CONVEXITY IN REAL ALGEBRAIC GEOMETRY
— PROJECT DESCRIPTION —

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Contents

Overview of the research area

The research presented here concerns questions from adjacent areas of mathematics, mainly **real and classical algebraic geometry**, **convex geometry** and **convex optimisation**. The unifying theme is the geometric study of convex semi-algebraic sets.

Semidefinite programming has developed into a versatile tool of convex optimisation. The objective of a semidefinite programme is optimising a linear function over a **spectrahedron**, i.e. an affine-linear slice of the cone of positive semidefinite matrices. Semidefinite programming is a particular instance of the more general framework of cone programming in convex optimisation, as described in the 1990s by Nemirovski [34]. The **spectrahedral cones** fall into the bigger class of **hyperbolicity cones**, which were first studied in the 1950s in connection with PDE theory. It is an open problem, known as the **Generalised Lax Conjecture**, whether every hyperbolicity cone is in fact spectrahedral. Work towards this conjecture is one of the underlying motivations for this research proposal.

The relation between the geometry of spectrahedral cones and semidefinite programming is analogous to that between the geometry of polyhedra (or polytopes) and classical linear programming. Algebraic geometry enters the picture because the boundary of a semi-algebraic cone (with non-empty interior) is contained in an algebraic hypersurface. Such cones (resp. their affine slices) thus correspond to certain projective (resp. affine) varieties. As an example, consider the so-called elliptope

$$\mathcal{E}_3 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{bmatrix} \text{ is positive semidefinite} \right\}.$$

This spectrahedron (together with its higher-dimensional analogues) plays an important role in combinatorial optimisation. Its boundary is contained in the surface in affine-three space given by the equation $1 + 2xyz - x^2 - y^2 - z^2 = 0$. The homogenisation $p = w^3 + 2xyz - x^2w - y^2w - z^2w$, which defines the famous Cayley cubic in $\mathbb{P}^3$, is a **hyperbolic**

polynomial in (w,x,y,z) with respect to the point $e = (1,0,0,0)$. This means that all zeros of the univariate polynomial $p(te-v)$ are real, for fixed $v \in \mathbb{R}^4$. In other words, p behaves like a generalised characteristic polynomial of a symmetric matrix. This is reflected in the **symmetric linear determinantal representation**

$$p = \det \begin{bmatrix} w & x & y \\ x & w & z \\ y & z & w \end{bmatrix}.$$

Determinantal representations (with additional properties) present the algebro-geometric approach to the study of spectrahedral cones. In improving our understanding of the geometry of spectrahedra, the search for the big picture, as in the Generalised Lax Conjecture, goes hand in hand with the study of the geometry in low dimensions, as is customary in algebraic geometry.

Spectrahedral cones are intimately related to **positive polynomials** and **sums of squares**, another central theme of real algebraic geometry. The cone $\mathcal{P}(\mathbb{R}^n)$ of polynomials in n variables that take only non-negative values on all of $\mathbb{R}^n$ is a notoriously inaccessible, "unknowable" object. Given a polynomial f, it is generally hard to decide whether it belongs to $\mathcal{P}(\mathbb{R}^n)$ or not. By contrast, the smaller cone Σ of sums of squares of polynomials, which is contained in $\mathcal{P}(\mathbb{R}^n)$, is dual to a spectrahedral cone if one restricts to a fixed degree. Deciding membership in Σ can therefore be carried out by a semidefinite programme.

This is generalised from $\mathbb{R}^n$ to basic closed semi-algebraic sets by passing from sums of squares to **weighted sums of squares**: Given a set $S = \{u \in \mathbb{R}^n \mid g_1(u) \geqslant 0, \ldots, g_r(u) \geqslant 0\}$ defined by polynomials $g_1, \ldots, g_r$, one considers the cone M generated by the expressions $f^2 g_i$ for all polynomials f. It is contained in the cone $\mathcal{P}(S)$ of polynomials that are non-negative on S. Representing and approximating elements of $\mathcal{P}(S)$ by elements in M is the goal of the various **Positivstellensätze** of real algebraic geometry. The idea, as in the global case, is that, for a given polynomial, it should be easier to test for membership in M than to test for non-negativity on S. If one bounds the degree of the expressions $f^2 g_i$ above, the resulting finite-dimensional cone is again dual to a spectrahedral cone and therefore amenable to semidefinite programming.

Cones that are dual to spectrahedral cones are particular instances of **projected spectrahedra** (also called spectrahedral shadows or semidefinitely representable sets), which are the images of spectrahedra under linear maps. Explicitly, such cones and their slices are defined by **lifted linear matrix inequalities**. Projected spectrahedra present a much wider class of convex sets. In fact, the **Helton-Nie Conjecture** says that *every* convex semi-algebraic set is a projected spectrahedron. A general machinery for producing approximations or representations of basic closed semi-algebraic sets as projected spectrahedra is the so-called **Lasserre Relaxation**, which is based on representations supporting hyperplanes by weighted sums of squares. Proving the convergence and exactness of the Lasserre Relaxation is equivalent to establishing refined Positivstellensätze for linear functions. A proof of the full Helton-Nie Conjecture would have far-reaching theoretical consequences for convex optimisation (namely that several classes of cone programmes coincide). Equally important is the study of the complexity of representations by projected spectrahedra and finding constructive methods for particular classes of sets as explicitly as possible.

The blend of convexity and algebraic geometry that gives rise to the mathematics in this proposal has received much attention from several research groups worldwide in recent years. This is witnessed by the NSF Focus Research Group *Semidefinite Optimization and Convex Algebraic Geometry* (2008-2011) based at several U.S. universities and the new French National Research Agency (ANR) project *GEOLMI (Geometry and Algebra of Linear Matrix Inequalities with Systems Control Applications)* (since 2011). It has also been the subject of specialised workshops (Banff workshop *Convex Algebraic Geometry*, 2010) and featured prominently in larger programmes (*Modern Trends in Optimization and Its Application* at IPAM, UCLA, 2010; SIAM Conference on Optimization, Darmstadt, 2011).

The forthcoming book *Semidefinite Optimization and Convex Algebraic Geometry* (edited by Blekherman, Parillo and Thomas) [5] will be the first somewhat comprehensive reference on the subject and gives a good idea of current developments.

A. Spectrahedral cones and determinantal representations

Problem statement. A **spectrahedral cone** is the preimage of the cone of real symmetric positive semidefinite matrices under a linear map. Such a cone can be written in the form

$$S = \left\{ u \in \mathbb{R}^n \mid u_1 A_1 + \cdots + u_n A_n \text{ is positive semidefinite} \right\},$$

where $A_1, \ldots, A_n$ are real symmetric matrices of some size d. In other words, the spectrahedron S is defined by the **linear matrix inequality** $A(x) \geq 0$, where $A(x) = x_1 A_1 + \cdots + x_n A_n$ is a symmetric matrix whose entries are real linear forms. Spectrahedral cones are the domains of semidefinite programmes in convex optimisation, which has been the main motivation for their study in the last 15 years.

A principal theoretical problem is to characterise the spectrahedral cones among convex semi-algebraic cones. Let S be as above and assume that $A(e)$ is positive definite for some point $e \in \mathbb{R}^n$, so that e is an interior point of S. Consider

$$p = \det(x_1 A_1 + \cdots + x_n A_n),$$

a real homogeneous polynomial of degree d in the variables $x_1, \ldots, x_n$. The polynomial p is **hyperbolic** with respect to e, which means that the polynomial $p(te - v)$ in one variable t has only real zeros for all $v \in \mathbb{R}^n$. The boundary of S is contained in the **determinantal hypersurface** defined by the vanishing of p. Furthermore, S consists precisely of those points $v \in \mathbb{R}^n$ for which all zeros of $p(te - v)$ are non-negative. In particular, S can be characterised solely in terms of p and e. Thus, given any hyperbolic polynomial p with $p(e) > 0$, we consider the **hyperbolicity cone** $\mathcal{C}(p)$ **of** p, which is the set of points $v \in \mathbb{R}^n$ such that all zeros of $p(te - v)$ are non-negative. The problem of representing $\mathcal{C}(p)$ as a spectrahedral cone is closely related to the **determinantal representations** of p. Arguably the central open problem is the **Generalised Lax Conjecture**:

Every hyperbolicity cone is spectrahedral.

In terms of determinantal representations, this can be phrased as follows: Given a hyperbolic polynomial p, does there exist another hyperbolic polynomial q whose hyperbolicity cone contains that of p and such that pq admits a definite determinantal representation?

The theory of determinantal hypersurfaces is a classical subject of complex algebraic geometry. (See Beauville [1] and Catanese [9] for results in a modern language.) From this point of view, spectrahedra and determinantal representations of hyperbolic polynomials present additional challenges.

- The first is **reality**, i.e. one is interested in real symmetric (or complex hermitian) determinantal representations of real hypersurfaces.
- The second is **positivity**. A determinantal representation $p = \det(A(x))$ of a hyperbolic polynomial p as above will only relate to a spectrahedral representation of the hyperbolicity cone if $A(e)$ is positive definite.
- The hyperbolicity cone of a hyperbolic polynomial p may have a spectrahedral representation, even if p itself does not possess a determinantal representation. This is because a multiple of p may define the same hyperbolicity cone and admit a determinantal representation, as explained in the reformulation of the Generalised Lax Conjecture above. This possibility circumvents all known obstructions to determinantal representability, which apply in the irreducible case. It is necessary to study determinantal representations of **reducible** (or even non-reduced) hypersurfaces, which is beyond the scope of the classical results.

State of the art.

- Real and definite determinantal representations of plane curves were studied in the 1980s by Dubrovin and Vinnikov (see [11], [68], [69]), but the connection to spectrahedral cones has only been picked up more recently. The **Helton-Vinnikov Theorem** from 2004 states that every hyperbolic polynomial in three variables has a definite determinantal representation (see [21]). This was previously known as the **Lax Conjecture** (see [30]). In particular, every three-dimensional hyperbolicity cone is spectrahedral.
- In 2010, Brändén [7] showed that there exists a hyperbolic polynomial p (of degree 4 in 8 variables, recently reduced to 4 variables) such that no power p^r of p admits a definite determinantal representation. The counterexample is constructed by combinatorial methods and is based on the so-called Vámos cube from matroid theory.
- By contrast, Helton, McCullough and Vinnikov proved that every real homogeneous polynomial admits a symmetric (but not necessarily definite) determinantal representation after multiplying by a sufficiently high power of a linear form [18] (see also Quarez [55]).
- Netzer and Thom refined Brändén's result to show that , in a suitable sense, very few hyperbolic polynomials admit a definite determinantal representation [39] when degree and dimension are sufficiently high. (This is even true in an inhomogeneous setup, which corresponds to multiplication with a fixed linear form). They also related the problem to the representation theory of certain generalised Clifford algebras [40].

Preliminary work.

- In two joint papers with Sturmfels and Vinzant [51], [52], we investigated determinantal representations from a computational point of view. In particular, we studied the computation of determinantal representations of **plane quartic curves** over $\mathbb{Q}$ in exact arithmetic. We also examined the convex body of all sums of squares representations of a non-negative ternary quartic, the so-called **Gram spectrahedron**. While quartic curves represent an extremely special case, the beauty of classical geometry is striking, and I have subsequently found the wealth of concrete examples, where everything can be understood in great detail, to be very useful in studying more general questions.

• In a further paper with Vinzant [53], we revisted the classical construction of symmetric determinantal representations of plane curves due to Dixon. By adapting this construction to the Hermitian case, we gave a new proof of the existence of a definite Hermitian determinantal representation of a hyperbolic curve using only the elementary theory of plane curves and some topology. The key point is that the non-degeneracy of the construction (corresponding to the non-vanishing of an even theta-characteristic in the classical case) can be seen directly in the case of hyperbolic curves.

• Further algorithmic aspects of this construction are being explored in a joint project with Sinn, Speyer and Vinzant (a prelimanary version [50] is available from my homepage.) The method there is based on [53]. We have also begun to investigate the arithmetic of the problem, i.e. the question for the minimal field extension over which a Hermitian determinantal representation can be constructed. A different numerical approach, using homotopy continuation methods, has been explored in joint work with Anton Leykin [31]. Here, we compute real symmetric determinantal representations by exploiting the known structure of a certain covering map from pairs of matrices to hyperbolic polynomials.

• In a very recent paper with Kummer and Vinzant [25], we study systematically the relationship between a hyperbolic polynomial p and the cone of hyperbolic polynomials q of degree one less that interlace p, which means that the zeros of q are nested between those of p. We provide a description of the **interlacer cone** in terms of certain **Wronkians** and use this to realize the hyperbolicity cone as a slice of the cone of non-negative polynomials. We study sums-of-squares relaxations of this representation and connections to determinantal representability. The correspondence between hyperbolic polynomials and their interlacers is particularly interesting in the multiaffine case and ties in with the work of Bränden in [6] and [7], Wagner and Wei [71], as well as recent results due to Netzer and Sanyal [38] and Parrilo and Saunderson [43].

• In a joint work with Netzer and Thom [41], we present an approach to finding real definite determinantal representations based on **sums-of-squares decompositions of the parametrised Hermite matrix** of a polynomial in several variables. This Hermite matrix has itself polynomial entries, and it is everywhere positive semidefinite if and only if the polynomial is hyperbolic. We show that a definite determinantal representation of a power of a hyperbolic polynomial yields a decomposition of the parametrised Hermite matrix as a sum of matrix squares. We then investigate to what extent the converse holds, giving an algorithmic approach to the problem. In particular, we show that definite determinantal representations always exist if one allows for denominators.

B. Semidefinite representations and weighted sums of squares

Problem statement. The relation between **positive polynomials and sums of squares** pervades real algebraic geometry. Suppose that $g_1, \ldots, g_r$ are real polynomials in variables $x = (x_1, \ldots, x_n)$ and consider the **basic closed semi-algebraic set**

$$S = \{u \in \mathbb{R}^n \mid g_1(u) \geqslant 0, \ldots, g_r(u) \geqslant 0\}.$$

One would like to get a hold on the cone $\mathcal{P}(S) = \{f \in \mathbb{R}[x] \mid f(u) \geqslant 0 \text{ for all } u \in S\}$ of polynomials that take only non-negative values on S. Let Σ denote the cone of sums of squares in $\mathbb{R}[x]$. Then $\mathcal{P}(S)$ contains the cone

$$M = \{s_0 + \textstyle\sum_{i=1}^r s_i g_i \mid s_i \in \Sigma\}$$

of weighted sums of squares, called the **quadratic module** generated by $g_1, \ldots, g_r$. The role of quadratic modules in real algebraic geometry is somewhat similar to that of ideals in classical algebraic geometry, with the crucial difference that the full cone $\mathcal{P}(S)$ is usually not finitely generated as a quadratic module. Membership in M provides what is called a **certificate for positivity**: Instead of testing for non-negativity of a polynomial f on S, one may test for membership of f in M, which is often much easier. Moreover, if one can find a representation $f = s_0 + \sum_{i=1}^r s_i g_i \in M$, then the non-negativity of f on S has been made obvious. The usefulness of this approach depends on the following two questions:

- **Reliabilty.** How close is M to $\mathcal{P}(S)$?
- **Computability.** How hard is it to test for membership of a polynomial in M?

The answer to the first question is provided by various kinds of **Positivstellensätze**, of which there are several basic versions and many derived results. The answer to the second question depends mostly on the existence of suitable **degree bounds** for M. A key observation is that if $s = t_1^2 + \cdots + t_k^2$ is a sum of squares of real polynomials, then $2\deg(t_i) \leqslant \deg(s)$ for $i = 1, \ldots, k$, since **leading terms cannot cancel** in real sums of squares. However, leading terms might well cancel when dealing with weighted sums of squares. Let $\mathbb{R}[x]_d$ be the finite-dimensional space of polynomials of degree at most d and consider

$$M_d = \{s_0 + \textstyle\sum_{i=1}^r s_i g_i \mid s_i \in \Sigma \text{ and } \deg(s_0), \deg(s_i g_i) \leqslant d\} \subset \mathbb{R}[x]_d,$$

the **truncation** of M. Membership of polynomials in M_d can be tested, in practice rather efficiently, via semidefinite programming. However, because of the possible cancellation of leading terms, the finite-dimensional cone $M \cap \mathbb{R}[x]_k$ of polynomials in M of degree at most k might not be contained in M_d for any d. Given $k \geqslant 1$, we say that M is k-**stable** if there exists $d \geqslant 1$ such that $M \cap \mathbb{R}[x]_k \subset M_d$. If M is k-stable for all k, it is called **stable**.

Weighted sums of squares and stability questions are related to the spectrahedral cones discussed in topic **A** in several ways. Recall that a spectrahedron is an affine slice of a spectrahedral cone, and a **projected spectrahedron** is the image of a spectrahedron under a linear mapping. While spectrahedra present a very special class of convex sets, projected spectrahedra provide much more flexibility. Indeed, it has been conjectured by Helton and Nie that every convex semi-algebraic set is a projected spectrahedron. With a suitable choice of coordinates, a projected spectrahedron is represented by a linear matrix inequality involving additional variables. Such representations are also called **lifted linear matrix inequality representations** or **semidefinite representations**. Projected spectrahedra possess a number of good properties:

- They are closed under convex duality and other basic operations on convex sets. In particular, the dual of a spectrahedron is a projected spectrahedron.
- Semidefinite programming can be used for optimisation of linear functions on projected spectrahedra, by optimising over a defining spectrahedron.
- The convex hull of a compact basic closed semi-algebraic set can be approximated by a projected spectrahedron via a procedure called the **Lasserre Relaxation**.

The Lasserre Relaxation can be described in several ways. Within our setup we do as follows: Given a basic closed semi-algebraic set $S = \{u \in \mathbb{R}^n | g_1(u) \geqslant 0, \ldots, g_r(u) \geqslant 0\}$, let M_d be the truncated quadratic module, as above. The convex dual M_d^* consisting of all linear functionals $\ell \colon \mathbb{R}[x]_d \to \mathbb{R}$ with $\ell|_{M_d} \geqslant 0$ and $\ell(1) = 1$ is a spectrahedron in $\mathbb{R}[x]_d^*$. Under the projection $\pi \colon \mathbb{R}[x]_d^* \to \mathbb{R}^n$ given by $\ell \mapsto (\ell(x_1), \ldots, \ell(x_n))$, the projected spectrahedron $\pi(M_d^*)$ provides an outer approximation of the convex hull of S, called the d-th Lasserre Relaxation of S (with respect to $g_1, \ldots, g_r$). If S is compact (and M archimedean), Putinar's or Schmüdgen's Positivstellensatz will ensure that the Lasserre Relaxation converges, i.e. the convex hull of S is the intersection of all Lasserre Relaxations. If some Lasserre Relaxation actually coincides with the convex hull of S, the relaxation is said to be **exact**. This method lies at the heart of various approximation results in convex optimisation.

The exactness of the Lasserre Relaxation is very strongly related to the question of stability for quadratic modules discussed above. Namely, the Lasserre Relaxation becomes exact if and only if M contains all linear polynomials that are positive on S and M is 1-stable.

State of the art.

- The best known Positivstellensätze are those by Schmüdgen and Putinar, which assert that the quadratic module M above contains all polynomials that are strictly positive on S, provided that S is compact and M is closed under taking products (Schmüdgen), or that M is archimedean, which also implies compactness (Putinar). This is in stark contrast to the famous result of Hilbert that there exist positive polynomials in more than one variable that are not sums of squares [23].
- Several more refined results on the question whether the quadratic module of nonnegative polynomials on a semi-algebraic set S is finitely generated were obtained by Scheiderer, especially if S is contained in a curve or surface (see [60], [61], [63]).
- Semidefinite programming in convex optimisation was given a popular boost in the 1990s through the work of Nesterov and Nemirovski [34], [33]. The relation to sums of squares of polynomials was studied and exploited in several applications soon after, in particular in the extensive work of Lasserre (see for example [26], [27], [28], [29]) and Parrilo [42], [44], [45].
- The first systematic study of projected spectrahedra was undertaken by Helton and Nie in [19] and [20]. Their approach is based on proving the exactness of the Lasserre Relaxation under certain regularity conditions on the boundary. This is combined with a glueing result and local-global principle to cover a wider class of sets.
- Recently, Scheiderer proved the existence of degree bounds for quadratic modules supported on smooth compact curves [65], thereby proving the two-dimensional case of the Helton-Nie Conjecture.

• Netzer and Sanyal employed the results of Helton and Nie to show that every smooth hyperbolicity cone is a projected spectrahedron [38]. Also, Parrilo and Saunderson have produced explicit representations of certain hyperbolicity cones [43].

Preliminary work.

• In a joint paper with Netzer and Schweighofer [37], we provided an obstruction to the exactness of the Lasserre Relaxation in the case when the original inequalities describe the convex hull. In that case, we showed that the Lasserre Relaxation can only become exact if all faces are exposed. A simpler proof was later given by Gouveia and Netzer in [15]. However, the methods employed in our proof in [37] still have potential for further applications in the study of projected spectrahedra within this project.

• I have worked on certificates of positivity both before and after obtaining my PhD, in particular on sums of squares in coordinate rings of affine real varieties. My paper [48] deals with sums of squares on curves with several irreducible components. Such reducible curves typically show up as special fibres of morphisms, which are considered in the study of the moment problem in [67]. In [47], I studied certificates of positivity and degree bounds for sums-of-squares representations of piecewise polynomials on simplicial complexes, which are related to splines. This combines Positivstellensätze with combinatorial arguments.

• In recent work with Kummer and Vinzant in [25] we obtain approximations of the hyperbolicity cone of a hyperbolic polynomial by projected spectrahedra. This we do by representing the cone of interlacing polynomials as a slice of the cone of non-negative polynomials followed by a sums-of-squares relaxation.

• The construction of compactifications that imply the existence of degree bounds (stability) for quadratic modules was one of the subjects of my PhD thesis [46]. Parts of these results were refined in a joint paper with Scheiderer [49]. We studied the ring of bounded polynomials on a semi-algebraic set, which is strongly related to the construction of the compactifications in [54] and [46], which are used to prove stability of quadratic modules.

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