Definable henselian valuation rings

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Abstract We give model theoretic criteria for the existence of $\exists\forall$ and $\forall\exists$- formulas in the ring language to define uniformly the valuation rings $\mathcal{O}$ of models $(K,\mathcal{O})$ of an elementary theory Σ of henselian valued fields. As one of the applications we obtain the existence of an $\exists\forall$-formula defining uniformly the valuation rings $\mathcal{O}$ of valued henselian fields $(K,\mathcal{O})$ whose residue class field k is finite, pseudo-finite, or hilbertian. We also obtain $\forall\exists$-formulas φ_2 and φ_4 such that φ_2 defines uniformly $k[[t]]$ in $k((t))$ whenever k is finite or the function field of a real or complex curve, and φ_4 replaces φ_2 if k is any number field.

1 Introduction

Let $(K,\mathcal{O})$ be a field K together with a valuation ring $\mathcal{O}$. We call $\mathcal{O}$ *definable* in the ring language L if there exists an L-formula $\varphi(x)$ with x a only free variable and no parameter from K such that

$$\mathcal{O} = \{a \in K \mid \varphi(a) \text{ hold in } K\}.$$

We shall mainly be interested in L-formulas $\varphi(x)$ of the following types:

$$\begin{array}{rcl} \exists\text{- formula} & : & \exists y_1 \cdots y_n \chi(x,\overline{y}) \\ \forall\text{- formula} & : & \forall y_1 \cdots y_n \chi(x,\overline{y}) \\ \exists\forall\text{- formula} & : & \exists y_1 \cdots y_n \forall z_1 \cdots z_m \chi(x,\overline{y},\overline{z}) \\ \forall\exists\text{- formula} & : & \forall y_1 \cdots y_n \exists z_1 \cdots z_m \chi(x,\overline{y},\overline{z}) \end{array} .$$

The whole investigation generalizes in a straightforward way to more complex quantifier types (and is left to the interested reader). We shall prove (in Section 2) and apply (in Section 3 and 4) the following model theoretic criteria.

Characterization Theorem *Let Σ be a first order axiom system in the ring language L together with a unary predicate $\mathcal{O}$. Then there exists an L-formula $\varphi(x)$, defining uniformly in every model $(K,\mathcal{O})$ of Σ the set $\mathcal{O}$, of quantifier type*

$$\left.\begin{array}{lll} \exists & \text{iff} & (K_1 \subseteq K_2 \Rightarrow \mathcal{O}_1 \subseteq \mathcal{O}_2) \\ \forall & \text{iff} & (K_1 \subseteq K_2 \Rightarrow \mathcal{O}_2 \cap K_1 \subseteq \mathcal{O}_1) \\ \exists\forall & \text{iff} & (K_1 \overset{\exists}{\subseteq} K_2 \Rightarrow \mathcal{O}_1 \subseteq \mathcal{O}_2) \\ \forall\exists & \text{iff} & (K_1 \overset{\exists}{\subseteq} K_2 \Rightarrow \mathcal{O}_2 \cap K_1 \subseteq \mathcal{O}_1) \end{array}\right\} \text{ for all models } (K_1,\mathcal{O}_1),(K_2,\mathcal{O}_2) \text{ of } \Sigma.$$

Here $K_1 \overset{\exists}{\subseteq} K_2$ means that K_1 is existentially closed in K_2, i.e. every $\exists$-formula $\varrho(x_1,\ldots,x_m)$ with parameters from K_1 that holds in K_2 also holds in K_1.

If K_1 and K_2 are fields, this implies that K_1 is relatively algebraically closed in K_2. In particular, if $(K_2, \mathcal{O}_2)$ is henselian, then also $(K_1, \mathcal{O}_2 \cap K_1)$ is henselian in K_1. Thus if $(K_1, \mathcal{O}_1)$ is also henselian, we can apply the theory of henselian valuation rings on a field K_1 as explained in Section 4.4 [E-P]. This will yield a series of applications in Sections 3 and 4.

It is important to realize that the model theoretic criteria above do not give explicit L-formulas, rather only their existence. But the knowledge of the existence may help to construct such a formula. In many cases explicit formulas are already known. Let us mention here the papers [C-D-L-M], [A-K], [F], and [H]. These papers actually inspired us to look for general model theoretic criteria.

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2 Proof of the Characterization Theorem

Let Σ be a first order axiom system in the ring language L enlarged by a unary predicate $\mathcal{O}$. Moreover, fix a constant c. We denote by $L(\mathcal{O}, c)$ the enlarged language. An $L(\mathcal{O}, c)$-structure then looks like $(K, \mathcal{O}, a)$ where K is an L-structure, $\mathcal{O} \subseteq K$ and $a \in K$. Next let Φ be a subset of $L(c)$-sentences, i.e. formulas in the ring language L enlarged by c without free variables. We assume that Φ is closed by $\wedge$ and $\vee$. Examples of interest to us are the sets of $\exists, \forall, \exists\forall$, and $\forall\exists$-sentences in $L(c)$. We furthermore use the following abbreviation for subsets Φ of $L(\mathcal{O}, c)$-sentences: if $(K_1, \mathcal{O}_1, a_1)$ and $(K_2, \mathcal{O}_2, a_2)$ are two $L(\mathcal{O}, c)$-structures and every $\gamma \in \Phi$ that holds in $(K_1, \mathcal{O}_1, a_1)$ also holds in $(K_2, \mathcal{O}_2, a_2)$ we write

$$(K_1, \mathcal{O}_1, a_1) \overset{\Phi}{\rightsquigarrow} (K_2, \mathcal{O}_2, a_2).$$

Now Lemma 3.1.6 of [P-D], an easy consequence of the Compactness Theorem for 1-order logic, immediately gives:

Theorem 1. *Assume that for all models $(K_i, \mathcal{O}_i, a_i)$ of $\Sigma (i = 1, 2)$ we have the following implications:*

$$\textit{If } (K_1, \mathcal{O}_1, a_1) \overset{\Phi}{\rightsquigarrow} (K_2, \mathcal{O}_2, a_2) \textit{ then } (K_1, \mathcal{O}_1, a_1) \overset{\{c \in \mathcal{O}\}}{\rightsquigarrow} (K_2, \mathcal{O}_2, a_2).$$

Then there exists some $\varphi(c) \in \Phi$ such that

$$\forall x (x \in \mathcal{O} \Leftrightarrow \varphi(x))$$

holds in all models $(K, \mathcal{O})$ of Σ, i.e., φ defines $\mathcal{O}$ in K.

The philosophy behind Lemma 3.1.6 is: if a sentence behaves like all $\varphi \in \Phi$, it is equivalent to some fixed $\varphi \in \Phi \mod \Sigma$.

Proof of the Characterization Theorem: Let us first observe that the right hand side of the equivalences follow clearly from the corresponding definabilities. Now we can prove the other direction.

$\exists$-case: Let Φ be the set of all $\exists$-sentences in $L(c)$. Assume that for all models $(K_i, \mathcal{O}_i)(i = 1, 2)$ of Σ we have: $K_1 \subseteq K_2 \Rightarrow \mathcal{O}_1 \subseteq \mathcal{O}_2$.

Considering two models $(K_1, \mathcal{O}_1, a_1)$ and $(K_2, \mathcal{O}_2, a_2)$ of Σ satisfying

$$(K_1, \mathcal{O}_1, a_1) \overset{\Phi}{\rightsquigarrow} (K_2, \mathcal{O}_2, a_2)$$

we have to show if $a_1 \in \mathcal{O}_1$ then also $a_2 \in \mathcal{O}_2$. We assume that $a_2 \notin \mathcal{O}_2$ and consider the set

$$\Pi = \text{Diag } (K_1, a_1, (b)_{b \in K_1}) \cup Th(K_2, \mathcal{O}_2, a_2)$$

of sentences. We claim that Π is consistent. In fact, if Π were inconsistent, there would exist some elements $b_1, \ldots, b_n \in K_1$ and a quantifier free L-formula χ such that $\chi(a_1, b_1, \ldots, b_n) \in$ Diag of K_1 and $\{\chi\} \cup Th(K_2, \mathcal{O}_2, a_2)$ is inconsistent. Hence the $\exists$-formula $\varphi(c) \equiv \exists b_1, \ldots, b_n \chi(c, \overline{b})$ of $L(c)$ [1] holds in (K_1, a_1) and we have

$$Th(K_2, \mathcal{O}_2, a_2) \vdash \forall b_1 \cdots b_n \neg\chi(c, \overline{b}).$$

This is impossible as $\varphi(c)$ carries over from (K_1, a_1) to (K_2, a_2).

Therefore Π is consistent and hence has a model

$$(K_2^*, \mathcal{O}_2^*, a_2^*)$$

that is elementarily equivalent to $(K_2, \mathcal{O}_2, a_2)$ and contains an isomorphic copy of (K_1, a_1). After identifying (K_1, a_1) with its image in $(K_2^*, \mathcal{O}_2^*, a_2^*)$ we obtain $K_1 \subseteq K_2^*, a_1 \in \mathcal{O}_1$, and $a_1 = a_2^* \notin \mathcal{O}_2^*$. Hence $\mathcal{O}_1 \not\subseteq \mathcal{O}_2^*$. This contradicts our assumption, as $(K_2^*, \mathcal{O}_2^*)$ is also a model of Σ.

$\exists\forall$-case: Assume that for all models $(K_i, \mathcal{O}_i)$ of Σ we have: $K_1 \overset{\exists}{\subseteq} K_2 \Rightarrow \mathcal{O}_2 \subseteq \mathcal{O}_2$. Looking at the proof of the $\exists$-case, the only change we need is that (K_1, a_1) (after identification with its image) is existentially closed in (K_2^*, a_2^*). This is obtained by replacing Π by the set

$$\Pi_\forall = Th_\forall(K_1, a_1, (b)_{b \in K_1}) \cup Th(K_2, \mathcal{O}_2, a_2)$$

where $Th_\forall(K_1, a_1(b)_{b \in K_1})$ consists of all $\forall$-formulas

$$\varphi(x, \overline{b}) \equiv \forall y_1 \cdots y_n \chi(c, \overline{y}, \overline{b}),$$

where χ is quantifier free, that hold in $(K_1, a_1, (b)_{b \in K_1})$.

$\forall$-case and $\forall\exists$-case: Is obtained from the $\exists$-case and the $\exists\forall$-case just by replacing the sets Π and $\Pi_\forall$ by

$$\Pi' = \text{Diag } (K_2, a_2, (b)_{b \in K_2}) \cup Th(K_1, \mathcal{O}_1, a_1)$$

and

$$\Pi'_\forall = Th_\forall(K_2, a_2, (b)_{b \in K_2}) \cup Th(K_1, \mathcal{O}_1, a_1)$$

respectively. $\square$

[1] Now $b_1 \cdots b_n$ play the role of variables.

3 $\exists\forall$-definable henselian valuation rings

In our applications we shall concentrate here on henselian valued fields $(K, \mathcal{O})$. The maximal ideal of $\mathcal{O}$ is denoted by M, the residue field by $k = \mathcal{O}/M$, and the value group by $v(K)$. If we deal with several valued fields $(K_i, \mathcal{O}_i)$ we use corresponding indices for the residue fields k_i and the value groups $v_i(K_i)$. Of particular interest will be the henselian valuation ring $k[[t]]$ of the fields $k((t))$ of formal Laurent series and p-adic number fields.
Before we proceed to concrete results let us quote some facts about henselian valued fields from [E-P].

A valued field $(K, \mathcal{O})$ is called *henselian* if the valuation ring $\mathcal{O}$ of K extends uniquely to the separable closure K^s of K. Note that the trivial valuation $\mathcal{O} = K$ always is henselian, its residue class field is K. Two valuation rings $\mathcal{O}_1$ and $\mathcal{O}_2$ of the same field K are called comparable if $\mathcal{O}_1 \subseteq \mathcal{O}_2$ or $\mathcal{O}_2 \subseteq \mathcal{O}_1$; the upper one is called *coarser*. Here are some important facts.

Fact 1. *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are henselian on K and at least one of the residue fields is not separably closed then $\mathcal{O}_1$ and $\mathcal{O}_2$ are comparable. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are not comparable, then the residue field of $\mathcal{O}_1, \mathcal{O}_2$, and of the smallest common coarsening of $\mathcal{O}_1$ and $\mathcal{O}_2$, all are separably closed.* (Theorem 4.4.2 in [E-P]).

Now let $\mathcal{O}_1$ and $\mathcal{O}_2$ be comparable valuation rings of K, say $\mathcal{O}_1 \subseteq \mathcal{O}_2$. (Hence $M_2 \subseteq M_1$.) Then $\mathfrak{o} = \mathcal{O}_1/M_2$ is a valuation ring of the residue class field $k = \mathcal{O}_2/M_2$. Then we obtain from Section 2.3 in [E-P] and Corollary 4.1.4:

Fact 2. *The value group of $(k, \mathfrak{o})$ is isomorphic to a convex subgroup Δ of $v(K_1)$ and $v_2(K_2) \cong v_1(K_1)/\Delta$.*

Fact 3. *$(K_1, \mathcal{O}_1)$ is henselian if and only if $(K_1, \mathcal{O}_2)$ and $(k, \mathfrak{o})$ are both henselian.*

Now let us consider a first order axiom system Σ for henselian valued fields $(K, \mathcal{O})$ such that the residue field $k = \mathcal{O}/M$

(1) is not separably closed

(2) does not carry a proper henselian valuation.

We shall then prove for any two models $(K_1, \mathcal{O}_1)$ and $(K_2, \mathcal{O}_2)$ of Σ:

$$K_1 \overset{\exists}{\subseteq} K_2 \Rightarrow \mathcal{O}_1 \subseteq \mathcal{O}_2.$$

Then by the Characterization Theorem there exists an $\exists\forall$-formula $\varphi(x)$ in the ring language that defines $\mathcal{O}$ in every model $(K, \mathcal{O})$ of Σ.

For the proof assume that $K_1 \overset{\exists}{\subseteq} K_2$ and $\mathcal{O}_1 \not\subseteq \mathcal{O}_2$. As K_1 is separably closed in K_2, it follows that $\mathcal{O} := K_1 \cap \mathcal{O}_2$ is a henselian valuation ring of K_1. Since by (1) the residue field of $\mathcal{O}_1$ is not separably closed, Fact 1 implies $\mathcal{O} \subsetneq \mathcal{O}_1$. Now Fact 3 implies that $\mathcal{O}/M_1$ is a proper henselian valuation of $\mathcal{O}_1/M_1$. This contradicts (2). Hence $\mathcal{O}_1 \subseteq \mathcal{O}_2$, and we are done.
As an application we obtain

Theorem 1. *There is an $\exists\forall$-formula $\varphi(x)$ defining uniformly the valuation rings of henselian fields $(K,\mathcal{O})$ if the residue field k of $\mathcal{O}$ is finite, pseudo-finite, or hilbertian.*[2]

Proof: The class of finite and pseudo-finite fields is the model class of the theory of finite fields, hence an elementary class. The class of hilbertian fields is as well elementary. The union of the elementary classes is again elementary. Let Σ' be a first order axiom system for the union. Then let Σ express the fact that its models $(K,\mathcal{O})$ are henselian valued fields (not excluding the trivial valuation) such that the residue field $\mathcal{O}/M$ satisfies the axioms for Σ'.

We have to check (1) and (2) from above. (1) is clear. (2) is clearly true for finite fields k. If k is pseudo-finite it is a PAC-field (see [A2], Section 6, Lemma 2), and PAC-fields do not carry a proper henselian valuation (unless they are separably closed). This old result of the author can be found in [F-J], Corollary 11.5.5. Thus it remains to prove that a hilbertian field k does not allow a proper henselian valuation ring $\mathfrak{o}$. For contradiction assume $\mathfrak{o} \subsetneqq k$ is a henselian valuation ring of k. We then choose a separable polynomial $f(x) \in k[X]$ without zero in k. As $\mathfrak{o}$ is henselian, the set $f(k)$ stays away from 0, say $f(k) \cap m = \emptyset$ (m the maximal ideal of $\mathfrak{o}$). We then choose $\pi \in m \setminus \{0\}$ and consider the polynomial

$$g(X,Y) = f(X)Y^2 + f(X)Y + \pi.$$

Replacing Y by Z^{-1} and applying Eisenstein, we see that $g(X,Y)$ is absolutely irreducible. Now let x be any element of k. Then $Y^2 + Y + \frac{\pi}{f(x)}$ maps to $Y(Y+1)$ in $\mathcal{O}/M$. Now by Hensel's Lemma $g(x,Y)$ has a zero in k, thus is not irreducible. This contradicts the assumption that k is hilbertian. □

Theorem 1 covers all completions of finite number fields, i.e. finite extension of the p-adic number fields $\mathbb{Q}_p$ for any prime p. Moreover, it covers all fields $k_0((t))$ of Laurent series with k_0 finite, pseudo-finite, or hilbertian. It even covers any such fields k_0 together with the trivial valuation as (k_0, k_0) then is a model of Σ. Thus the sentence $\forall x \varphi(x)$ is true in all finite, pseudo-finite, and hilbertian fields k_0.

Remark 1. *In [A1] Ax gives a $\exists\forall\exists\forall$-formula that defines $k[[t]]$ uniformly in $k((t))$ for all fields k. Theorem 1 gives an improvement in case k is finite, pseudo-finite, or hilbertian. In the next Section we shall consider classes of fields k for which $k[[t]]$ is uniformly $\forall\exists$-definable. We shall also explain an example k^* of A. Fehm for which $k^*[[t]]$ is not $\forall\exists$-definable in $k^*((t))$.*

[2] A field k is called *hilbertian* if every finite sequence of irreducible polynomials $g_i(X,Y) \in K[X,Y]$ admits a substitution $x \in k$ such that all $g_i(x,Y)$ remain irreducible in $k[Y]$. All number fields and all function fields are hilbertian (c.f. [R]).

4 $\forall\exists$-definable henselian valuation rings

For our next theorem we shall need some preparation. As usual we consider henselian valued fields $(K, \mathcal{O})$. This time, however, we shall require that the value group $v(K)$ is like a discrete value group in not admitting a convex 2-divisible subgroup $\Delta \neq \{0\}$, like discrete value groups do. This property is easily expressed in the elementary ring language L together with a predicate $\mathcal{O}$. In fact, $v(K)$ is order-isomorphic to $K^\times/\mathcal{O}^\times$, and expressing the existence of a proper convex, 2-divisible subgroup of $(v(K), \leq)$ can be done by saying in $(v(K), \leq)$:

$$\exists\gamma(0 < \gamma \wedge \forall\delta(0 \leq \delta \leq \gamma \Rightarrow \exists\varepsilon\ \delta = 2\varepsilon)).$$

Next we need to talk about the u-invariant of the residue field k of $(K, \mathcal{O})$ and again be able to do this is first order logic. We shall make use of the language of quadratic forms as found e.g. in [E-P], Section 6.3 or in [P-D], Chapter 3.
The *u-invariant* of a field k is defined to be the maximal dimension $n \in \mathbb{N} \cup \{0\}$ of an anisotropic, quadratic form $\varrho =< a_1, \ldots, a_n >$ with $a_i \in k \setminus \{0\}$ that has total signature zero (see [E-K-M], Chapter VI). For example, $u(k) = 4$ for every finite number field, and $u(\mathbb{C}) = 1$. The u-invariant of a real field clearly has to be even, e.g. $u(\mathbb{R}) = 0$. Here ϱ is said to be of *total signature* zero over k, if for any ordering $\leq$ of k, one half of the a_i is positive and the other half is negative. There is a quantifier free formula $\zeta(a_1, \ldots, a_n)$ expressing in the real closure of $(k, \leq)$ that ϱ is not of signature zero. Thus we have to say that there does not exist an ordering $\leq$ of k such that ζ holds in $(k, \leq)$. Using the theory of pre-orderings (see [P-D]) this can be done in the language of k in case the Pythagoras number $P(k)$ is finite. The Pythagoras number $P(k)$ is the smallest $m \in \mathbb{N} \cup \{0\}$ such that every sum of squares in k equals a sum of m squares. Clearly, if $u(k)$ is finite, then also $P(k)$ is finite.

Theorem 2. *For every non-zero $n \in \mathbb{N}$ there is an $\forall\exists$-formula $\varphi_n(x)$ defining uniformly the valuation ring $\mathcal{O}$ of henselian fields $(K, \mathcal{O})$ if the value group $v(k)$ does not admit a convex 2-divisible subgroup $\Delta \neq \{0\}$, char $\mathcal{O}/M \neq 2$, and the u-invariant of the residue field $\mathcal{O}/M$ is n.*

Proof: We want to apply the Characterization Theorem to the models $(K, \mathcal{O})$ of a first order axiom system Σ expressing that the value group $v(k)$ does not admit a convex 2-divisible subgroup $\Delta \neq \{0\}$ and that $u(\mathcal{O}/M) = n$. Let $(K_1, \mathcal{O}_1)$ and $(K_2, \mathcal{O}_2)$ be models of Σ and let $K_1 \overset{\exists}{\subseteq} K_2$. We then have to prove that $\mathcal{O} := K_1 \cap \mathcal{O}_2 \subseteq \mathcal{O}_1$.

Since K_1 is existentially closed in K_2 it follows that $\mathcal{O}$ is henselian and thus $\mathcal{O}$ and $\mathcal{O}_1$ are comparable. If not, Fact 1 implies that the residue field of $\mathcal{O}_1, \mathcal{O}$, and the smallest common coarsening $\mathcal{O}'$ of $\mathcal{O}_1$ and $\mathcal{O}$, all have separably closed residue field As $\mathcal{O}_1 \subsetneqq \mathcal{O}', \mathcal{O}_1/M'$ is a proper henselian valuation ring of $\mathcal{O}'/M'$, that has a value group Δ, not divisible by 2. This follows from the assumption of the theorem and Fact 2. On the other hand as $\mathcal{O}'/M'$ is separably closed, the value group Δ of $\mathcal{O}_1/M'$ has to be divisible, a contradiction. Therefore $\mathcal{O}$ and $\mathcal{O}_1$ are comparable. It thus remains to exclude $\mathcal{O}_1 \subsetneqq \mathcal{O}$.

Let us assume $\mathcal{O}_1 \subsetneqq \mathcal{O}$. Then $\mathcal{O}_1/M = \mathfrak{o}$ is a proper henselian valuation ring on $\mathcal{O}/M$ by Fact 3. The value group Δ of $\mathfrak{o}$ is a convex subgroup $\neq \{0\}$ of $v_1(K_1)$ and hence by Σ not 2-divisible. The residue field of $\mathfrak{o}$ equals that of $\mathcal{O}_1$ (Fact 2). As $u(\mathcal{O}_1/M_1) = n$ there exists a quadratic form $\overline{\varrho} =< a_1 + M, \ldots, a_n + M >$ with $a_i \in \mathcal{O}^\times$ such that $\overline{\varrho}$ is of total signature zero but not isotropic in $\mathcal{O}_1/M_1$. We then choose some $b \in \mathcal{O}^\times$ such that with respect to the valuation of $\mathfrak{o}$ its value is not 2-divisible in Δ. Then one can easily check that the quadratic form

$$\overline{\varrho}_b :=< a_1 + M, \ldots, a_n + M, a_1 b + M, \ldots, a_n b + M >$$

cannot be isotropic in $\mathcal{O}/M$. Moreover, $\overline{\varrho}_b$ is of total signature zero in $\mathcal{O}/M$. At this point we use Lemma 4.3.6 and Theorem 2.2.5 of [E-P] to see that every ordering of $\mathcal{O}/M$ maps to some ordering of the residue field $\mathcal{O}_1/M_1$ of $\mathcal{O}_1/M$.

On the other hand, by $u(\mathcal{O}_2/M_2) = n$ we know that $\overline{\varrho}_b$ is isotropic in the extension $\mathcal{O}_2/M_2$ of $\mathcal{O}/M$. As $(K_2, \mathcal{O}_2)$ is henselian and char $\mathcal{O}_2/M_2 \neq 2$, it follows from Hensel's Lemma that the quadratic form

$$\varrho_b :=< a_1, \ldots, a_n, a_1 b, \ldots, a_n b >$$

is isotropic in K_2. Now, since K_1 is existentially closed in K_2, ϱ_b is also isotropic in K_1. This, however, clearly implies that $\overline{\varrho}_b$ is isotropic in $\mathcal{O}/M$. This contradiction implies that $\mathcal{O}_1 \subsetneqq \mathcal{O}$ cannot hold. Hence $\mathcal{O}_2 \cap K_1 = \mathcal{O} \subseteq \mathcal{O}_1$, and we are done. □

As an application of Theorem 2 we see that the henselian valuation rings $k[[t]]$ are uniformly definable in $k((t))$ by some $\forall\exists$-formula φ_n in case[3]

- k is $\mathbb{C}(n = 1)$;
- k is a finite field or the function field of a real or complex curve $(n = 2)$;
- k is a finite number field $(n = 4)$;
- k is the function field of a complex variety $V(n = u(k) \leq 2^d$ where d is the dimension of $V)$.

There are, however, fields k such that $k[[t]]$ is not $\forall\exists$-definable in $k((t))$. Here is an example suggested by A. Fehm:
Let $k^* = \bigcup_{n\geq 1} k_n$ with k arbitrary and $k_n = k((t_n))\ldots((t_1))$.
Clearly $k_n \subsetneqq k_{n+1}$. It is also clear that k^* is isomorphic to $K = k^*((t))$ by sending t_1 to t and t_{n+1} to t_n for $n \geq 1$. The pre-image $\mathfrak{o}$ of the henselian valuation ring $\mathcal{O} = k^*[[t]]$ of K is again a henselian valuation ring of k^*. Note that the restriction of $\mathcal{O}$ to the subfield k^* of K is the trivial valuation on k^*.

[3]As $u(\mathbb{R}) = 0$, this theorem does not cover the case of $\mathbb{R}[[t]]$. Replacing, however, the u-invariant by the number of square classes of the residue field (which for $\mathbb{R}$ is 2) similar arguments as in the proof of Theorem 2 give as well an $\forall\exists$-formula.

We now use the fact that k^* is existentially closed in K (see Proposition 2' in [E]). We then see that $\mathcal{O}$ cannot be $\forall\exists$-definable in K. Assume some $\forall\exists$-formula $\varphi(x)$ would define $\mathcal{O}$ in K. Then the same formula would define $\mathfrak{o}$ in k^*. Now $t_1^{-1} \in \mathcal{O}$ implies that $\varphi(t_1^{-1})$ holds in K. As k^* is existentially closed in K, we would also get $\varphi(t_1^{-1})$ in k^*. But then $t_1^{-1} \in \mathfrak{o}$, a contradiction.

5 $\exists$-definable henselian valuation rings

Again we assume that Σ is a first order axiom system for henselian valued fields $(K, \mathcal{O})$. We consider two models $(K_1, \mathcal{O}_1), (K_2, \mathcal{O}_2)$ of Σ and assume $K_1 \subseteq K_2$ (as fields). In order to get uniform $\exists$-definability for all rings $\mathcal{O}$ of the models $(K, \mathcal{O})$ of Σ, we have to show that $\mathcal{O}_1 \subseteq \mathcal{O}_2$. Now we can no longer assume that $\mathcal{O} := \mathcal{O}_2 \cap K_1$ is a henselian valuation ring of K_1. Thus we pass to the henselian closure K^h of K_1 inside K_2 with respect to $\mathcal{O}_2$. The valuation $\mathcal{O}_1$ being henselian on K_1 uniquely extends to a henselian valuation $\mathcal{O}_1'$ on K^h. Thus we have now $\mathcal{O}^h = \mathcal{O}_2 \cap K^h$ and $\mathcal{O}_1'$ as henselian valuations on K^h. In the next application we shall fix a condition such that Fact 1 yields comparability of $\mathcal{O}^h$ and $\mathcal{O}_1'$. This clearly implies comparability of $\mathcal{O}$ and $\mathcal{O}_1$. Thus it remains to exclude $\mathcal{O} \subsetneqq \mathcal{O}_1$.

Theorem 3. *Let Σ be a first order axiom system for henselian valued fields $(K, \mathcal{O})$ such that the residue field $\mathcal{O}/M = k$ is finite or PAC and the fixed polynomial $f(x) \in \mathbb{Z}[X]$ has no zero in k. Then there is an $\exists$-formula φ_f defining uniformly the rings $\mathcal{O}$ of models $(K, \mathcal{O})$ of Σ.*

Proof: If $\mathcal{O}^h$ and $\mathcal{O}_1'$ were not comparable by Fact 1 both would have a separably closed residue field. But then f would have a zero in $\mathcal{O}_2/M_2$. Thus we get comparability of $\mathcal{O}$ and $\mathcal{O}_1$. We want to exclude $\mathcal{O} \subsetneqq \mathcal{O}_1$. In case it holds, $\mathcal{O}/M_1$ is a proper valuation of $\mathcal{O}_1/M_1$. It then follows from Corollary 11.5.5 in [F-J], that the residue field $\mathcal{O}/M$ being the residue field of $\mathcal{O}/M_1$ w.r.t. M/M_1 is separably closed. (Note that henselianity of $\mathcal{O}/M_1$ is not needed.) But then again f would have a zero in $\mathcal{O}_2/M_2$, a contradiction. $\square$

The result of Theorem 3 is due to A. Fehm. In [F] he explicitly constructs an $\exists$-formula φ_f.

References

[A-K] W. Anscombe, J. Koenigsmann: *An existential $\emptyset$-definition of $\mathbb{F}_q[[t]]$ in $\mathbb{F}_q((t))$*, ArXiv: 1306.6/60v1.

[A1] J. Ax: *On the undecidability of power series fields*, Proc. Amer. Math. Soc 16:846 (1965).

[A2] J. Ax: *The elementary theory of finite fields*, Ann. of Math. 88 (1968), 239-271.

[C-D-L-M] R. Cluckers, J. Deraskhshan, E. Leenknegt, A. Macintyre: *Uniformly defining valuation rings in henselian valued fields with finite or pseudo-finite residue fields*, Preprint.

[E-K-M] R. Elman, N. Karpenko, A. Merkurjev: *The algebraic and geometric theory of quadratic forms*, AMS, Coll Pub, 56 (2008).

[E-P] A. Engler, A. Prestel: *Valued Fields*, Springer Monographs in Mathematics (2005).

[E] Y.L. Ershov: *Relative regular closedness and π-valuations*, Algebra and Logic, 31(6) (1992), 140-146.

[F] A. Fehm: *Existential $\emptyset$-definability of henselian valuation rings*, Preprint, ArXiv 1307.1956v2.

[F-J] M.D. Fried, M. Jarden: *Field arithmetic*, 2nd Ed., Springer Ergebnisse (2005).

[H] J. Hong: *Definable non-divisible Henselian valuations*, Bull. Lond. Math. Soc. 46 (2014), No 1, 1418.

[P-D] A. Prestel, C.N. Delzell: *Positive polynomials*, Springer Monographs in Mathematics (2001).

[R] P. Roquette: *Nonstandard aspects of Hilbert's Irreducibility Theorem*, Springer Lecture Notes in Mathematics 498 (1975), 231-275.

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