Polynomial stability in two-dimensional magneto-elasticity*

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Abstract: For a class of bounded reference configurations that are of partially rectangular type, the linear system of magneto-elasticity in two space dimensions is considered and a polynomial decay rate of the energy as time tends to infinity is proved.

AMS subject classification: 73 C 99, 35 Q 99

1 Introduction

We consider the following linear initial boundary value problem in magneto-elasticity which describes, for a homogenous isotropic medium with bounded reference configuration $\Omega \subset R^2$, the interaction between elastic movements and a magnetic field. The governing differential equations for the displacement vector $u = (u^1, u^2, 0)' = u(t,x)$ depending on the time variable $t \geq 0$ and on the space variable $x \in \Omega$, and for the magnetic field $h = (h^1, h^2, 0)' = h(t,x)$ are

$$u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div}\, u - \alpha[\nabla \times h]\times \vec{H} \;=\; 0, \tag{1.1}$$

$$h_t - \Delta h - \beta\nabla \times [u_t \times \vec{H}] \;=\; 0, \tag{1.2}$$

cp. [3]. Here λ, μ and κ are positive constants. The coupling constants α, β satisfy $\alpha\beta > 0$. $\vec{H} = (H,0,0)'$ is a constant vector with $H \neq 0$.

Additionally, one has initial conditions

$$u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \;\; h(0,x) = h_0(x) \tag{1.3}$$

and the following classical Dirichlet-type boundary conditions

$$u = 0, \quad \nu \times (\nabla \times h) = 0, \;\; \nu \cdot h = 0 \quad \text{ on } \Gamma := \partial\Omega. \tag{1.4}$$

Moreover,

$$\mathrm{div}\, h = 0, \tag{1.5}$$

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which follows from (1.2) if div $h_0 = 0$. The vector $\nu = (\nu_1, \nu_2, 0)' = \nu(x)$ denotes the exterior normal vector in $x \in \Gamma$, the boundary of Ω.

Note that the system above is very similar to the isotropic thermoelastic system. The main difference is that now the dissipation is given by the magnetic field instead of the temperature. In thermoelasticity it is well known by now that for materials configurated in bounded domains of R^n, for $n > 1$, there is no decay in general. This ineffectiveness of the thermal dissipation is due essentially to the degree of freedom of the displacement vector field, which is greater than the degree of freedom of the temperature when the dimension is greater than $n = 1$. In magneto-elasticity the situation is different, because the dissipative mechanism and the displacement have the same degree of freedom. But now, this system introduces an additional difficulty given by the coupling term. Making a closer analysis of the system (see equations (2.3)–(2.6) of section 2) we see that the dissipative term is coupled only to the gradient of the second component of the displacement, while the first component is coupled to the second one only by equations (2.3), (2.4) thru its mixed derivatives, which is very subtle. The main question here is if the dissipation given by the magnetic field is strong enough to produce a uniform rate of decay for the whole system. If so, what type of rate of decay can we expect? It seems to us that there is not yet any result about this topic. To fill part of this gap we study this topic here.

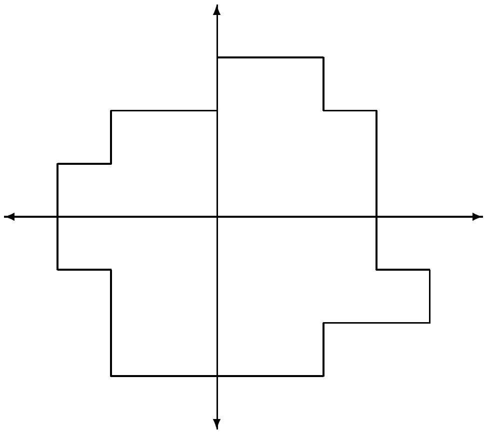

Figure 1.1: Type I

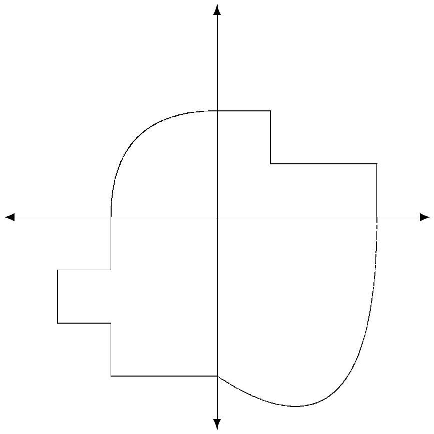

Figure 1.2: Type II

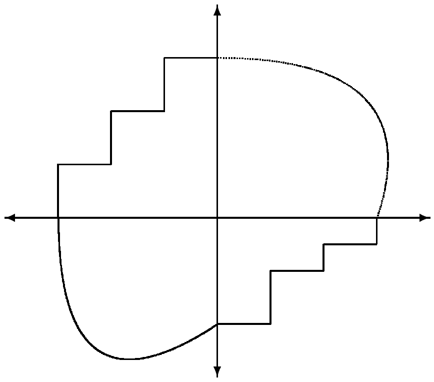

Figure 1.3: Type III

The main result of this paper is to show that the solution of the magneto-elastic system decays polynomially as time goes to infinity. To show this we assume that the boundary Γ is smooth with the exception of at most finitely many points. This is in accordance with the following assumption on the two-dimensional domain Ω: Without loss of generality we may assume that $0 \in \Omega$ and additionally that Ω is connected and of *partially rectangular type* which means that it is homeomorphic to the unit ball and one of the following three conditions I–III is satisfied:

I: Ω is the union of finitely many rectangles with axes paralles to the x_1- and x_2-axes, respectively, see Figure 1.1.

II: Ω satisfies $\nu_1\nu_2 = 0$ in the first quadrant (where $x_1 \geq 0$ and $x_2 \geq 0$) and in the third quadrant (where $x_1 \leq 0$ and $x_2 \leq 0$). In the second and fourth quadrant Ω satisfies $x\nu \geq \delta_0 > 0$, for some δ_0, see Figure 1.2.

III: Ω satisfies $\nu_1\nu_2 = 0$ in the second and fourth quadrant. In the first and third quadrant Ω satisfies $x\nu \geq \delta_0 > 0$, for some δ_0, see Figure 1.3.

Remark: By domains of partially rectangular type I, all sufficiently smoothly bounded, connected domains can be exhausted, also all connected Jordan measurable sets.

The energy $E = E(t)$(of first order) associated to the equations (1.1), (1.2) is given by

$$E(t) := E(t; u, h) := \frac{1}{2} \int\limits_{\Omega} \left(|u_t|^2 + \mu|\nabla u|^2 + (\mu + \lambda)|\mathrm{div} u|^2 + \frac{\alpha}{\beta}|h|^2 \right) (t, x) dx. \tag{1.6}$$

We shall also use energy terms of higher order given for $j \in N$ by

$$E_j(t) := E(t; \partial_t^j u, \partial_t^j h). \tag{1.7}$$

Then it will be proved that the energy $E_1(t)$ decays like t^{-1}. More precisely, the main theorem is the following

Theorem 1.1 *Let (u, h) be the solution to the initial boundary value problem (1.1)–(1.5). Then the energy E_1 defined in (1.6), (1.7) decays polynomially,*

$$\exists d > 0 \quad \forall t \geq 0 : \quad E_1(t) \leq \frac{d}{t} \sum_{j=0}^{7} E_j(0).$$

This result presents a polynomial decay that is uniform with respect to initial data but involves derivatives at time $t = 0$ higher than those estimated for $t > 0$. Indeed, it is open whether there is a uniform exponential decay of the associated semigroup, and our calculations do not assist this possibility.

The method we use is an energy method, looking for appropriate multipliers and Lyapunov functionals.

Remark: The existence of solutions to the initial boundary value problem is simply assumed. It is a standard procedure to obtain solutions in Sobolev spaces — e.g. via semigroup theory, cp. [9]. The assumed smoothness of the initial data is described by the finiteness of the right-hand side in the estimate in Theorem 1.1.

The time-asymptotic behavior for the initial boundary value problem has been studied by Perla Menzala & Zuazua [9]. They proved the decay of $E(t)$ to zero for fixed initial data, no uniformity was given, but more general domains, also in three space dimensions, were considered. For a damping boundary condition of memory type, replacing $u = 0$, the authors proved the exponential stability in [8]. We also mention the results on polynomial decay for the corresponding Cauchy problem contained in the paper by Andreou & Dassios [1] and in [8]. For earlier papers on magneto-elasticity e.g for plane waves see the references in [9, 8].

Remark: We remark that an additional damping as in magneto-*thermo*-elasticity (see [8]) is of course likely to lead to a similar result. The damping will certainly not give worse decay, but we conjecture that it will also not improve the decay essentially.

In connection with the question of optimal decay rates it should be mentioned that for the related thermo-elastic system exponential decay can only be expected in special situations like radial symmetry, for example, see Jiang & the authors [4], but not in general, see Koch [5] or Lebeau & Zuazua [6].

In Section 2 we shall present appropriate multipliers and the essential estimates for the components of the final Lyapunov functional. Theorem 1.1 is then proved in Section 3.

2 Multipliers and energy estimates

It is easy to verify that

$$\frac{d}{dt}E(t;u,h) = -\frac{\alpha}{\beta}\int\limits_\Omega |\nabla \times h|^2 dx$$

and, in general,

$$\frac{d}{dt}E_j(t) = -\frac{\alpha}{\beta}\int\limits_\Omega |\nabla \times \partial_t^j h|^2 dx.$$

Using the fact that Ω is simply connected, we conclude

$$\int\limits_\Omega |h|^2 dx + \int\limits_\Omega |\nabla h|^2 dx \leq c \int\limits_\Omega |\nabla \times h|^2 dx$$

(cf. page 356 in [2] or page 157 in [7]), hence

$$\frac{d}{dt}E_j(t) \leq -c\int\limits_\Omega |\nabla \partial_t^j h|^2 dx, \quad j = 0, \ldots, 7. \tag{2.1}$$

Here and in the sequel c will denote various positive constants not depending on t or on the initial data. Using the notation

$$\partial_j := \frac{\partial}{\partial x_j}, \qquad j = 1, 2,$$

simple calculations give

$$\begin{aligned} \nabla \times h &= (0, 0, \partial_1 h^2 - \partial_2 h^1)', \\ (\nabla \times h) \times \vec{H} &= H(0, \partial_1 h^2 - \partial_2 h^1, 0)', \end{aligned}$$

$$
\begin{aligned}
u_t \times \vec{H} &= H(0,0,-u_t^2)', \\
\nabla \times (u_t \times \vec{H}) &= H(-\partial_2 u_t^2,\ \partial_1 u_t^2, 0)', \\
\nu \times (\nabla \times h) &= (\nu_2(\partial_1 h^2 - \partial_2 h^1), -\nu_1(\partial_1 h^2 - \partial_2 h^1), 0)'.
\end{aligned}
$$

In particular, the boundary condition $\nu \times (\nabla \times h)_{|\Gamma} = 0$ turns into

$$
(\partial_1 h^2 - \partial_2 h^1)_{|\Gamma} = 0. \tag{2.2}
$$

The function

$$
\omega := \partial_1 h^2 - \partial_2 h^1
$$

denotes the two-dimensional rotation of $(h^1, h^2)'$, and the differential equations (1.1), (1.2) can be written as

$$
\begin{aligned}
u_{tt}^1 - \mu \Delta u^1 - (\mu+\lambda)\partial_1^2 u^1 - (\mu+\lambda)\partial_1\partial_2 u^2 &= 0, && (2.3) \\
u_{tt}^2 - \mu \Delta u^2 - (\mu+\lambda)\partial_2^2 u^2 - (\mu+\lambda)\partial_1\partial_2 u^1 &= \alpha H \omega, && (2.4) \\
h_t^1 - \Delta h^1 &= -\beta H \partial_2 u_t^2 && (2.5) \\
h_t^2 - \Delta h^2 &= \beta H \partial_1 u_t^2. && (2.6)
\end{aligned}
$$

We introduce the following functionals as abbreviations:

$$
\begin{aligned}
A_1 u^1 &:= -\mu \Delta u^1 - (\mu+\lambda)\partial_1^2 u^1, \\
A_2 u^2 &:= -\mu \Delta u^2 - (\mu+\lambda)\partial_2^2 u^2, \\
\mathcal{E}^1(t) &:= \frac{1}{2}\int_\Omega |u_t^1|^2 + \mu |\nabla u^1|^2 + (\mu+\lambda)|\partial_1 u^1|^2 dx.
\end{aligned}
$$

In the next Lemma we define the functional which shows the dissipative properties of the second component of the displacement.

Lemma 2.1 *Let*

$$
\Phi_1 := \Phi_1(t) := \int_\Omega \omega u_t^2 + \omega \Delta u^2 - \beta H(\mu+\lambda)\partial_2 u^2 \partial_1 u^1 dx - \beta H \mathcal{E}^1(t).
$$

Then

$$
\frac{d}{dt}\Phi_1 = -\beta H \int_\Omega |\nabla u_t^2|^2 dx + \int_\Omega \omega_t \Delta u^2 dx + \alpha H \int_\Omega |\omega|^2 dx + \int_\Omega \omega A^2 u^2 dx + (\mu+\lambda)\int_\Omega h_t^1 \partial_1 u^1 dx.
$$

Proof: From the equations (2.5) and (2.6) we obtain

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega h^1 \partial_2 u_t^2 dx &= \int_\Omega h_t^1 \partial_2 u_t^2 + h^1 \partial_2 u_{tt}^2 dx \\
&= \int_\Omega \Delta h^1 \partial_2 u_t^2 dx - \beta H \int_\Omega |\partial_2 u_t^2|^2 dx - \int_\Omega \partial_2 h^1 u_{tt}^2 dx
\end{aligned}
$$

$$
\begin{aligned}
&= \int_\Omega \Delta h^1 \partial_2 u_t^2 - \beta H \int_\Omega |\partial_2 u_t^2|^2 dx - \int_\Omega \partial_2 h^1 A_2 u^2 dx \\
&\quad -(\mu+\lambda)\int_\Omega \partial_1\partial_2 u^1 \partial_2 h^1 dx - \alpha H \int_\Omega \partial_2 h^1 \omega dx.
\end{aligned}
$$

Similarly, using the equation (2.3), (2.6), we get

$$
\begin{aligned}
-\frac{d}{dt}\int_\Omega h^2 \partial_1 u_t^2 dx &= -\int_\Omega \Delta h^2 \partial_1 u_t^2 dx - \beta H \int_\Omega |\partial_1 u_t^2|^2 dx \\
&\quad + \int_\Omega \partial_1 h^2 A_2 u^2 dx + (\mu+\lambda)\int_\Omega \partial_1\partial_2 u^1 \partial_1 h^2 dx + \alpha H \int_\Omega \partial_1 h^2 \omega dx.
\end{aligned}
$$

Summing up the last two identities we obtain

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega \omega u_t^2 dx &= \frac{d}{dt}\int_\Omega h^1 \partial_2 u_t^2 - h^2 \partial_1 u_t^2 dx \\
&= \underbrace{\int_\Omega \nabla\omega \nabla u_t^2 dx}_{=: I_1} - \beta H \int_\Omega |\nabla u_t^2|^2 dx \\
&\quad + \int_\Omega \omega A_2 u^2 dx + \underbrace{(\mu+\lambda)\int_\Omega \omega \partial_1\partial_2 u^1 dx}_{=: I_2} + \alpha H \int_\Omega |\omega|^2 dx. \qquad (2.7)
\end{aligned}
$$

Note that, since $\omega_{|\Gamma} = 0$,

$$
I_1 = -\int_\Omega \omega \Delta u_t^2 dx = -\frac{d}{dt}\int_\Omega \omega \Delta u^2 dx + \int_\Omega \omega_t \Delta u^2 dx, \qquad (2.8)
$$

and

$$
I_2 = -(\mu+\lambda)\int_\Omega \partial_2 \omega \partial_1 u^1 dx.
$$

Since

$$
\operatorname{div} \mathrm{h} = 0
$$

we have

$$
\partial_2 \omega = -\Delta h^1,
$$

therefore we get

$$
\begin{aligned}
I_2 &= (\mu+\lambda)\int_\Omega \Delta h^1 \partial_1 u^1 dx \\
&= (\mu+\lambda)\int_\Omega (h_t^1 + \beta H \partial_2 u_t^2)\partial_1 u^1 dx \\
&= (\mu+\lambda)\int_\Omega h_t^1 \partial_1 u^1 dx + \beta H(\mu+\lambda)\int_\Omega \partial_2 u_t^2 \partial_1 u^1 dx
\end{aligned}
$$

$$= \quad (\mu+\lambda)\int_\Omega h_t^1\partial_1 u^1 dx + \beta H(\mu+\lambda)\frac{d}{dt}\int_\Omega \partial_2 u^2\partial_1 u^1 dx$$
$$-\beta H(\mu+\lambda)\int_\Omega \partial_1 u_t^1\partial_2 u^2 dx. \tag{2.9}$$

Multiplying equation (2.3) by u_t^1 we get

$$\frac{d}{dt}\mathcal{E}^1(t) = -(\mu+\lambda)\int_\Omega \partial_1 u_t^1\partial_2 u^2 dx.$$

Substituting this identity into (2.9) we obtain

$$I_2 = (\mu+\lambda)\int_\Omega h_t^1\partial_1 u^1 dx + \frac{d}{dt}\{\beta H(\mu+\lambda)\int_\Omega \partial_2 u^2\partial_1 u^1 dx + \beta H(\mu+\lambda)\mathcal{E}^1(t)\}. \tag{2.10}$$

Recalling the definition of Φ_1 the conclusion follows from (2.7), (2.8), (2.10).

Q.E.D.

Following Lemma 2.1 and since (u_t,h_t) also satisfies the equations (1.1), (1.2) and (1.4), (1.5) we can introduce functionals Φ_2 and Φ_3 in the following way. Writing

$$\Phi_1(t) = \Phi_1(t;u,h)$$

we have (see Lemma 2.1)

$$\begin{aligned}
\frac{d}{dt}\Phi_1(t;u_t,h_t) \quad &= \quad -\beta H\int_\Omega |\nabla u_{tt}^2|^2 dx + \int_\Omega \omega_{tt}\Delta u_t^2 dx \\
&\quad +\alpha H\int_\Omega |\omega_t|^2 dx + \int_\Omega \omega_t A_2 u_t^2 dx + (\mu+\lambda)\int_\Omega h_{tt}^1\partial_1 u_t^1 dx \\
&= \quad -\beta H\int_\Omega |\nabla u_{tt}^2|^2 dx + \frac{d}{dt}\int_\Omega \omega_{tt}\Delta u^2 dx \\
&\quad -\int_\Omega \omega_{ttt}\Delta u^2 dx + \alpha H\int_\Omega |\omega_t|^2 dx + \frac{d}{dt}\int_\Omega \omega_t A_2 u_2 dx \\
&\quad -\int_\Omega \omega_{tt}A_2 u^2 + (\mu+\lambda)\frac{d}{dt}\int_\Omega h_{tt}^1\partial_1 u^1 dx - (\mu+\lambda)\int_\Omega \partial_1 u^1 dx
\end{aligned}$$

which implies for

$$\Phi_2(t) := \Phi_2(t;u,h) := \Phi_1(t;u_t,h_t) - \int_\Omega \omega_{tt}\Delta u^2 + \omega_t A_2 u^2 - (\mu+\lambda)h_{tt}^1\partial_1 u^1 dx$$

the identity

$$\begin{aligned}
\frac{d}{dt}\Phi_2(t) \quad &= \quad -\beta h\int_\Omega |\nabla u_{tt}^2|^2 dx - \int_\Omega \omega_{ttt}\Delta u^2 dx + \alpha H\int_\Omega |\omega_t|^2 dx \\
&\quad -\int_\Omega \omega_{tt}A_2 u^2 dx - (\mu+\lambda)\int_\Omega h_{ttt}^1\partial_1 u^1 dx.
\end{aligned} \tag{2.11}$$

Similarly we can define

$$\Phi_3(t) := \Phi_2(t; u_t, h_t) - \int\limits_\Omega \omega_{tttt}\Delta u^2 + \omega_{ttt}A_2 u^2 + (\mu+\lambda)h^1_{tttt}\partial_1 u^1 dx.$$

Then we have

$$\begin{aligned}\frac{d}{dt}\Phi_3(t) &= -\beta H\int\limits_\Omega |\nabla u^2_{ttt}|^2 dx + \int\limits_\Omega \omega_{ttttt}\Delta u^2 dx \\ &\quad +\alpha H\int\limits_\Omega |\omega_{tt}|^2 dx + \int\limits_\Omega \omega_{tttt}A_2 u^2 dx + (\mu+\lambda)\int\limits_\Omega h^1_{ttttt}\partial_1 u^1 dx. \end{aligned} \tag{2.12}$$

The following lemma plays an important role in the sequel because it will connect the dissipative properties of the magnetic field with the first component of the displacement vector field.

Lemma 2.2 *Let*

$$q(x) := (x_2 - \delta x_1,\, x_1 - \delta x_2)'$$

for some $\delta \in R$, and let

$$F(t) := \int\limits_\Omega u^2_{tt} q\nabla u_t + A_2 u^2_t q\nabla u^1 dx.$$

Then for $|\delta|$ sufficiently large — only depending on the domain Ω —, we have

$$\begin{aligned}\frac{d}{dt}F(t) &\le -(\mu+\lambda)\int\limits_\Omega |\nabla u^1_t|^2 dx + c\int\limits_\Gamma \left|\frac{\partial u^2_{tt}}{\partial\nu}\right|\left|\frac{\partial u^1}{\partial\nu}\right| ds \\ &\quad + c\int\limits_\Omega |\nabla u^2_{tt}||\Delta u^1| dx + c\int\limits_\Omega |\omega_t||\nabla u^1_t| dx.\end{aligned}$$

Proof: The main idea to show the above inequality is to use equations (2.3)–(2.6) to connect the dissipation given by h to u^1. Then we will use the geometrical properties of the domain to eliminate unpleasant boundary terms in u^1. In fact let us consider

$$\begin{aligned}\frac{d}{dt}\int\limits_\Omega u^2_{tt}q_1\partial_1 u^1_t dx &= \int\limits_\Omega u^2_{ttt}q_1\partial_1 u^1_t dx + \int\limits_\Omega u^2_{tt}q_1\partial_1 u^1_{tt} \\ &= -\int\limits_\Omega A_2 u^2_t q_1 \partial_1 u^1_t dx + \frac{\mu+\lambda}{2}\int\limits_\Omega q_1\partial_2|\partial_1 u^1_t|^2 dx + \alpha H\int\limits_\Omega \omega_t q_1\partial_1 u^1_t dx \\ &= -\frac{d}{dt}\int\limits_\Omega A_2 u^2_t q_1\partial_1 u^1 dx + \int\limits_\Omega A_2 u^2_{tt} q_1 \partial_1 u^1_t dx \\ &\quad + \frac{\mu+\lambda}{2}\int\limits_\Gamma q_1\nu_2|\partial_1 u^1_t|^2 ds - \frac{\mu+\lambda}{2}\int\limits_\Omega \partial_2 q_1|\partial_1 u^1_t|^2 dx \\ &\quad + \alpha H\int\limits_\Omega \omega_t q_1\partial_1 u^1_t dx.\end{aligned} \tag{2.13}$$

Note that

$$\int_\Omega A_2 u_{tt}^2 q_1 \partial_1 u^1 dx = -\mu \int_\Gamma \frac{\partial u_{tt}^2}{\partial \nu} q_1 \partial_1 u^1 ds - (\mu+\lambda) \int_\Gamma \nu_2 \partial_2 u_{tt}^2 q_1 \partial_1 u^1 ds$$
$$+(\mu+\lambda) \int_\Omega \partial_2 u_{tt}^2 [\partial_2 q_1 \partial_1 u^1 + q_1 \partial_1 \partial_2 u^1] dx$$
$$+\mu \int_\Omega \nabla u_{tt}^2 [\nabla q_1 \partial_1 u^1 + q_1 \partial_1 \nabla u^1] dx$$

from where we conclude

$$\frac{d}{dt}\left\{ \int_\Omega u_{tt}^2 q_1 \partial_1 u_t^1 dx + \int_\Omega A_2 u_t^2 q_1 \partial_1 u^1 dx \right\}$$
$$= \frac{\mu+\lambda}{2} \int_\Gamma q_1 \nu_2 |\partial_1 u_t^1|^2 ds - \mu \int_\Gamma \frac{\partial u_{tt}^2}{\partial \nu} q_1 \partial_1 u^1 ds$$
$$-(\mu+\lambda) \int_\Gamma \nu_2 q_1 \partial_2 u_{tt}^2 \partial_1 u^1 dx + (\mu+\lambda) \int_\Omega \partial_2 u_{tt}^2 [\partial_2 q_1 \partial_1 u^1 + q_1 \partial_1 \partial_2 u^1] dx$$
$$+\mu \int_\Omega \nabla u_{tt}^2 [\nabla q_1 \partial_1 u^1 + q_1 \partial_1 \nabla u^1] dx - (\mu+\lambda) \int_\Omega \partial_2 q_1 |\partial_1 u_t^1|^2 dx$$
$$+\alpha H \int_\Omega \omega_t q_1 \partial_1 u_t^1 dx$$
$$= \frac{\mu+\lambda}{2} \int_\Gamma q_1 \nu_1^2 \nu_2 \left| \frac{\partial u_t^1}{\partial \nu} \right| ds - \mu \int_\Gamma \frac{\partial u_{tt}^2}{\partial \nu} q_1 \nu_1 \frac{\partial u^1}{\partial \nu} dx$$
$$-(\mu+\lambda) \int_\Gamma \nu_2^2 q_1 \nu_1 \frac{\partial u_{tt}^2}{\partial \nu} \frac{\partial u^1}{\partial \nu} ds$$
$$+(\mu+\lambda) \int_\Omega \partial_2 u_{tt}^2 [\partial_2 q_1 \partial_1 u^1 + q_1 \partial_1 \partial_2 u^1] dx$$
$$+\mu \int_\Omega \nabla u_{tt}^2 [\nabla q_1 \partial_1 u^1 + q_1 \partial_1 \nabla u^1] dx - (\mu+\lambda) \int_\Omega \partial_2 q_1 |\partial_1 u_t^1| dx$$
$$+\alpha H \int_\Omega \omega_t q_1 \partial_1 u_t^1 dx. \tag{2.14}$$

Similarly we obtain

$$\frac{d}{dt}\left\{ \int_\Omega u_{tt}^2 q_2 \partial_2 u^1 dx + \int_\Omega A_2 u_t^2 q_2 \partial_2 u^1 dx \right\} =$$
$$\frac{\mu+\lambda}{2} \int_\Gamma q_2 \nu_1 \nu_2^2 \left| \frac{\partial u_t^1}{\partial \nu} \right|^2 ds - \mu \int_\Gamma \frac{\partial u_{tt}^2}{\partial \nu} q_2 \nu_2 \frac{\partial u^1}{\partial \nu} ds$$
$$-(\mu+\lambda) \int_\Gamma q_2 \nu_1^2 \nu_2 \frac{\partial u_{tt}^2}{\partial \nu} \frac{\partial u^1}{\partial \nu} ds + (\mu+\lambda) \int_\Omega \partial_2 u_{tt}^2 [\partial_2 q_2 \partial_2 u^1 + q_2 \partial_2^2 u^1] dx$$

$$+\mu\int_\Omega \nabla u_{tt}^2[\nabla q_2\partial_2 u^1 + q_2\partial_2\nabla u^1]dx - (\mu+\lambda)\int_\Omega \partial_1 q_2|\partial_2 u_t^1|^2 dx$$

$$+\alpha H\int_\Omega \omega_t q_2\partial_2 u_t^1 dx \tag{2.15}$$

Summing up (2.14) and (2.15) we get

$$\begin{aligned}
\frac{d}{dt}F(t) &= \frac{d}{dt}\left\{\int_\Omega u_{tt}^2 q\nabla u_t dx + \int_\Omega A_2 u_t^2 q\nabla u^1 dx\right\}\\
&= \frac{\mu+\lambda}{2}\int_\Gamma \nu_1\nu_2[\nu_1 q_1+\nu_2 q_2]\left|\frac{\partial u_t^1}{\partial\nu}\right|^2 ds\\
&\quad -\mu\int_\Gamma \frac{\partial u_{tt}^2}{\partial\nu}q\nu\frac{\partial u^1}{\partial\nu}ds - (\mu+\lambda)\int_\Gamma \nu_1\nu_2[q_1\nu_2+q_2\nu_1]\frac{\partial u_{tt}^2}{\partial\nu}\frac{\partial u^1}{\partial\nu}ds\\
&\quad +(\mu+\lambda)\int_\Omega \partial_2 u_{tt}^2[\partial_2 q_1\partial_1 u^1 + \partial_2 q_2\partial_2 u^1 + q_1\partial_1\partial_2 u^1 + q_2\partial_2^2 u^1]dx\\
&\quad +\mu\int_\Omega \nabla u_{tt}^2[\nabla q_1\partial_1 u^1 + \nabla q_2\partial_2 u^1 + q_1\partial_1\nabla u^1 + q_2\partial_2\nabla u^1]dx\\
&\quad -(\mu+\lambda)\int_\Omega \partial_2 q_1|\partial_1 u_t^1|^2 + \partial_1 q_2|\partial_2 u_t^1|dx\\
&\quad +\alpha H\int_\Omega \omega_t[q_1\partial_1 u_t^1 + q_2\partial_2 u_t^1]dx,
\end{aligned}$$

which implies

$$\begin{aligned}
\frac{d}{dt}F(t) \le\ & \underbrace{\frac{\mu+\lambda}{2}\int_\Gamma \nu_1\nu_2\nu\cdot q\left|\frac{\partial u_t^1}{\partial\nu}\right|^2 ds}_{=:I_3}\\
& -(\mu+\lambda)\int_\Omega |\nabla u_t^1|^2 dx + c\int_\Gamma \left|\frac{\partial u_{tt}^2}{\partial\nu}\right|^2\left|\frac{\partial u^1}{\partial\nu}\right| ds\\
& +c\int_\Omega |\nabla u_{tt}^2||\Delta u^1|dx + c\int_\Omega |\omega_t||\nabla u_t^1|dx.
\end{aligned} \tag{2.16}$$

Now we use that Ω is of partial rectangular type I, II or III.
For type I we have $\nu_1\nu_2 = 0$, hence

$$I_3 = 0.$$

For type II we have for δ positive and sufficiently large (only depending on Ω)

$$I_3 \le 0.$$

For type III we have for δ negative and $|\delta|$ sufficiently large (only depending on Ω)

$$I_3 \le 0.$$

That is, for all types I, II, III we have

$$I_3 \leq 0, \tag{2.17}$$

provided δ is chosen appropriately. The assertion of the lemma now follows from (2.16) and (2.17).

Q.E.D.

Lemma 2.2 gives a good estimate for u^1, except for the boundary term which contains the normal derivative on u^2_{tt}. To estimate this surface integral we will use the following Lemma applied to u^1_t and u^2_{tt}.

Lemma 2.3 *Let $g \in C^1(R^2)$, $g = (g_1, g_2)'$ such that $g\nu \geq \gamma > 0$ for some γ. Then*

$$\begin{aligned}
-\frac{d}{dt}\int_\Omega u_t^2 g_k \partial_k u^2 dx \quad &\leq \quad -\frac{\mu}{2}\int_\Gamma g_k \nu_k \left|\frac{\partial u^2}{\partial \nu}\right| ds \\
&+c\int_\Omega |u_t^2|^2 + |\nabla u^2|^2 dx - (\mu+\lambda)\int_\Omega g_k \partial_1 \partial_2 u^1 \partial_k u^2 dx \\
&+\alpha H \int_\Omega \omega g_k \partial_k u^2 dx.
\end{aligned}$$

Proof:

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u_t^2 g_k \partial_k u^2 dx \quad &= \quad \int_\Omega u_{tt}^2 g_k \partial_k u^2 dx + \int_\Omega u_t^2 g_k \partial_k u_t^2 dx \\
&= \mu \int_\Omega \Delta u^2 g_k \partial_k u^2 dx + (\mu+\lambda)\int_\Omega \partial_2^2 u^2 g_k \partial_k u^2 dx \\
&+\frac{1}{2}\int_\Omega g_k \partial_k |u_t^2|^2 dx + (\mu+\lambda)\int_\Omega \partial_1\partial_2 u^1 g_k \partial_k u^2 dx + \alpha H \int_\Omega \omega g_k \partial_k u^2 \\
&= \mu \int_\Gamma \frac{\partial u^2}{\partial \nu} g_k \partial_k u^2 ds + (\mu+\lambda)\int_\Gamma \nu_2 \partial_2 u^2 g_k \partial_k u^2 ds \\
&-\mu \int_\Omega \nabla u_2 \nabla g_k \partial_k u^2 dx - \mu \int_\Omega \nabla u^2 g_k \partial_k \nabla u^2 dx \\
&-(\mu+\lambda)\int_\Omega \partial_2 u^2 \partial_2 g_k \partial_k u^2 dx - (\mu+\lambda)\int_\Omega \partial_2 u^2 g_k \partial_2 \partial_k u^2 dx \\
&-\frac{1}{2}\int_\Omega \partial_k g_k |u_t^2|^2 dx + \frac{1}{2}\int_\Gamma g_k \nu_k |u_t^2|^2 ds \\
&+(\mu+\lambda)\int_\Omega g_k \partial_1 \partial_2 u^1 \partial_k u^2 dx + \alpha H \int_\Omega \omega g_k \partial_k u^2 dx.
\end{aligned}$$

This implies

$$\frac{d}{dt}\int_\Omega u_t^2 g_k \partial_k u^2 dx \quad = \quad \frac{\mu}{2}\int_\Gamma g_k \nu_k \left|\frac{\partial u^2}{\partial \nu}\right|^2 ds + \frac{\mu+\lambda}{2}\int_\Gamma g_k \nu_k |\partial_2 u^2|^2 ds$$

$$
\begin{aligned}
&-\frac{1}{2}\int\limits_\Omega \partial_k g_k\{|u_t^2|^2-\mu|\nabla u^2|^2-(\mu+\lambda)|\partial_2 u^2|^2\}dx \\
&-\mu\int\limits_\Omega \nabla u^2\nabla g_k\partial_k u^2 dx-(\mu+\lambda)\int\limits_\Omega \partial_2 u^2\partial_2 g_k\partial_k u^2 dx \\
&+(\mu+\lambda)\int\limits_\Omega g_k\partial_1\partial_2 u^1\partial_k u^2 dx+\alpha H\int\limits_\Omega \omega g_k\partial_k u^2 dx \\
&+\frac{1}{2}\int\limits_\Gamma g_k\nu_k|u_t^2|^2 ds.
\end{aligned}
$$

This implies

$$
\begin{aligned}
-\frac{d}{dt}\int\limits_\Omega u_t^2 g_k\partial_k u^2 dx \quad\leq\quad & -\frac{\mu}{2}\int\limits_\Gamma g_k\nu_k\left|\frac{\partial u^2}{\partial\nu}\right|^2 ds+c\int\limits_\Omega |u_t^2|^2+|\nabla u^2|^2 dx \\
&-(\mu+\lambda)\int\limits_\Omega g_k\partial_1\partial_2 u^1\partial_k u^2 dx-\alpha H\int\limits_\Omega \omega g_k\partial_k u^2 dx
\end{aligned}
$$

which completes the proof of Lemma 2.3.

Q.E.D.

Let us extend Lemma 2.3 to time derivatives of u and h. Since (u_t, h_t) satisfy essentially the same equations as (u, h), we have

$$
\begin{aligned}
-\frac{d}{dt}\int\limits_\Omega u_{tt}^2 g_k\partial_k u_t^2 dx \quad\leq\quad & -\frac{\mu}{2}\int\limits_\Gamma g_k\nu_k\left|\frac{\partial u_t^2}{\partial\nu}\right|^2 ds+c\int\limits_\Omega |u_{tt}^2|^2+|\nabla u_t^2|^2 dx \\
&-(\mu+\lambda)\int\limits_\Omega g_k\partial_1\partial_2 u_t^1\partial_k u_t^2 dx-\alpha H\int\limits_\Omega \omega_t g_k\partial_k u_t^2 dx. \qquad (2.18)
\end{aligned}
$$

Denoting

$$
J(t;u,h) := -\int\limits_\Omega u_t^2 g_k\partial_k u^2 dx
$$

we conclude from (2.18)

$$
\begin{aligned}
\frac{d}{dt}J(t;u_t,h_t) \quad\leq\quad & -\frac{\mu}{2}\int\limits_\Gamma g_k\nu_k\left|\frac{\partial u_t^2}{\partial\nu}\right|^2 ds \\
&+c\int\limits_\Omega |u_{tt}^2|^2+|\nabla u_t^2|^2 dx-(\mu+\lambda)\frac{d}{dt}\int\limits_\Omega g_k\partial_1\partial_2 u^1\partial_k u_t^2 dx \\
&+(\mu+\lambda)\int\limits_\Omega g_k\partial_1\partial_2 u^1\partial_k u_{tt}^2 dx-\alpha H\int\limits_\Omega \omega_t g_k\partial_k u_t^2
\end{aligned}
$$

which implies for the functional

$$
J_2(t) := J_2(t;u,h) := J(t;u_t,h_t)+(\mu+\lambda)\int\limits_\Omega g_k\partial_1\partial_2 u^1\partial_k u_t^2 dx
$$

that

$$\frac{d}{dt}J_2(t) \;\; \leq \;\; -\frac{\mu}{2}\int\limits_{\Gamma} g_k\nu_k\left|\frac{\partial u_t^2}{\partial\nu}\right|^2 ds + c\int\limits_{\Omega}|u_{tt}^2|^2+|\nabla u_t^2|^2dx$$
$$+(\mu+\lambda)\int\limits_{\Omega} g_k\partial_1\partial_2 u^1\partial_k u_{tt}^2 dx - \alpha H\int\limits_{\Omega}\omega_t g_k\partial_k u_t^2 dx. \qquad (2.19)$$

Similarly, defining

$$J_3(t) := J_2(t;u_t,h_t) + (\mu+\lambda)\int\limits_{\Omega} g_k\partial_1\partial_2 u^1\partial_k u_{ttt}^2 dx,$$

we conclude that

$$\frac{d}{dt}J_3(t) \;\; \leq \;\; -\frac{\mu}{2}\int\limits_{\Gamma} g_k\nu_k\left|\frac{\partial u_{tt}^2}{\partial\nu}\right|^2 ds + c\int\limits_{\Omega}|u_{ttt}^2|^2+|\nabla u_{tt}^2|^2dx$$
$$-(\mu+\lambda)\int\limits_{\Omega} g_k\partial_1\partial_2 u^1\partial_k u_{tttt}^2 dx - \alpha H\int\limits_{\Omega}\omega_{tt} g_k\partial_k u_{tt}^2 dx. \qquad (2.20)$$

If we define the functional (cp. (2.12))

$$\Phi_4(t) := \Phi_3(t;u_t,h_t) - \int\limits_{\Omega}\omega_{tttttt}\Delta u^2 dx - \int\limits_{\Omega}\omega_{ttttt}A_2u^2dx - (\mu+\lambda)\int\limits_{\Omega}h_{ttttt}^1\partial_1 u^1 dx,$$

we get for $N > N_1 > 0$ sufficiently large (observe $g_k\nu_k \geq \gamma > 0$):

$$\frac{d}{dt}\{N_1J_3(t)+N\Phi_3(t)+N\Phi_4(t)\} \leq$$
$$-\frac{N_1\mu}{4}\gamma\int\limits_{\Gamma}\left|\frac{\partial u_{tt}^2}{\partial\nu}\right|^2 ds - \frac{N}{2}\int\limits_{\Omega}|u_{ttt}^2|^2+|\nabla u_{ttt}^2|^2+|\nabla u_{tttt}^2|^2dx$$
$$+c\varepsilon N\left(\int\limits_{\Omega}|\Delta u^1|^2+|\Delta u^2|^2dx\right)+C_\varepsilon N\int\limits_{\Omega}\left(\sum_{j=2}^{7}|\partial_t^j w|^2+\sum_{j=3}^{5}|\partial_t h|^2\right)dx, \qquad (2.21)$$

where $\varepsilon > 0$ is small and $C_\varepsilon = C(\varepsilon)$ is a positive constant. Using Lemma 2.1 and Lemma 2.2 we conclude

$$\frac{d}{dt}\{F(t)+N_1J_3(t)+N\Phi_3(t)+N\Phi_4(t)\} \leq$$
$$-\frac{N_1\mu\gamma}{4}\int\limits_{\Gamma}\left|\frac{\partial u_{tt}^2}{\partial\nu}\right|^2 ds - \frac{N}{4}\int\limits_{\Omega}|u_{ttt}^2|^2+|\nabla u_{ttt}^2|^2+|\nabla u_{tttt}^2|^2dx$$
$$-\frac{\mu+\lambda}{4}\int\limits_{\Omega}|\nabla u_t^1|^2dx + 2\varepsilon\int\limits_{\Omega}|\Delta u|^2dx - \frac{cN}{2}\int\limits_{\Omega}|\nabla u_t^2|^2dx$$
$$+C_\varepsilon\int\limits_{\Omega}\left(\sum_{j=0}^{7}|\partial_t^j w|^2+\sum_{j=3}^{5}|\partial_t^j h|^2)dx\right). \qquad (2.22)$$

Finally we obtain, using the differential equations once more, and denoting $Au := (A_1u^1,\ A_2u^2)'$,

$$\frac{d}{dt}\int_\Omega u_t Au\,dx = \int_\Omega \mu|\nabla u_t|^2 + (\mu+\lambda)|\mathrm{div}\,u_t|^2 dx - \int_\Omega |Au|^2 dx + \alpha H \int_\Omega \omega Au$$

from where we get

$$\frac{d}{dt}\int_\Omega u_t Au\,dx \le c\int_\Omega |\nabla u_t|^2 dx - \frac{1}{2}\int_\Omega |Au|^2 + c\int_\Omega |\omega|^2 dx. \tag{2.23}$$

3 Proof of Theorem 1.1

Let L denote the following Lyapunov functional:

$$L(t) := \sum_{k=0}^{7} M_k E_k(t) + N(\Phi_1(t) + \Phi_3(t) + \Phi_4(t)) + F(t) + N_1 J_3(t) + \int_\Omega u_t Au\,dx,$$

with $M_k > 0$. Then we obtain ($\varepsilon > 0$ sufficiently small in (2.22)) for sufficiently large M_k, combining (2.1), Lemma 2.1, (2.22) and (2.23), that there exists $k_0 > 0$ such that for $t \ge 0$:

$$\begin{aligned}\frac{d}{dt}L(t) &\le -k_0E_1(t) - k_0\int_\Gamma \left|\frac{\partial u_{tt}^2}{\partial\nu}\right|^2 ds - k_0\sum_{j=0}^{7}\int_\Omega |\partial_t^j\omega|^2 dx \\ &\le -k_0E_1(t)\end{aligned}$$

which implies, since $L(t) \ge 0$ for large M_k,

$$\int_0^t E_1(s)ds \le \frac{1}{k_0}L(0) \le d\sum_{j=0}^{7} E_j(0) \tag{3.24}$$

for some $d > 0$. Observing (2.1)we have

$$\frac{d}{dt}(tE_1(t)) = E_1(t) + t\frac{d}{dt}E_1(t) \le E_1(t)$$

which implies, using (3.24),

$$E_1(t) \le \frac{d}{t}\sum_{j=0}^{7} E_j(0).$$

Q.E.D.

References

[1] Andreou, E., Dassios, G.: Dissipation of energy for magnetoelastic waves in a conductive medium. *Quart. Appl. Math.* **55** (1997), 23–39.

[2] Duvaut, G, Lions, J.L.: *Inequalities in Mechanics and Physics.* Grundlehren math. Wiss. **219**. Springer Verlag, Berlin et al. (1976).

[3] Eringen, C.A., Maugin, G.A.: *Electrodynamics of continua I.* Springer-Verlag, New York et al. (1990).

[4] Jiang, S., Muñoz Rivera, J.E., Racke, R.: Asymptotic stability and global existence in thermoelasticity with symmetry. *Quart. Appl. Math.* **56** (1998), 259–275.

[5] Koch, H.: Slow decay in linear thermo-elasticity. *Quart. Appl. Math.* (to appear).

[6] Lebeau, G., Zuazua, E.: Decay rates for the three-dimensional linear system of thermoelasticity. *Arch. Rational Mech. Anal.* **148** (1999), 179–231.

[7] Leis, R.: *Initial boundary value problems in mathematical physics.* B.G. Teubner-Verlag, Stuttgart; John Wiley & Sons, Chichester et al. (1986).

[8] Muñoz Rivera, J.E., Racke, R.: Magneto-thermo-elasticity — large time behavior for linear systems. *Konstanzer Schriften Math. Inf.* **102** (1999).

[9] Perla Menzala, G., Zuazua, E.: Energy decay of magnetoelastic waves in a bounded conductive medium. *Asymptotic Analysis* **18** (1998), 349–362.

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