \sum_i (s_i, s_i(P)) \underline{H}^i_{(m)}$, and $\underline{H}^i_{(m)} \in \mathcal{H}$ by hypothesis (4); thus $(f, f(P)) \in T$, as claimed. In the same way, we show $(f'(P), f') \in T$ for every $f' \in R[C']$ with $f' \geqslant 0$ on $S \cap C'(R)$, using the induction hypothesis instead of (1).

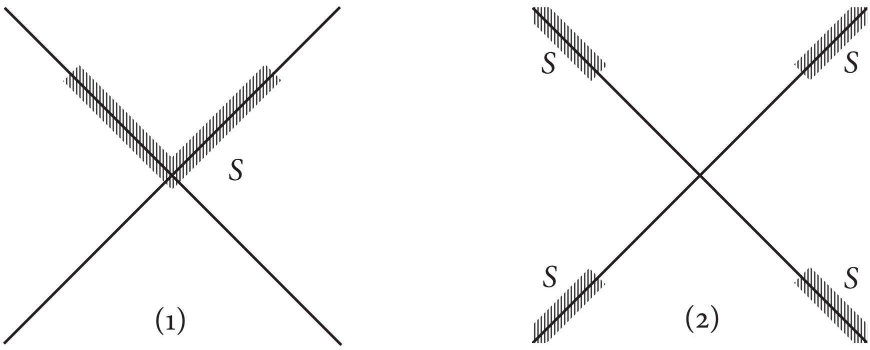
Now let $F = (f, f') \in R[C]$, and assume that $F \geqslant 0$ holds on S. If $F(P) = 0$, then $F = (f, 0) + (0, f') \in T$ by what we have just shown. If $F(P) \neq 0$, then $F(P) > 0$ by hypothesis (2), and we can write $F = (f, f(P))(1, f(P)^{-1} f') \in T$. □

Remark — Note that we also could have proved Prop. 3.6 by the same method employed here.

Examples 3.26 — (1) Let C be the plane curve $\{xy = 0\}$ in $\mathbf{A}^2_R$. Write $R[C] = \{(f, g) \in R[u] \times R[v] \mid f(0) = g(0)\}$, and consider the preordering $T = \mathrm{PO}((u, 0), (0, v))$ defining the semialgebraic set $S = \mathcal{S}(T) = \{(x, y) \in C(R) \mid x \geqslant 0 \wedge y \geqslant 0\}$. By the proposition, T is saturated. However, the proposition does not apply to the preodering $\mathrm{PO}((u, v))$ that defines the same set, because condition (4) is violated. In fact, $(u, 0) \notin \mathrm{PO}((u, v))$ since $(u, 0) = (s_1, t_1) + (s_2, t_2)(u, v)$ with $s_i \in \sum R[u]^2$ and $t_i \in \sum R[v]^2$ would imply $t_1 = t_2 = s_1 = 0$ by degree considerations and $s_2 = 1$, but $(1, 0) \notin R[C]$. Thus $\mathrm{PO}((u, v))$ is not saturated.

(2) Let C be as before, and let $T = \mathrm{PO}((u^2 - 1, v^2 - 1))$ so that $S = \mathcal{S}(T) = \{(x, y) \in C(R) \mid |x| \geqslant 1 \wedge |y| \geqslant 1\}$. Here, the intersection point $(0, 0)$ is not contained in S, and the proposition does not apply. In fact, T is not saturated: Let $f \in R[u]$ be any quadratic polynomial that is non-negative on $S \cup \{(0, 0)\}$ but not psd (i.e. not a sum of squares in $R[u]$). For example, $f = (u - \frac{1}{4})(u - \frac{3}{4})$ will do. Then $(f, f(0))$ is a function in $R[C]$ that is non-negative on S but not contained in T: For $(f, f(0)) =$

$(s_1, t_1) + (s_2, t_2)(u^2 - 1, v^2 - 1)$ would imply $t_1 = f(0)$ and $t_2 = 0$ by degree considerations, so we must have $s_2(0) = 0$. On the other hand, $s_2 \in R$ again because of the degree of f, so $(f, f(0)) = (s_1, t_1)$, a contradition. It is not clear whether the preordering $\mathcal{P}_C(S)$ is finitely generated. The fact that the intersection point is not contained in S makes condition (4) in the proposition unfulfillable, and it is not clear what a suitable replacement should look like. The best guess for a set of generators in this particular instance seems to be $\{(u^2 - 1, v^2 - 1), (u(u-1), 0), (u(u+1), 0), (0, v(v-1)), (0, v(v+1))\}$ but I have been unable to verify this.



While our results for preorderings on reducible curves are not sufficient to give a complete solution of the moment problem in dimension 1, we can still use Schmüdgen's fibre theorem to reduce to the case where there do not exist any bounded functions, just as for sums of squares. We need the following

Lemma 3.27 — *Let C be an affine curve over R, S a semialgebraic subset of $C(R)$, and C_i an irreducible component of C such that $B_{C_i}(S \cap C_i(R)) \neq R$. Then there exists $f \in B_C(S)$ such that $f|_{C_i} \notin R$. In particular, $B_C(S) \neq R$.*

Proof — Let $C_1, \dots, C_m$ be the irreducible components of C, and put $C' = \bigcup_{j \neq i} C_j$. Let $I = \mathcal{I}_C(C')$ be the vanishing ideal of C' in C. It suffices to find an element $f \in B_{C_i}(S \cap C_i(R))$ that lies in the image of I under the restriction map $R[C] \to R[C_i]$. Let $\varphi\colon \widetilde{C_i} \to C_i$ be the normalization of C_i, let E be the ramification divisor of φ and let $D = \sum\{P \mid \varphi(P) \in C' \cap C_i\}$. The hypothesis $B_{C_i}(S \cap C_i(R)) \neq R$ implies that C_i has a point Q at infinity that is not contained in the closure of $S \cap C_i(R)$ (see Cor. 1.14). Thus any $f \in R(C_i)$ with poles only in Q and zeros in $n(D+E)$ with n sufficiently large will do what we want. Such f exists by the Riemann-Roch theorem. □

Remark — Note that the restriction map $B_C(S) \to B_{C_i}(S \cap C_i(R))$ is not surjective in general.

Theorem 3.28 — *Let C be an affine curve over $\mathbf{R}$, let T be a finitely generated preordering in $\mathbf{R}[C]$, and put $S = \mathcal{S}(T)$. Let C' be the union of all irreducible components C_i of C for which $B_{C_i}(S \cap C_i(\mathbf{R})) = \mathbf{R}$. The following are equivalent:*

(1) *T has the strong moment property;*
(2) *$T|_{C'}$ has the strong moment property;*
(3) *$T|_{C'}$ is saturated.*

Proof — (1) implies (2) since the moment property is preserved when passing to a closed subvariety (Lemma 2.3). Assume that (2) holds, then $T|_{C'}$ is stable by the Powers-Scheiderer theorem (see Cor. 2.10), hence closed in $\mathbf{R}[C']$, as explained in section 2.2. Therefore, saturatedness and the strong moment property are equivalent for $T|_{C'}$. Finally, (2) implies (1), for if C_i is any component of C such $B_{C_i}(S \cap C_i(\mathbf{R})) \neq \mathbf{R}$, put $C'' = C' \cup C_i$, and take $f \in \mathbf{R}[C'']$ as in the preceding lemma. The fibres of $f: C'' \to \mathbf{A}^1_{\mathbf{R}}$ are the points of C_i and the curve C', so Schmüdgen's fibre theorem 2.4 implies that $T|_{C''}$ has the strong moment property. Now continue inductively until $C'' = C$. □

3.4. EXTENDING PSD FUNCTIONS

The general problem of extending psd functions can be stated as follows: Let V be an affine R-variety, S a semialgebraic subset of $V(R)$, and Z a closed subvariety of V. We say that $f \in R[V]$ extends $g \in R[Z]$ if $f|_Z = g$, and we ask whether it is possible to extend a function in $R[Z]$ that is non-negative on $S \cap Z(R)$ to a function in $R[V]$ that is non-negative on S. Since the restriction map $R[V] \to R[Z]$ is surjective, it is equivalent to start with $f \in R[V]$ such that $f|_Z$ is psd on $S \cap Z(R)$ and ask whether there exists $h \in \mathcal{I}_V(Z) = \{h \in R[V] \mid h|_Z = 0\}$ such that $f + h$ is psd on S. In this generality, not much is known about this problem. In [27] §5, Scheiderer has shown the following for the case $S = V(R)$:

(1) Every $g \in R[Z]$ that is strictly positive on $Z(R)$ can be extended to a strictly positive function on $V(R)$, with no conditions on V and Z.
(2) If V is non-singular and Z is a non-singular curve, then every function that is non-negative on $Z(R)$ can be extended to a non-negative function on $V(R)$.

The aim of this section is to generalise these results to arbitrary semialgebraic subsets S. We begin with some terminology: Let $P \in V(R)$ be a point. A *germ of semialgebraic sets around P* is an equivalence class of semialgebraic sets containing P where two such sets S and S' are equivalent if there exists an open neighbourhood U of P in $V(R)$ (with respect to the strong topology) such that $S \cap U = S' \cap U$. We will denote the equivalence class of S by $[S]_P$.

Let $f \in \mathcal{O}_{V,P}$ be a function germ, and let W be a Zariski-open neighbourhood of P in V such that f is regular on W. Then we will write $\mathcal{S}_P(f)$ for the germ of semialgebraic sets $[\mathcal{S}_V(f)]_P = [\{Q \in W(R) \mid f(Q) \geqslant 0\}]_P$ around P, which is clearly independent of the choice of W. Given a semialgebraic subset S of $V(R)$ containing P, we will say that *f is psd on $[S]_P$* or that *f is psd on S around P* if there exists a semialgebraic subset S' of $W(R)$ containing P such that $[S']_P = [S]_P$ and such that f is psd on S'. Equivalently, f is psd on S around P if and only if P is not contained in the closure of $\{Q \in S \cap W(R) \mid f(Q) < 0\}$.

Recall that a function $f\colon R^n \to R^m$ is called *nash* if it is semialgebraic and of class $\mathcal{C}^\infty$ (see Bochnak, Coste, and Roy [**6**], ch. 8). A nash function $\gamma\colon (-\varepsilon, \varepsilon) \to V(R)$, $\varepsilon > 0$, such that $\gamma(0) = P$ is called a *nash curve in V centered in P*. Given such γ, taking the Taylor expansion of γ in 0 yields a homomorphism $\varphi\colon \mathcal{O}_{V,P} \to R[[t]]_{\mathrm{alg}}$, where $R[[t]]_{\mathrm{alg}}$ is the ring of algebraic power series in t (see *ibid.*, Cor. 8.1.6). Conversely, any such homomorphism defines a nash curve $(-\varepsilon, \varepsilon) \to V(R)$ centered in P for some $\varepsilon > 0$. This constitutes a natural correpondence between germs of nash curves in V centered in P and R-algebra homomorphisms $\varphi\colon \mathcal{O}_{V,P} \to R[[t]]_{\mathrm{alg}}$. For any power series $0 \neq f = \sum_i a_i t^i \in R[[t]]$, we put $\mathrm{ord}(f) = \min\{i \geqslant 0 \mid a_i \neq 0\}$, the order of f, and $\mathrm{lc(f)} = \mathrm{a_{ord(f)}}$, the leading coefficient of f. We also define $\mathrm{ord}(0) = \infty$.

Lemma 3.29 (Nash curve selection lemma) — *Let V be an R-variety, S a semialgebraic subset of $V(R)$, and P a point in the closure of S in $V(R)$. Then there exists a nash curve $\gamma\colon (-1,1) \to V(R)$ centered in P such that $\gamma\big((0,1)\big) \subset S$.*

Proof — see Bochnak, Coste, and Roy [**6**], Prop. 8.1.13. □

Corollary 3.30 — *Let $f \in R[V]$, S a semialgebraic subset of $V(R)$, and $P \in V(R)$ a point. The following are equivalent:*

(1) *f is psd on S around P.*
(2) *For every nash curve $\gamma\colon (-1,1) \to V(R)$ centered in P such that $\gamma\big((0,1)\big) \subset S$ there exists $\varepsilon > 0$ such that $f(\gamma(a)) \geqslant 0$ for all $0 \leqslant a < \varepsilon$.*

Proof — The implication (1)⇒(2) is clear. Conversely, suppose that f is not psd on S around P. This means that P is contained in the closure of $S \cap \mathcal{Y}(-f)$ in $V(R)$, hence (2) fails by the nash curve selection lemma. □

Note that if $0 \neq h \in R[[t]]_{\mathrm{alg}}$ is an algebraic power series and $h\colon (-\varepsilon, \varepsilon) \to R$ the corresponding nash function, then $\mathrm{lc(h)} > 0$ precisely if there exists $0 < \delta < \varepsilon$ such that $h(a) > 0$ for all $0 < a < \delta$. We will now use this property to relate the behaviour of a function $f \in \mathcal{O}_{V,P}$ around a non-singular point $P \in V(R)$ to its leading form: For $f \in \mathcal{O}_{V,P}$, let $\ell_P(f)$ denote the *leading form of f in P*, defined as follows: Let $\mathrm{Gr}(\mathcal{O}_{V,P}) = \bigoplus_{n \geqslant 0} \mathfrak{m}_P^n/\mathfrak{m}_P^{n+1}$ where $\mathfrak{m}_P$ is the maximal ideal of $\mathcal{O}_{V,P}$. Since P is non-singular, we can choose a regular system of parameters $x_1, \dots, x_d \in \mathfrak{m}_P \setminus \mathfrak{m}_P^2$, where d is the local dimension of V in P. Put $\xi_i = x_i + \mathfrak{m}_P^2$, then $\mathrm{Gr}(\mathcal{O}_{V,P}) \cong R[\xi_1, \dots, \xi_d]$ as graded R-algebras (see Matsumura [17], Thm. 35, p. 120). Let $n = \mu_P(f) = \max\{i \geqslant 0 \mid f \in \mathfrak{m}_P^i\}$. Now $\ell_P(f)$ is defined to be the image of $f + \mathfrak{m}_P^{n+1} \in \mathfrak{m}_P^n/\mathfrak{m}_P^{n+1}$ in $\mathrm{Gr}(\mathcal{O}_{V,P})$, a homogeneous polynomial of degree n.

Lemma 3.31 — *Let V be an affine R-variety of dimension $d \geqslant 1$, let $P \in V(R)$ be a non-singular point, and let $x, z_2, \dots, z_d$ be a regular system of parameters for $\mathcal{O}_{V,P}$. Write $\xi = \ell_P(x)$ and $\zeta_i = \ell_P(z_i)$ for $i \in \{2, \dots, d\}$, so that* $\mathrm{Gr}(\mathcal{O}_{V,P}) \cong R[\xi, \zeta_2, \dots, \zeta_d]$. *For $f \in \mathcal{O}_{V,P}$ write $n = \mu_P(f)$ and $L = \ell_P(f)$, a homogeneous polynomial of degree n in $\xi, \zeta_1, \dots, \zeta_d$.*

(1) *If f is psd on $\mathcal{S}_P(x)$, then $L(a) \geqslant 0$ for all $a = (a_1, \dots, a_d) \in R^d$ with $a_1 > 0$.*
(2) *If L is positive definite, i.e. $L(a) > 0$ for all $a = (a_1, \dots, a_d) \neq (0, \dots, 0)$, then f is psd on $V(R)$ around P.*
(3) *If $L = c \cdot \xi^n$ for some $c > 0$, and $M = \ell_P(f - cx^n)$ satisfies $M(0, a_2, \dots, a_d) > 0$ for all $(a_2, \dots, a_d) \neq (0, \dots, 0)$, then f is psd on $\mathcal{S}_P(x)$.*

Proof — Since all statements are local in P and P is a non-singular point, we can assume that V is irreducible and that $x, z_2, \dots, z_d$ are regular on all of V (after replacing V by the component containing P and a suitable Zariski-open neighbourhood of P therein if necessary). Write $U = \{Q \in V(R) \mid x(Q) > 0\}$ so that $\mathcal{S}_P(x) = [\mathrm{clos}(U)]_P$.

(1) Suppose that there exists $a = (a_1, \dots, a_d) \in R^d$, $a_1 > 0$, such that $L(a) < 0$. We will construct a nash curve $\gamma\colon (-1,1) \to V(R)$ centered in P with $\gamma\big((0,1)\big) \subset U$ such that $f(\gamma(a)) < 0$ for all $0 < a < \varepsilon$ for some $\varepsilon > 0$. This will show that f is not psd on $\mathcal{S}_P(x)$ by Cor. 3.30 above. By the preceding discussion, this

amounts to giving a homomorphism $\varphi\colon \mathcal{O}_{V,P} \to R[[t]]_{\mathrm{alg}}$ such that $\mathrm{lc}(\varphi(\mathrm{x})) > 0$ and $\mathrm{lc}(\varphi(\mathrm{f})) < 0$. Since P is non-singular, the completion $\widehat{\mathcal{O}}_{V,P}$ of the local ring $\mathcal{O}_{V,P}$ of V in P is isomorphic to the power series ring $R[[\xi, \zeta_2 \dots, \zeta_d]]$. Let φ be the homomorphism $\mathcal{O}_{V,P} \to R[[t]]_{\mathrm{alg}}$ obtained by composing the completion map $\mathcal{O}_{V,P} \to \widehat{\mathcal{O}}_{V,P}$ with the homomorphism

$$\begin{array}{ccc} R[[\xi, \zeta_2, \dots, \zeta_d]] & \to & R[[t]] \\ \xi & \mapsto & a_1 t \\ \zeta_i & \mapsto & a_i t. \end{array}$$

Then for any $g \in \mathcal{O}_{V,P}$, we have $\varphi(g) = \ell_P(g)(a_1, \dots, a_d)t^{\mu(g)}$+(terms of order $> \mu(g)$). From this we see $\mathrm{lc}(\varphi(\mathrm{x})) > 0$ and $\mathrm{lc}(\varphi(\mathrm{f})) < 0$, as desired.

(3) Let $\gamma\colon (-1, 1) \to V(R)$ be a nash curve through P such that $\gamma\big((0,1)\big) \subset U$ corresponding to a homomorphism $\varphi\colon \mathcal{O}_{V,P} \to R[[t]]_{\mathrm{alg}}$ such that $\mathrm{lc}(\varphi(\mathrm{x})) > 0$. We want to apply Cor. 3.30 again. We can assume $\varphi(f) \neq 0$ and have to show that $\mathrm{lc}(\varphi(\mathrm{f})) > 0$. Let $a = \mathrm{lc}(\varphi(\mathrm{x})) > 0$ and $k = \mathrm{ord}(\varphi(x))$. Write $b_i = \mathrm{lc}(\varphi(\mathrm{z_i}))$ for all $i \in \{2, \dots, d\}$ such that $\varphi(z_i) \neq 0$. There are two cases to consider:

(a) $k \leqslant \mathrm{ord}(\varphi(z_i))$ for all $i \in \{2, \dots, d\}$. After rearranging $z_2, \dots, z_d$, we can assume that $\mathrm{ord}(\varphi(z_2)) = \dots = \mathrm{ord}(\varphi(z_e)) = k$ and $\mathrm{ord}(z_i) > k$ for all $i > e$ for some $1 \leqslant e \leqslant d$. Put

$$\begin{aligned} f_0 &= L(x, z_2, \dots, z_e, 0, \dots, 0), \\ f_1 &= L(x, z_2, \dots, z_d) - f_0, \text{ and} \\ f_2 &= f - L(x, z_2 \dots, z_d), \end{aligned}$$

then $f = f_0 + f_1 + f_2$, and $\mathrm{lc}(\varphi(\mathrm{f_0})) = \mathrm{lc}(\mathrm{L}(\varphi(\mathrm{x}), \varphi(\mathrm{z_2}), \dots, \varphi(\mathrm{z_e}), 0, \dots, 0)) = \mathrm{L}(\mathrm{a}, \mathrm{b_2}, \dots, \mathrm{b_e}, 0, \dots, 0) = \mathrm{ca^n} > 0$ by hypothesis; furthermore, $\mathrm{ord}(\varphi(f_0)) = kn$, while $\mathrm{ord}(\varphi(f_1)) > kn$ since every monomial in f_1 must involve one of $z_{e+1}, \dots, z_d$. Finally, $f_2 \in \mathfrak{m}_P^{n+1}$ which implies $\mathrm{ord}(\varphi(f_2)) \geqslant k(n+1)$. It follows that $\mathrm{lc}(\varphi(\mathrm{f})) = \mathrm{lc}(\varphi(\mathrm{f_0})) > 0$.

(b) There exists $i \in \{2, \dots, d\}$ such that $\mathrm{ord}(\varphi(z_i)) < k$. Again, we can rearrange $z_2, \dots, z_d$ and assume $\mathrm{ord}(\varphi(z_2)) = \dots = \mathrm{ord}(\varphi(z_e)) = l$ while $\mathrm{ord}(\varphi(z_i)) > l$ for all $e < i \leqslant d$ for some $2 \leqslant e \leqslant d$. Let $g = f - L(x, z_2, \dots, z_d)$. Since $\mathrm{lc}(\varphi(\mathrm{L}(\mathrm{x}, \mathrm{z_2}, \dots, \mathrm{z_d})) = \mathrm{lc}(\mathrm{c}\varphi(\mathrm{x})^{\mathrm{n}}) = \mathrm{ca^n} > 0$, by hypothesis, it suffices to show that $\mathrm{lc}(\varphi(\mathrm{g})) > 0$. Put

$$\begin{aligned} M &= \ell_P(g), \\ g_0 &= M(0, z_2, \ldots, z_e, 0, \ldots, 0), \\ g_1 &= M(x, z_2, \ldots, z_d) - g_0, \text{ and} \\ g_2 &= g - M(x, z_2 \ldots, z_d), \end{aligned}$$

then $g = g_0 + g_1 + g_2$ and $\mathrm{lc}(\mathrm{g}_0) = \mathrm{M}(0, \mathrm{b}_2, \ldots, \mathrm{b}_\mathrm{e}, 0, \ldots, 0) > 0$ by hypothesis, and we can conclude $\mathrm{lc}(\varphi(\mathrm{g})) = \mathrm{lc}(\varphi(\mathrm{g}_0)) > 0$ as in (a).

(2) As in the proof of (3), let $\varphi: \mathcal{O}_{V,P} \to R[[t]]_{\mathrm{alg}}$ be such that $\varphi(f) \neq 0$, then we have to show that $\mathrm{lc}(\varphi(\mathrm{f})) > 0$. This can be argued in the same way as for g in case (b) above. □

Next, we will need a local-global principle. This is most easily carried out in the real spectrum. All we require can be copied almost word-by-word from Scheiderer [27], 5.3-5.5: Let A be a ring, X a constructible subset of $\mathrm{Sper}(A)$, $\alpha \in \mathrm{Sper}(A)$ a point, and $f \in A$. We say that f is locally psd on X around α if α is not contained in the closure of $\{\beta \in \mathrm{Sper}\, A \mid f(\beta) < 0\}$ or, equivalently, if $f(\beta) \geqslant 0$ holds for every generalisation $\beta \in X$ of α. (We could have developed the entire section in this language which has the advantage of being more adept to the local case; e.g. there would have been no need to drag along the variety V in the statement of Lemma 3.31.)

Lemma 3.32 — *Let I be an ideal of A, and suppose that $f \in A$ is psd on $X \cap \mathcal{Z}(I)$. Assume that for every $\alpha \in X \cap \mathcal{Z}(I + (f))$ there exists $h_\alpha \in I$ such that h_α is psd on X and $f + h_\alpha$ is locally psd on X around α. Then there is $h \in I$ such that $f + h$ is psd on X.*

Proof — Let $\alpha \in X$. We will show that there is $h_\alpha \in I$ such that h_α is psd on X and $(f + h_\alpha)(\alpha) \geqslant 0$. Suppose first that α specialises to a point in $\mathcal{Z}(I)$, i.e. there exists $\beta \in \mathcal{Z}(I)$ with $\beta \in \overline{\{\alpha\}}$. Now if $f(\beta) = 0$, the existence of h_α is part of the hypothesis while if $f(\beta) > 0$ we may take $h_\alpha = 0$ (note that β must lie in X). Thus we are left with the case where $\overline{\{\alpha\}} \cap \mathcal{Z}(I) = \varnothing$. This implies the existence of $g \in I$ with $g(\alpha) \geqslant 1$ (see Lemma 2.16). Put $h_\alpha = (1 + f^2)g^2$, then h_α is psd even on all of $\mathrm{Sper}(A)$ and $h_\alpha(\alpha) > |f(\alpha)|$.

Now since X is compact with respect to the constructible topology, there are finitely many points $\alpha_1, \ldots, \alpha_n \in X$ such that $X = \bigcup_{i=1}^n \mathcal{X}(f + h_\alpha)$. Put $h = \sum_{i=1}^n h_\alpha$, then $h \in I$ and $f + h \geqslant 0$ on X. □

Corollary 3.33 — *For every $f \in A$ which is strictly positive on $X \cap Z(I)$, there is $h \in I$ such that $f + h$ is strictly positive on X.*

Proof — Proceed as in the proof of the Lemma but choose h_α such that $(f + h_\alpha)(\alpha) > 0$ for every $\alpha \in X$. □

Going back to the original geometric question, we can apply the above propositions to $\widetilde{S}$, the constructible subset of $V_r = \operatorname{Sper}(R[V])$ associated with S. We first note the following corollary:

Corollary 3.34 — *Let V be an affine R-variety, Z a closed subvariety of V, and S a semialgebraic subset of $V(R)$. Then every function in $R[Z]$ that is strictly positive on $S \cap Z(R)$ can be extended to a function in $R[V]$ that is strictly positive on S.* □

Theorem 3.35 — *Let V be a non-singular affine R-variety, C a non-singular curve on V, and S a semialgebraic subset of $V(R)$. Assume that for every point $P \in \partial(S \cap C(R))$ there exists $x \in \mathcal{O}_{V,P}$ such that x is psd on $[S]_P$ and such that the image of x in $\mathcal{O}_{C,P}$ is a regular parameter of $\mathcal{O}_{C,P}$. Then every function in $R[C]$ that is psd on $S \cap C(R)$ can be extended to a function in $R[V]$ that is psd on S.*

Proof — As V is non-singular, we can assume that it is also irreducible since otherwise we can argue on each component separately. Let $f \in R[V]$ be psd on $S \cap C(R)$. If f vanishes identically on C, then $f^2|_C = f|_C$, and we are done. So we may assume that f has only finitely many real zeros $P_1, \dots, P_r$ in $S \cap C(R)$. Let $I = \mathcal{I}_V(C)$ be the prime ideal of $R[V]$ corresponding to C. We will prove the following:

Claim: For each $i \in \{1, \dots, r\}$ there exists an integer $k_i \geqslant 1$ and $h_i \in I \cdot \mathcal{O}_{V,P_i}$ such that for every $h \in \mathcal{O}_{V,P_i}$ with $h - h_i \in \mathfrak{m}_{P_i}^{k_i}$, the function $f + h$ is psd on S around P_i.

The claim will imply the theorem as follows: By the subsequent lemma, we can find $h \in I$ with $h - h_i \in \mathfrak{m}_{P_i}^{k_i}$ for all $i \in \{1, \dots, r\}$ simultaneously, so that $f + h$ is psd on S around P_i for all $i \in \{1, \dots, r\}$. Now apply Lemma 3.32 to $f + h$ with $A = R[V]$ and $X = \widetilde{S}$. (To see that the hypotheses of the lemma are indeed satisfied, note that every $\alpha \in \widetilde{S} \cap \mathcal{Z}(I + (f))$ must correspond to one of $P_1, \dots, P_r$, since f does not vanish identically on C).

To prove the claim, fix some $i \in \{1, \dots, r\}$ and write $P = P_i$. For $g \in \mathcal{O}_{V,P}$, denote the image of g in $\mathcal{O}_{C,P} \cong \mathcal{O}_{V,P}/I \cdot \mathcal{O}_{V,P}$ by $\overline{g}$. Let $x \in \mathcal{O}_{V,P}$ be such that $\overline{x}$ is a regular parameter in $\mathcal{O}_{C,P}$, and, using the hypothesis, we choose

x to be psd on S around P if $P \in \partial(S \cap C(R))$. Note that in the latter case we have $[\mathcal{S}(x) \cap C(R)]_P = [S \cap C(R)]_P$. Now choose generators $z_2, \dots, z_d \in \mathcal{O}_{V,P}$ of $I \cdot \mathcal{O}_{V,P}$ such that $x, z_2, \dots, z_d$ is a regular system of parameters (see Matsumura [17], Thm. 36, p. 121), i.e. $\mathfrak{m}_P = (x, z_2, \dots, z_d)$. Write $\xi = \ell_P(x)$, $\zeta_i = \ell_P(z_i)$, and $\overline{\xi} = \ell_P(\overline{x})$. Since f is psd on $\mathcal{S}(x) \cap C(R)$, Lemma 3.31 (1) implies $\ell_P(\overline{f}) = c\overline{\xi}^n$ for some $c > 0$, $n \geqslant 0$. Thus there is $g' \in I \cdot \mathcal{O}_{V,P}$ such that $\ell_P(f + g') = c\xi^n + A(\xi, \zeta_2, \dots, \zeta_n)$ where A is a homogeneous polynomial of degree n over R such that each term of A is divisible by one of $\zeta_2, \dots, \zeta_n$. Put $g = g' - A(x, z_2, \dots, z_n) \in I\mathcal{O}_{V,P}$. What we have accomplished so far is that

$$\ell_P(f + g) = c\xi^n.$$

Now there are two cases to consider:

(1) $n = 2m$ is even. In this case, put $h' = g + z_2^{2m} + \dots + z_d^{2m}$. Then for every $h \in R[V]$ with $h - h' \in \mathfrak{m}_P^{2m+1}$ we have

$$\ell_P(f + h) = c\xi^{2m} + \zeta_2^{2m} + \dots + \zeta_d^{2m}$$

which is a positive definite form. Therefore, $f + h$ is psd on $V(R)$ around P and hence on $[S]_P$ by Lemma 3.31 (2), which proves the claim.

(2) $n = 2m - 1$ is odd. Note that this case can only occur if $P \in \partial(S \cap C(R))$. We have $\mu_P(f + g - cx^n) \geqslant 2m$, and we can decompose the leading form as $\ell_P(f + g - cx^n) = \xi \cdot B_1(\xi, \zeta_2, \dots, \zeta_d) + B_2(\zeta_2, \dots, \zeta_d)$, with forms B_1 and B_2, $\deg B_2 \geqslant 2m$. Put $h' = g - B_2(z_2, \dots, z_d) + z_2^{2m} + \dots + z_d^{2m}$. Now things are arranged in such a way that whenever $h \in R[V]$ satisfies $h - h' \in \mathfrak{m}_P^{2m+1}$, then $\ell_P(f + h) = \ell_P(f + g) = c\xi^n$ and

$$\ell_P((f + h) - cx^n) = \xi C(\xi, \zeta_2, \dots, \zeta_d) + \zeta_2^2 + \dots + \zeta_d^2$$

for some form C (in fact, $C = B_1$ if $\mu_P(f + g - cx^n) = 2m$ and $C = 0$ otherwise), so that the hypotheses of Lemma 3.31 (3) are satisfied. Thus $f + h$ is psd on $\mathcal{S}_P(x)$ and hence on $[S]_P$, as desired.

□

Lemma 3.36 — *Let A be a ring, $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ maximal ideals, and I any ideal of A. There is a canonical isomorphism*

$$\frac{I}{I \cap \bigcap_i \mathfrak{m}_i^{k_i}} \cong \prod_i \frac{IA_{\mathfrak{m}_i}}{(I \cap \mathfrak{m}_i^{k_i})A_{\mathfrak{m}_i}},$$

for all integers $k_i \geqslant 1$, $i \in \{1, \dots, r\}$.

Proof — Assume first that $r = 1$ and put $\mathfrak{m} = \mathfrak{m}_1$, $k = k_1$. Define a homomorphism $\varphi\colon I/(I \cap \mathfrak{m}^k) \to IA_\mathfrak{m}/(I \cap \mathfrak{m}^k)A_\mathfrak{m}$, $a + (I \cap \mathfrak{m}^k) \mapsto \frac{a}{1} + (I \cap \mathfrak{m}^k)A_m$. It is not hard to check that φ is well-defined and injective. For surjectivity, note that every $s \in A \setminus \mathfrak{m}$ is a unit in $A/\mathfrak{m}^k$: For if $t \in \mathfrak{m}$, then $(1-t)^{-1} \equiv \sum_{i \geqslant 0}^{k-1} t^i$ modulo $\mathfrak{m}^k$. Since $\mathfrak{m}$ is maximal, we can find $t \in \mathfrak{m}$ such that $st + 1 \in \mathfrak{m}$, so s is indeed a unit in $A/\mathfrak{m}^k$. This shows that $\frac{a}{s} + (I \cap \mathfrak{m}^k)A_\mathfrak{m} = \varphi(\frac{a}{s})$ for all $a \in I$, $s \in A \setminus \mathfrak{m}$, so that φ is surjective. For the general case, apply the Chinese remainder theorem. (That the ideal I is not present in the usual form of the theorem makes no difference to the proof.) □

For surfaces, we obtain the following corollary to the theorem:

Corollary 3.37 — *Let V be a non-singular, affine surface over R, $h_1, \dots, h_r \in R[V]$, and $S = \mathcal{S}(h_1, \dots, h_r)$. Let C be a non-singular curve on V. If for every point $P \in \partial(S \cap C(R))$ there exists $i \in \{1, \dots, r\}$ such that C and $\mathcal{V}_V(h_i)$ intersect transversally in P, then every function in $R[C]$ that is psd on $S \cap C(R)$ can be extended to a function in $R[V]$ that is psd on S.* □

Examples 3.38 — Let $V = \mathbf{A}_R^2$ be the affine plane with coordinates x, y, and let C be the curve $\{x = 0\}$ so that $R[C] \cong R[y]$. For $h_1, \dots, h_r \in R[x, y]$ write $S = \mathcal{S}(h_1, \dots, h_r)$.

(1) Put $r = 1$, $h_1 = x^2 + y^2 - 1$. Then the corollary applies, showing that every polynomial in y that is psd on $S \cap C(R)$ can be extended to a polynomial in x, y that is psd on S.

(2) Put $r = 1$, $h_1 = y^3 - x^2$. Then the corollary does not apply since the intersection of $\mathcal{V}(h_1)$ and C in the point $(0, 0)$ is not transversal. However, the function y is psd on S, and $\mathcal{V}(y)$ and C intersect transversally in $(0, 0)$, so we can add y to the description of S and the corollary applies again.

(3) Put $r = 1$, $h_1 = y^3 + x^2$. This time, there is no way to fix the non-transversal intersection of $\mathcal{V}(h_1)$ and C. In fact, it is easy to see that the polynomial y, which is psd on $S \cap C(R)$, cannot be extended to a a polynomial in x, y that is psd on S.

CHAPTER 4

SURFACES

4.1. PRELIMINARIES

In this chapter, we will always work in the following setup: Let R be a real closed field, and let V be an affine surface over R. Let $h_1, \dots, h_r \in R[V]$ be given, and put $S = \mathcal{S}(h_1, \dots, h_r)$ and $T = \mathrm{PO}(h_1, \dots, h_r)$. We seek ways to decide whether T has the strong moment property (or is even saturated). Assume that V is normal and irreducible (i.e. $R[V]$ is an integrally closed domain) and that S is regular at infinity (see Def. 1.20). Under these hypotheses, we have shown that V possesses a good compactification $V \hookrightarrow X$ with dense boundaries for S (Thm. 1.22) and that the ring $B_V(S)$ of bounded functions on S is a finitely generated R-algebra (Cor. 1.28). Write $W = \operatorname{Spec} B$, then W is an irreducible, normal, affine variety over R of dimension 0, 1, or 2, i.e. W is a point, a non-singular curve, or a normal surface. Let $\varphi\colon V \to W$ denote the canonical bounded morphism induced by the inclusion $B \subset R[V]$. If $R = \mathbf{R}$ and W is a curve or a surface, we want to use Schmüdgen's fibre theorem to solve the moment problem for T by studying the restrictions of T to the fibres of φ. The dimension of W, while not strictly indicative of the solvability of the moment problem, can be taken as a rough guide for what kind of phenomena are to be expected. Before we subdivide into the cases $\dim(W) = 0, 1, 2$, we explain how $\dim(W)$ can sometimes be calculated from the good completion $V \hookrightarrow X$.

Assume that V is non-singular. By Thm. 1.22, X can be chosen to be non-singular, too. Let $D = X \setminus V$, then D is a reduced effective divisor on X; divide $D = D_0 + D_1$ such that $B = \mathcal{O}_X(X - D_0)$ (Prop. 1.12). Write $D_0 = Z_1 + \cdots + Z_r$ with all Z_i irreducible and distinct, and let M_{D_0} be the intersection matrix of D_0; its (i, j)-entry is $(Z_i.Z_j)$ where $(-.-)$ denotes the intersection pairing on $\mathrm{Pic}(V)$. For $i \neq j$, the integer $(Z_i.Z_j)$ is just the number of intersection points of Z_i and Z_j in V while the definition of the self-intersection $Z_i^2 = (Z_i.Z_i)$ is only possible in terms of divisor classes (see e.g. Hartshorne [**9**], ch. V, §1). The significance of the matrix M_D for our purposes is the following:

(1) If M_D is negative definite, then W has dimension 0.
(2) If M_D has a positive eigenvalue, then W has dimension 2.
(3) If W has dimension 1, then M_D is negative semidefinite.

(see Iitaka [**11**], §8.3, for the proof of (1) and (2) which imply (3); (1) will also follow from a more general statement proved below (Lemma 4.2)). The converse is not true in any of these cases.

4.2. THE CASE DIM(W)=0

This is equivalent to $B_V(S) = R$ (since $B_V(S)$ is integrally closed in $R(V)$). Here, one generally expects that T does not have the moment property. The strongest indication comes from the Powers-Scheiderer theorem (Cor. 2.10): If the good completion $V \hookrightarrow X$ is such that $B = \mathcal{O}_X(X) = R$, i.e. if $\mathrm{clos}_{X(R)}(S) \cap Z(R)$ is Zariski-dense in every irreducible component Z of $X \setminus V$, then $\dim(W) = 0$ and T is stable, hence T cannot have the moment property by Scheiderer's theorem (Thm. 2.12). Unfortunately, it may happen that $B = \mathcal{O}_X(X - D_0) = R$ with a non-zero divisor D_0 on X. In this section, we show that the generalised form of the Powers-Scheiderer theorem (Thm. 2.9) can still be applied, provided that the intersection matrix of M_{D_0} is negative definite. We need some preparations:

Lemma 4.1 — *Let $r \geqslant 1$, and let $x_1, \dots, x_r$ be a basis of $\mathbf{R}^r$. Then there exist integers $m_1, \dots, m_r > 0$ such that the vector $x = \sum_{i=1}^r m_i x_i$ satisfies $(x.x_i) > 0$ for all $i \in \{1, \dots, r\}$ (where $(x.y)$ denotes the euclidean inner product on $\mathbf{R}^r$).*

Proof — More generally, the following holds: Let K be an open convex cone in $\mathbf{R}^n$ with vertex 0 that does not contain any line, and let $K^* = \{x \in \mathbf{R}^n \mid (x.y) > 0 \text{ for all } y \in \overline{K}, y \neq 0\}$, the dual cone of K. Then $K \cap K^* \neq \varnothing$. For the proof, see for example Ochiai [**22**], Lemma B. Apply this to $K = \{\sum_i \lambda_i x_i \mid \lambda_i > 0 \text{ for all } i\}$, and the lemma is proved. (Note that $K \cap K^*$ is again an open cone, so it contains a point with integer coordinates, as claimed.) □

Lemma 4.2 — *Let X be a non-singular, projective surface over a field k, $\mathcal{L}$ an invertible sheaf on X, and let $C = \sum_{i=1}^r C_i$ be a reduced curve in X such that the intersection matrix $\mathcal{M}(C) = (C_i.C_j)$ is negative definite (or $C = \varnothing$). Then the space of sections $\Gamma(X \setminus C, \mathcal{L})$ is finite-dimensional over k.*

Proof — The case when $C = \varnothing$ is a standard result in algebraic geometry (see e.g. Hartshorne [**9**], ch. II, Prop. 5.19), so assume $C \neq \varnothing$. Applying embedded resolution of singularities for curves in surfaces to X and C (Thm. A.9), we may assume without loss of generality that C is a divisor with normal crossings on X. Let $D = \sum m_i C_i$ be any effective divisor supported on C with $m_i \geqslant 1$ for all $i \in \{1, \dots, r\}$. Tensoring the canonical injection of the ideal sheaf $\mathcal{O}_X(-D) = \mathcal{I}_X(D) \hookrightarrow \mathcal{O}_X$ with $\mathcal{O}_X((n+1)D) \otimes \mathcal{L}$ and taking global sections, gives injections

$$\alpha_n^D \colon \Gamma(X, \mathcal{L} \otimes \mathcal{O}_X(nD)) \hookrightarrow \Gamma(X, \mathcal{L} \otimes \mathcal{O}_X((n+1)D))$$

for all $n \geqslant 0$. On the other hand, we have injections $\Gamma(X, \mathcal{L} \otimes \mathcal{O}_X(nD)) \hookrightarrow \Gamma(X \smallsetminus C, \mathcal{L})$ for all $n \geqslant 0$, since the restriction of $\mathcal{O}_X(nD)$ to $X \smallsetminus C$ is trivial. Furthermore,

$$\Gamma(X \smallsetminus C, \mathcal{L}) = \bigcup_{n \geqslant 0} \Gamma(X, \mathcal{L} \otimes \mathcal{O}_X(nD))$$

holds with respect to these injections. Since X is projective, the k-vector spaces $\Gamma(X, \mathcal{L} \otimes \mathcal{O}_X(nD))$ are all finite-dimensional (see Hartshorne [**9**], Prop. II.5.19). Therefore, the lemma will be proved if we can show that the injections α_n^D are surjective for some suitable choice of D and all sufficiently large n.

To prove this, we choose D as follows: Since the intersection matrix $M = \mathcal{M}_X(D)$ is negative definite by hypothesis, there exists a basis $x_1, \dots, x_r$ of $\mathbf{R}^r$ such that the matrix $A = (x_1, \dots, x_r) \in \mathrm{GL_r}(\mathbf{R})$ satisfies $-M = A^t A$. Now apply Lemma 4.1 and choose $x = \sum_i m_i x_i$ with integers $m_i > 0$ such that $(x.x_i) > 0$ holds for all $i \in \{1, \dots, r\}$. Put $D = \sum m_i C_i$, then $(D.C_i) = (\sum_j m_j e_j)^t M e_i = (\sum_j m_j x_j . x_i) > 0$ holds for all $i \in \{1, \dots, r\}$. Since C has normal crossings, the intersection number $(D.C_i)$ coincides with the degree of the invertible sheaf $\mathcal{O}_X(D)$ restricted to C_i (see Hartshorne [**9**], Lemma V.1.3), so D satisfies

$$\deg_{C_i}(\mathcal{O}_{C_i}(D)) < 0$$

for all $i = 1, \dots, r$ (where $\mathcal{O}_{C_i}(D) = \mathcal{O}_X(D) \otimes \mathcal{O}_{C_i}$).
Now choose n so large that

$$n \cdot \deg_{C_i}(\mathcal{O}_{C_i}(D)) < \deg_{C_i}(\mathcal{L} \otimes \mathcal{O}_{C_i}(\textstyle\sum a_i C_i))$$

is satisfied for all r-tuples $(a_1, \dots, a_r)$ of integers with $0 \leqslant a_i \leqslant m_i$ and all $i \in \{1, \dots, r\}$. To prove that α_n^D is indeed surjective, we show that all maps

$$\Gamma(X, \mathcal{L} \otimes \mathcal{O}_X(nD)) \hookrightarrow \Gamma(X, \mathcal{L} \otimes \mathcal{O}_X(nD + \sum_j a_j C_j))$$

with $0 \leqslant a_i \leqslant m_i$ are surjective. This we prove by induction on $\sum_j a_j$: If $\sum_j a_j = 0$, there is nothing to show; so assume $\sum_j a_j > 0$ and fix $i \in \{1, \dots, r\}$ with

$a_i > 0$. From the exact sequence $0 \to \mathcal{O}_X(-C_i) \to \mathcal{O}_X \to \iota^*\mathcal{O}_{C_i} \to 0$ (where ι is the embedding of C_i into X) we obtain an exact sequence of sections

$$0 \to \Gamma(X, \mathcal{L} \otimes \mathcal{O}_X(nD + \sum_j a_jC_j - C_i)) \to \Gamma(X, \mathcal{L} \otimes \mathcal{O}_X(nD + \sum_j a_jC_j))$$
$$\to \Gamma(C_i, \mathcal{L} \otimes \mathcal{O}_{C_i}(nD + \sum_j a_jC_j)).$$

By our choice of n, the invertible sheaf $\mathcal{L} \otimes \mathcal{O}_{C_i}(nD + \sum_j a_jC_j)$ has negative degree on C_i, so the lower term in the exact sequence above is 0. From this and the induction hypothesis we see that the composition

$$0 \to \Gamma(X, \mathcal{L} \otimes \mathcal{O}_X(nD)) \to \Gamma(X, \mathcal{L} \otimes \mathcal{O}_X(nD + \sum_j a_jC_j - C_i))$$
$$\to \Gamma(X, \mathcal{L} \otimes \mathcal{O}_X(nD + \sum_j a_jC_j))$$

is surjective as claimed. □

Remark 4.3 — For $k = \mathbf{C}$, the lemma can also be directly deduced from results in complex geometry: Namely, if X is a non-singular projective surface over $\mathbf{C}$, then, by a theorem of Grauert, any curve C on X with negative definite intersection matrix can be contracted to a normal singularity in the category of complex spaces, i.e. there exists a compact 2-dimensional complex space W with at most normal singularities and a bimeromorphic map $\varphi\colon X \to W$ such that $\varphi(C) = \{P\}$ is a point, and φ is biholomorphic on $X \setminus C$ (see Barth, Peters, van de Ven [**2**], Thm. III.2.1). This implies $\Gamma(X \setminus C, \mathcal{F}) = \Gamma(W \setminus P, \varphi^*\mathcal{F}) = \Gamma(W, \varphi^*\mathcal{F})$ for any coherent sheaf $\mathcal{F}$ on X. Here, the last equality is due to Hartog's theorem: Since W is normal, any section of $\varphi^*\mathcal{F}$ over $W \setminus P$ will extend over P. Now $\Gamma(W, \varphi^*\mathcal{F})$ is finite-dimensional over $\mathbf{C}$ since $\varphi^*\mathcal{F}$ is coherent and W is compact.

We are now ready to prove the following stronger version of the Powers-Scheiderer theorem (Cor. 2.10) for surfaces:

Theorem 4.4 — *Let V be a non-singular, real, affine surface over R, and let $h_1, \dots, h_r \in R[V]$; put $S = \mathcal{S}_V(h_1, \dots, h_r)$ and $T = \mathrm{PO}(h_1, \dots, h_r)$. Assume that $V \hookrightarrow X$ is a good completion of V with dense boundaries for S, and let D_0 be the union of all irreducible components Z of $X \setminus V$ that satisfy $S \cap Z(R) = \varnothing$. If the intersection matrix of the divisor D_0 is negative definite (or if $D_0 = \varnothing$), then $B_V(S) = \mathcal{O}_X(X - D_0) = R$ and T is stable.*

Proof — We have $B_V(S) = \mathcal{O}_X(X - D_0)$ by Prop. 1.12. If $\mathcal{O}_X(X - D_0) \neq R$, it would follow that $\mathcal{O}_X(X - D_0)$ contains an element that is transcendental over R so that $\mathcal{O}_X(X-D_0)$ would be infinite-dimensional over R, contradicting Lemma 4.2 above. That T is stable follows from the general form of the Powers-Scheiderer theorem (Thm. 2.9), using Lemma 4.2 to show that the spaces U_n are finite-dimensional. □

Examples 4.5 — We give a number of examples in the affine plane: Let $V = \mathbf{A}^2_R$ with coordinates u, v, and let $h_1, \dots, h_r \in R[u, v]$, and put $S = \mathcal{S}(h_1, \dots, h_r)$, $T = \mathrm{PO}(h_1, \dots, h_r)$.

(1) The simplest type of example (already discussed in section 2.2) is when S contains an open cone, e.g. $r = 1$ and $h_1 = 1$ or $h_1 = xy$. In this case, the standard embedding $\mathbf{A}^2_R \hookrightarrow \mathbf{P}^2_R$, for which the complement of $\mathbf{A}^2_R$ in $\mathbf{P}^2_R$ is a line l_∞, is a good completion, and $S \cap l_\infty(R)$ contains a non-empty interval. Thus the Powers-Scheiderer theorem (Cor. 2.10), which is included in the above theorem as the case $D_0 = \varnothing$, applies, showing that T is stable and hence cannot have the strong moment property by Theorem 2.12.

(2) Let $r = 3$, $h_1 = 1 - (u-1)(v-1)$, $h_2 = u$, $h_3 = v$. Choose homogeneous coordinates x, y, z on $\mathbf{P}^2_R$ and put $u = \frac{x}{z}$, $v = \frac{y}{z}$ so that $\mathbf{A}^2_R$ is identified with the subset $\{z \neq 0\}$ in $\mathbf{P}^2_R$. A good completion X of $V = \mathbf{A}^2_R$ with dense boundaries for S is obtained by blowing up $\mathbf{P}^2_R$ in the points $(1:0:0)$ and $(0:1:0)$. Thus $X \smallsetminus V$ consists of three lines E_1, E_2 and L where L is the strict transform of $\{z = 0\}$ in X and E_1, E_2 are the exceptional divisors. Now $\mathrm{clos}_{X(R)}(S)$ has dense intersection with $E_1(R)$ and $E_2(R)$ but empty intersection with $L(R)$. Thus $D_0 = L$ in this case. Since we have performed two blow-ups in points of $\{z = 0\}$ to obtain L, the self-intersection L^2 is equal to -1 which is a negative-definite 1×1-matrix. Therefore, the theorem applies, showing that T is stable.

(3) For a slightly more complicated version of the preceding example, put $r = 3$, $h_1 = 1 - (u-1)v$, $h_2 = u$, $h_3 = v$. Starting with the embedding $V \hookrightarrow \mathbf{P}^2_R$ as before, a good completion $V \hookrightarrow X$ is obtained by blowing up in $(1:0:0)$ and $(0:1:0)$ giving E_1 and E_2, and then doing another blow-up on E_1 in the intersection point of the strict transforms of $\overline{\mathcal{V}(h_1)}$ and $\overline{\mathcal{V}(h_3)}$. The configuration obtained is the following: The divisor $X \smallsetminus V$ is $L + E_1 + E_2 + E_3$, where E_3 is the exceptional divisor of the third blow-up; the sets $\mathrm{clos}(S) \cap E_2(R)$ resp. $\mathrm{clos}(S) \cap E_3(R)$ are Zariski-dense in E_2 resp. E_3 while $\mathrm{clos}(S) \cap L(R) = \mathrm{clos}(S) \cap E_1(R) = \varnothing$. Thus $D_0 = L + E_1$

and the intersection matrix of D_0 is

$$\begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$$

which is again negative definite so that T is stable.

4.3. THE CASE DIM(W)=1

In this case, W is a non-singular affine curve. Unlike the previous case, it is much harder to make any general predictions about the the moment problem. Kuhlmann, Marshall, and Schwartz ([**14**], [**15**]) have proved several results in the case of cylinders in $\mathbf{R}^n$ with $n-1$-dimensional compact cross-section. For $n=2$, they give several examples that also fit in our setup. We will give a few examples of a similar kind. Just like theirs, they clearly show that the solvability of the moment problem in this situation depends sensitively on very particular properties of the set S and the equations that describe it.

Note first that there is a non-empty Zariski-open subset U of $W(R)$ such that all fibres $C_u := \varphi^{-1}(u)$, $u \in U$, are non-singular irreducible curves over R of a fixed genus g. If we can show that $T|_{C_u}$ has property (SMP) for all $u \in U$, then there remain only the finitely many non-generic fibres C_w, $w \in W(R) \setminus U$ to check for (SMP) in order to apply Schmüdgen's fibre theorem. The methods for discussing these exceptional fibres are essentially the same as in the case $\dim W = 2$ below, so it makes sense to restrict attention to the generic fibres.

Assume that $R = \mathbf{R}$ and that all C_u, $u \in U$, are non-singular rational curves. Then the moment problem for $S \cap C_u(\mathbf{R})$ is finitely solvable, i.e. for every $u \in U$ there are finitely many elements $h_{1,u}, \ldots, h_{s,u} \in \mathbf{R}[C_u]$ such that the preordering $\mathrm{PO}_{\mathbf{R}[C_u]}(h_{1,u}, \ldots, h_{s,u}) \subset \mathbf{R}[C_u]$ has property (SMP) and $\mathcal{S}_{C_u}(h_{1,u}, \ldots, h_{s,u}) = S \cap C_u(\mathbf{R})$. What remains to check, however, is whether the given generators $h_1, \ldots, h_r \in \mathbf{R}[V]$ will have this property when restricted to U or, failing that, whether suitable generators for T in $\mathbf{R}[V]$ exist. We give a number of examples:

Examples 4.6 — Let always $V = \mathbf{A}^2_{\mathbf{R}}$ with coordinates x, y.

(1) The first and best-known example is the following: Let $T = \mathrm{PO}(1-x^2) \subset \mathbf{R}[x,y]$, $S = \mathcal{S}(T)$. Then $B(S) = \mathbf{R}[x]$ and the canonical bounded morphism is the projection $(x,y) \mapsto x$. All fibres $C_u = \mathcal{V}(x-u)$, $u \in \mathbf{R}$, are isomorphic to $\mathbf{A}^1_{\mathbf{R}}$ with a natural isomorphism $\mathbf{R}[C_u] \cong \mathbf{R}[y]$. We have $T_{C_u} = \sum \mathbf{R}[y]^2$ for all $u \in [-1,1]$, and all of these have property (SMP) (they are even saturated), hence so does T by Schmüdgen's fibre

theorem. Kuhlmann and Marshall have also proved this (even a stronger statement) by more direct means, much as for the cylinder in three-space (c.f. Example 2.5). The question whether T is saturated or even whether T contains all polynomials that are strictly positive on $[-1,1]\times\mathbf{R}$ remains an open problem.

(2) For a similar example where the fibres are not lines, let $T = \mathrm{PO}(1-x^2y^2)$, $S = \mathcal{S}(T)$. It is easy to see that $B(S) = \mathbf{R}[xy]$ and the canonical bounded morphism is the map $\mathbf{R}^2 \to \mathbf{R}$, $(x,y) \mapsto xy$. All fibres $C_u = \mathcal{V}(xy-u)$ are hyperbolas for $0 \neq u \in [-1,1]$ and $T|_{C_u} = \sum \mathbf{R}[C_u]^2$. Since psd=sos holds in $\mathbf{R}[C_u]$ (Thm. 3.16), all T_{C_u}, $0 \neq u \in [-1,1]$ have property (SMP). The reducible fibre $C_0 = \mathcal{V}(xy)$ is a pair of crossing lines and the induced preordering $T|_{C_0}$ is just $\sum \mathbf{R}[C_0]^2$ so that $T|_{C_0}$ is saturated, too (see Example 3.7). Hence T has property (SMP) by Schmüdgen's fibre theorem.

(3) If we restrict to the first quadrant in the previous example, i.e. if we consider $T = \mathrm{PO}(x, y, 1-xy)$, the same argument shows that T has property (SMP). Here we have to use the fact that $T|_{C_0}$ is the preordering generated by x and y: Under the natural isomorphism $\mathbf{R}[C_0] = \mathbf{R}[x,y]/(xy) \cong \mathbf{R}[t] \times \mathbf{R}[u]$, this means that $T|_{C_0}$ is the preordering generated by $(t,0)$ and $(0,u)$ which is saturated (see Example 3.26 (1)).

On the other hand, if $T = \mathrm{PO}(x+y, 1-x^2y^2)$, then T does not have property (SMP), since the restriction $T|_{C_0}$ is the preordering generated by (t,u) which is not saturated by the discussion in Example 3.26 (1). It follows that $T|_{C_0}$ cannot have property (SMP) either (c.f. Thm. 3.28).

(4) Let $S = \mathcal{S}(1-x^2, y^3-x^2) \subset \mathbf{R}^2$. We have $B(S) = \mathbf{R}[x]$, and all fibres $C_u = \mathcal{V}(x-u)$, $u \in \mathbf{R}$, of the projection $(x,y) \mapsto x$ are isomorphic to $\mathbf{A}^1_{\mathbf{R}}$. Therefore, the strong moment problem is solvable for $S \cap C_u(\mathbf{R})$ for all $u \in \mathbf{R}$. However, there does not exist a finitely generated preordering T with $\mathcal{S}(T) = S$ such that $T|_{C_u}$ has the strong moment property for all $u \in \mathbf{R}$ simultaneously, in other words the strong moment problem for S is not finitely solvable. The reasoning is as follows: Let $h_1,\ldots,h_r \in \mathbf{R}[x,y]$ be such that $S = \mathcal{S}(h_1,\ldots,h_r)$, and put $T = \mathrm{PO}(h_1,\ldots,h_r)$. Since $S \cap C_u(\mathbf{R}) = \{(u,a) \in \mathbf{R}^2 \mid a \in \mathbf{R}, a^3 \geqslant u^2\}$ for all $u \in [-1,1]$, it follows from the results of Kuhlmann and Marshall (Thm. 3.23) that the strong moment problem for $S \cap C_u(\mathbf{R})$ is solved by $T|_{C_u}$ if and only if there exists an index $j \in \{1,\ldots,r\}$ and $\lambda \in \mathbf{R}$ such that

$$(*) \qquad h_j \equiv \lambda^2(y - \sqrt[3]{u^2}) \quad \mathrm{mod}(x-u).$$

Suppose that this holds for all $u \in [-1,1]$. Then there must be one index j such that h_j has property $(*)$ for infinitely many $u \in [-1,1]$. Therefore, h_j must share infinitely many points with the real reduced curve $\mathcal{V}(y^3 - x^2)$, hence h_j is a multiple of $y^3 - x^2$. But this contradicts property $(*)$. (Note that the cusp singularity in the origin in this example is not responsible for the failure of the strong moment property. In fact, $\mathcal{S}(1-x^2, y^3+y-x^2)$ shows the same behaviour.)

It is not quite clear in general what are suitable conditions to prevent the problem in the last example. If the intersection of a set $S = \mathcal{S}(T)$ with a non-singular rational fibre C is unbounded and has many connected components, it is hard to make the condition more explicit that the induced preordering $T|_C$ contain the proper generators to be saturated. But with suitable assumptions, one can prove specific statements like the following:

Proposition 4.7 — *Let* $T = \mathrm{PO}(h_1, \ldots, h_r)$ *be a finitely generated preordering of* $\mathbf{R}[V]$*, and let* $S = \mathcal{S}_V(T)$*. Assume that the following conditions are satisfied:*

(1) $B_V(S)$ *is of dimension* 1.
(2) *All fibres* C_u*,* $u \in \varphi_{\mathbf{R}}(S)$*, of the canonical bounded morphism* $\varphi\colon V \to W = \mathrm{Spec}(B_V(S))$ *are isomorphic to* $\mathbf{A}^1_{\mathbf{R}}$.
(3) *All sets* $S \cap C_u(\mathbf{R})$*,* $u \in \varphi_{\mathbf{R}}(S)$*, are compact or connected.*

Then T *has property (SMP) if and only if the following holds: For every* $u \in W(\mathbf{R})$ *such that* $S \cap C_u(\mathbf{R})$ *is different from* $C_u(\mathbf{R})$ *and non-compact, there exists* $i \in \{1, \ldots, r\}$ *such that the boundary point of* $S \cap C_u(\mathbf{R})$ *is the only intersection point of the curves* $\mathcal{V}(h_i)$ *and* C_u*, and that intersection is transversal.*

Proof — Assume that the stated condition is satisfied. Let $u \in \varphi_{\mathbf{R}}(S)$. By Schmüdgen's fibre theorem and Lemma 2.3, T has property (SMP) if and only if $T|_{C_u}$ has property (SMP) for all $u \in \varphi_{\mathbf{R}}(S)$. If $S \cap C_u(\mathbf{R})$ is compact, then $T|_{C_u}$ has property (SMP) by Schmüdgen's Positivstellensatz. If $S \cap C_u(\mathbf{R}) = C_u(\mathbf{R})$, then $T|_{C_u}$ also has property (SMP) since psd=sos holds in $\mathbf{R}[C_u]$. Otherwise, $S \cap C_u(\mathbf{R})$ has by assumption a unique boundary point P. Let $i \in \{1, \ldots, r\}$ be such that $\mathcal{V}(h_i) \cap C_u = \{P\}$. Write $\mathbf{R}[C_u] \cong \mathbf{R}[t]$. Then $g_i := h_i|_{C_u}$ is a polynomial in $\mathbf{R}[t]$ whose only zero is the point $\alpha \in \mathbf{R}$ corresponding to P. Thus $g_i = \lambda(t - \alpha)$ for some $\lambda \in \mathbf{R}$, and the sign of λ must be such that g_i is non-negative on $S \cap C_u(\mathbf{R})$. Hence g_i is a "natural generator" for $S \cap C_u(\mathbf{R})$ so that $T|_{C_u}$ is saturated by the result of Kuhlmann and Marshall (Thm. 3.23) and thus has property (SMP). By the same theorem (3.23), this condition is also necessary which proves the converse. □

4.4. THE CASE DIM(W)=2

In this case, we have shown in Cor. 1.5 that the canonical bounded map φ is automatically birational so that there exist only finitely many fibres of φ that consist of more than one point. Let $C_1, \dots, C_r$ be all the irreducible curves in V contracted by φ. Let $C = C_1 \cup \cdots \cup C_r$, and let C' be the subcurve of C which is the union of all those C_i for which $B_{C_i}(S \cap C_i(\mathbf{R})) = \mathbf{R}$. By Schmüdgen's fibre theorem, T has property (SMP) if and only if $T|_C$ has (SMP) which in turn is equivalent to $T|_{C'}$ being saturated (see Thm. 3.28). The only case where this is known to be possible is when all irreducible components of C' are open subcurves of $\mathbf{A}^1_{\mathbf{R}}$ and all intersection points of C' are real ordinary double points. Sufficient conditions for $T|_{C'}$ to be saturated in this case were discussed in section 3.3.

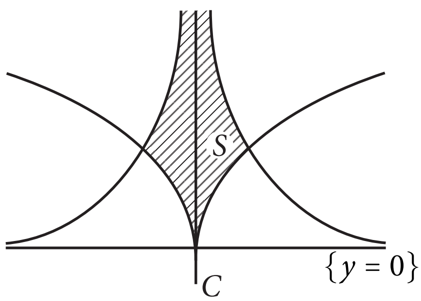
If C' has an irreducible component C_i that is non-singular and of positive genus, then $T|_{C_i}$ does not have property (SMP) by Thm. 3.22. Examples of this kind for $V = \mathbf{A}^2_{\mathbf{R}}$ can be easily constructed: Let $f \in \mathbf{R}[x,y]$ be a polynomial such that the curve $D = \mathcal{V}(f)$ has the required properties, i.e. $B(D) = \mathbf{R}$, and D is non-singular, irreducible, and of positive genus. For example, $f = y^2 - x^3 - x$ will do. Then there exists a semialgebraic set $S \subset \mathbf{R}^2$ containing $D(\mathbf{R})$ such that $B(S)$ has transcendence degree 2 and such that D is contracted under the universal bounded morphism $\mathbf{A}^2_{\mathbf{R}} \to W$: Let g be any polynomial in $\mathbf{R}[x,y]$ algebraically independent from f (for example x for f as above) and consider the set $S = \mathcal{S}(1-f^2, 1-(fg)^2)$. Clearly, $D(\mathbf{R}) \subset S$ and $f, fg \in B(S)$, so $B(S)$ has transcendence degree 2 as claimed. On the other hand, any bounded polynomial map must be constant on D by hypothesis, so D is a component of C'.

Examples 4.8 — Consider the following examples in the plane. Let $h_1, \dots, h_r \in \mathbf{R}[x,y]$, $T = \mathrm{PO}(h_1, \dots, h_r)$, and $S = \mathcal{S}(T)$.

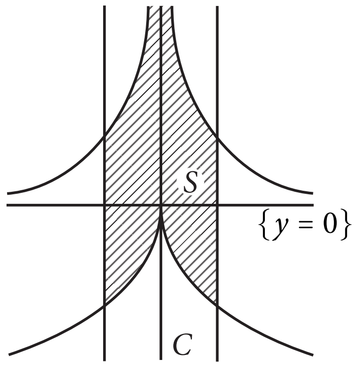
(1) Put $r = 2$, $h_1 = y^3 - x^2$, $h_2 = 1 - x^2 y$ (see illustration below). The preordering T does not have the strong moment property, since the restriction of T to $\mathcal{V}(x)$ is the preordering $\mathrm{PO}(y^3)$ in $\mathbf{R}[y]$ which does not have the strong moment property (by Thm. 3.23). On the other hand, we may add y to the description of S, and applying Schmüdgen's fibre theorem 2.4 to the projection $\pi\colon (x,y) \mapsto x$ onto the first factor, we see that the preordering $T' = \mathrm{PO}(h_1, h_2, y)$, which describes the same set S, does

have the strong moment property: Indeed, the intersection of the fibres $\pi^{-1}(a)$ with S is compact for all $a \neq 0$ while $T|_{\varphi^{-1}(0)} = T|_{\mathcal{V}(x)} = \mathrm{PO}(y)$ has the strong moment property (It is even saturated by Thm. 3.23).

(2) Put $r = 3$, $h_1 = y^3 + x^2$, $h_2 = 1 - x^2 y$, $h_3 = 1 - x^2$. With π as above, $\pi^{-1}(0) = \mathcal{V}(x)$ is once again the only fibre for which the intersection with S is non-compact. So if there were a finitely generated preordering T with $\mathcal{S}(T) = S$ such that $T|_{\varphi^{-1}(0)}$ has the strong moment property, then T would have the strong moment property by Schmüdgen's fibre theorem. But such T does not exist, since (SMP) for $T|_{\varphi^{-1}(0)}$ would mean that one of the generators of T must restrict to the function y on $\mathcal{V}(x)$ (again by Thm. 3.23) which is impossible. Therefore, the moment problem for S is not finitely solvable.



(1)



(2)

In the birational setting, it is sometimes also possible to say something about saturatedness of T rather than just the moment problem, using Scheiderer's results for the compact case. The rest of this section will be devoted to this topic. First, we reprove a theorem due to Roggero, concerning the divisor class group of a real variety of dimension at least 2.

We briefly recall the basic notion of linear systems on an algebraic variety (see also Hartshorne [**9**], ch. II, §§6,7): Let V be a normal, irreducible variety over a field k, and let $D = \sum a_Z Z$ be a Weil-divisor on V (i.e. a finite linear combination of prime diviors Z on V with coefficients in $\mathbf{Z}$.) For any point $P \in V$, we write $P \in D$ if P is contained in some prime divisor Z occuring in D, i.e. with $a_Z \neq 0$. Put $L(D) = \{f \in k(V)^\times \mid \mathrm{div}_V(f) \geqslant -D\} \cup \{0\}$. The set $L(D)$ is a linear subspace of $k(V)$. The *complete linear system* $|D|$ associated with D is the set of all effective divisors that are linearly equivalent to D. It is easy to see that $|D|$ is in canonical bijection with $(L(D) \setminus \{0\})/k^*$, which is just the projective space associated to the vector space $L(D)$, namely $|D| = \{D + \operatorname{div}(f) \mid 0 \neq f \in L(D)\}$. In this way, $|D|$ carries the structure of a projective space over k, and the dimension of $|D|$ is defined to be the projective dimension of that space, i.e. $\dim_k L(D) - 1$. A *linear system* is any projective subspace of a complete linear system. The set of base points of a linear system Λ is defined as $\bigcap \Lambda$, the set of points common to all divisors in Λ. The system is said to be *free from fixed components* if the base points do not contain any divisor.

A divisor D is said to be *compatible* with a semialgebraic subset S of $V(R)$ if $S \cap Z(R)$ is Zariski-dense in Z for every prime divisor Z occuring in D. A divisor is called *totally real* if all its prime divisors are real varieties. Hence D is totally real if and only if D is compatible with $V(R)$.

Theorem 4.9 (Roggero [26]) — *Let V be a real, irreducible, normal, affine variety over R of dimension at least 2. Let $S \subset V(R)$ be a Zariski-dense semialgebraic subset. Then every divisor on V is linearly equivalent to an effective divisor that is compatible with S. In particular, the divisor class group of V is generated by totally real divisors.*

We include a somewhat modified version of the proof for the sake of completeness and also because the set S is not present in Roggero's original theorem. We first need the following

Lemma 4.10 — *Let V be a real, irreducible, normal variety over R, and let Λ be a linear system of positive dimension on V. Then there exists a non-empty Zariski-open subset U of V such that every point of $U(R)$ is a non-singular point of some $D \in \Lambda$.*

Proof — Let $\Lambda \subset |D_0|$, and let L be the linear subspace of $L(D_0)$ corresponding to Λ. Clearly, after passing to a possibly smaller linear system, we may assume that Λ is one-dimensional, i.e. L is of dimension 2. Let $B = \bigcap \Lambda$ denote the set of base points of Λ and V_{sing} the singular locus of V. By Bertini's first theorem (see for example Zariski [39], §5, for the variant employed here), there are finitely many divisors $E_1, \ldots, E_s \in \Lambda$, $s \geqslant 0$, such that all $D \in \Lambda$, $D \neq E_1, \ldots, E_s$, have no singular points except possibly in $B \cup V_{\text{sing}}$. Now put

$$U = V \setminus \big((\textstyle\sum E_i + D_0) \cup B \cup V_{\text{sing}} \big).$$

Then $U(R)$ is non-empty since V is real, and choosing a basis $\{f_0, f_1\}$ of L, we see that every $P \in U(R)$ is contained in $\mathcal{V}(f_1(P)f_0 + f_0(P)f_1)$, hence $P \in (D_0 + \mathrm{div}(f_1(P)f_0 + f_0(P)f_1))$. □

Proof of Roggero's theorem. — Let D be any divisor on V. Fix an open dense embedding $V \hookrightarrow X$ into a normal projective variety X, and let H be the ample divisor $X \setminus V$. There is a canonical surjection of divisor class groups $\mathrm{Cl}(X) \to \mathrm{Cl}(V)$ which sends the class $[E]$ of a divisor $E = \sum n_i Z_i$ on X to the class $[E \cap V] = [\sum n_i(Z_i \cap V)]$ where it is understood that those Z_i for which $Z_i \cap V = \varnothing$ are omitted. The kernel is generated by the components of H. Let D' be the closure of D in X. Since H is ample, there is $m \geqslant 0$ and an effective divisor E on X such that $[-D' + mH] = [E]$ holds in $\mathrm{Cl}(X)$, hence $[D] = [D' \cap V] = -[E \cap V]$ in $\mathrm{Cl}(V)$. Therefore, to prove the assertion of the theorem for D, it suffices to prove it for $-D_0$, with D_0 a prime divisor on V. Given such D_0, we first show the following

Claim. There exist $f_0, f_1 \in R[V]$ such that $\mathrm{div}_V(a_0 f_0 + a_1 f_1) = D_0 + D_a$ for all but finitely many $a = (a_0 : a_1) \in \mathbf{P}^1(R)$, where the D_a are prime divisors on V.

To prove the claim, let $\mathfrak{p}_0$ be the prime ideal of $R[V]$ corresponding to D_0, and choose any non-zero $f_0 \in \mathfrak{p}_0 \setminus \mathfrak{p}_0^2$. Let B be the subring of all elements in $R[V]$ that are algebraically dependent on f_0 (i.e. B is the intersection of the relative algebraic closure of $R(f_0)$ in $R(V)$ with $R[V]$). Furthermore, let $\mathfrak{p}_0, \mathfrak{p}_1, \dots, \mathfrak{p}_r$, $r \geqslant 0$, be all the distinct prime ideals of height one in $R[V]$ that contain f_0. Now $B, \mathfrak{p}_0, \dots, \mathfrak{p}_r$ are all linear subspaces of $R[V]$ without any pairwise inclusions (note that $\mathfrak{p}_i \subset B$ for some i would imply $\mathrm{qf}(B) = R(V)$ which is impossible since B has only transcendence degree 1). Since R is an infinite field, it follows that there exists $f_1 \in \mathfrak{p}_0$, $f_1 \notin B \cup \mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_r$. Now to see that f_0 and f_1 have the desired properties, consider the one-dimensional linear system

$$\Lambda = \{-D_0 + \mathrm{div}_V(a_0 f_0 + a_1 f_1) \mid (a : b) \in \mathbf{P}^1(R)\} \subset |-D_0|.$$

From the choice of f_0, f_1, it is clear that Λ is free from fixed components. Furthermore, since f_0 and f_1 are algebraically independent, it follows that $R(a_0 f_0 + a_1 f_1)$ is relatively algebraically closed in $R(V)$ for all but finitely many $(a_0 : a_1) \in \mathbf{P}^1(R)$ (see Zariski [38], Lemma 5). Hence the linear system Λ is not composite with another pencil (see [38], p. 61), and therefore the divisors $D_a = -D_0 + \mathrm{div}_V(a_0 f_0 + a_1 f_1)$ are irreducible for all but finitely many $a = (a_0 : a_1) \in \mathbf{P}^1(R)$ by Bertini's second theorem for pencils (see [38], §10). This proves the claim.

The theorem now follows by applying Lemma 4.10 to the linear system Λ: Let U be as in the Lemma. Then $\operatorname{int}(S) \cap U(R)$ is non-empty. Thus there exists $P \in \operatorname{int}(S) \cap U(R)$ such that P is a non-singular point of some irreducible $D \in \Lambda$, so P is also an interior point of $S \cap D(R)$. Hence D is compatible with S and linearly equivalent to $-D_0$, which completes the proof. □

Remark 4.11 — In the situation of Roggero's theorem, one can require in addition that any finite collection $E_1, \ldots, E_r$ of prime divisors on V does not occur in the resulting compatible effective divisors. In fact, it suffices to apply the theorem to $S \smallsetminus (\bigcup_{i=1}^r E_i(R))$ in place of S.

It should be noted that the statement of Roggero's theorem is also true in dimension 1, though the proof is an entirely different one since Bertini's theorems cannot be applied (see Scheiderer [**27**], Thm. 2.7, and Monnier [**19**], Thm. B).

We will now apply Roggero's theorem to the study of preorderings on surfaces, yielding the following result:

Theorem 4.12 — *Let $\varphi\colon V \to W$ be a birational morphism of normal, irreducible, affine surfaces over R. Assume that there exists a totally real divisor D on V such that φ restricts to an isomorphism $V \smallsetminus D \xrightarrow{\sim} \varphi(V \smallsetminus D)$.*
Given $h_1, \ldots, h_r \in R[W]$, assume that the following conditions are satisfied:

(1) $\mathcal{U}_W(h_1, \ldots, h_r) \neq \varnothing$;
(2) $\varphi(D(R)) \subset \mathcal{U}_W(h_1, \ldots, h_r)$;
(3) $\mathrm{PO}_{R[W]}(h_1, \ldots, h_r)$ *is saturated.*

Then the preordering $\mathrm{PO}_{R[V]}(\varphi^\# h_1, \ldots, \varphi^\# h_r)$ of $R[V]$ is saturated, too.

Proof — Note first that the morphism φ is dominant by assumption so that $\varphi^\#\colon R[W] \to R[V]$ is injective. We will therefore consider $R[W]$ as a subring of $R[V]$ (inside the common function field $R(V)$) and drop $\varphi^\#$ from the notation. Write $S = \mathcal{S}_W(h_1, \ldots, h_r)$ and assume that $f \in R[V]$ is non-negative on $\mathcal{S}_V(h_1, \ldots, h_r) = \varphi_R^{-1}(S)$. Let $\operatorname{div}_W(f) = D_0 - D_1$ with D_0, D_1 effective divisors on W without common components. By Roggero's theorem 4.9, there exists $g \in R(W)$ and an effective divisor E that is compatible with S such that $\operatorname{div}_W(g) = E + D_1$. Additionally, we may assume that E has no components in common with the divisors $\operatorname{div}_W(h_i)$ (see Remark 4.11). It follows that $g \in R[W]$ and $\operatorname{div}_W(g^2 f) = D_0 + 2E + D_1$, hence $g^2 f \in R[W]$, too. Furthermore, $g^2 f$ is non-negative on S, so by hypothesis (3) there exists an expression

$$(*) \qquad g^2 f = \sum_{i \in \{0,1\}^r} s_i \underline{h}^i$$

with all $s_i \in \sum R[W]^2$. We seek to eliminate the denominator g^2.

Let $E' = \mathrm{div}_V(g)$ be the divisor of g on V. We claim that E' is compatible with $\varphi_R^{-1}(S)$. Indeed, the image of E' under φ is contained in $E \cup \varphi(D)$ (where D is the given totally real divisor where φ is not necessarily an isomorphism; note that $\varphi^{-1}(D_1) \subset D$ by the definition of D_1, since f was regular on V). In other words, E' must be supported on $E_0 + D$ where E_0 is the closure of $\varphi^{-1}(E) \cap (V \setminus D)$ in V. Both E_0 and D are compatible with $\varphi^{-1}(S)$, because E is compatible with S and $D(R) \subset \varphi^{-1}(S)$ by hypothesis (2), hence so is E'.

Let Z be a component of E'. Since Z is compatible with $\varphi^{-1}(S)$ and $\varphi^{-1}(S)$ has non-empty interior by hypothesis (1), there exists an ordering α of $R(V)$ with $h_1, \dots, h_r \in \alpha$ and such that α is compatible with the valuation ν_Z. Going back to the expression $(*)$, this implies $\nu_Z(g^2 f) = \min_{i \in \{0,1\}^r} \{\nu_Z(s_i) + \nu_Z(\underline{h}^i)\}$. But $\nu_Z(\underline{h}^i) = 0$ by our choice of E and hypothesis (2). Therefore, $\nu_Z(s_i g^{-2}) \geqslant 0$ for all $i \in \{0,1\}^r$, hence $s_i g^{-2} \sum R[V]^2$ which proves the claim. □

Corollary 4.13 — *Let W be an irreducible, normal, affine surface over R, and let $V \subset W$ be an open affine subvariety. Given $h_1, \dots, h_r \in R[W]$, assume that $\mathcal{S}_W(h_1, \dots, h_r)$ is Zariski-dense in W and that the preodering* $\mathrm{PO}_W(h_1, \dots, h_r)$ *of $R[W]$ is saturated. Then* $\mathrm{PO}_V(h_1, \dots, h_r) \subset R[V]$ *is saturated, too.*

Proof — Apply the theorem to the inclusion morphism $\varphi \colon V \hookrightarrow W$. □

Note that in the proof of the theorem we never made use of the assumption that V has dimension 2. In fact, the theorem is true in any dimension, but the hypothesis that $T \subset R[W]$ be saturated is never satisfied if the dimension is greater than 2 (Thm. 2.1). (For the case of curves, the theorem reduces to the analogue of the above corollary, which is true, though the proof is different; see Lemma 2.16 in Scheiderer [27].)
In dimension 2 however, Scheiderer's results for the compact case give many examples for finitely generated saturated preoderings, and our theorem can then be used to produce new non-compact examples. The combined result can be stated easily for sums of squares:

Corollary 4.14 — *Let $\varphi \colon V \to W$ be a birational morphism of non-singular, irreducible, affine surfaces over $\mathbf{R}$. Assume that $W(\mathbf{R})$ is compact and that the divisor $\varphi^{-1}(W \setminus \mathrm{dom}(\varphi^{-1}))$ is totally real. Then psd=sos holds in both $\mathbf{R}[W]$ and $\mathbf{R}[V]$.*

Proof — Since W is non-singular and $W(\mathbf{R})$ is compact, psd=sos holds in $\mathbf{R}[W]$ by Cor. 3.4 in Scheiderer [30]. Note that $D := \varphi^{-1}(W \setminus \mathrm{dom}(\varphi^{-1}))$ is indeed a divisor on V, i.e. of pure dimension 1 (see Lemma II.9 in Beauville [3]). Since D is totally real by hypothesis, Thm. 4.12 applies so that psd=sos holds in $\mathbf{R}[V]$. □

Examples 4.15 — (1) Let T be the preordering $\mathrm{PO}(1-u^2, 1-v^2)$ of $\mathbf{R}[u,v]$, defining a square in the plane centered around the origin. By Cor. 3.5. in [30], T is saturated. Let $\varphi\colon \mathbf{A}^2_R \to \mathbf{A}^2_R$ be the blow-up map $(x,y) \mapsto (x, xy)$, and let $D = \mathcal{V}(x)$ be the exceptional divisor. Then Thm. 4.12 applies, showing that the preordering $\mathrm{PO}(1-x^2, 1-x^2y^2)$ is saturated, too. The corresponding subset of $\mathbf{R}^2$ is non-compact, since it contains the line $L(\mathbf{R})$.

(2) Fix affine coordinates u, v on $\mathbf{A}^2_R$ and x, y, z on $\mathbf{A}^3_R$. Let $V = \mathbf{A}^2_R \setminus \mathcal{V}(u^2 + v^2+1)$, and let $W = \mathcal{V}(1-x^2-y^2-z^2) \subset \mathbf{A}^3_R$. Consider the open immersion

$$\varphi\colon \begin{cases} V & \longrightarrow & W \\ (u,v) & \longmapsto & \left(\frac{u^2+v^2-1}{u^2+v^2+1}, \frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}\right), \end{cases}$$

the inverse of the stereographic projection $W \setminus \mathcal{V}(z) \to \mathbf{A}^2_R$. Since $W(\mathbf{R})$ is compact and W is non-singular, psd=sos holds in $\mathbf{R}[W]$ by Scheiderer's result cited above. Hence psd=sos also holds in $\mathbf{R}[V]$ by Cor. 4.13. Note that $V(\mathbf{R}) = \mathbf{R}^2$, in particular, $V(\mathbf{R})$ is far from compact even though $B(V)$ has transcendence degree 2. (This result can also be deduced directly from Scheiderer's results without the use of Roggero's theorem; see Scheiderer [30]. It can be seen as an extension of Reznick's "uniform denominator theorem" to the semidefinite-case in dimension 2; see Reznick [25].)

Going back to our initial situation, let V be an irreducible, normal, affine surface over $\mathbf{R}$, and let $g_1, \ldots, g_r \in \mathbf{R}[V]$. Put $S = \mathcal{S}_V(g_1, \ldots, g_r)$ and $T = \mathrm{PO}_V(g_1, \ldots, g_r)$. Assume that S is regular at infinity and that $B := B_V(S)$ has dimension 2. Then the canonical bounded map $\varphi\colon V \to W = \mathrm{Spec}(B)$ is a birational morphism of surfaces, and one would like to apply Thm. 4.12 "upside down" in this situation, rephrasing the hypotheses of the theorem in terms of data on V rather than on W. Unfortunately, there are severe problems in putting this into practice:

(1) We have to assume the existence of the divisor D, which is essentially an assumption on the fibres of φ.
(2) We may attempt to apply the theorem to either the semialgebraic set $S' := \mathrm{clos}_{W(\mathbf{R})}(\varphi(S))$ or to $\mathcal{S}_W(T \cap B)$. The two are not necessarily equal (only

the inclusion $S' \subset \mathcal{S}_W(T \cap B)$ is automatic; see section 2.3). In any case, $\varphi^{-1}(S')$ and $\varphi^{-1}(\mathcal{S}_W(T \cap B))$ may both be strictly larger than S.

(3) The hypotheses of the theorem would ask that we exhibit a finite set of generators of the saturated preordering $\mathcal{P}(S')$ in B or to assert that $T \cap B$ is finitely generated and saturated. As for the latter, I have no idea in general how to find generators for $T \cap B$ or even how to decide whether $T \cap B$ is finitely generated. In order to then apply Scheiderer's results for saturatedness, we would have to know that S' or $\mathcal{S}_W(T \cap B)$ are sufficiently regular (and possibly even that W is non-singular), both of which may be hard to decide by just looking at the initial data.

CHAPTER 5

OPEN PROBLEMS

Here are three problems that I would still like to solve or see solved. Let always V be a non-singular $\mathbf{R}$-variety, $T \subset \mathbf{R}[V]$ a finitely generated preordering, and $S = \mathcal{S}(T)$ the corresponding semialgebraic subset of $V(\mathbf{R})$.

(1) Assume that S is regular at infinity. If $\dim(S) \geqslant 2$ and $B_V(S) = \mathbf{R}$, does it follow that T is stable and hence, by Thm. 2.12, that the moment problem for S is not finitely solvable? This is true in all known examples but there is no proof in the general situation, not even in the case $V = \mathbf{A}^2_{\mathbf{R}}$ (see section 4.2). It is not yet clear how to construct suitable filtrations of $\mathbf{R}[V]$ by finite-dimensional subspaces. A solution of this problem would nicely complement Schmüdgen's fibre theorem, which applies in the case $B_V(S) \neq \mathbf{R}$.

(2) Provided that S is regular at infinity, does V always possess a good completion with dense boundaries for S? This we have proved if $\dim(V) \leqslant 2$. The approach in the case of surfaces (Thm. 1.22) could perhaps be made to work in arbitrary dimensions, but, starting from dimension 3, the problem cannot anymore be reduced to embedded resolution of singularities alone. This is because the constructed resolution $\widetilde{X} \to X$ of a completion $V \hookrightarrow X$ has to be an isomorphism over V. If this statement were proved, it might help to generalise certain results from dimension 2 to higher dimensions, in particular in the case $B_V(S) = \mathbf{R}$. Completions with dense boundaries are also appealing in themselves and might have applications to other problems.

(3) We still lack much understanding of the preordering $T \cap B_V(S)$, in spite of the impressive results of Becker-Powers, Monnier, and Schweighofer mentioned in section 2.3. In particular, one would like to know whether it is finitely generated and how to find the generators if it is. A related

problem is to understand when the equality $\operatorname{clos}(\varphi_{\mathbf{R}}(S)) = \mathcal{S}_W(T \cap B_V(S))$ holds, where $\varphi\colon V \to W = \operatorname{Spec}\big(B_V(S)\big)$ is the canonical bounded morphism. Results in this direction would probably lead to new Positivstellensätze in the non-compact situation, as indicated in section 2.3 and at the end of section 4.4.

APPENDIX

A.1. REAL VARIETIES

Let k be a topological field, and let X be an algebraic k-variety. The topology of k induces a topology on $X(k)$. If k is an ordered field, we always equip it with the order topology and call the resulting topology on $X(k)$ the *strong topology*, because it is finer than the Zariski topology. Thus we have two topologies on $X(k)$. When no topology is mentioned, topological statements about subsets of $X(k)$ always refer to the strong topology. Note however, that on X we only consider the Zariski topology, so an open subvariety of X necessarily means a Zariski-open subvariety etc.

Proposition A.1 — *Let R be a real closed field and X an irreducible R-variety. The following are equivalent:*

(1) *The function field $R(X)$ of X is real;*
(2) *$X(R)$ is Zariski-dense in X;*
(3) *X has an R-rational non-singular point;*
(4) *the semialgebraic dimension of $X(R)$ is equal to the Krull dimension of X.*

Proof — All statements are local in nature, so we may assume that X is affine. For this case we refer to section 7.6. in Bochnak, Coste, and Roy [**6**]: By Prop. 7.6.4 *ibid.*, the closure of Sper $R(X)$ inside X_r is the constructible subset of X_r associated with $\mathrm{clos}_{X(R)}(X_{\mathrm{reg}}(R))$. In particular, this shows that (1) $\Leftrightarrow$ (3). By Prop. 7.6.2, the local semialgebraic dimension of $X(R)$ in a point x coincides with the Krull dimension of X if and only if x is the specialisation of a point of Sper $R(X)$ in X_r, which shows that (1) $\Leftrightarrow$ (4). We have (4) $\Rightarrow$ (2), since a semialgebraic set of dimension d contains an open ball of dimension d which must be Zariski-dense in any variety of dimension d. Finally, (2) $\Rightarrow$ (1), for if

$R(X)$ is not real, then -1 is a sum of squares in $R(X)$, say $-1 = \sum f_i^2$. Let U be an open subset of X where all the f_i are regular, then clearly $U(R) = \varnothing$. Hence $X(R) \subset (X \setminus U)(R)$ cannot be Zariski-dense. □

Definition — Let R be a real closed field. An R-variety is called *real* if all its irreducible components satisfy the equivalent conditions of the preceding proposition.

Lemma A.2 — *Let k be a field that is not algebraically closed, and let X be a quasi-projective k-variety. Then there exists an open affine subvariety U of X such that $U(k) = X(k)$.*

Proof — We first show that for every $n \geqslant 1$ there exists a homogeneous polynomial $F_n \in k[x_0, \ldots, x_n]$ such that $(0, \ldots, 0)$ is the only zero of F_n in k^{n+1}. If k is a real field, one can take $F_n = x_0^2 + \cdots + x_n^2$. In general, since k is not algebraically closed, there exists a non-constant polynomial $f \in k[t]$ such that $f(\alpha) \neq 0$ for all $\alpha \in k$. Let $F_1 \in k[x_0, x_1]$ be the homogenisation of $f(x_0)$, then $(0, 0)$ is the only zero of F_1 in $k \times k$. For any $n > 1$, put $F_n(x_0, \ldots, x_n) = F_1(F_{n-1}(x_0, \ldots, x_{n-1}), x_n^{\deg F_{n-1}})$. The F_n have the desired property.

Choose an embedding of X into $\mathbf{P}^n$. Now $W = \mathbf{P}^n \setminus \mathcal{V}_+(F_n)$ is an affine variety with $W(k) = \mathbf{P}^n(k)$, hence $(X \cap W)(k) = X(k)$. Upon replacing X by $X \cap W$, we may therefore assume that X is quasi-affine. Thus there exists an affine variety V and $g_1, \ldots, g_r \in k[V]$ such that $X = V \setminus \mathcal{V}(g_1, \ldots, g_r)$. Then $U = V \setminus \mathcal{V}(F_{r-1}(g_1, \ldots, g_r))$ is an affine subvariety of X with $U(k) = X(k)$. □

With a different notion of isomorphism of real varieties often used in real geometry, the lemma says that "every real variety is affine" (see e.g. Thm. 3.4.4 in Bochnak, Coste, and Roy [**6**]).

A.2. SEMIALGEBRAIC SETS

Let R be a real closed field, and let V be an affine R-variety. A subset S of $V(R)$ is called *basic closed (in V)* if there exist $h_1, \ldots, h_r \in R[V]$, $r \geqslant 0$, such that

$$S = \mathcal{S}_V(h_1, \ldots, h_r) = \{x \in V(R) \mid \forall i\colon h_i(x) \geqslant 0\}.$$

A *semialgebraic* subset of $V(R)$ is any finite boolean combination of basic closed sets, i.e. any subset obtained by taking complements and finite unions and intersections of basic closed subsets of $V(R)$. More explicitly, put

$$\mathcal{U}_V(h_1, \ldots, h_r) = \{x \in V(R) \mid \forall i\colon h_i(x) > 0\}$$

and

$$\mathcal{V}_V^R(h_1,\dots,h_r) = \{x \in V(R) \mid \forall i\colon h_i(x) = 0\}.$$

Then a subset S of $V(R)$ is semialgebraic if and only if it can be written as

$$S = \bigcup_{i=1}^{m} \big(\mathcal{U}_V(h_{i,1},\dots,h_{i,r}) \cap \mathcal{V}_V^R(g_i)\big)$$

for some $m, r \geqslant 0$, $h_{i,j}, g_i \in R[V]$. To prove this, just check that the class of subsets of the above form is closed under finite boolean operations.

Now let X be an R-variety, and let $x \in X(R)$. A subset S of $X(R)$ is called *locally semialgebraic in x* if there exists an open affine subvariety $U \subset X$ with $x \in U(R)$ such that $S \cap U(R)$ is semialgebraic in U.

Proposition A.3 — *Let X be an R-variety. A subset $S \subset X(R)$ is locally semialgebraic in every point of $X(R)$ if and only if $S \cap U(R)$ is semialgebraic in U for every open affine subvariety U of X.*

Proof — Assume that S is locally semialgebraic in every point of $X(R)$, and let U be an open affine subvariety of X. Since X is quasi-compact, there is a finite cover $U_1,\dots,U_k$ of X by open affine subvarieties such that $S \cap U_i(R)$ is semialgebraic in U_i. Since X is separated, $U_i \cap U$ is again affine, so we may assume $U_i \subset U$ for all $i = 1,\dots,k$. Since $S \cap U(R) = \bigcup_{i=1}^{r}(S \cap U_i(R))$, it suffices to show that $S \cap V(R)$ is semialgebraic in U for every open affine subvariety V of U. We may further assume that $V = D_U(s)$ for $s \in R[U]$, so that $V(R) = \mathcal{U}_U(s^2)$ and $R[V] = R[U]_s$. Now let $S \cap V(R) = \bigcup_{i=1}^{m}\big(\mathcal{U}_V(\widetilde{h}_{i,1},\dots,\widetilde{h}_{i,r}) \cap \mathcal{V}_V^R(\widetilde{g}_i)\big)$ as above, with $\widetilde{h}_{i,j}, \widetilde{g}_i \in R[V]$. Write $\widetilde{h}_{i,j} = h_{i,j} \cdot s^{-d_{i,j}}$ with $h_{i,j} \in R[U]$, $d_{i,j} \geqslant 0$ and $\widetilde{g}_i = g_i \cdot s^{-e_i}$ with $g_i \in R[U]$, $e_i \geqslant 0$, then $S \cap V(R) = \bigcup_{i=1}^{m}\big(\mathcal{U}_U(h_{i,1}s^{d_{i,1}},\dots,h_{i,r}s^{d_{i,r}}) \cap \mathcal{V}_U^R(g_i)\big) \cap \mathcal{U}_U(s^2)$, so $S \cap V(R)$ is semialgebraic in U as claimed. The converse is trivial. □

Definition — Let X be an R-variety. A subset of $X(R)$ is called *semialgebraic* if it satisfies the equivalent conditions of the preceding proposition.

Assume that X is quasi-projective. By Lemma A.2, there exists an open affine subvariety V of X such that $V(R) = X(R)$. The above proposition implies that a subset $S \subset X(R)$ is semialgebraic if and only if $S \cap V(R)$ is semialgebraic in V. Therefore, semialgebraic subsets of quasi-projective varieties behave exactly like semialgebraic subsets of affine varieties.

We recall the following facts and definitions, referring to Bochnak, Coste, and Roy [**6**] for details: Let X be a quasi-projective R-variety and S a semialgebraic subset of $X(R)$.

(1) If X' is another R-variety and S' a semialgebraic subset of $X'(R)$, a map $f: S \to S'$ is called *semialgebraic* if the graph $\Gamma_f \subset (X \times X')(R)$ is semialgebraic.
(2) The *(semialgebraic) dimension* of S is the largest integer d such that there exists a semialgebraic embedding $B^d \hookrightarrow S$ where B^d is the unit ball $\{x \in R^{d+1} \mid \|x\| < 1\}$ in R^{d+1}. The closure $\mathrm{clos}_{X(R)}(S)$ and the interior $\mathrm{int}_{X(R)}(S)$ of a semialgebraic set are again semialgebraic. It is not hard to see that $\dim(\mathrm{clos}(S)) = \dim(S)$, and if $\mathrm{int}(S) \neq \varnothing$, then $\dim(\mathrm{int}(S)) = \dim(S)$. The (semialgebraic) dimension of S is also equal to the (Krull-) dimension of its Zariski-closure in X.
(3) A semialgebraic subset of R^n is called *semialgebraically compact* (or *complete*) if it is closed and bounded. The image of a semialgebraically compact set under a continuous semialgebraic map is again semialgebraically compact. Therefore, we say that $S \subset X(R)$ is semialgebraically compact if S is semialgebraically compact for some (equivalently: for any) semialgebraic embedding of $X(R)$ into R^n, $n \geqslant 0$. A more intrinsic characterisation, that generalises sequential compactness, is the following: S is semialgebraically compact if and only if every semialgebraic path $[0,1) \to S$ extends to a semialgebraic path $[0,1] \to S$.

Proposition A.4 — *Let X be a projective R-variety. Then $X(R)$ is semialgebraically compact.*

Proof — Projective space $\mathbf{P}^n(R)$ is the image of the unit sphere $\{x \in R^{n+1} \mid \|x\| = 1\}$ under the quotient map $R^{n+1} \smallsetminus \{0\} \to \mathbf{P}^n(R)$ and is therefore semialgebraically compact. Since $X(R)$ is a closed subset of $\mathbf{P}^n(R)$ for some n, $X(R)$ is semialgebraically compact. □

A.3. DIVISORS AND DISCRETE VALUATIONS

Let X be an integral, noetherian scheme, and let Z be a closed subscheme of X. We say that X is *normal along Z* if all local rings $\mathcal{O}_{X,x}$ with $x \in Z$ are integrally closed domains.

Proposition A.5 — *Let X be an integral, noetherian scheme, and let $Z_1, \dots, Z_r$ be closed, integral subschemes of codimension* 1 *in X. If X is normal along Z_i for all $i = 1, \dots, r$, then*

$$\mathcal{O}_X(X) = \mathcal{O}_X\big(X \smallsetminus (Z_1 \cup \cdots \cup Z_r)\big) \cap \mathcal{O}_{X,Z_1} \cap \cdots \cap \mathcal{O}_{X,Z_r}.$$

This statement is sometimes referred to as Hartog's theorem, in analogy with complex analysis.

Proof — Put $Z = Z_1 \cup \cdots \cup Z_r$. We only have to show that an element $f \in \mathcal{O}_X(X \smallsetminus Z) \cap \bigcap_i \mathcal{O}_{X,Z_i}$ is regular in any point $x \in Z$. Since X is normal along Z, the local ring $\mathcal{O}_{X,x}$ is integrally closed and therefore equals the intersection of all its localisations in prime ideals of height 1 (see e.g. Bourbaki [**8**], §1, no. 6, Thm. 4). In other words, we have $\mathcal{O}_{X,x} = \bigcap \mathcal{O}_{X,y}$ where the intersection is taken over all points $y \in X$ of codimension 1 in X such that $x \in \overline{\{y\}}$. But such y is either contained in $X \smallsetminus Z$, which implies $\mathcal{O}_X(X \smallsetminus Z) \subset \mathcal{O}_{X,y}$, or it is the generic point of some Z_i so that $\mathcal{O}_{X,y} = \mathcal{O}_{X,Z_i}$. In either case, we find $f \in \mathcal{O}_{X,y}$. This shows $f \in \mathcal{O}_{X,x}$, as claimed. □

Write K for the function field of X and $\mathcal{K}$ for the locally constant sheaf with values in K. For any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, let $\mathcal{F}^\vee$ denote $\mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$, the dual sheaf of $\mathcal{F}$. If $\mathcal{I}$ is a coherent sheaf of ideals on X, then $\mathcal{I}$ can be identified with a subsheaf of $\mathcal{K}$ via the inclusion $\mathcal{I} \subset \mathcal{O}_X \subset \mathcal{K}$. Then $\mathcal{I}^\vee$ is also identified with a subsheaf of $\mathcal{K}$, in such a way that for any open affine subset U of X we have $\mathcal{I}^\vee(U) = (\mathcal{O}_X(U) : \mathcal{I}(U)) = \{f \in K \mid f\mathcal{I}(U) \subset \mathcal{O}_X(U)\}$, the ideal quotient of $\mathcal{I}(U)$ in $\mathcal{O}_X(U)$. In particular, $\mathcal{I}^\vee$ is again coherent.

Recall that a noetherian local domain of dimension 1 is integrally closed if and only if it is a discrete valuation ring (see Bourbaki [**8**], §1, no. 7, Prop. 11). Thus if Z is an integral subscheme of codimension 1 in X and X is normal along Z, then the local ring $\mathcal{O}_{X,Z}$ is a discrete valuation ring, and we denote the corresponding valuation by v_Z. We then have the following relation between the valuation v_Z and sections of the ideal sheaf of Z:

Lemma A.6 — *Let X be an integral, noetherian scheme, and let Z be a closed integral subscheme of codimension* 1 *in X with ideal sheaf $\mathcal{I}_Z$. Assume that X is normal along Z. Then for any open subset U of X and any $n \geqslant 0$, we have*

$$(\mathcal{I}_Z^n)^\vee(U) = \begin{cases} \{f \in \mathcal{O}_X(U - Z) \mid v_Z(f) \geqslant -n\} & U \cap Z \neq \varnothing \\ \mathcal{O}_X(U) & U \cap Z = \varnothing. \end{cases}$$

(Here, the notation $\mathcal{I}_Z^n$ refers to the n-fold tensor power $\mathcal{I}_Z^{\otimes n}$ of $\mathcal{I}_Z$ over $\mathcal{O}_X$ which can be identified with the sheaf $U \mapsto \mathcal{I}_Z(U)^n \subset \mathcal{O}_X(U)$ given by the usual ideal power.)

Proof — After taking an open affine cover of U, we easily reduce to the case where U is affine. Thus $(\mathcal{I}_Z^n)^\vee(U) = (\mathcal{O}_X(U) : \mathcal{I}_Z^n(U))$, as noted above. If $U \cap Z = \varnothing$, it is clear that $(\mathcal{I}_Z^n)^\vee(U) = \mathcal{O}_X(U)$. If $U \cap Z \neq \varnothing$, choose $t \in \mathcal{I}_Z(U) \cap \mathcal{O}_{X,Z}$ with $v_Z(t) = 1$, a local parameter for $\mathcal{O}_{X,Z}$. We have $(\mathcal{O}_X(U) : \mathcal{I}_Z^n(U)) \subset \mathcal{O}_X(U - Z)$: for given $f \in (\mathcal{O}_X(U) : \mathcal{I}_Z^n(U))$ and x a point in $U - Z$, there exists $h \in \mathcal{I}_Z^n(U)$ such that $h(x) \neq 0$, so $fh = g \in \mathcal{O}_X(U)$ implies $f = g/h \in \mathcal{O}_{X,x}$. Furthermore, $ft^n \in \mathcal{O}_X(U)$, so $v_Z(f) \geqslant -n$. Thus we have shown $(\mathcal{O}_X(U) : \mathcal{I}_Z^n(U)) \subset \{f \in \mathcal{O}_X(U - Z) \mid v_Z(f) \geqslant -n\}$. Conversely, let $f \in \mathcal{O}_X(U - Z)$ with $v_Z(f) \geqslant -n$, and let $g \in \mathcal{I}_Z^n(U)$. Then $v(g) \geqslant n$, hence $fg \in \mathcal{O}_X(U) \cap \mathcal{O}_{X,Z}$. Since $\mathcal{O}_X(U) \cap \mathcal{O}_{X,Z} = \mathcal{O}_X(U)$ by Prop. A.5, we are done. □

Recall that Z is a *Cartier divisor* if its ideal sheaf $\mathcal{I}_Z$ is invertible, i.e. if Z is locally defined by a single equation. In this case, $(\mathcal{I}_Z^n)^\vee$ together with its embedding into $\mathcal{K}$ is the invertible subsheaf of $\mathcal{K}$ that is denoted by $\mathcal{L}(nZ)$ in Hartshorne [**9**], ch. II.6, and the lemma above reduces to the usual correspondence between Cartier divisors and invertible subsheaves of $\mathcal{K}$.

A.4. RESOLUTION OF SINGULARITIES

Some of our constructions rely heavily on resolution of singularities in characterestic zero. Even though we only need standard results, it seems worthwhile to state them here in the required form. The original reference for resolution of singularities is Hironaka's paper [**10**].

Theorem A.7 (Resolution of singularities) — *Let k be a field of characteristic zero, and let X be a k-variety. Then there exists a non-singular k-variety $\widetilde{X}$ together with a regular map $\varphi\colon \widetilde{X} \to X$ (given by finitely many blow-ups with non-singular centers) such that φ restricts to an isomorphism $\varphi^{-1}(X_{\mathrm{reg}}) \xrightarrow{\sim} X_{\mathrm{reg}}$.*

See Main Thm. I in Hironaka [**10**].

Recall that a Cartier divisor D on a non-singular variety X is said to have *normal crossings* if all of its irreducible components are non-singular, and whenever r components $D_1, \dots, D_r$ meet in a closed point $x \in X$, their local equations $f_1, \dots, f_r$ in x are linearly independent modulo m_x^2, which means that they can be extended to a regular system of parameters of $\mathcal{O}_{X,x}$.

Theorem A.8 (Embedded resolution of singularities) — *Let k be a field of characteristic zero, let X be an irreducible, non-singular k-variety, and let $Z \subsetneq X$ be a closed subvariety. Then there exists an irreducible, non-singular k-variety $\widetilde{X}$ together with a regular map $\varphi\colon \widetilde{X} \to X$ such that φ restricts to an isomorphism $\varphi^{-1}(X \smallsetminus Z) \xrightarrow{\sim} X \smallsetminus Z$ and such that $\varphi^{-1}(Z)$ is a Cartier divisor with normal crossings in $\widetilde{X}$.*

See Cor. 3 to Main Thm. II in Hironaka [**10**].

In the simpler case of resolution of curves in surfaces, we need the following more precise version:

Theorem A.9 (Embedded resolution of curves in surfaces)

Let k be a field, X a non-singular surface over k, C a curve in X, and T a finite set of closed points on C. Then there exists a sequence $\widetilde{X} \to X_{n-1} \to \cdots \to X_1 \to X$ of blow-ups in closed points such that if $\varphi\colon \widetilde{X} \to X$ is their composition, then

(1) $\varphi^{-1}(X \smallsetminus T) \xrightarrow{\sim} X \smallsetminus T$;
(2) *all intersection points of the divisor $\varphi^{-1}(C)$ lying in $\varphi^{-1}(T)$ are normal crossings*
(3) *all irreducible components of $\varphi^{-1}(C)$ are non-singular in all points of $\varphi^{-1}(T)$.*

The proof is no different from the one of Thm. V. 3.7 in Hartshorne [**9**] where all singularities of C are resolved simultaneously, i.e. where $T = C_{\mathrm{sing}}$.

ZUSAMMENFASSUNG AUF DEUTSCH

Übersetzung der Einleitung

Es seien $h_1, \ldots, h_r$ Polynome in n Veränderlichen $x = (x_1, \ldots, x_n)$ mit reellen Koeffizienten, und es sei

$$S = \left\{a \in \mathbf{R}^n \,\middle|\, h_1(a) \geqslant 0, \ldots, h_r(a) \geqslant 0\right\}$$

die durch sie bestimmte einfach abgeschlossene semialgebraische Menge. Wir untersuchen den Kegel

$$\mathcal{P}(S) = \left\{f \in \mathbf{R}[x] \,\middle|\, \forall a \in S\colon f(a) \geqslant 0\right\}$$

aller Polynome, die auf S nicht-negativ sind. Insbesondere interessieren wir uns für den Zusammenhang zwischen $\mathcal{P}(S)$ und dem von $h_1, \ldots, h_r$ in $\mathbf{R}[x]$ erzeugten Kegel (richtiger: endlich-erzeugte Präordnung):

$$T = \mathrm{PO}(h_1, \ldots, h_r) = \left\{\sum_{i \in \{0,1\}^r} s_i h_1^{i_1} \cdots h_r^{i_r} \,\middle|\, s_i \in \sum \mathbf{R}[x]^2\right\},$$

wobei $\sum \mathbf{R}[x]^2 = \{\sum_{j=1}^k g_j^2 \mid g_1, \ldots, g_k \in \mathbf{R}[x], k \geqslant 0\}$ die Menge aller Quadratsummen in $\mathbf{R}[x]$ bezeichnet. Die Inklusion $T \subset \mathcal{P}(S)$ ist offenkundig, und es ist eine natürliche Frage, ob Gleichheit gilt, oder man doch wenigstens der Gleichheit nahe kommen kann. Derartige Fragen sind in den letzten 15 Jahren intensiv untersucht worden. Ein klassischer Satz von Hilbert besagt, dass nicht jedes auf $\mathbf{R}^n$ positive Polynom als Quadratsumme in $\mathbf{R}[x]$ geschrieben werden kann, sobald $n \geqslant 2$ ist. Unter Benutzung dieser Tatsache hat Scheiderer gezeigt, dass $\mathcal{P}(S)$ nicht endlich-erzeugt ist, also nicht mit einem Kegel der Form $\mathrm{PO}(h_1, \ldots, h_r)$ übereinstimmen kann, sobald die Dimension von S mindestens 3 ist. Gleichheit zwischen T und $\mathcal{P}(S)$ ist somit ein Phänomen, dass ausschließlich in niedriger Dimension auftreten kann. Andererseits lassen sich schwächere Aussagen in beliebiger Dimension beweisen: Schmüdgen bewies im Jahr 1991, dass jedes auf S strikt positive Polynom in T liegt, falls S kompakt ist. Allgemeiner definierte er die *Momenteneigenschaft* für T, ausgehend vom Momentenproblem der Funktionalanalysis, in dessen Zusammenhang er seinen Satz bewies. Die Momenteneigenschaft fordert, dass T in $\mathcal{P}(S)$ dicht liege im dem Sinn, dass T und $\mathcal{P}(S)$ nicht durch ein lineares Funktional getrennt werden können. Sie kann auch als eine Approximationsbedingung für Elemente in $\mathcal{P}(S)$ durch Elemente in T angesehen werden (siehe Abschnitt 2.1 für exakte Formulierungen und Referenzen).

Im Hinblick auf die Momenteneigenschaft bewies Schmüdgen vor einigen Jahren eine weitaus stärkere Form seines Satzes, nämlich:

Satz (Schmüdgen 2003) — *Es gebe eine polynomiale Abbildung $\varphi\colon \mathbf{R}^n \to \mathbf{R}^m$ derart, dass das Bild $\varphi(S)$ in $\mathbf{R}^m$ beschränkt ist. Genau dann hat T die Momenteneigenschaft, wenn die Einschränkung von T auf jede Faser $\varphi^{-1}(a)$, $a \in \mathbf{R}^m$ die Momenteneigenschaft besitzt.*

(Die Einschänkung einer Präordnung definieren wie in Abschnitt 2.1). Um Schmüdgens neuen Satz anwenden zu können, muss man zunächst wissen, ob eine Abbildung φ mit der geforderten Eigenschaft existiert. Daher betrachten wir den Teilring der auf S beschränkten Polynome in $\mathbf{R}[x]$:

$$B(S) = \left\{ f \in \mathbf{R}[x] \,\middle|\, \exists \lambda \in \mathbf{R} \forall a \in S\colon |f(a)| \leqslant \lambda \right\}.$$

Offenbar gilt $B(S) = \mathbf{R}[x]$ genau dann, wenn S kompakt ist. Allgemeiner kann die Größe des Rings $B(S)$ als ein Maß für die „Kompaktheit" von S angesehen werden. Schmüdgens Satz deutet darauf hin, dass je größer $B(S)$ ist, umso dichter liegen T und $\mathcal{P}(S)$ bei einander. Diese Sichtweise ist mir für die vorliegende Arbeit nahegelegt worden. Abgesehen von Schmüdgens Sätzen ist sie auch durch Ergebnisse im eindimensionalen Fall motiviert, wenn also S in einer algebraischen Kurve enthalten ist. Kuhlmann, Marshall und Scheiderer haben hier beinahe alle soeben diskutierten Fragen beantwortet und in verschiedenen Aspekten eine Dichotomie zwischen den Fällen $B(S) = \mathbf{R}$ und $B(S) \neq \mathbf{R}$ festgestellt.

Es folgt nun eine kurze Übersicht über Fragen und Resultate in dieser Arbeit:

(1) Eine erste Aufgabe liegt darin, $B(S)$ zu bestimmen und insbesondere zu entscheiden, welcher der Fälle $B(S) = \mathbf{R}$ oder $B(S) \neq \mathbf{R}$ vorliegt. Dies gelingt in einem gewissen Umfang dadurch, dass $B(S)$ als ein Ring

von Polynomfunktionen auf einer geeigneten algebraischen Kompaktifizierung von $\mathbf{R}^n$ aufgefasst wird. (Prop. 1.12). Diese Konstruktion wird in Kapitel 1 erklärt. Das stärkste Ergebnis beweisen wir jedoch nur für den Fall, dass die Dimension von S höchstens 2 ist (Thm. 1.22). Wir diskutieren auch die Frage, ob $B(S)$ eine endlich-erzeugte $\mathbf{R}$-Algebra ist. Wir zeigen, dass dies der Fall ist, wenn S ausreichend regulär und von Dimension höchstens 2 ist. In höheren Dimensionen gibt es jedoch Gegenbeispiele, auch solche, die ansonsten gute Eigenschaften haben (Abschnitt 1.6).

(2) Ist $B(S) = \mathbf{R}$, so gibt Schmüdgens Satz keine Auskunft, und es beginnt die Suche nach anderen Kriterien zur Entscheidung des Momentenproblems. Dazu benutzen (und erweitern) wir ein Ergebnis von Powers und Scheiderer, dass zeigt, dass die Präordnung T im Fall $B(S) = \mathbf{R}$ häufig stabil ist, was grob gesagt der Existenz von Gradschranken für Darstellungen in T entspricht. Diese Eigenschaft schließt die Momenteneigenschaft aus. Diese Begriffe und Resultate sind in Abschnitt 2.2 dargestellt.

(3) Ist $B(S) \neq \mathbf{R}$, so kann man Schmüdgens Satz anwenden und durch Einschränkung auf jede Faser das Momentenproblem im Prinzip durch einen induktiven Prozess zu entscheiden versuchen. Wir tun dies so explizit wie möglich für den Fall, dass S die Dimension 2 hat. Die Fasern sind dann üblicherweise Kurven, und man kann Scheiderers umfangreiche Resultate für irreduzible Kurven anwenden. Allerdings sind die als Fasern vorkommenden Kurven selbst unter günstigen Voraussetzungen nicht notwendig irreduzibel. Wir versuchen daher, Scheiderers Ergebnisse auf den reduziblen Fall auszudehnen. Dies gelingt vollständig für das Momentenproblem für Quadratsummen (Abschnitt 3.2), während wir für allgemeine Präordnungen nur einige Spezialfälle behandeln (Abschnitt 3.3). Diese Ergebnisse verwenden wir in Kapitel 4, um neue Beispiele und Kriterien in Dimension 2 zu produzieren.

(4) Über das Momentenproblem hinaus können wir unter bestimmten Voraussetzungen auch Aussagen zur Saturiertheit von Präordnungen, also zur Gleichheit $\mathcal{P}(S) = \mathrm{PO}(h_1, \ldots, h_r)$, treffen, wenn S von Dimension 1 oder 2 ist. Für Kurven nämlich betreffen viele der erwähnten Resultate auf reduziblen Kurven diese stärkere Eigenschaft. In Dimension 2 beweisen wir einen Satz von Roggero, der die Divisorklassengruppe einer reellen Varietät betrifft (Thm. 4.9). Diesen verwenden wir in Verbindung mit Scheiderers Ergebnissen im kompakten Fall. Dies führt zu einer

recht allgemeinen Methode, nicht-kompakte Beispiele zu produzieren, in denen die zugehörige Präordnung saturiert ist (Thm. 4.12).

(5) In einigen eindimensionalen Beispielen weiß man, dass die Frage nach der Momenteneigenschaft für eine endlich-erzeugte Präordnung $T = \mathrm{PO}(h_1, \ldots, h_r)$ von der Wahl der Erzeuger $h_1, \ldots, h_r$ abhängt. Falls nun geeignete Erzeuger für die Einschränkung von T auf die Fasern von φ gefunden werden können, ergibt sich das Problem, diese Erzeuger in geeigneter Weise zu Erzeugern von T fortzusetzen. Dies führt auf das folgende allgemeine Fortsetzungsproblem: Gegeben eine abgeschlossene Untervarietät Z in $\mathbf{R}^n$ (oder einer beliebigen umgebenden Varietät) und ein Polynom $f \in \mathbf{R}[x]$, dass auf $S \cap Z$ nicht negativ ist. Gibt es dann ein $g \in \mathbf{R}[x]$ derart, dass g nicht-negativ auf S ist und f und g auf Z übereinstimmen? Hierzu verallgemeinern wir ein Ergebnis von Scheiderer, wenn Z eine nicht-singuläre Kurve ist, vom globalen Fall auf beliebige semialgebraische Mengen (Thm. 3.35).

Zwei technische Bemerkungen erscheinen angebracht: In dieser Zusammenfassung haben wir die Ausgangssituation im affinen Raum betrachtet, also für $S \subset \mathbf{R}^n$ und $h_1, \ldots, h_r \in \mathbf{R}[x]$. Es ist jedoch natürlicher und technisch vorteilhaft, den affinen Raum durch den Zariski-Abschluss V von S zu ersetzen, eine affine $\mathbf{R}$-Varietät. Der Vorteil liegt vor allem darin, dass S in $V(\mathbf{R})$ stets nichtleeres relatives Inneres hat. Diese Voraussetzung wird sehr häufig benötigt. Die zweite Bemerkung betrifft den Grundkörper: Einige der genannten Resultate verwenden die Topologie und die archimedische Eigenschaft des Körpers der reellen Zahlen, während andere über jedem reell abgeschlossenen Körper richtig bleiben. Diese Unterscheidung ist von Bedeutung für Verbindungen zur Modelltheorie. Auch wenn dies für uns keine unmittelbare Rolle spielt, verwenden wir die klassischen reellen Zahlen nur dort, wo es technisch nötig ist.

LIST OF NOTATIONS

$k[V]$	coordinate ring of the affine k-variety V (see p. 11)
$V(R)$	set of R-rational points of the variety V (see p. 11)
$\mathrm{Spec}(A)$	Zariski spectrum of the ring A
$\mathcal{I}_V(M)$	vanishing ideal of the set M in $R[V]$ (see p. 11)
φ_R	map $V(R) \to W(R)$ induced by $\varphi\colon V \to W$ (see p. 11)
$\varphi^\#$	homomorphism $R[W] \to R[V]$ induces by $\varphi\colon V \to W$ (see p. 11)
$\overline{S}$	Zariski closure of $S \subset V$ or $S \subset V(R)$
$\mathrm{clos}(S)$	closure of S in the strong topology (see p. 12)
$B_V(S)$	ring of functions in $R[V]$ that are bounded on S (see p. 12)
$\mathrm{qf}(A)$	field of fractions of the domain A
$\mathcal{V}_V(M)$	subvariety of V defined by $M \subset R[V]$
$\mathcal{V}_V^R(M)$	R-rational points of $\mathcal{V}_V(M)$
$R(X)$	function field of the irreducible R-variety X
v_Z	discrete valuation of $R(X)$ associated with the divisor $Z \subset X$
$\mathcal{O}_{X,Z}$	local ring of X in the closed subvariety Z
$\mathcal{O}_X(U)$	ring of regular functions of X on the open subset U
$\mathrm{dom}(f)$	maximal domain of the rational map f
$A^\times$	group of multiplicative units in the ring A
$\mathbf{A}_R^n$, $\mathbf{P}_R^n$	affine resp. projective n-space over R
V_{reg}	regular locus of the variety V
V_{sing}	singular locus of the variety V
$\mathrm{int}(S)$	interior of S (in the strong topology)
$\mathrm{End}(A)$	endomorphism ring of the ring A
$\mathrm{trdeg}_R(A)$	transcendence degree of the R-algebra A
$\mathrm{PO}_A(G)$	preordering of the ring A generated by $G \subset A$ (see p. 31)
$\underline{h}^i$	short for $h_1^{i_1} \cdots h_r^{i_r}$, $i \in \{0,1\}^r$ (see p. 31)

$\operatorname{supp}(T)$	$T \cap (-T)$; support of the preordering T (see p. 31)
$\mathcal{P}_V(S)$	preordering of functions in $R[V]$ non-negative on S (see p. 32)
MP, SMP	(strong) moment property (see p. 33)
$\sqrt[\mathrm{re}]{I}$	real radical of the ideal I
$V^\vee$	(algebraic) dual space of the vector space V
$T\|_Z$	restriction of the preordering T of $R[V]$ to $Z \subset V$ (see p. 34)
$\sum A^2$	preordering of sums of squares in the ring A
$\Gamma(U, \mathcal{F})$	sections the sheaf $\mathcal{F}$ on the open set U
$K[[x]]$	ring of formal power series over K in the variables $x = (x_1, \ldots, x_n)$
Γ_C	graph associated with the curve C (see p. 46)
$\operatorname{div}_V(f)$	divisor of the function $f \in R(V)$ on the variety V
psd=sos	$\mathcal{P}_V(V(R)) = \sum R[V]^2$ holds for the affine R-variety V in question
$\mathcal{S}_V(M)$	set of pts. in $V(R)$ where all elements of $M \subset R[V]$ are non-negative
$\mathcal{U}_V(M)$	set of points in $V(R)$ where all elements of $M \subset R[V]$ are positive
$\mathcal{X}_A(M)$	set of points in Sper A where all elements of $M \subset A$ are non-negative
$\mathcal{Y}_A(M)$	set of points in Sper A where all elements of $M \subset A$ are positive
$\mathcal{Z}_A(M)$	set of points in Sper A where all elements of $M \subset A$ vanish
$\operatorname{Pic}(V)$	picard group of the variety V

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