Energy decay for Timoshenko systems of memory type *

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Abstract: Linear systems of Timoshenko type equations for beams including a memory term are studied. The exponential decay is proved for exponential kernels, while polynomial kernels are shown to lead to a polynomial decay. The optimality of the results is also investigated.

AMS subject classification: 73 C 99, 35 Q 99

1 Introduction

In this paper we consider linear systems of Timoshenko type with memory, which are written as

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0 \quad \text{in} \quad (0,L)\times(0,\infty), \tag{1.1}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + g*\psi_{xx} + k(\varphi_x+\psi) = 0 \quad \text{in} \quad (0,L)\times(0,\infty) \tag{1.2}$$

where ρ_1, k, ρ_2, b and L are positive constants. The functions ϕ and ψ describe the transverse displacement of the beam and the rotation angle of a filament, respectively. The boundary conditions we consider here are given by

$$\varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, \quad t \geq 0. \tag{1.3}$$

The initial conditions are

$$\varphi(\cdot,0) = \varphi_0, \quad \varphi_t(\cdot,0) = \varphi_1, \quad \psi(\cdot,0) = \psi_0, \quad \psi_t(\cdot,0) = \psi_1 \quad \text{in} \quad (0,L). \tag{1.4}$$

The usual convolution term

$$g*\psi_{xx}(x,t) = \int_0^t g(t-s)\psi_{xx}(x,s)ds$$

represents the memory effect with a real-valued C^2-function g.

Our main interest concerns the asymptotic behavior of the solution of the system above. That is, whether the dissipation given by the memory effect in equation (1.2) is strong enough to stabilize the whole system. Another natural question concerning the asymptotic behavior is about the rate of decay of the solution. That is, what type of rate of decay may we expect?

*Supported by a CNPq-DLR grant

(If there exist one). How can the damping mechanism given by the memory effect through the relaxation function g be effective to produce uniform stabilization?

Let us mention some known results about related viscoelastic systems. Dafermos [3] proved that the solutions to viscoelastic systems tend to zero as time tends to infinity, but without giving explicit rates of decay. Lagnese [8] considered a linear viscoelastic equation obtaining uniform rates of decay but introducing additional damping terms acting on the boundary. Greenberg [5] and Hrusa [6] proved an exponential rate of decay for the nonlinear viscoelastic equation when the relaxation function g is of the form $g(t) = e^{-\mu t}$. In this case using the fact that $g'(t) = -\mu g(t)$ the convolution term is eliminated by differentiation, therefore the resulting equation has no integral term, hence this method can not be used for a more general class of relaxation functions even for those which are a linear combination of exponential terms with varying rates of decay. A similar result was obtained by Dassios and Zafiropoulos [4] for homogeneous and and isotropic viscoleastic materials which occupy the whole three-dimensional space. They proved that the longitudinal and transverse waves decay to zero uniformly like $t^{-m-3/2}$, where m increases depending on the symmetry of the initial data, provided the relaxation is an exponential function like $t \mapsto \mu_0 e^{-\gamma t}$. The method the authors used is based on the study of the roots of the characteristic polynomial associated to the ordinary differential equation, which is obtained by taking Fourier transform of the system and then differentiating the resulting equation with respect to time. By using the fact that the kernel g is an exponential function, that is $g'(t) = -\gamma g(t)$, the convolution term is eliminated, so the authors work with the resulting purely ordinary differential equation. In [9, 11, 12] was proved that the rate of decay of the solution depends on the rate of decay of the relaxation function, that is if the relaxation function decays exponentially then the solution decays exponentially, while if the relaxation function decays polynomially then the solution decays also polynomially with the same rate. For localized damping in viscoelasticity see Rivera and Peres [13] where it is shown that the first order energy decays exponentially to zero provided the relaxation kernel also decays exponentially to zero. When the kernel decays polynomially, the problem is open.

The main result of this paper is that the whole system decays uniformly if and only if the coefficients satisfy

$$\frac{\rho_1}{\rho_2} = \frac{k}{b}. \tag{1.5}$$

Concerning the rate of decay, we will show that the solution decays exponentially to zero provided the kernel tends to zero also exponentially. When the kernel decays to zero polynomially, the solution also decays polynomially with the same rate. More precisely: If g is of exponential type, i.e. if the following assumption

$$\left.\begin{aligned} g > 0, \quad \exists k_0, k_1, k_2 > 0 :\ -k_0 g \le g' \le -k_1 g, \quad |g''| \le k_2 g, \\ \lambda := b - \int_0^\infty g(s)ds > 0 \end{aligned}\right\} \tag{1.6}$$

is satisfied, then the exponential decay of the energy

$$E(t) = \frac{1}{2}\int_0^L \rho_1|\varphi_t|^2 + \rho_2|\psi_t|^2 + (b - \int_0^t g\, d\tau)|\psi_x|^2 + k|\varphi_x + \psi|^2 + g\Box\psi_x\, dx$$

for a solution (ϕ, ψ) as time tends to infinity will be proved if and only if the coefficients satisfy (1.5). The symbol $\Box$ denotes the following convolution:

$$(g\Box f)(t) := \int_0^t g(t-s)|f(s) - f(t)|^2 ds.$$

If g is of polynomial type, i.e. if it satisfies

$$\left.\begin{array}{r} 0 < g(t) \leq b_0(1+t)^{-p}, \\ -b_1 g(t)^{\frac{p+1}{p}} \leq g'(t) \leq -b_2 g(t)^{\frac{p+1}{p}}, \\ -b_3|g'(t)|^{\frac{p+2}{p+1}} \leq g''(t) \leq -b_4|g'(t)|^{\frac{p+2}{p+1}}, \end{array}\right\} \tag{1.7}$$

with positive constants b_0, b_1, b_2, b_3, b_4 and $p > 2$, then the polynomial decay of the energy will be proved. This result is also shown to be optimal in the sense, that there cannot occur an exponential decay. The typical example $\bar{g}$ satisfying (1.7) is of course

$$\bar{g}(t) = b_0(1+t)^{-p}.$$

In Sections 2 and 3 we consider exponential kernels showing the exponential decay result under assumption (1.5) and that there is no uniform decay if this assumption is not satisfied, respectively. In Sections 4 and 5 polynomial kernels are studied and the polynomial decay of the energy (under assumption (1.5)) is proved as well as the optimality, i.e. non exponential decay, respectively. The results in Sections 2,4,5 are proved by energy methods, using suitably sophisticated estimates for multipliers, while Section 3 also uses sharp perturbation arguments for the spectral radius of a semigroup.

Remark: Timoshenko *plates* can be dealt with in a similar manner as the Timoshenko beams discussed here.

The uniform stabilization of Timoshenko beams with the memory term $g * \psi_{xx}$ in equation (1.2) replaced by some control function f was studied by Soufyane [18]. He showed the exponential decay of the associated energy for

$$f(x,t) = b(x)\psi_t(x,t)$$

and also if and only if the assumption (1.5) is satisfied. Previous work of different authors considered two boundary control functions, like Kim & Renardy [7], or two forces, see Taylor [19]. In our paper the first results are presented for a memory type control term, both for exponential and for polynomial kernels.

In the sequel we shall always assume the unique existence of strong solutions to the initial-boundary value problem under consideration, cp. for example [18], [16]. The problem is well-posed for data $((\varphi_0, \varphi_1), (\psi_0, \psi_1))$ in the Sobolev space $[H^2((0,L)) \times H_0^1((0,L))]^2$. Weak solutions and the energy are well defined also in $[H_0^1((0,L)) \times L^2((0,L))]^2$.

2 Exponential decay

First we consider exponential kernels of type (1.6) and we look for the exponential decay of the energy

$$E(t) := \frac{1}{2}\int_0^L \rho_1|\varphi_t|^2 + \rho_2|\psi_t|^2 + (b - \int_0^t g\, d\tau)|\psi_x|^2 + k|\varphi_x + \psi|^2 + g\Box\psi_x\, dx. \tag{2.1}$$

Using the following simple lemma (cf. Lemma 3.2 from [14])

Lemma 2.1 *For $f, h \in C^1([0,\infty),\mathbb{R})$ we have*

$$2(f*h)(t)h_t(t) = (f'\Box h)(t) + \frac{d}{dt}\left\{\int_0^t f(s)ds|h(t)|^2 - (f\Box h)(t)\right\} - f(t)|h(t)|^2.$$

we easily conclude that the energy decays:

$$\frac{d}{dt}E(t) = -\frac{1}{2}g(t)\int_0^L |\psi_x|^2\,dx + \frac{1}{2}\int_0^L g'\Box\psi_x\,dx \le 0. \tag{2.2}$$

The main point to show the exponential decay is to construct a Lyapunov functional $\mathcal{L}$ satisfying

$$\beta_1 E(t) \le \mathcal{L}(t) \le \beta_2 E(t)$$

for all $t \ge 0$ and some positive constants β_1, β_2, and

$$\frac{d}{dt}\mathcal{L}(t) \le -\alpha\mathcal{L}(t)$$

for some $\alpha > 0$. To achieve this we will use the multiplicative technique, and our starting point will be the multiplier $(g*\psi)_t$ to deal with the functional I given by

$$\begin{aligned} I(t) \quad := \quad & \int_0^L \rho_2\psi_t(g*\psi)_t\,dx + b\int_0^L \psi_x(g*\psi_x)\,dx + k\int_0^L \psi(g*\psi)\,dx \\ & -\frac{1}{2}\int_0^L |g*\psi_x|^2\,dx - \frac{b}{2}(\int_0^t g\,d\tau)\int_0^L |\psi_x|^2\,dx \\ & +\frac{b}{2}\int_0^L g\Box\psi_x\,dx + \frac{k}{2}\int_0^L g\Box\psi\,dx - \frac{k}{2}\left(\int_0^t g\,d\tau\right)\int_0^L |\psi|^2\,dx. \end{aligned}$$

To simplify notations let us introduce the symbol $\diamond$ by

$$(g\diamond h)(t) := \int_0^t g(t-s)\left\{h(t) - h(s)\right\}ds.$$

The following straightforward formulae will be used frequently:

$$\begin{aligned} (f*h)_t(t) &= f(0)h(t) + (f'*h)(t) \\ &= f(t)h(t) - (f'\diamond h)(t). \\ (f*h)(t) &= \left(\int_0^t f(s)ds\right)h(t) - (f\diamond h)(t). \end{aligned}$$

Then we have

Lemma 2.2 *There are $c > 0$ and for any $\varepsilon > 0$ a positive constant C_ε such that for $t \geq 0$*

$$\begin{aligned}
-\frac{d}{dt}I(t) &\leq -\frac{\rho_2}{2}g(0)\int_0^L |\psi_t|^2\, dx + C_\varepsilon(|g'|+g)\int_0^L |\psi_x|^2 dx \\
&\quad + c\int_0^L |g'|\Box\psi_x\, dx + c\int_0^L g''\Box\psi\, dx + \varepsilon\int_0^L |\varphi_x|^2\, dx. \qquad (2.3)
\end{aligned}$$

PROOF:

Multiplying equation (1.2) by $(g*\psi)_t$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_0^L \rho_2\psi_t(g*\psi)_t\, dx = \\
&\quad \rho_2\int_0^L \psi_{tt}(g*\psi)_t\, dx + \rho_2\int_0^L \psi_t\left(g(0)\psi + g'*\psi\right)_t\, dx \\
&\quad = b\int_0^L \psi_{xx}(g*\psi)_t\, dx - \int_0^L g*\psi_{xx}(g*\psi)_t\, dx - k\int_0^L (\varphi_x+\psi)(g*\psi)_t\, dx \\
&\quad + \rho_2 g(0)\int_0^L |\psi_t|^2\, dx + \rho_2 g'(0)\int_0^L \psi_t\psi\, dx + \rho_2\int_0^L \psi_t(g''*\psi)\, dx \\
&\quad = \frac{d}{dt}\left\{-b\int_0^L \psi_x(g*\psi_x)_t\, dx + \frac{1}{2}\int_0^L |g*\psi_x|^2\, dx - k\int_0^L \psi(g*\psi)\, dx\right\} \\
&\quad + b\int_0^L \psi_{xt}(g*\psi_x)\, dx + k\int_0^L \psi_t(g*\psi)\, dx \\
&\quad + \rho_2 g(0)\int_0^L |\psi_t|^2\, dx + \rho_2\int_0^L g'\psi_t\psi\, dx - \rho_2\int_0^L \psi_t(g''\diamond\psi)\, dx \\
&\quad - k\int_0^L \varphi_x\left\{g(0)\psi + g'*\psi\right\}\, dx.
\end{aligned}$$

Observing

$$\begin{aligned}
b\int_0^L \psi_{xt}(g*\psi_x)dx &= \frac{b}{2}\frac{d}{dt}\int_0^L\left(\int_0^t g ds\right)|\psi_x|^2 dx - \frac{b}{2}\int_0^L g|\psi_x|^2 dx \\
&\quad -\frac{b}{2}\frac{d}{dt}\int_0^L g\Box\psi_x dx + \frac{b}{2}\int_0^L g'\Box\psi_x dx
\end{aligned}$$

and

$$\begin{aligned}
k\int_0^L \psi_t(g*\psi)dx &= \frac{k}{2}\frac{d}{dt}\int_0^L\left(\int_0^t g ds\right)|\psi|^2 dx - \frac{k}{2}\int_0^L g|\psi|^2 dx \\
&\quad -\frac{k}{2}\frac{d}{dt}\int_0^L g\Box\psi dx + \frac{k}{2}\int_0^L g'\Box\psi dx
\end{aligned}$$

we conclude

$$\begin{aligned}
\frac{d}{dt}I(t) &= \rho_2 g(0)\int_0^L |\psi_t|^2\,dx + \rho_2\int_0^L g'\psi_t\psi\,dx\\
&-\rho_2\int_0^L \psi_t(g''\diamond\psi)\,dx + k\int_0^L \varphi_x\left(g\psi - g'\diamond\psi\right)\,dx\\
&-\frac{b}{2}\int_0^L g|\psi_x|^2 dx + \frac{b}{2}\int_0^L g'\Box\psi_x dx\\
&-\frac{k}{2}\int_0^L g|\psi|^2 dx + \frac{k}{2}\int_0^L g'\Box\psi dx
\end{aligned}$$

which implies the assertion of the lemma using the Cauchy-Schwarz inequality.

Q.E.D.

Now we introduce the multiplier w given by the solution of the Dirichlet problem

$$-w_{xx} = \psi_x, \qquad w(0) = w(L) = 0,$$

and we introduce the functional

$$J_1(t) := \rho_2\int_0^L \psi_t\psi\,dx + \rho_1\int_0^L \varphi_t w\,dx.$$

Lemma 2.3 *For any $\varepsilon_1 > 0$ there exists a positive constant $C_{\varepsilon_1} > 0$ such that for $t \geq 0$:*

$$\frac{d}{dt}J_1(t) \leq C_{\varepsilon_1}\int_0^L |\psi_t|^2\,dx - \frac{\lambda}{2}\int_0^L |\psi_x|^2\,dx + C_{\varepsilon_1}\int_0^L g\Box\psi_x\,dx + \varepsilon_1\int_0^L |\varphi_t|^2\,dx. \tag{2.4}$$

PROOF: Multiplying equation (1.2) by ψ we get

$$\begin{aligned}
\frac{d}{dt}\int_0^L \rho_2\psi_t\psi\,dx &= \int_0^L \rho_2\psi_{tt}\psi\,dx + \rho_2\int_0^L |\psi_t|^2\,dx\\
&= \rho_2\int_0^L |\psi_t|^2\,dx - b\int_0^L |\psi_x|^2\,dx - k\int_0^L |\psi|^2\,dx - k\int_0^L \varphi_x\psi\,dx\\
&+\int_0^L (g*\psi_x)\psi_x\,dx\\
&= \rho_2\int_0^L |\psi_t|^2\,dx - (b - \int_0^t g\,d\tau)\int_0^L |\psi_x|^2\,dx - k\int_0^L |\psi|^2\,dx\\
&-k\int_0^L \varphi_x\psi\,dx - \int_0^L (g\diamond\psi_x)\psi_x\,dx.
\end{aligned} \tag{2.5}$$

Multiplying equation (1.1) by w we obtain

$$\begin{aligned}
\frac{d}{dt}\int_0^L \rho_1\varphi_t w\,dx &= \int_0^L \rho_1\varphi_{tt}w\,dx + \int_0^L \rho_1\varphi_t w_t\,dx\\
&= k\int_0^L (\varphi_x + \psi)_x w\,dx + \rho_1\int_0^L \varphi_t w_t\,dx\\
&= k\int_0^L \varphi w_{xx}\,dx - k\int_0^L w_{xx}w\,dx + \rho_1\int_0^L \varphi_t w_t\,dx\\
&= -k\int_0^L \varphi\psi_x\,dx + k\int_0^L |w_x|^2\,dx + \rho_1\int_0^L \varphi_t w_t\,dx.
\end{aligned} \tag{2.6}$$

The equations (2.5), (2.6) lead to

$$\begin{aligned}\frac{d}{dt}J_1(t) &= \rho_2\int_0^L |\psi_t|^2\,dx-(b-\int_0^t g\,d\tau)\int_0^L|\psi_x|^2\,dx-k\int_0^L|\psi|^2\,dx\\&+k\int_0^L|w_x|^2\,dx+\rho_1\int_0^L\varphi_t w_t\,dx-\int_0^L(g\diamond\psi_x)\psi_x\,dx.\end{aligned}$$

Noting that

$$\int_0^L|w_x|^2\,dx\le\int_0^L|\psi|^2\,dx\le c\int_0^L|\psi_x|^2\,dx \tag{2.7}$$

with a positive constant c, and, for $\delta>0$,

$$|\int_0^L(g\diamond\psi_x)\psi_x dx|\le C_\delta\int_0^L g\Box\psi_x dx+\delta\int_0^L|\psi_x|^2dx,$$

our conclusion follows.

Q.E.D.

Let $\mathcal{E}_1(t)$ denote the functional

$$\mathcal{E}_1(t):=N_1E(t)-N_2I(t)+N_3J_1(t). \tag{2.8}$$

Using Lemma 2.2, Lemma 2.3 and the assumption (1.6) on g, it follows for sufficiently large $N_1^{\varepsilon_1}>N_2^{\varepsilon_1},N_3^{\varepsilon_1}>1$ that $\mathcal{E}_1(t)$ satisfies

$$\frac{d}{dt}\mathcal{E}_1(t)\le-\frac{N_2^{\varepsilon_1}}{2}\int_0^L\left(|\psi_t|^2+|\psi_x|^2+g\Box\psi_x\right)\,dx-\frac{N_1^{\varepsilon_1}}{2}\int_0^L g|\psi_x|^2\,dx+\varepsilon_1\int_0^L\left(|\varphi_t|^2+|\varphi_x|^2\right)\,dx. \tag{2.9}$$

Let us introduce the functional

$$K(t):=\int_0^L\rho_2\psi_t(\varphi_x+\psi)\,dx+\rho_2\int_0^L\psi_x\varphi_t\,dx-\frac{\rho_1}{k}\int_0^L(g*\psi_x)\varphi_t\,dx. \tag{2.10}$$

Lemma 2.4 *Assume (1.5), i.e.*

$$\frac{\rho_1}{\rho_2}=\frac{k}{b}.$$

Then there exists for any $\varepsilon>0$ a constant $C_\varepsilon>0$ such that for $t\ge0$:

$$\begin{aligned}\frac{d}{dt}K(t) &\le [(b\psi_x-g*\psi_x)\varphi_x]_{x=0}^{x=L}-k\int_0^L|\varphi_x+\psi|^2\,dx\\&+\varepsilon\int_0^L|\varphi_t|^2\,dx+C_\varepsilon\int_0^L|g'|\Box\psi_x+g|\psi_x|^2\,dx\\&+\rho_2\int_0^L|\psi_t|^2\,dx.\end{aligned}$$

Proof: Multipliying equation (1.2) by $\psi+\varphi_x$ and using equation (1.1) we get

$$\frac{d}{dt}\int_0^L\rho_2\psi_t(\varphi_x+\psi)\,dx=$$

$$
\begin{aligned}
&\int_0^L \rho_2\psi_{tt}(\varphi_x+\psi)\;dx+\int_0^L \rho_2\psi_t(\varphi_x+\psi)_t\;dx\\
&= b\int_0^L \psi_{xx}(\varphi_x+\psi)\;dx-k\int_0^L |\varphi_x+\psi|^2\;dx-\int_0^L g*\psi_{xx}(\varphi_x+\psi)\;dx\\
&\quad+\int_0^L \rho_2\psi_t(\varphi_x+\psi)_t\;dx\\
&= [(b\psi_x-g*\psi_x)\varphi_x]_{x=0}^{x=L}-b\frac{\rho_1}{k}\int_0^L \psi_x\varphi_{tt}\;dx-k\int_0^L|\varphi_x+\psi|^2\;dx\\
&\quad+\frac{\rho_1}{k}\int_0^L (g*\psi_x)\varphi_{tt}\;dx+\rho_2\int_0^L \psi_t(\varphi_x+\psi)_t\;dx\\
&= [(b\psi_x-g*\psi_x)\varphi_x]_{x=0}^{x=L}-b\frac{\rho_1}{k}\int_0^L \psi_x\varphi_{tt}\;dx-k\int_0^L|\varphi_x+\psi|^2\;dx\\
&\quad+\frac{d}{dt}\left\{\frac{\rho_1}{k}\int_0^L (g*\psi_x)\varphi_t\;dx\right\}-\frac{\rho_1}{k}\int_0^L\left\{g(0)\psi_x+g'*\psi_x\right\}\varphi_t\;dx\\
&\quad+\rho_2\int_0^L|\psi_t|^2\;dx+\rho_2\int_0^L \psi_t\varphi_{xt}\;dx.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\int_0^L \psi_t\varphi_{xt}\;dx &= \frac{d}{dt}\int_0^L \psi\varphi_{xt}\;dx-\int_0^L \psi\varphi_{xtt}\;dx\\
&= -\frac{d}{dt}\int_0^L \psi_x\varphi_t\;dx+\int_0^L \psi_x\varphi_{tt}\;dx.
\end{aligned}
$$

This implies, using the assumption (1.5),

$$
\begin{aligned}
\frac{d}{dt}K(t) &= [(b\psi_x-g*\psi_x)\varphi_x]_{x=0}^{x=L}-k\int_0^L|\varphi_x+\psi|^2\;dx\\
&\quad-\frac{\rho_1}{k}\int_0^L (g\psi_x+g'\diamond\psi_x)\varphi_t\;dx+\rho_2\int_0^L|\psi_t|^2\;dx
\end{aligned}
$$

from where our conclusion follows.

Q.E.D.

The last lemma implies the estimate

$$
\begin{aligned}
\frac{d}{dt}K(t) &\le C_\varepsilon\left\{|b\psi_x(L,t)-(g*\psi_x)(L,t)|^2+|b\psi_x(0,t)-(g*\psi_x)(0,t)|^2\right\}\\
&\quad+\varepsilon\left\{|\varphi_x(L,t)|^2+|\varphi_x(L,t)|^2\right\}-k\int_0^L|\varphi_x+\psi|^2\;dx\\
&\quad+\varepsilon\int_0^L|\varphi_t|^2\;dx+C_\varepsilon\int_0^L|g'|\Box\psi_x+g|\psi_x|^2\;dx\\
&\quad+\rho_2\int_0^L|\psi_t|^2dx.
\end{aligned}
\tag{2.11}
$$

The next functional is denoted by $\mathcal{N}(t)$ and is given by

$$
\mathcal{N}(t):=\int_0^L |\psi_t|^2+(b-\int_0^t g ds)|\psi_x|^2+g\Box\psi_x\;dx.
$$

Lemma 2.5 *Let $q \in C^1([0,L])$ satisfy $q(0) = -q(L) = 2\gamma > 0$. Then then there exist $C_1 > 0$ and for any $\tilde{\varepsilon} > 0$ a positive constant $C_{\tilde{\varepsilon}}$ such that for $t \geq 0$ we have*

$$(i)\ \frac{d}{dt}\int_0^L \rho_2\psi_t q(b\psi_x - g*\psi_x)\,dx \leq -\gamma\left\{|b\psi_x(L,t) - (g*\psi_x)(L,t)|^2 + |b\psi_x(0,t) - (g*\psi_x)(0,t)|^2\right\} + \tilde{\varepsilon}\int_0^L |\varphi_x|^2\,dx + C_{\tilde{\varepsilon}}\mathcal{N}(t).$$

$$(ii)\ \frac{d}{dt}\int_0^L \rho_1\varphi_t q\varphi_x\,dx \leq -k\gamma\left\{|\varphi_x(L,t)|^2 + |\varphi_x(0,t)|^2\right\} + C_1\int_0^L |\varphi_t|^2 + |\varphi_x|^2 + |\psi_x|^2\,dx.$$

PROOF: With equation (1.2) we obtain

$$\begin{aligned}
\frac{d}{dt}\int_0^L \rho_2\psi_t q(b\psi_x - g*\psi_x)\,dx &= \int_0^L (b\psi_{xx} - g*\psi_{xx})q(b\psi_x - g*\psi_x)\,dx \\
&\quad -k\int_0^L (\varphi_x + \psi)q(b\psi_x - g*\psi_x)\,dx \\
&\quad +\rho_2\int_0^L \psi_t q(b\psi_x - g*\psi_x)_t\,dx \\
&= \frac{1}{2}\left[q(b\psi_x - g*\psi_x)^2\right]_{x=0}^{x=L} - \frac{1}{2}\int_0^L q_x(b\psi_x - g*\psi_x)^2\,dx \\
&\quad -k\int_0^L (\varphi_x + \psi)q(b\psi_x - g*\psi_x)\,dx + \frac{1}{2}b\rho_2\int_0^L q\frac{d}{dx}|\psi_t|^2\,dx \\
&\quad -\rho_2\int_0^L \psi_t q(g(0)\psi_x + g'*\psi_x)\,dx \\
&\leq -\gamma\left\{|b\psi_x(L,t) - (g*\psi_x)(L,t)|^2\right. \\
&\quad \left.+|b\psi_x(0,t) - (g*\psi_x)(0,t)|^2\right\} + \tilde{\varepsilon}\int_0^L |\varphi_x|^2\,dx + C_{\tilde{\varepsilon}}\mathcal{N}(t)
\end{aligned}$$

where we used the assumption (1.6) on g. This proves (i). The estimate (ii) is proved, using equation (1.1), as follows:

$$\begin{aligned}
\frac{d}{dt}\int_0^L \rho_1\varphi_t q\varphi_x\,dx &= k\int_0^L q\varphi_{xx}\varphi_x\,dx + \rho_1 k\int_0^L q\psi_x\varphi_x\,dx \\
&\quad +\frac{1}{2}\rho_1\left[q|\varphi_t|^2\right]_{x=0}^{x=L} - b\rho_1\int_0^L q_x|\varphi_t|^2\,dx \\
&\leq -k\gamma\left\{|\varphi_x(L,t)|^2 + |\varphi_x(0,t)|^2\right\} + C_1\int_0^L |\varphi_t|^2 + |\varphi_x|^2 + |\psi_x|^2\,dx.
\end{aligned}$$

Q.E.D.

For $\delta > 0$ and $N_3 > 1$ let

$$L(t) := K(t) + N_3\int_0^L \rho_2\psi_t q(b\psi_x - g*\psi_x)\,dx + \delta\int_0^L \rho_1\varphi_t q\varphi_x\,dx. \tag{2.12}$$

Observing

$$-\frac{k}{2}\int_0^L |\varphi_x+\psi|^2\,dx \le -\frac{k}{4}\int_0^L |\varphi_x|^2\,dx + C\int_0^L |\psi_x|^2\,dx,$$

for some positive constant C, we conclude from Lemma 2.5 and (2.11) that for sufficiently large N_3 and sufficiently small δ we have for $0<\tau<1$ and some $C_\tau>0$ and $C_2>0$ that

$$\frac{d}{dt}L(t) \le -\frac{k}{2}\int_0^L |\varphi_x+\psi|^2\,dx + C_2\tau\int_0^L |\varphi_t|^2\,dx + C_\tau\mathcal{N}(t) \tag{2.13}$$

where we used (1.6) again. Here, one can choose first δ of order τ, then ε small enough, then N_3 large enough, then $\tilde{\varepsilon}$ small enough.

Finally, let us introduce the functional

$$J_2(t) := \int_0^L \rho_1\varphi_t\varphi + \rho_2\psi_t\psi\,dx. \tag{2.14}$$

Lemma 2.6 *There exists a positive constant c satisfying*

$$-\frac{d}{dt}J_2(t) \le -\rho_1\int_0^L |\varphi_t|^2\,dx - \rho_2\int_0^L |\psi_t|^2\,dx + k\int_0^L |\varphi_x+\psi|^2\,dx + c\mathcal{N}(t).$$

PROOF:

$$\begin{aligned}\frac{d}{dt}\int_0^L \rho_1\varphi_t\varphi\,dx &= \int_0^L \rho_1\varphi_{tt}\varphi\,dx + \int_0^L \rho_1|\varphi_t|^2\,dx\\ &= -k\int_0^L (\varphi_x+\psi)\varphi_x\,dx + \rho_1\int_0^L |\varphi_t|^2\,dx. \end{aligned}\tag{2.15}$$

Similarly,

$$\begin{aligned}\frac{d}{dt}\int_0^L \rho_2\psi_t\psi\,dx &= \int_0^L \rho_2\psi_{tt}\psi\,dx + \int_0^L \rho_2|\psi_t|^2\,dx\\ &= -(b-\int_0^t g\,ds)\int_0^L |\psi_x|^2\,dx + \int_0^L (g\diamond\psi_x)\psi_x\,dx \qquad (2.16)\\ &\quad -k\int_0^L (\varphi_x+\psi)\psi\,dx + \rho_2\int_0^L |\psi_t|^2\,dx. \qquad (2.17)\end{aligned}$$

Summing up (2.15), (2.17) the assertion follows.

Q.E.D.

Lemma 2.6 and (2.13) yield, choosing τ small enough,

$$\frac{d}{dt}\left\{L(t) - \frac{2C_2\tau}{\rho_1}J_2(t)\right\} \le -\frac{k}{4}\int_0^L |\varphi_x+\psi|^2\,dx - C_2\tau\int_0^L |\varphi_t|^2\,dx + C_\tau\mathcal{N}(t). \tag{2.18}$$

Now we are in the position to show the main result of this section:

Theorem 2.7 *Let us suppose that the initial data satisfy*

$$\varphi_0,\psi_0 \in H_0^1((0,L)), \qquad \varphi_1,\psi_1 \in L^2((0,L)),$$

and that the coefficients of the system (1.1), (1.2) satisfy (1.5), i.e.,

$$\frac{\rho_1}{\rho_2} = \frac{k}{b}.$$

Moreover assume that the kernel g is of exponential type satisfying (1.6). Then the energy $E(t)$ decays exponentially as time tends to infinity, that is, there exist positive constants C and α, being independent of the initial data, such that for $t \geq 0$:

$$E(t) \leq CE(0)e^{-\alpha t}.$$

PROOF: Let the final Lyapunov functional be defined by

$$\mathcal{L}(t) := \mathcal{E}_1(t) + L(t) - \frac{2C_2\tau}{\rho_1}J_2(t)$$

where $\mathcal{E}_1(t)$, $L(t)$ and $J_2(t)$ were defined in (2.8), (2.12) and (2.14), respectively. With (2.9) and (2.18) we conclude for sufficiently small ε_1 and some $\beta_0 > 0$ that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\beta_0 E(t).$$

Moreover, there are positive constants β_1, β_2 such that for $t \geq 0$

$$\beta_1 E(t) \leq \mathcal{L}(t) \leq \beta_2 E(t)$$

whence

$$\frac{d}{dt}\mathcal{L}(t) \leq -\alpha\mathcal{L}(t)$$

for $\alpha := \beta_0/\beta_2$, and hence our conclusion follows.

Q.E.D.

3 Parameter optimality

The condition (1.5), i.e.

$$\frac{\rho_1}{\rho_2} = \frac{k}{b}$$

turned out to be sufficient to prove the exponential stability in the previous section. Now we shall demonstrate that it is also a necessary condition in general.

3.1 Approximation of the problem

Let us denote the energy defined in (2.1) by

$$E(t) =: E_g(\varphi, \psi) =: E_g(t). \tag{3.1}$$

In order to approximate the function g, we first introduce a sequence $(\beta_i)_{i\geq 1} \subset \mathbb{R}$ such that

$$\beta_j > \beta_i > 0, \quad \forall j > i \geq 1, \quad \lim_{i\to\infty} \beta_i = \infty,$$

$$\sum_{i\geq 1}\frac{1}{\beta_i}=\infty, \tag{3.2}$$

and let's put

$$e_i(t)=e^{-\beta_i t}\,,\ \ \forall t\geq 0,\ i\geq 1. \tag{3.3}$$

We recall Müntz's theorem:

Theorem 3.1 *(Müntz, [17, Chap. I, p. 24]) The functions e_i defined in (3.3) are complete*[1] *in $L^p((0,\infty))$ for all $1\leq p<\infty$ if and only if the sequence $(\beta_i)_{i\geq 0}\subset\mathbb{R}$ satisfies (3.2).*

In this section, we will assume that:

$$\begin{aligned} g\ &\in\ C^1([0,\infty))\cap W^{1,1}((0,\infty))\,, &(3.4)\\ g\ &>\ 0,\ g'\leq 0\quad on\ \ (0,\infty)\,, &(3.5)\\ \|g\|_{L^1(0,\infty)}\ &<\ b. &(3.6)\end{aligned}$$

According to Theorem 3.1, since g satisfies (3.4):

$$\forall\varepsilon>0\,\exists\, n=n(\varepsilon)\geq 1\,\exists\, h_\varepsilon=\sum_{i=1}^{n}\alpha_i^\varepsilon e^{-\beta_i t}\,:\ \left\|g'-h_\varepsilon'\right\|_{L^1((0,\infty))}<\varepsilon. \tag{3.7}$$

The function h_ε in (3.7) will be called an $\varepsilon-$approximation of g. Our main assumption in this section will be

$(\mathbf{H}_g)$ *g satisfies assumptions (3.4)–(3.6) and there exists an ε-approximation h_ε of g such that:*

$$\begin{aligned} \alpha_i^\varepsilon\ &>\ 0, &(3.8)\\ \exists\varepsilon_0\ &>\ 0\,\exists\delta>0\,\forall\varepsilon\in(0,\varepsilon_0)\,:\ \|h_\varepsilon\|_{L^1(0,\infty)}\leq\delta<b. &(3.9)\end{aligned}$$

Let us denote by $(\overline{\varphi},\overline{\psi})$ the solution of the initial boundary value problem (1.1)–(1.4) with the same initial data but with g replaced by some approximation $h=h_\varepsilon$ corresponding to some fixed $\varepsilon>0$.

Lemma 3.2 *Assume that g satisfies $(\mathbf{H}_g)$ and that there exists a positive function $d\in L^1((0,\infty))$ such that $\lim_{t\to\infty}d(t)=0$ and for all data and $t\geq 0$*

$$E_g(\varphi,\psi)(t)\leq d^2(t)E_g(\varphi,\psi)(0). \tag{3.10}$$

Then, with the previous notations, there exists $C>0$, such that for all $\varepsilon>0$ sufficiently small and all $t\geq 0$

(i)

$$E_h(\varphi-\overline{\varphi},\psi-\overline{\psi})(t)\leq C\varepsilon E_h(0).$$

(ii)

$$\left|E_g(\varphi,\psi)(t)-E_h(\overline{\varphi},\overline{\psi})(t)\ \right|\leq C\varepsilon^{1/2}E_h(0).$$

[1] the linear span is dense

Proof: Let us denote by $z := \varphi - \overline{\varphi}$ and by $w := \psi - \overline{\psi}$. Then z and w satisfy

$$\begin{gathered}
\rho_1 z_{tt} - k(z_x + w)_x = 0, \\
\rho_2 w_{tt} - b w_{xx} + h * w_{xx} + k(z_x + w) = \ (h-g) * \psi_{xx}, \\
z(0,t) = z(L,t) = w(0,t) = w(L,t) = 0, \\
z(0) = 0,\ w(0) = 0.
\end{gathered}$$

Then we get

$$\frac{d}{dt} E_h(z,w)(t) = -\frac{1}{2} h(t) \int_0^L w_x^2 dx + \frac{1}{2} \int_0^L h' \Box w_x dx + \int_0^L \left((h-g) * \psi_{xx}\right) w_t dx.$$

Integration with respect to time yields

$$E_h(z,w)(t) \leq \int_0^t \int_0^L ((h-g) * \psi_{xx}) w_t dx ds =: I(t). \tag{3.11}$$

Integration by parts in time leads to

$$\begin{aligned}
I(t) \quad &= \quad \left[\int_0^L (h-g) * \psi_{xx} w dx \right]_0^t - \int_0^t \int_0^L \frac{d}{dt} \left((h-g) * \psi_{xx}\right) w dx ds \\
&= \quad \int_0^L ((h-g) * \psi_{xx})(t) w(t) dx \ - \int_0^t \int_0^L \ (h-g)(0) \psi_{xx} w dx ds \\
&\quad\quad - \int_0^t \int_0^L \ ((h'-g') * \psi_{xx}) w dx ds.
\end{aligned}$$

Integration by parts in space and time yields

$$\begin{aligned}
I(t) \quad &= \quad - \int_0^L ((h-g) * \psi_x(t) w_x(t) dx + \int_0^t \int_0^L \ (h-g)(0) \psi_x w_x dx ds \\
&\quad\quad + \int_0^t \int_0^L \ ((h'-g') * \psi_x) w_x dx ds \\
&\leq \quad (|\ h-g| * \|\psi_x\|) \|w_x\| + \int_0^t \int_0^L \ (h-g)(0) \psi_x w_x dx ds \\
&\quad\quad + \int_0^t \left(\left|\ h'-g' \right| * \ \|\psi_x\| \right) \|w_x\| \, ds
\end{aligned} \tag{3.12}$$

where we introduced the notation

$$\|f\|^2 (t) := \int_0^L |f(t,x)|^2 dx.$$

On the other hand, there exists a constant $C>0$, independent of h, such that

$$\begin{aligned} \|w_x\|^2(t) &\leq CE_h(z,w)(t), & (3.13)\\ \|\psi_x\|^2(t) &\leq CE_g(\varphi,\psi)(t). & (3.14)\end{aligned}$$

Note also that, since $g-h\in W^{1,1}((0,\infty))$, for all $t\geq 0$:

$$\begin{aligned} |(g-h)(t)| &= \left|\int_t^\infty (g'-h')(s)ds\right| \\ &\leq \|g'-h'\|_{L^1(0,\infty)} \\ &< \varepsilon \end{aligned}$$

which implies

$$\|g-h\|_{L^\infty(0,\infty)}<\varepsilon. \tag{3.15}$$

Using the general decay of the energy (see (2.2)), assumption ($\mathbf{H}_g$), (3.13), (3.14) and (3.10) we arrive at (denoting by C various positive constants independent of t and ε all along this proof):

$$\begin{aligned} (|\ h-g|*\|\psi_x\|(t))\|w_x\|(t) &\leq C\left(|\ h-g|*\sqrt{E_g(\varphi,\psi)(t)}\right)\sqrt{E_h(z,w)(t)}, \\ &\leq C|\ h-g|*d(t)\sqrt{E_g(\varphi,\psi)(0)}\sqrt{E_h(z,w)(t)} \\ &\leq C\|\ h-g\|_{L^\infty(\mathbb{R}^+)}\|d\|_{L^1(\mathbb{R}^+)}(E_g(\varphi,\psi)(0)+E_h(z,w)(t)) \\ &\leq C\varepsilon(E_g(\varphi,\psi)(0)+E_h(z,w)(t)) \qquad (3.16) \end{aligned}$$

Using (3.15), (3.10) and the assumption on d, we also get:

$$\begin{aligned} \int_0^t\int_0^L (h-g)(0)\psi_x.w_x dx ds &\leq \varepsilon\int_0^t \sqrt{E_g(\varphi,\psi)(s)}.\sqrt{E_h(z,w)(s)}ds \\ &\leq \varepsilon\sqrt{E_g(\varphi,\psi)(0)}\int_0^t d(s).\sqrt{E_h(z,w)(s)}ds \\ &\leq \frac{\varepsilon}{2}\int_0^t d(s)\left(E_g(\varphi,\psi)(0)+E_h(z,w)(s)\right)ds \\ &\leq \frac{\varepsilon}{2}\left(\|d\|_{L^1(0,\infty)}E_g(\varphi,\psi)(0)+\int_0^t d(s)E_h(z,w)(s)ds\right) \\ &\leq C\varepsilon\left(E_g(\varphi,\psi)(0)+\int_0^t d(s)E_h(z,w)(s)ds\right) \qquad (3.17) \end{aligned}$$

Returning to the estimation of the third term on the right-hand side of inequality (3.12), using the previous estimate and (3.10) again, we also get

$$\begin{aligned}\int_0^t \left(\left| h'-g'\right| * \|\psi_x\|\right)\|w_x\|\,ds &\leq \int_0^t \left(\left| h'-g'\right| * \sqrt{E_g(\varphi,\psi)}\right)(s)\sqrt{E_h(z,w)(s)}ds \\ &\leq \sqrt{E_g(\varphi,\psi)(0)}\left(\int_0^t \left| h'-g'\right| * d(s)ds\right)^{1/2} \times \\ &\quad \left(\int_0^t \left| h'-g'\right| * d(s)E_h(z,w)(s)ds\right)^{1/2}\end{aligned}$$

for all $t \geq 0$. But now, we have by (H_g) and the assumption on d:

$$\begin{aligned}\int_0^t\int_0^s \left| h'-g'\right|(s-\tau)d(\tau)\,d\tau ds &\leq \left\|\left| h'-g'\right| * d\right\|_{L^1(\mathbb{R}^+)} \\ &\leq \left\| h'-g'\right\|_{L^1(\mathbb{R}^+)}\| d\|_{L^1(\mathbb{R}^+)} \\ &\leq C\varepsilon \end{aligned} \tag{3.18}$$

Thus, setting

$$k(s) = \int_0^s \left| h'-g'\right|(s-\tau)d(\tau)\,d\tau \qquad s \geq 0$$

we have

$$\begin{aligned}\int_0^t \left(\left| h'-g'\right| * \|\psi_x\|\right)\|w_x\|\,ds &\leq C\sqrt{\varepsilon E_g(\varphi,\psi)(0)}\left(\int_0^t k(s)E_h(z,w)(s)ds\right)^{1/2} \\ &\leq C\left(\varepsilon E_g(\varphi,\psi)(0) + \int_0^t k(s)E_h(z,w)(s)ds\right).\end{aligned} \tag{3.19}$$

Using (3.12), (3.16), (3.17) and (3.19), inequality (3.11) becomes

$$E_h(z,w)(t) \leq C\varepsilon E_g(\varphi,\psi)(0) + C\varepsilon E_h(z,w)(t) + C\int_0^t (d(s)+k(s))\,E_h(z,w)(s)ds$$

and, for $\varepsilon > 0$ sufficiently small,

$$E_h(z,w)(t) \leq \frac{C\varepsilon}{1-C\varepsilon}E_g(\varphi,\psi)(0) + \frac{C}{1-C\varepsilon}\int_0^t (d(s)+k(s))\,E_h(z,w)(s)ds$$

Applying Gronwall's inequality to $E_h(z,w)(t)$ yields

$$E_h(z,w)(t) \leq \frac{C\varepsilon}{1-C\varepsilon} E_g(\varphi,\psi)(0) \ \exp\left(\frac{C}{1-C\varepsilon} \int_0^t \ (d(s)+k(s))\, ds \right)$$

From (3.18), it follows that $\int_0^t (d(s)+k(s))\, ds \leq C(\varepsilon+1)$ uniformly in t and then we conclude that

$$\begin{aligned} E_h(z,w)(t) &\leq \frac{C\varepsilon}{1-C\varepsilon} E_g(\varphi,\psi)(0) \ \exp\left(\frac{C(\varepsilon+1)}{1-C\varepsilon} \right) \\ &\leq C\varepsilon E_g(\varphi,\psi)(0) = C\varepsilon E_h(0) \end{aligned}$$

which is claim (i) of the lemma (Notice that $E_g(0)$ does not depend on g : $E_g(0) = E_h(0) = E(0)$ for the same initial data).

Let us prove claim (ii) of our lemma. Using the definition, the general decay of the energy and (3.15), we get:

$$\begin{aligned} \left| E_g(\varphi,\psi) - E_h(\varphi,\psi) \right|(t) &= \frac{1}{2} \left| \int_0^L \left(\int_0^t (h-g)\,(s) ds |\psi_x|^2 + (g-h) \Box \psi_x \right) dx \right| \\ &\leq C \int_0^L \left(\int_0^t |h-g|\,(s) ds |\psi_x|^2 + (|g-h| * |\psi_x|^2 \right) dx \\ &\leq C \left(\int_0^t |h-g|\,(s) ds \ E_g(\varphi,\psi)(t) + |h-g| * E_g(\varphi,\psi) \right) \\ &\leq C \left(\int_0^t |h-g|\,(s) \sqrt{E_g(\varphi,\psi)(s)} ds \ + |h-g| * \sqrt{E_g(\varphi,\psi)} \right) \sqrt{E_g(\varphi,\psi)}(0) \\ &\leq C \left(\int_0^t |h-g|\,(s) d(s) ds \ + |h-g| * d \right) E_g(\varphi,\psi)(0) \\ &\leq C \left\| h-g \right\|_{L^\infty((0,\infty))} \left\| d \right\|_{L^1(0,\infty)} E_g(\varphi,\psi)(0) \\ &\leq C\varepsilon E_g(\varphi,\psi)(0) \end{aligned}$$

So, from this last inequality and claim (i), we get

$$\left| E_g(\varphi,\psi)(t) - E_h(\overline{\varphi},\overline{\psi})(t) \ \right| \ \leq \ \left| \ E_g(\varphi,\psi)(t) - E_h(\varphi,\psi)(t) \right| + \left| E_h(\varphi,\psi)(t) - E_h(\overline{\varphi},\overline{\psi})(t) \ \right|$$

$$
\begin{aligned}
&\leq C(\varepsilon E_g(0) + 2\sqrt{E_g(\varphi,\psi)(0)}\sqrt{E_h(\varphi-\overline{\varphi},\psi-\overline{\psi})(t)}) \\
&\leq \left(C(\varepsilon+\sqrt{\varepsilon})\right) E_g(0) \\
&\leq C\varepsilon^{1/2} E_g(0).
\end{aligned}
$$

which proves claim (ii). Q.E.D.

3.2 Non uniform decay for the approximated problem

In this section we will prove that, whenever the wave speeds are different, i.e. when

$$\frac{k}{\rho_1} \neq \frac{b}{\rho_2}$$

then the associated energy does not decay uniformly for a large class of kernels.

We consider an ε-approximation h of g (ε will be fixed later):

$$h(t) = \sum_{i=1}^{n} \alpha_i e^{-\beta_i t}, \ (\alpha_i > 0 < \beta_i, \beta_i \neq \beta_j \, \forall i \neq j)$$

satisfying (3.8),(3.9) and we set:

$$h * \overline{\psi}_{xx} = \sum_{i=1}^{n} \frac{\partial}{\partial x} y_i(t,x)$$

where for $i = 1, \ldots, n$:

$$
\begin{aligned}
\frac{\partial y_i}{\partial t} &= -\beta_i y_i + \alpha_i \overline{\psi}_x, \\
y_i(x,0) &= 0, \quad x \in (0,L).
\end{aligned}
$$

The initial-boundary value problem for $(\overline{\varphi},\overline{\psi})$ is then equivalent to the following one:

$$
\left.
\begin{array}{c}
\rho_1 \overline{\varphi}_{tt} - k(\overline{\varphi}_x + \overline{\psi})_x = 0, \\[2ex]
\rho_2 \overline{\psi}_{tt} - b\overline{\psi}_{xx} + k(\overline{\varphi}_x + \overline{\psi}) + BY_x = 0, \\[2ex]
Y'(t) = AY(t) + D\overline{\psi}_x, \\[2ex]
\overline{\varphi}(0,t) = \overline{\varphi}(L,t) = \overline{\psi}(0,t) = \overline{\psi}(L,t) = 0, \\[2ex]
Y(0) = 0
\end{array}
\right\} \tag{3.20}
$$

where

$$
Y(t,x) := \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}(t,x), \quad A := \begin{pmatrix} -\beta_1 & 0 & 0 \\ \ldots & \ldots & \ldots \\ 0 & 0 & -\beta_n \end{pmatrix}
$$

and

$$B := \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}, \quad D := \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

With system (3.20), we associate the energy:

$$E(\overline{\varphi}, \overline{\psi}, Y)(t) := \frac{1}{2} \int_0^L \{\rho_1 \overline{\varphi}_t^2 + \rho_2 \overline{\psi}_t^2 + b \overline{\psi}_x^2 + k(\overline{\varphi}_x + \overline{\psi})^2 + |Y|^2\} dx(t) \tag{3.21}$$

In the energy space $H = \left(H_0^1 \times L^2\right)^2 \times \left(L^2\right)^n$, it is not difficult to associate a semigroup with system (3.20).

Now, the new variables

$$Z \equiv \begin{pmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_5 \end{pmatrix} := \begin{pmatrix} \sqrt{\rho_1}\, \overline{\varphi}_t - \sqrt{k}\left(\overline{\varphi}_x + \overline{\psi}\right) \\ \sqrt{\rho_2}\, \overline{\psi}_t - \sqrt{b}\, \overline{\psi}_x + \frac{1}{\sqrt{b}} BY \\ \sqrt{\rho_1}\, \overline{\varphi}_t + \sqrt{k}\left(\overline{\varphi}_x + \overline{\psi}\right) \\ \sqrt{\rho_2}\, \overline{\psi}_t + \sqrt{b}\, \overline{\psi}_x - \frac{1}{\sqrt{b}} BY \\ Y \end{pmatrix} \tag{3.22}$$

satisfy the system

$$Z_t = \Lambda Z_x + MZ \tag{3.23}$$

with the boundary conditions

$$\left(z_i + z_{i+2}\right)(0, t) = \left(z_i + z_{i+2}\right)(L, t) = 0, \quad i = 1, 2, \tag{3.24}$$

in the space $G = \left(L^2\right)^4 \times \left(L^2\right)^n$. Here

$$\begin{aligned} \Lambda :=& \quad diag\left(-\sqrt{\frac{k}{\rho_1}}, -\sqrt{\frac{b}{\rho_2}}, \sqrt{\frac{k}{\rho_1}}, \sqrt{\frac{b}{\rho_2}}, 0_n\right), \\ 0_n :=& \quad (0, \ldots, 0) \in \mathbb{R}^n \end{aligned}$$

and

$$M := \begin{pmatrix} M_4 & M_{4n} \\ N_{n4} & A + \frac{1}{b} DB \end{pmatrix}$$

where

$$M_4 := \begin{pmatrix} 0 & -\frac{1}{2}\sqrt{\frac{k}{\rho_2}} & 0 & -\frac{1}{2}\sqrt{\frac{k}{\rho_2}} \\ \frac{1}{2}\sqrt{\frac{k}{\rho_2}} & -\frac{\sum_{i=1}^n \alpha_i}{2b} & -\frac{1}{2}\sqrt{\frac{k}{\rho_2}} & \frac{\sum_{i=1}^n \alpha_i}{2b} \\ 0 & \frac{1}{2}\sqrt{\frac{k}{\rho_2}} & 0 & \frac{1}{2}\sqrt{\frac{k}{\rho_2}} \\ \frac{1}{2}\sqrt{\frac{k}{\rho_2}} & \frac{\sum_{i=1}^n \alpha_i}{2b} & -\frac{1}{2}\sqrt{\frac{k}{\rho_2}} & -\frac{\sum_{i=1}^n \alpha_i}{2b} \end{pmatrix}.$$

The matrices M_{4n} and N_{n4} will not play any role in the sequel but for completeness we give their elements.

$$M_{4n} := \begin{pmatrix} 0 \\ \frac{1}{b^{3/2}}\left(BDB + bBA\right) \\ 0 \\ -\frac{1}{b^{3/2}}\left(BDB + bBA\right) \end{pmatrix}; \quad N_{n4} := \begin{pmatrix} 0 & -\frac{D}{2b^{1/2}} & 0 & \frac{D}{2b^{1/2}} \end{pmatrix}$$

Note here that (3.22) can be written as

$$Z = P^{-1}\begin{pmatrix} \sqrt{\rho_1}\,\overline{\varphi}_t \\ \sqrt{k}\,(\overline{\varphi}_x+\overline{\psi}) \\ \sqrt{\rho_2}\,\overline{\psi}_t \\ \sqrt{b}\,\overline{\psi}_x \\ Y \end{pmatrix}$$

with

$$P^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & \frac{1}{\sqrt{b}}B \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -\frac{1}{\sqrt{b}}B \\ 0 & 0 & 0 & 0 & I \end{pmatrix} ; P = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 & 0 \\ -1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & -1/2 & 0 & 1/2 & \frac{1}{\sqrt{b}}B \\ 0 & 0 & 0 & 0 & I \end{pmatrix}. \tag{3.25}$$

If we forget for a moment the condition $Y(x,0) = 0$ and replace it by any initial data, we may associate to (3.20) a C_0-semigroup $e^{(\Lambda\partial_x+M)t}$. A result of Neves, Ribeiro and Lopes [15] (actually, an extension of their result to the case where there are null and multiple eigenvalues in the diagonal matrix Λ, see Ammar-Khodja & Bader [1, 2]) asserts that, since $\frac{k}{\rho_1} \neq \frac{b}{\rho_2}$, if we consider the semigroup $e^{(\Lambda\partial_x+M_0)t}$ associated to the following system:

$$Z_t = \Lambda Z_x + M_0 Z, \tag{3.26}$$

$$(z_i + z_{i+2})(0,t) = (z_i + z_{i+2})(L,t) = 0, \quad i = 1,2.$$

$$M_0 := diag\left(0, -\frac{\sum_{i=1}^n \alpha_i}{2b}, 0, -\frac{\sum_{i=1}^n \alpha_i}{2b}, A + \frac{1}{b}DB\right)$$

with the same boundary conditions, then $e^{(\Lambda\partial_x+M)t} - e^{(\Lambda\partial_x+M_0)t}$ is a *compact operator*. Now, for this last system, the eigenvalues can be easily computed by solving the diagonal differential system:

$$\lambda Z = \Lambda Z_x + M_0 Z$$

$$(z_i + z_{i+2})(0) = (z_i + z_{i+2})(L) = 0, \quad i = 1,2.$$

Indeed we get

$$\sigma(\Lambda\partial_x + M_0) = \sigma(A + \frac{1}{b}DB) \cup \left\{ m\sqrt{\frac{k}{\rho_1}}\frac{\pi}{L}i, m \in \mathbb{Z} \right\} \cup \left\{ -\frac{\sum_{i=1}^n \alpha_i}{2b} + m\sqrt{\frac{b}{\rho_2}}\frac{\pi}{L}i,\ m \in \mathbb{Z} \right\}. \tag{3.27}$$

We now compare the energy $E(\overline{\varphi},\overline{\psi},Y)(t)$ defined in (3.21) and $E_h(\overline{\varphi},\overline{\psi})(t)$

Lemma 3.3 *With the previous notations, assume that h satisfies $(\mathbf{H}_g)$ and let*

$$Y(t,x) = \int_0^t e^{A(t-s)} D\overline{\psi}_x(x,s)ds$$

Then

$$\exists C > 0\, \forall t \geq 0: \; E(\overline{\varphi}, \overline{\psi}, Y)(t) \leq C E_h(\overline{\varphi}, \overline{\psi})(t).$$

PROOF: Using the definition of Y and (3.9), we get

$$\begin{aligned}
\int_0^L |Y|^2\, dx(t) &= \int_0^L \sum_{i=1}^n \alpha_i^2 \left(\int_0^t e^{-\beta_i(t-s)} \overline{\psi}_x(s) ds \right)^2 dx \\
&\leq \int_0^L \sum_{i=1}^n \alpha_i^2 \int_0^t e^{-\beta_i(t-s)} ds \int_0^t e^{-\beta_i(t-s)} \overline{\psi}_x^2 ds dx \\
&\leq \int_0^L \int_0^\infty h(s) ds \int_0^t h(t-s) \overline{\psi}_x^2 ds dx \\
&\leq b \int_0^L \int_0^t h(t-s) \overline{\psi}_x^2(s) ds dx \\
&\leq 2b \int_0^L \int_0^t h(t-s) \left(\overline{\psi}_x(s) - \overline{\psi}_x(t) \right)^2 ds + 2b^2 \left\| \overline{\psi}_x(t) \right\|^2 \\
&= 2b \int_0^L h \Box \overline{\psi}_x(t) dx + 2b^2 \left\| \overline{\psi}_x(t) \right\|^2.
\end{aligned}$$

Consequently, from the definition of $E(\overline{\varphi}, \overline{\psi}, Y)$ in (3.21) and this last inequality, we get:

$$\begin{aligned}
E(\overline{\varphi}, \overline{\psi}, Y)(t) &\leq \frac{1}{2} \int_0^L \left\{ \rho_1 |\overline{\varphi}_t|^2 + \rho_2 |\overline{\psi}_t|^2 + (b - \int_0^t h\, d\tau) |\overline{\psi}_x|^2 + k |\overline{\varphi}_x + \overline{\psi}|^2 + h \Box \overline{\psi}_x \right\} dx \\
&\quad + (b^2 + b/2) \left\| \overline{\psi}_x(t) \right\|^2 + \frac{1}{2} \int_0^L h \Box \overline{\psi}_x\, dx \\
&\leq E_h(\overline{\varphi}, \overline{\psi})(t) + \frac{(b^2 + b/2)}{b - \delta} \left(b - \int_0^t h\, d\tau \right) \left\| \overline{\psi}_x(t) \right\|^2 + \frac{1}{2} \int_0^L h \Box \overline{\psi}_x\, dx \\
&\leq (1 + C) E_h(\overline{\varphi}, \overline{\psi})(t)
\end{aligned}$$

where $C = 1 + 2\frac{(b^2 + b/2)}{b - \delta}$ (δ is defined in (3.9)).

Q.E.D.

We are now ready to state and prove the main result of this section:

Theorem 3.4 *Assume that g satisfies ($\mathbf{H}_g$). Assume moreover that*

$$\frac{\rho_1}{\rho_2} \neq \frac{k}{b}$$

Then the energy $E_g(\varphi, \psi)$ does not decay uniformly in the initial data to zero as time tends to infinity, i. e., there does not exist $d \in L^1((0, \infty))$ such that $\lim_{t \to \infty} d(t) = 0$ and for all $t \geq 0$:

$$E_g(t) \leq d^2(t) E_g(0).$$

PROOF:

Assume to the contrary that there exists $d \in L^1((0,\infty))$ such that $\lim_{t\to\infty} d(t) = 0$ and

$$E_g(t) \leq d^2(t) E_g(0). \tag{3.28}$$

Using (ii) in the Lemma 3.2, we get, h being the $\varepsilon-$approximate function of g given in $(\mathbf{H}_g)$:

$$\begin{aligned} E_h(\overline{\varphi},\overline{\psi})(t) &\leq E_g(\varphi,\psi)(t) + C\varepsilon^{1/2} E_g(\varphi,\psi)(0) \\ &\leq \left(d^2(t) + C\varepsilon^{1/2}\right) E_g(\varphi,\psi)(0) \end{aligned} \tag{3.29}$$

From (3.29) and Lemma 3.3, it follows for $Y(t,x) = \int_0^t e^{A(t-s)} D\overline{\psi}_x ds$

$$E(\overline{\varphi},\overline{\psi},Y)(t) \leq C\left(d^2(t) + C\varepsilon^{1/2}\right) E_g(\varphi,\psi)(0) \tag{3.30}$$

Let 's now introduce the natural norm defined for $Z = (z_1,\cdots,z_4,Y) \in \left(L^2\right)^4 \times \left(L^2\right)^n$ by:

$$\|Z\|^2 = \int_0^L \left(\sum_{i=1}^4 |z_i|^2 + |Y|^2\right) dx$$

With this norm, it is not difficult to check that for any solution Z of (3.23), (3.24) corresponding to initial data of the form $Z_0 = (z_{10},\cdots,z_{40},0)$,

$$\|Z(t)\|^2 = E_h(\overline{\varphi},\overline{\psi},Y)(t)$$

where $(\overline{\varphi},\overline{\psi},Y)(t)$ is the corresponding solution of (3.20). Remember that in our previous notations we also have

$$Z(t) = e^{(\Lambda\partial_x + M)t} Z_0.$$

Now, let 's choose as initial data the particular sequence $(Z_0^m)_{m\in\mathbb{Z}}$ of the eigenfunctions of the operator $\Lambda\partial_x + M_0$, associated with the eigenvalues $im\sqrt{\frac{k}{\rho_1}}\frac{\pi}{L}$, $m \in \mathbb{Z}$ (see (3.27)), which are given by

$$Z_0^m = \frac{1}{\sqrt{2L}}\left(e^{-im\frac{\pi}{L}x}, 0, -e^{im\frac{\pi}{L}x}, 0, 0\right), \quad m \in \mathbb{Z}.$$

Clearly, we have:

$$\begin{aligned} \|Z_0^m\|^2 &= 1, \quad m \in \mathbb{Z}, \\ Z_0^m &\overset{weakly}{\rightharpoonup} 0 \quad \text{in } \left(L^2\right)^4 \times \left(L^2\right)^n. \end{aligned} \tag{3.31}$$

Now (3.30) and Lemma 3.2 yield

$$\begin{aligned} \left\|e^{(\Lambda\partial_x + M_0)t} Z_0^m\right\| &\leq \left\|\left(e^{(\Lambda\partial_x + M_0)t} - e^{(\Lambda\partial_x + M)t}\right) Z_0^m\right\| + \left\|e^{(\Lambda\partial_x + M)t} Z_0^m\right\| \\ &\leq \left\|\left(e^{(\Lambda\partial_x + M_0)t} - e^{(\Lambda\partial_x + M)t}\right) Z_0^m\right\| + \left\|P^{-1}\right\| \sqrt{E(\overline{\varphi}^m,\overline{\psi}^m,Y^m)(t)} \\ &\leq \left\|\left(e^{(\Lambda\partial_x + M_0)t} - e^{(\Lambda\partial_x + M)t}\right) Z_0^m\right\| + \\ &\quad C\sqrt{\left(d^2(t) + C\varepsilon^{1/2}\right) E_g(\varphi^m,\psi^m)(0)} \end{aligned} \tag{3.32}$$

where $(\overline{\varphi}^m, \overline{\psi}^m, Y^m)$ (resp. (φ^m, ψ^m)) is a solution of the system (3.20) (resp. (1.1)–(1.4)) with initial data deduced from Z_0^m with the help of (3.22). On the other hand, since Z_0^m is an eigenfunction of the operator $\Lambda\partial_x + M_0$ associated with the eigenvalue $im\sqrt{\frac{k}{\rho_1}}\frac{\pi}{L}$,

$$\begin{aligned}\left\| e^{(\Lambda\partial_x + M_0)t} Z_0^m \right\| &= \left\| e^{im\sqrt{\frac{k}{\rho_1}}\frac{\pi}{L}t} Z_0^m \right\| \\ &= 1 \quad \forall m \in \mathbb{Z}.\end{aligned}$$

Now we arrive at a contradiction, choosing ε sufficiently small and observing:

1. $\lim\limits_{t\to\infty} d(t) = 0$.
2. $\left\| \left(e^{(\Lambda\partial_x + M_0)t} - e^{(\Lambda\partial_x + M)t}\right) Z_0^m \right\| \to 0$ as $m \to \infty$ since $e^{(\Lambda\partial_x + M_0)t} - e^{(\Lambda\partial_x + M)t}$ is a compact operator and (3.31) holds.

Q.E.D.

4 Polynomial decay

Here we shall show the polynomial decay of the solution when the kernel g decays polynomially. More precisely, we use the assumption (1.7) to prove the polynomial rate of decay of the first-order energy. The method used is essentially the same as in Section 2, but there exist some major points in some estimates which demand a different procedure. Therefore the proof has to be adapted to the case of polynomially decaying kernels, and we have to discuss the points that need a different argument. We follow the approach in [14] and shall prove the following Theorem.

Theorem 4.1 *Let us suppose that the initial data satisfy*

$$\varphi_0, \psi_0 \in H_0^1((0,L)), \qquad \varphi_1, \psi_1 \in L^2((0,L)),$$

and that the coefficients of the system (1.1), (1.2) satisfy (1.5), i.e.,

$$\frac{\rho_1}{\rho_2} = \frac{k}{b}.$$

Moreover assume that the kernel g is of polynomial type satisfying (1.7) with $p > 2$. Then the energy $E(t)$ decays polynomially as time tends to infinity, that is, there exists a positive constant C, being independent of the initial data, such that for $t \geq 0$:

$$E(t) \leq \frac{C}{(1+t)^p} E(0).$$

For the proof we need the following versions of three lemmas from [14] (based on [10]) which we state for the sake of completeness with the short proofs.

Lemma 4.2 *Let m and h be integrable functions, and let $0 \le r < 1$ and $q > 0$. Then, for $t \ge 0$:*

$$\int_0^t |m(t-\tau)\,h(\tau)|d\tau \le$$

$$\left(\int_0^t |m(t-\tau)|^{1+\frac{1-r}{q}}|h(\tau)|d\tau\right)^{\frac{q}{q+1}} \left(\int_0^t |m(t-\tau)|^r|h(\tau)|d\tau\right)^{\frac{1}{q+1}}.$$

PROOF: Define

$$v(\tau) := |m(t-\tau)|^{1-\frac{r}{q+1}}|h(\tau)|^{\frac{q}{q+1}},$$
$$w(\tau) := |m(t-\tau)|^{\frac{r}{q+1}}|h(\tau)|^{\frac{1}{q+1}}.$$

Then, for fixed $t \ge 0$,

$$|m(t-\tau)h(\tau)| = |v(\tau)\,h(\tau)|.$$

An application of Hölder's inequality with exponents

$$\delta = \frac{q}{q+1} \qquad \text{for } v$$
$$\delta^* = q+1 \qquad \text{for } w$$

gives the assertion of Lemma 4.2.

Q.E.D.

Lemma 4.3 *Let $p > 1$, $0 \le r < 1$, $t \ge 0$ and $z \in L^\infty((0,T), H^1((0,L)))$ for any $T > 0$. Then we have for $r > 0$:*

$$\int_0^L g\Box z_x\,dx \le$$

$$2\left(\int_0^t |g(\tau)|^r d\tau \|z\|^2_{L^\infty((0,t),H^1((0,L)))}\right)^{\frac{1}{1+(1-r)p}} \left(\int_0^L |g|^{1+\frac{1}{p}}\Box z_x\,dx\right)^{\frac{(1-r)p}{1+(1-r)p}},$$

and for $r = 0$:

$$\int_0^L g\Box z_x\,dx \le$$

$$2\left(\int_0^t \|z_x(\tau,\cdot)\|^2 d\tau + t\|z_x(t,\cdot)\|^2\right)^{\frac{1}{p+1}} \left(\int_0^L |g|^{1+\frac{1}{p}}\Box z_x\,dx\right)^{\frac{p}{p+1}}.$$

PROOF: Apply Lemma 4.2 with $m(\tau) := |g(\tau)|$, $h(\tau) := \int_0^L |z_x(t) - z_x(\tau)|^2 dx$ and $q := (1-r)p$, for fixed t. This proves Lemma 4.3.

Q.E.D.

Lemma 4.4 *Let $f \geq 0$ be differentiable, let $\alpha > 0$ and let f satisfy*

$$f'(t) \leq \frac{-\bar{c}_1}{f(0)^{1/\alpha}} f(t)^{1+\frac{1}{\alpha}} + \frac{\bar{c}_2}{(1+t)^\beta} f(0)$$

for $t \geq 0$, positive constants $\bar{c}_1, \bar{c}_2$ and

$$\beta \geq \alpha + 1.$$

Then there exists a constant $\bar{c}_3 > 0$ such that for $t \geq 0$:

$$f(t) \leq \frac{\bar{c}_3}{(1+t)^\alpha} f(0).$$

Proof: Let $t \geq 0$ and

$$F(t) := f(t) + \frac{2\bar{c}_2}{\alpha}(1+t)^{-\alpha} f(0).$$

Then

$$\begin{aligned} F' &= f' - 2\bar{c}_2(1+t)^{-(\alpha+1)} f(0) \\ &\leq \frac{-\bar{c}_1}{f(0)^{1/\alpha}} f^{1+\frac{1}{\alpha}} - \bar{c}_2(1+t)^{-(\alpha+1)} f(0), \end{aligned}$$

where we used $\beta \geq \alpha + 1$. Hence

$$\begin{aligned} F' &\leq \frac{-c}{f(0)^{1/\alpha}} \left(f^{1+\frac{1}{\alpha}} + (1+t)^{-(\alpha+1)} f(0)^{1+\frac{1}{\alpha}} \right) \\ &\leq \frac{-c}{F(0)^{1/\alpha}} F^{1+\frac{1}{\alpha}}. \end{aligned}$$

Integration yields

$$F(t) \leq \frac{F(0)}{(1+ct)^\alpha} \leq \frac{c}{(1+t)^\alpha} f(0)$$

whence

$$f(t) \leq \frac{\bar{c}_3}{(1+t)^\alpha} f(0)$$

follows for some $\bar{c}_3 > 0$, which proves Lemma 4.4.

Q.E.D.

Proof of Theorem 4.1:
Lemma 4.3 yields

$$\int_0^L g \Box \psi_x \, dx \leq c E(0)^{\frac{1}{1+(1-r)p}} \left(\int_0^L g^{1+\frac{1}{p}} \Box \psi_x \, dx \right)^{\frac{(1-r)p}{1+(1-r)p}}, \tag{4.1}$$

for $0 < r < 1$ with

$$rp > 1.$$

From the proof of Lemma 2.2 we get

$$
\begin{aligned}
-\frac{d}{dt}I(t) &= -\rho_2 g(0)\int_0^L |\psi_t|^2\,dx - \rho_2\int_0^L g'\psi_t\psi\,dx \\
&\quad +\rho_2\int_0^L \psi_t(g''\diamond\psi)\,dx - k\int_0^L \varphi_x\left(g\psi - g'\diamond\psi\right)\,dx \\
&\quad \frac{b}{2}\int_0^L g|\psi_x|^2dx - \frac{b}{2}\int_0^L g'\Box\psi_x dx \\
&\quad +\frac{k}{2}\int_0^L g|\psi|^2dx - \frac{k}{2}\int_0^L g'\Box\psi dx.
\end{aligned}
$$

Hypothesis (1.7) implies that

$$
|g'(t)| \le c g^{1+\frac{1}{p}}(t), \qquad |g''(t)| \le c g^{1+\frac{1}{p}}(t).
$$

Therefore we have that

$$
\rho_2\int_0^L \psi_t(g''\diamond\psi)\,dx \le c\left(\int_0^L g^{1+\frac{1}{p}}\Box\psi\,dx\right)^{1/2}\left(\int_0^L |\psi_t|^2\,dx\right)^{1/2}.
$$

Similarly

$$
k\int_0^L \varphi_x(g'\diamond\psi)\,dx \le c\left(\int_0^L g^{1+\frac{1}{p}}\Box\psi\,dx\right)^{1/2}\left(\int_0^L |\varphi_x|^2\,dx\right)^{1/2}.
$$

Using these relations and Poincaré's inequality we have

$$
\begin{aligned}
-\frac{d}{dt}I(t) &\le -\frac{1}{2}\rho_2 g(0)\int_0^L |\psi_t|^2\,dx + c_\epsilon(|g'|+|g|)\int_0^L |\psi_x|^2\,dx + \epsilon\int_0^L |\varphi_x|^2\,dx \\
&\quad +c_\epsilon\int_0^L g^{1+\frac{1}{p}}\Box\psi_x\,dx.
\end{aligned}
$$

On the other hand, from the proof of Lemma 2.3 we have that

$$
\begin{aligned}
\frac{d}{dt}J_1(t) &= \rho_2\int_0^L |\psi_t|^2\,dx - (b-\int_0^t g\,d\tau)\int_0^L |\psi_x|^2\,dx - k\int_0^L |\psi|^2\,dx \\
&\quad +k\int_0^L |w_x|^2\,dx + \rho_1\int_0^L \varphi_t w_t\,dx - \int_0^L (g\diamond\psi_x)\psi_x\,dx.
\end{aligned} \tag{4.2}
$$

Since

$$
\begin{aligned}
g\diamond\psi_x(x,t) &= \int_0^t g^{\frac{1}{2}-\frac{1}{2p}}g^{\frac{1}{2}+\frac{1}{2p}}\left\{\psi_x(x,s)-\psi_x(x,t)\right\}\,ds \\
&= \left(\int_0^t g^{1-\frac{1}{p}}\,ds\right)^{1/2}\left\{\int_0^t g^{1+\frac{1}{p}}|\psi_x(x,\tau)-\psi_x(x,t)|^2\,ds\right\}^{1/2} \\
&= \left(\int_0^t g^{1-\frac{1}{p}}\,ds\right)^{1/2}\left\{g^{1+\frac{1}{p}}\Box\psi_x\right\}^{1/2}
\end{aligned}
$$

which implies

$$|\int\limits_0^L (g\diamond\psi_x)\psi_x dx| \le C_\delta \int\limits_0^L g^{1+\frac{1}{p}}\Box\psi_x dx + \delta\int\limits_0^L |\psi_x|^2 dx.$$

Recalling from (2.7) that

$$\int_0^L |w_x|^2\, dx \le \int_0^L |\psi|^2\, dx \le c\int_0^L |\psi_x|^2\, dx$$

the identity (4.2) can be rewritten as

$$\frac{d}{dt}J_1(t) \le C_{\varepsilon_1}\int_0^L |\psi_t|^2\, dx - \frac{\lambda}{2}\int_0^L |\psi_x|^2\, dx + C_{\varepsilon_1}\int_0^L g^{1+\frac{1}{p}}\Box\psi_x\, dx + \varepsilon_1\int_0^L |\varphi_t|^2\, dx. \tag{4.3}$$

As in Section 2 we consider

$$\mathcal{E}_1(t) = N_1 E(t) - N_2 I(t) + N_3 J_1(t).$$

From the inequalities above we conclude that

$$\frac{d}{dt}\mathcal{E}_1(t) \le -\frac{N_2^{\varepsilon_1}}{2}\int_0^L \left(|\psi_t|^2 + |\psi_x|^2 + g^{1+\frac{1}{p}}\Box\psi_x\right)\, dx - \frac{N_1^{\varepsilon_1}}{2}\int_0^L g|\psi_x|^2\, dx + \varepsilon_1\int_0^L \left(|\varphi_t|^2 + |\varphi_x|^2\right)\, dx. \tag{4.4}$$

Using the same reasoning as above we can show that Lemma 2.4 and Lemma 2.5 imply

$$\begin{aligned}
\frac{d}{dt}K(t) \quad &\le \quad [(b\psi_x - g*\psi_x)\varphi_x]_{x=0}^{x=L} - k\int_0^L |\varphi_x+\psi|^2\, dx \\
&\quad + \varepsilon\int_0^L |\varphi_t|^2\, dx + C_\varepsilon\int_0^L g^{1+\frac{1}{p}}\Box\psi_x + g|\psi_x|^2\, dx \\
&\quad + \rho_2\int_0^L |\psi_t|^2\, dx,
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\int_0^L \rho_2\psi_t q(b\psi_x - g*\psi_x)\, dx \quad &\le \quad -\gamma\left\{|b\psi_x(L,t) - (g*\psi_x)(L,t)|^2\right. \\
&\quad \left. + |b\psi_x(0,t) - (g*\psi_x)(0,t)|^2\right\} + \tilde{\varepsilon}\int_0^L |\varphi_x|^2\, dx + C_{\tilde{\varepsilon}} \\
&\quad + C_\varepsilon\int_0^L |\psi_t|^2 + (b - \int\limits_0^t g ds)|\psi_x|^2 + g^{1+\frac{1}{p}}\Box\psi_x dx\, dx, \\
\frac{d}{dt}\int_0^L \rho_1\varphi_t q\varphi_x\, dx \quad &\le \quad -k\gamma\left\{|\varphi_x(L,t)|^2 + |\varphi_x(0,t)|^2\right\} \\
&\quad + C_1\int_0^L |\varphi_t|^2 + |\varphi_x|^2 + |\psi_x|^2\, dx.
\end{aligned}$$

Denoting by $L(t)$ again the functional

$$L(t) = K(t) + N_3\int_0^L \rho_2\psi_t q(b\psi_x - g*\psi_x)\, dx + \delta\int_0^L \rho_1\varphi_t q\varphi_x\, dx$$

we get

$$\frac{d}{dt}L(t) \le -\frac{k}{2}\int_0^L |\varphi_x + \psi|^2\, dx + C_2\tau \int_0^L |\varphi_t|^2\, dx + +C_\tau \mathcal{N}_p(t) \tag{4.5}$$

where

$$\mathcal{N}_p(t) := \left\{ \int_0^L |\psi_t|^2 + (b - \int\limits_0^t g ds)|\psi_x|^2 + g^{1+\frac{1}{p}} \Box \psi_x \, dx \right\}.$$

Finally, the functional J_2 defined in Section 2 satisfies

$$-\frac{d}{dt}J_2(t) \le -\rho_1 \int_0^L |\varphi_t|^2\, dx - \rho_2 \int_0^L |\psi_t|^2\, dx + k\int_0^L |\varphi_x + \psi|^2\, dx + C_\tau \mathcal{N}_p(t). \tag{4.6}$$

From the inequalities (4.5), (4.6) we get

$$\frac{d}{dt}\left\{ L(t) - \frac{2C_2\tau}{\rho_1} J_2(t) \right\} \le -\frac{k}{4}\int_0^L |\varphi_x + \psi|^2\, dx - C_2\tau \int_0^L |\varphi_t|^2\, dx + C_\tau \mathcal{N}_p(t).$$

Now using again the functional

$$\mathcal{L}(t) = \mathcal{E}_1(t) + L(t) - \frac{2C_2\tau}{\rho_1} J_2(t)$$

it is not difficult to see that

$$\frac{d}{dt}\mathcal{L}(t) \le -\beta_0 \left\{ \frac{1}{2}\int_0^L \rho_1|\varphi_t|^2 + \rho_2|\psi_t|^2 + |\psi_x|^2 + k|\varphi_x + \psi|^2 + g^{1+\frac{1}{p}} \Box \psi_x \, dx \right\}.$$

Let us denote by $\mathcal{E}_0(t)$ the functional

$$\mathcal{E}_0(t) := \frac{1}{2}\int_0^L \rho_1|\varphi_t|^2 + \rho_2|\psi_t|^2 + |\psi_x|^2 + k|\varphi_x + \psi|^2\, dx.$$

Since the energy is bounded, Lemma 4.3 implies

$$\mathcal{E}_0(t) \ge c\mathcal{E}_0(t)^{\frac{1+(1-r)p}{(1-r)p}} E(0)^{\frac{-1}{(1-r)p}},$$

$$\int\limits_0^L g^{1+\frac{1}{p}} \Box \psi_x dx \ge c \left\{ \int\limits_0^L g \Box \psi_x dx \right\}^{\frac{1+(1-r)p}{(1-r)p}} E(0)^{\frac{-1}{(1-r)p}}.$$

Observing that that $\mathcal{L}$ satisfies (cp. Section 2)

$$\beta_1 E(t) \le \mathcal{L}(t) \le \beta_3 \left\{ \mathcal{E}_0(t) + \int\limits_0^L g^{1+\frac{1}{p}} \Box \psi_x dx \right\}^{\frac{(1-r)p}{1+(1-r)p}} E(0)^{\frac{1}{1+(1-r)p}} \tag{4.7}$$

with some $\beta_3 > 0$, it follows that

$$\frac{d}{dt}\mathcal{L}(t) \le -c\mathcal{L}(t)^{\frac{1+(1-r)p}{(1-r)p}} \mathcal{L}(0)^{\frac{-1}{(1-r)p}}$$

and hence, by Lemma 4.4,

$$\mathcal{L}(t) \le C\frac{1}{(1+t)^{(1-r)p}}\mathcal{L}(0).$$

This implies for $r > 0$

$$\begin{aligned}
\int_0^t \|\psi(\tau)\|^2_{H^1((0,L))} d\tau &\leq c\int_0^t \mathcal{L}(\tau)d\tau \\
&\leq \int_0^t \frac{1}{(1+\tau)^(1-r)p} d\tau \mathcal{L}(0) \\
&\leq c\mathcal{L}(0)
\end{aligned}$$

if

$$(1-r)p > 1$$

which together with the previous condition

$$rp > 1$$

can be satisfied since $p > 2$. Moreover, we conclude

$$t\|\psi(\cdot,t)\|^2_{H^1((0,L))} \leq ct\mathcal{L}(t) \leq c\mathcal{L}(0).$$

and hence, using Lemma 4.3 now with $r = 0$,

$$\mathcal{E}_0(t) \geq c\mathcal{E}_0(t)^{\frac{1+p}{p}} E(0)^{\frac{-1}{p}}$$

and

$$\int_0^L g\Box\psi_x dx \geq c\left\{\int_0^L g\Box\psi_x dx\right\}^{\frac{1+p}{p}} E(0)^{\frac{-1}{p}}.$$

Repeating the same reasoning as before we now get

$$\frac{d}{dt}\mathcal{L}(t) \leq -c\mathcal{L}(t)^{1+\frac{1}{p}}\mathcal{L}(0)^{\frac{-1}{p}}$$

which implies by Lemma 4.4

$$\mathcal{L}(t) \leq \frac{C}{(1+t)^p}\mathcal{L}(0)$$

from where our result follows. The proof is now complete.

Q.E.D.

5 Decay rate optimality

Already for the system of (magneto-thermo-) elasticity with memory type boundary conditions it was shown in [14] that a merely polynomial kernel cannot lead to an exponential decay result for the energy in general. In a similar manner we are now able to prove that the decay rate for polynomial kernels can not be of exponential type.

We take the kernel

$$g(t) = \frac{1}{(1+t)^p}$$

for some $p > 1$.
For the initial data we assume

$$\psi_0 = 0, \quad \varphi_0, \varphi_1, \psi_1 \in C_0^\infty((0,L)), \quad \int_0^L \psi_1 dx \neq 0. \tag{5.1}$$

Then we shall demonstrate that the assumption of exponential decay,

$$\exists c > 0 \ \exists \delta > 0 \ \forall t \geq 0: \quad E(t) \leq c e^{-\delta t} E(0) \tag{5.2}$$

leads to a contradiction. With the choice of the initial conditions as in (5.1), $(v, w) := (\varphi_t, \psi_t)$ satisfies the same differential equations and boundary conditions as (φ, ψ). Hence also the energy associated to (v, w) decays exponentially, which implies, using the differential equation, that there is a constant c_0 depending on the initial data such that for all $t \geq 0$:

$$|\int_0^L (g * \psi_{xx}(x, \cdot))(t) dx| \leq c_0 e^{-\delta t/2}.$$

which is equivalent to

$$|\int_0^t \frac{1}{(1+t-s)^p} \underbrace{(\psi_x(L,s) - \psi_x(0,s))}_{=:h(s)} ds| \leq c_0 e^{-\delta t/2}. \tag{5.3}$$

On the other hand, since h decays exponentially by Sobolev's imbedding theorem and the assumption on exponential decay of the energy (applied to (φ_t, ψ_t)), it can be easily seen that for any $m > 1$

$$|\int_0^t \frac{1}{(1+t-s)^m} h(s) ds| \leq \frac{c_m}{(1+t)^m} \tag{5.4}$$

for some constant c_m. For $t \geq 0$ and $\beta \geq 0$ let

$$G_\beta(t) := \int_{t+\beta}^\infty h(s) ds.$$

Then

$$\begin{aligned} \int_0^t \frac{1}{(1+t-s)^p} h(s) ds &= \left[\frac{1}{(1+s)^p} G_\beta(t-s) \right]_{s=0}^{s=t} + p \int_0^t \frac{1}{(1+s)^{p+1}} G_\beta(t-s) ds \\ &= \frac{G_\beta(0)}{(1+t)^p} - G_\beta(t) + \mathcal{O}\left(\frac{1}{(1+t)^{p+1}} \right), \end{aligned} \tag{5.5}$$

where we used (5.4) for $m = p+1$.
Case 1: $\exists \tilde{\beta} \in [0, \infty]: \quad G_{\tilde{\beta}}(0) \neq 0$.
Thus, from (5.5),

$$\lim_{t \to \infty} \left| \int_0^t \frac{1}{(1+t-s)^p} h(s) ds \right| (1+t)^p = G_{\tilde{\beta}}(0) \neq 0$$

which is a contradiction to (5.3).
Case 2: $\forall \beta \in [0, \infty] : \quad G_\beta(0) = 0.$
This implies

$$\forall t \geq 0 : \quad h(t) = 0$$

which means that ψ satisfies additionally the boundary conditions

$$\psi_x(L, t) = \psi_x(0, t), \qquad t \geq 0.$$

Using this we conclude after integration of both sides of the differential equation (1.2) that

$$\rho_2 \frac{d^2}{dt^2} \int_0^L \psi(x,t)dx + k \int_0^L \psi(x,t)dx = 0$$

which implies, for $\nu := \sqrt{k/\rho_2}$,

$$\int_0^L \psi(x,t)dx = \int_0^L \psi_1(x)dx \sin(\nu t)$$

which is a contradiction to the assumption of exponential decay of the energy.

References

[1] Ammar-Khodja, F., Bader, A.: Sur le comportement asymptotique de solutions de systèmes hyperboliques. *C. Rend. Acad. Sci. Paris* **329, Série 1** (1999), 957–960.

[2] Ammar-Khodja, F., Bader, A.: Stabilizability of systems of one-dimensional wave equations by one internal or boundary control force. *Prépublications du laboratoire de mathématiques de Besançon* **2000/04**.

[3] Dafermos, C.M.: An abstract Volterra equation with application to linear viscoelasticity. *J. Differential Equations* **7** (1970), 554–589.

[4] Dassios, G., Zafiropoulos, F.: Equipartition of energy in linearized 3-d viscoelasticity. *Quart. Appl. Math.* **48** (1990), 715–730.

[5] Greenberg, J.M.: A priori estimates for flows in dissipative materials. *J. Math. Anal. Appl.* **60** (1977), 617-630

[6] Hrusa, W.J.: Global existence and asymptotic stability for a semilinear Volterra equation with large initial data. *SIAM J. Math. Anal.* **16** 1985, 110–134.

[7] Kim, J.U., Renardy, Y.: Boundary control of the Timoshenko beam. *SIAM J. Control Optim.* **25** (1987), 1417–1429.

[8] Lagnese, J.E.: *Asymptotic energy estimates for Kirchhoff plates subject to weak viscoelastic damping.* Intern. series of numerical mathematics **91**, Birkäuser Verlag, Basel (1989).

[9] Muñoz Rivera, J.E., Barreto, R.K.: Uniform rates of decay in nonlinear viscoelasticity for polynomial decaying kernels. *Applicable Analysis* **60** (1996), 341–357.

[10] Muñoz Rivera, J.E., Barreto, R.K.: Decay rates of solutions to thermoviscoelastic plates with memory. *IMA J. Appl. Math.* **60** (1998), 263–283.

[11] Muñoz Rivera, J.E., Cabanillas, E.: Decay rates of solutions of an anisotropic inhomogeneous n-dimensional viscoelastic equation with polynomial decaying kernels. *Communications Math. Phys.* **177** (1996), 583–602.

[12] Muñoz Rivera, J.E., Jiang, S.: A global existence theorem for the Dirichlet problem in nonlinear n-dimensional viscoelasticity. *Differential Integral Equations* **9** (1996), 791–810.

[13] Muñoz Rivera, J.E., Peres Salbatierra, A.: Asymptotic behaviour of the energy to partially viscoelastic materials. *To appear in Quart. Appl. Math.*

[14] Muñoz Rivera, J.E., Racke, R.: Magneto-thermo-elasticity — large time behavior for linear systems. *Adv. Diff. Equations* (to appear).

[15] Neves, A.F., Ribeiro, H. de S., Lopes, O.: On the spectrum of evolution operators generated by hyperbolic systems. *J. Functional Anal.* **67** (1986), 320–344.

[16] Renardy, M., Rogers, R.C.: *An introduction to partial differential equations.* Texts Appl. Math. **13**, Springer-Verlag, New York (1992).

[17] Schwartz, L.. *Etude des sommes d'exponentielles réelles.* Hermann, Paris (1943).

[18] Soufyane, A.: Stabilisation de la poutre de Timoshenko. *C. R. Acad. Sci. Paris, Sér. I* **328** (1999), 731–734.

[19] Taylor, S.W.: Boundary control of the Timoshenko beam with variable physical characteristics. *Research Report, Dept. Math.. Univ. Auckland* **356** (1998).

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