Asymptotic behavior of solutions in linear 2- or 3-d thermoelasticity with second sound

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Abstract: We consider thermoelastic systems in two or three space dimensions where thermal disturbances are modeled popagating as wavelike pulses traveling at finite speed. This is done using Cattaneo's law for heat conduction instead of Fourier's law. For Dirichlet type boundary conditions, the exponential stability of the now purely, but slightly damped, hyperbolic system is proved in the radially symmetric case.

1 Introduction

We consider two- or three-dimensional thermoelastic bodies including the second sound effect. The latter turns the classical thermoelastic system of hyperbolic-parabolic coupled type into a purely hyperbolic, but damped hyperbolic one. This way the paradox of infinite propagation speed of heat pulses is overcome, which is regarded as being important in some applications like pulsed laser heating of solids, cp. [10, 17]. The governing differential equations for the displacement vector $u = u(t,x) \in \mathbb{R}^n$, $n = 2$ or 3, the temperature difference $\theta = \theta(t,x)$ and the heat flux vector $q = q(t,x) \in \mathbb{R}^n$, where $t \geq 0$ and $x \in \Omega \subset \mathbb{R}^4$, Ω denoting the bounded reference configuration with smooth boundary $\partial\Omega$, are the following, cp. [2, 9, 13, 15], for a homogeneous isotropic medium:

$$u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla \operatorname{div} u + \beta\nabla\theta = 0, \tag{1.1}$$

$$\theta_t + \gamma \operatorname{div} q + \delta \operatorname{div} u_t = 0, \tag{1.2}$$

$$\tau_0 q_t + q + \kappa\nabla\theta = 0. \tag{1.3}$$

These equations are completed by initial conditions

$$u(0,\cdot) = u_0,\ \ u_t(0,\cdot) = u_1,\ \ \theta(0,\cdot) = \theta_0,\ \ q(0,\cdot) = q_0 \tag{1.4}$$

and the following boundary conditions for a rigidly clamped body held at constant temperature on the boundary,

$$u(t,\cdot) = 0,\ \ \theta(t,\cdot) = 0 \text{ on } \partial\Omega,\ \ t \geq 0. \tag{1.5}$$

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The parameters $\mu, \lambda, \beta, \gamma, \delta, \tau_0$ and κ are positive constants, where μ and λ represent the Lamé moduli and τ_0 is the relaxation time, a small parameter compared to the others. For $\tau_0 = 0$ we return to classical thermoelasticity since in this case Cattaneo's law for heat conduction, expressed in (1.3) and responsible for the overall hyperbolic character of the system, turns into Fourier's law

$$q = -\kappa \nabla \theta$$

making (1.2), (1.3) to a parabolic equation for the temperature

$$\theta_t - \gamma\kappa\Delta\theta + \delta \operatorname{div} u_t = 0. \tag{1.6}$$

It is known for classical thermoelasticity (1.1), (1.6) that in general bounded domains, for instance in all domains that allow reflecting rays, there is not any decay rate for the energy, see [2, 9, 4]. On the other hand, it has been proved that whenever the rotation $\operatorname{rot} u$ vanishes identically — for instance in a radially symmetric domain Ω with radially symmetric data — then the energy decays exponentially, see [3, 4]. Now back to thermoelasticity with second sound, (1.1)–(1.3) what can one expect for this even less dissipative — while only damped hyperbolic — system? Will the dissipation which is still given through heat conduction be strong enough at least in radially symmetric situations to exponentially stabilize all components u, θ and q? We recall that in [12] the corresponding question was answered with yes even for nonlinear systems. Here we shall prove the same for the radially symmetric case after a discussion of the general asymptotic behavior in terms of possible oscillatory behavior.

There seems to be no result on the precise decay up to now, in [14] the stability of the null solution was investigated, cp. also [16] in one dimension. Thus we present the first description of the exponential stability. For the proof appropriate Lyapunov functions have to be found, combining techniques from classical thermoelasticity [3] and from one-dimensional situations [12].

We remark that corresponding nonlinear problems are of interest also for the applications, and are under investigation. Another class of problems concerns Cauchy problems, i.e. $\Omega = \mathbb{R}^m$, $m = 1, 2, 3$.

In section 2 we look at the well-posedness in suitable Sobolev spaces that will allow a clear description of the generic asymptotic behavior; in particular the connection to classical thermoelasticity will become apparent. In section 3 the exponential decay will be proved for systems where $\operatorname{rot} u = 0 = \operatorname{rot} q$, which has its application in section 4 for radially symmetric situations.

2 Well-posedness and remarks on general asymptotics

We transform the initial boundary value problem (1.1)–(1.5) for (u, θ, q) to a first-order system for

$$V \equiv \begin{pmatrix} V^1 \\ V^2 \\ V^3 \\ V^4 \end{pmatrix} := \begin{pmatrix} u \\ u_t \\ \theta \\ q \end{pmatrix} \in \mathbb{R}^{3n+1}, \quad V(0) \equiv V_0 := \begin{pmatrix} u_0 \\ u_1 \\ \theta_0 \\ q_0 \end{pmatrix}, \tag{2.1}$$

observing that (1.1)–(1.3) is equivalent to

$$\kappa\delta u_{tt} + \kappa\delta E u + \kappa\delta\beta\nabla\theta = 0, \tag{2.2}$$

$$\kappa\beta\theta_t + \kappa\beta\gamma \operatorname{div} q + \kappa\delta\beta \operatorname{div} u_t = 0, \tag{2.3}$$

$$\beta\gamma\tau_0 q_t + \beta\gamma q + \kappa\beta\gamma\nabla\theta = 0, \tag{2.4}$$

where

$$Eu := -\mu\Delta u - (\mu+\lambda)\nabla \operatorname{div} u. \tag{2.5}$$

Let

$$Q := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \kappa\delta & 0 & 0 \\ 0 & 0 & \kappa\beta & 0 \\ 0 & 0 & 0 & \beta\gamma\tau_0 \end{pmatrix}, \quad N := \begin{pmatrix} 0 & -1 & 0 & 0 \\ \kappa\delta E & 0 & \kappa\delta\beta\nabla & 0 \\ 0 & \kappa\delta\beta \operatorname{div} & 0 & \kappa\beta\gamma \operatorname{div} \\ 0 & 0 & \kappa\beta\gamma\nabla & \beta\gamma \end{pmatrix},$$

in obvious notation, for instance "0" can denote a column or row vector in $\mathbb{R}^n$ with zero entries or the null matrix in $\mathbb{R}^{n^2}$.

Then V satisfies

$$V_t(t) + Q^{-1}NV(t) = 0, \quad V(0) = V_0. \tag{2.6}$$

Solving (2.6) in appropriate spaces is equivalent to solving (1.1)–(1.4) in appropriate spaces.

Let the operator E, formally defined in (2.5), be given by

$$E : D(E) \subset (L^2(\Omega))^n \to (L^2(\Omega))^n,$$

$$D(E) := (H^2(\Omega) \cap H_0^1(\Omega))^n,$$

$$Eu := -\mu\Delta u - (\mu+\lambda)\nabla \operatorname{div} u.$$

Then E is positively definite and selfadjoint (see [7]), and we can define

$$\mathcal{H} := (H_0^1(\Omega))^n \times (L^2(\Omega))^n \times L^2(\Omega) \times (L^2(\Omega))^n$$

with inner product

$$\langle V, W\rangle_{\mathcal{H}} := \langle E^{1/2}V^1, E^{1/2}W^1\rangle + \sum_{j=2}^{4}\langle V^j, W^j\rangle$$

where $\langle\cdot,\cdot\rangle$ denotes the usual $L^2(\Omega)^{[n]}$ inner product and norm $\langle g, h\rangle = \int_\Omega g(x)\overline{h}(x)dx$ and $\|g\|$, respectively.

$$A : D(A) \subset \mathcal{H} \to \mathcal{H},$$

$$D(A) := \{V \in \mathcal{H} \,|\, V^1 \in (H^2(\Omega))^n, \quad V^2 \in (H_0^1(\Omega))^n, \ V^3 \in H_0^1(\Omega), V^4 \in \tilde{D}\},$$

where

$$\tilde{D} := \{G \in (L^2(\Omega))^n \,|\, \operatorname{div} G \in L^2(\Omega)\},$$

and

$$AV := Q^{-1}NV \text{ for } V \in D(A).$$

Then (2.6) turns into

$$V_t(t) + AV(t) = 0, \quad V(0) = V_0 \in D(A). \tag{2.7}$$

Lemma 2.1 *(i) $D(A)$ is dense in $\mathcal{H}$ and $-A$ is dissipative.*

(ii) $0 \in \varrho(A)(\equiv$ resolvent set of $A)$.

PROOF:

(i) The density of $D(A)$ is obvious. Let $W \in D(A)$, then

$$\mathrm{Re}\langle -AW, W\rangle_{\mathcal{H}} = -\beta\gamma \int_{\Omega} |W^4|^2 dx \leq 0. \tag{2.8}$$

(ii) Let $W \in D(A)$ with $AW = 0$. Then by (2.8) we conclude

$$W^4 = 0.$$

$(AW)^1 = 0$ implies that

$$W^2 = 0$$

then $(AW)^4 = 0$ implies that

$$\nabla W^3 = 0$$

whence, since $W^3 \in H_0^1(\Omega)$,

$$W^3 = 0$$

follows. Finally, we get from $(AW)^2 = 0$

$$EW^1 = 0, \quad W^1 \in D(E),$$

implying

$$W^1 = 0,$$

altogether: $W = 0$, i.e. A^{-1} exists.

Now we solve $AW = F$ for arbitrary $F \in \mathcal{H}$.

$$AW = F$$

is equivalent to

$$-W^2 = F^1, \tag{2.9}$$

$$EW^1 + \beta\nabla W^3 = F^3, \tag{2.10}$$

$$\beta \operatorname{div} W^2 + \gamma \operatorname{div} W^4 = F^3, \tag{2.11}$$

$$\kappa\nabla W^3 + W^4 = \tau_0 F^4. \tag{2.12}$$

In view of (2.9) we define

$$W^2 := -F^1 \in H_0^1(\Omega)$$

yielding also

$$\|W^2\| = \|F^1\| \leq \|F\|. \tag{2.13}$$

In view of (2.11) W^4 should satisfy

$$\gamma \operatorname{div} W^4 = F^3 - \beta \operatorname{div} F^1 \tag{2.14}$$

or, looking at (2.12), W^3 should satisfy

$$\gamma\kappa\Delta W^3 = \gamma\tau_0 \operatorname{div} F^4 - F^3 - \beta \operatorname{div} F^1, \tag{2.15}$$

where the right-hand side belongs to $H^{-1}(\Omega)$ which is the dual space of $H_0^1(\Omega)$.

For (2.15) there exists a unique solution $W^3 \in H_0^1(\Omega)$ satisfying

$$\|W^3\| + \|\nabla W^3\| \leq c(\|F^4\| + \|F^3\| + \|F^1\|) \leq c\|F\|, \tag{2.16}$$

where c here and in the sequel will denote a generic constant possibly varying from estimate to estimate. Now let

$$W^4 := \tau_0 F_4 - \kappa\nabla W^3.$$

Then $W^4 \in \tilde{D}$ and

$$\|W^4\| \leq c\|F\|. \tag{2.17}$$

Finally, let $W^1 \in D(E)$ be the unique solution to

$$EW^1 = -\beta\nabla W^3 + F_2$$

which satisfies

$$\|E^{1/2}W^1\| \leq c\|F\|. \tag{2.18}$$

Then the constructed $W = (W^1, W^2, W^3, W^4)'$ satisfies $W \in D(A)$, $AW = F$, and by (2.13), (2.16)–(2.18),

$$\|W\|_{\mathcal{H}} \leq c\|F\|_{\mathcal{H}}, \tag{2.19}$$

i.e. the range of A is all $\mathcal{H}$ and A^{-1} is continuous, which proves $0 \in \varrho(A)$.

Q.E.D.

As a corollary (cp. e.g. Theorem 1.2.4 in [8]) we obtain that $-A$ generates a C_0-semigroup of contractions $\{e^{-tA}\}_{t\geq 0}$ on $\mathcal{H}$ and we solve (2.7) as follows.

Theorem 2.2 *The unique solution $V \in C_0([0,\infty), D(A)) \cap C^1([0,\infty), \mathcal{H})$ to (2.7)*

$$V_t(t) + AV(t) = 0, \;\; V(0) = V_0 \in D(A)$$

is given by

$$V(t) = e^{-tA}V_0.$$

Higher regularity is obtained observing $V(t) \in D(A^m)$ if $V_0 \in D(A^m)$, $m \in \mathbb{N}$.

We observe that A^{-1} is not compact in general:
Let

$$F_n := \begin{pmatrix} 0 \\ 0 \\ 0 \\ \operatorname{rot} \psi_n \end{pmatrix}, \quad \psi_n \in C^1(\overline{\Omega}), \quad \|\operatorname{rot} \psi_n\| := 1.$$

Let $AU_n = F_n$. Then we see that

$$U_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \tau_0 \operatorname{rot} \psi_n \end{pmatrix}$$

but $(U_n)_n$ does not have a convergent subsequence in general.

Purely oscillatory behavior is observed if there exist purely imaginary eigenvalues of A because if

$$AW = i\xi W, \;\; W \in D(A) \setminus \{0\}, \;\; \xi \in \mathbb{R} \setminus \{0\} \tag{2.20}$$

then

$$\forall t \geq 0: \;\; \|e^{-tA} W\|_{\mathcal{H}} = \|e^{-i\xi t} W\|_{\mathcal{H}} = \|W\|. \tag{2.21}$$

We remark that the spaces and the norms have been chosen such that $\frac{1}{2}\|V(t)\|^2_{\mathcal{H}}$ equals the energy of first order to be studied in the next sections. Hence (2.21) expresses conservation of energy for data W satisfying (2.20), in particular no decay rate at all, in this case.

Let W satisfy (2.20). Then we conclude

$$\beta\gamma \|W^4\|^2 = \operatorname{Re}\,(i\xi \|W\|^2_{\mathcal{H}}) = 0$$

hence

$$W^4 = 0$$

which implies

$$\begin{aligned} -W^2 &= i\xi W^1, \\ EW^1 + \beta \nabla W^3 &= i\xi W^2, \\ \beta \operatorname{div} W^2 &= i\xi W^3 \\ \kappa \nabla W^3 &= 0, \end{aligned}$$

from which we conclude successively

$$W^3 = 0$$

and

$$EW^1 = |\xi|^2 W^1, \quad W^1 \in D(A), \tag{2.22}$$

$$\operatorname{div} W^1 = 0. \tag{2.23}$$

That is, there are purely imaginary eigenvalues if and only if the eigenvalue problem (2.22) has a solution satisfying side condition (2.23). This is the same situation as in classical thermoelasticity ($\tau_0 = 0$), see e.g. [11, 4] and leads to the conjecture that the asymptotic behavior in thermoelasticity with second sound is essentially the same as that in classical thermoelasticity, not worse in general. We shall prove in the next sections that this is true concerning exponential stability in symmetrical situations.

We remark that (2.22), (2.23) has or has no solutions depending on the domain, e.g. in balls there are solutions and hence an oscillating behavior of solutions is possible *in general* see for instance [11, 4]. Another comparison with the limit $\tau_0 \to 0$ is given in the next section.

3 Exponential stability if $\operatorname{rot} u = \operatorname{rot} q = 0$

In this section we *assume a priori* that the solution (u, θ, q) to (1.1)–(1.5) additionally satisfies

$$\operatorname{rot} u = \operatorname{rot} q = 0 \quad \text{ in } [0,\infty) \times \Omega, \tag{3.1}$$

$$\nu \times q = 0 \quad \text{ in } [0,\infty) \times \partial\Omega, \tag{3.2}$$

where the rotation is defined for $n = 2$ as the scalar

$$\operatorname{rot} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} := \partial_1 u^2 - \partial_2 u^1,$$

and $\nu = \nu(x)$ denotes the exterior normal vector in $x \in \partial\Omega$. If the rotation of a scalar field f is defined in $\mathbb{R}^2$ as

$$\operatorname{rot} f := \begin{pmatrix} \partial_2 f \\ -\partial_1 f \end{pmatrix},$$

then the formula

$$\Delta = \nabla \operatorname{div} - \operatorname{rot} \operatorname{rot}$$

holds in two and three space dimensions, moreover, if $\operatorname{rot} u = 0$ and $u \in (H^2(\Omega) \cap H_0^1(\Omega))^n$ then

$$\nabla \operatorname{div} u = \Delta u \quad \text{ and } \quad \|\nabla u\| = \|\operatorname{div} u\|. \tag{3.3}$$

We remark that the condition $\operatorname{rot} q = 0$ is compatible with classical thermoelasticity where $q = -\kappa\nabla\theta$, and not a serious restriction, while the condition $\operatorname{rot} u = 0$ will be satisfied in the radially symmetric case. From section 2 we know that we cannot expect a decay of solutions in general.

Let the energy terms of first and second order be defined as

$$\begin{aligned} E_1(t) &:= \frac{1}{2}\int_\Omega \left\{ \kappa\delta|u_t|^2 + \kappa\delta\alpha|\nabla u|^2 + \kappa\beta|\theta|^2 + \tau\beta\gamma|q|^2 \right\}(t,x)dx \\ &\equiv E(t; u, \theta, q), \\ E_2(t) &:= E(t; u_t, \theta_t, q_t), \end{aligned}$$

where

$$\alpha := 2\mu + \lambda.$$

We shall prove the exponential decay of $E_1(t) + E_2(t)$ as $t \to \infty$. For this purpose we shall have to combine multiplier techniques and boundary control estimates used in [3] for classical thermoelasticity with those used in [12], the latter of course adapted to the multi- dimensional case requiring additional considerations.

Theorem 3.1 *Let (u, θ, q) be the solution to (1.1)–(1.5) satisfying (3.1), (3.2). Then the associated energy*

$$E(t) := E_1(t) + E_2(t)$$

decays exponentially, i.e.

$$\exists d_0,\, C_0 > 0 \quad \forall t \geq 0: \quad E(t) \leq C_0 e^{-d_0 t} E(0).$$

PROOF: One easily computes

$$\frac{d}{dt} E_1(t) = -\beta\gamma \|q\|^2, \tag{3.4}$$

$$\frac{d}{dt} E_2(t) = -\beta\gamma \|q_t\|^2. \tag{3.5}$$

Of course, this corresponds to the dissipativity of the operator $-A$ in (2.8).

The differential equation (1.3) yields

$$\|\nabla\theta\|^2 \leq \frac{2\tau_0^2}{\kappa^2}\|q_t\|^2 + \frac{2}{\kappa^2}\|q\|^2. \tag{3.6}$$

Multiplying (1.1) by $\frac{1}{\alpha}\Delta u$ — observing (3.3) — we get

$$\begin{aligned}
\|\Delta u\|^2 &= \frac{1}{\alpha}\langle u_{tt}, \Delta u\rangle + \frac{\beta}{\alpha}\langle \nabla\theta, \Delta u\rangle \\
&\leq -\frac{1}{\alpha}\frac{d}{dt}\langle \nabla u_t, \nabla u\rangle + \frac{1}{\alpha}\|\nabla u_t\|^2 + \frac{3}{4}\frac{\beta^2}{\alpha^2}\|\nabla\theta\|^2 + \frac{1}{3}\|\Delta u\|^2
\end{aligned}$$

implying

$$\frac{2}{3}\left\|\Delta u\right\|^2 + \frac{1}{\alpha}\frac{d}{dt}\langle \nabla u_t, \nabla u\rangle \leq \frac{1}{\alpha}\left\|\nabla u_t\right\|^2 + \frac{3}{4}\frac{\beta^2}{\alpha^2}\|\nabla\theta\|^2. \tag{3.7}$$

Multiplying (1.2) by $\frac{3}{\alpha\delta}\operatorname{div} u_t$ we obtain

$$\begin{aligned}
\frac{3}{\alpha}\|\operatorname{div} u_t\|^2 &= -\frac{3\gamma}{\alpha\delta}\langle \operatorname{div} q, \operatorname{div} u_t\rangle - \frac{3}{\alpha\delta}\langle \theta_t, \operatorname{div} u_t\rangle \\
&= \frac{3\gamma}{\alpha\delta}\langle q, \nabla \operatorname{div} u_t\rangle - \frac{3\gamma}{\alpha\delta}\langle \nu q, \operatorname{div} u_t\rangle_{\partial\Omega} + \frac{3}{\alpha\delta}\langle \nabla\theta_t, u_t\rangle
\end{aligned}$$

where $\langle\cdot,\cdot\rangle_{\partial\Omega}$ denotes the $L^2(\partial\Omega)$-inner product with norm $\|\cdot\|_{\partial\Omega}$.

Thus, using the equations (1.1), (1.3) again,

$$
\begin{aligned}
\frac{3}{\alpha}\|\operatorname{div} u_t\|^2 &= \frac{3\gamma}{\alpha\delta}\frac{d}{dt}\langle q, \Delta u\rangle - \frac{3\gamma}{\alpha\delta}\langle q_t, \Delta u\rangle + \frac{3\gamma}{\alpha\delta}\frac{d}{dt}\langle \nabla\theta, u_t\rangle \\
&\quad - \frac{3}{\alpha\delta}\langle \nabla\theta, u_{tt}\rangle - \frac{3\gamma}{\alpha\delta}\langle \nu q, \operatorname{div} u_t\rangle_{\partial\Omega} \\
&= \frac{3\gamma}{\alpha^2\delta}\frac{d}{dt}\langle q, u_{tt}\rangle + \frac{3\beta\gamma}{\alpha^2\delta}\frac{d}{dt}\langle q, \nabla\theta\rangle - \frac{3\gamma}{\alpha\delta}\langle q_t, \Delta u\rangle \\
&\quad + \frac{3}{\alpha\delta}\frac{d}{dt}\langle \nabla\theta, u_t\rangle - \frac{3}{\delta}\langle \nabla\theta, \Delta u\rangle + \frac{3\beta}{\alpha\delta}\|\nabla\theta\|^2 - \frac{3\gamma}{\alpha\delta}\langle \nu q, \operatorname{div} u_t\rangle_{\partial\Omega} \\
&\le \frac{3\gamma}{\alpha^2\delta}\frac{d}{dt}\langle q, u_{tt}\rangle - \frac{3\beta\gamma\tau_0}{\alpha^2\delta\kappa}\frac{d}{dt}\langle q, q_t\rangle - \frac{3\beta\gamma}{\alpha^2\delta\kappa}\frac{d}{dt}\|q\|^2 \\
&\quad + \frac{27\gamma^2}{\alpha^2\delta^2}\|q_t\|^2 + \frac{1}{12}\|\Delta u\|^2 - \frac{3\tau_0}{\alpha\delta\kappa}\frac{d}{dt}\langle q_t, u_t\rangle - \frac{3}{\alpha\delta\kappa}\frac{d}{dt}\langle q, u_t\rangle \\
&\quad + \frac{27}{\delta^2}\|\nabla\theta\|^2 + \frac{1}{12}\|\Delta u\|^2 + \frac{3\beta}{\alpha\delta}\|\nabla\theta\|^2 - \frac{3\gamma}{\alpha\delta}\langle \nu q, \operatorname{div} u_t\rangle_{\partial\Omega}.
\end{aligned} \tag{3.8}
$$

Combining (3.7) and (3.8) we obtain observing (3.3)

$$
\begin{aligned}
\frac{2}{\alpha}\|\nabla u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{d}{dt}G_1(t) &\le \left(\frac{3}{4}\frac{\beta^2}{\alpha^2} + \frac{27}{\delta^2} + \frac{3\beta}{\alpha\delta}\right)\|\nabla\theta\|^2 \\
&\quad + \frac{27\gamma^2}{\alpha^2\delta^2}\|q_t\|^2 - \frac{3\gamma}{\alpha\delta}\langle \nu q, \operatorname{div} u_t\rangle_{\partial\Omega},
\end{aligned} \tag{3.9}
$$

where

$$
\begin{aligned}
G_1(t) &:= \frac{1}{\alpha}\langle \nabla u_t, \nabla u\rangle - \frac{3\gamma}{\alpha^2\delta}\langle q, u_{tt}\rangle + \frac{3\beta\gamma\tau_0}{\alpha^2\delta\kappa}\langle q, q_t\rangle + \frac{3\beta\gamma}{\alpha^2\delta\kappa}\|q\|^2 \\
&\quad + \frac{3\tau_0}{\alpha\delta\kappa}\langle q_t u_t\rangle + \frac{3}{\alpha\delta\kappa}\langle q, u_t\rangle.
\end{aligned} \tag{3.10}
$$

Using the first Poincaré estimate for u_t and θ, as well as (1.1) for u_{tt}, we obtain

$$
\|u_{tt}\|^2 + \|u_t\|^2 + \|\theta\|^2 \le c(\|\Delta u\|^2 + \|\nabla\theta\|^2 + \|\nabla u_t\|^2), \tag{3.11}
$$

where $c > 0$ will again denote a constant that may take different values at different places, $c = c(\alpha, \beta, \gamma, \delta, \tau_0, \Omega)$.

Multiplying (1.1) by u we get

$$
\alpha\|\nabla u\|^2 = \langle u_{tt}, u\rangle + \beta\langle \nabla\theta, u\rangle
$$

which gives, using Poincaré again,

$$
\frac{\alpha}{2}\|\nabla u\|^2 \le c(\|u_{tt}\|^2 + \|\nabla\theta\|^2). \tag{3.12}
$$

A multiplication of (1.2) by θ_t yields

$$
\begin{aligned}
\|\theta_t\|^2 &= -\gamma\langle \operatorname{div} q, \theta_t\rangle - \delta\langle \operatorname{div} u_t, \theta_t\rangle \\
&\le \gamma\langle q, \nabla\theta_t\rangle + \frac{\delta^2}{2}\|\operatorname{div} u_t\|^2 + \frac{1}{2}\|\theta_t\|^2 \\
&= \frac{d}{dt}\langle q, \nabla\theta\rangle - \gamma\langle q_t, \nabla\theta\rangle + \frac{\delta^2}{2}\|\operatorname{div} u_t\|^2 + \frac{1}{2}\|\theta_t\|^2
\end{aligned}
$$

whence, for arbitrary $\tilde{\mu} > 0$,

$$\tilde{\mu}\|\theta_t\|^2 - 2\tilde{\mu}\frac{d}{dt}\langle q, \nabla\theta\rangle \leq \tilde{\mu}\gamma\|q_t\|^2 + \tilde{\mu}\gamma\|\nabla\theta\|^2 + \delta^2\tilde{\mu}\|\operatorname{div} u_t\|^2 \tag{3.13}$$

follows, and $\tilde{\mu}$ will be determined later.

The boundary term appearing in (3.9) will be treated next, in various steps using boundary control estimates, cp. [3] for the simpler situation in classical thermoelasticity.

$$\left|\frac{3\gamma}{\alpha\delta}\langle \nu q, \operatorname{div} u_t\rangle_{\partial\Omega}\right| \leq \frac{c_1}{\hat{\varepsilon}}\, \|\nu q\|^2_{\partial\Omega} + \hat{\varepsilon}\, \|\operatorname{div} u_t\|^2_{\partial\Omega}\,, \tag{3.14}$$

where $1 > \hat{\varepsilon} > 0$ is still arbitrary and will be determined later, and c_1 (similarly $c_2, c_3, \ldots$) denotes a fixed constant.

Let $\sigma \in (C^1(\overline{\Omega}))^3$ be such that $\sigma = (\sigma_i)_{i=1,\cdot,n}$ with $\sigma_k = \nu_k$ on $\partial\Omega$, and let $\partial_k := \frac{\partial}{\partial x_k}$, $k = 1, \cdot, n$.

Multiplying (1.2) by $-\frac{\beta}{\delta}\sigma_k\partial_k\theta_t$ (summation convention) we obtain

$$\begin{aligned}
0 &= -\frac{\beta}{\delta}\langle \theta_t, \sigma_k\partial_k\theta_t\rangle - \frac{\beta\gamma}{\delta}\langle \operatorname{div} q,\, \sigma_k\partial_k\theta_t\rangle - \beta\langle \operatorname{div} u_t, \sigma_k\partial_k\theta_t\rangle \\
&= -\frac{\beta}{\delta}\|\theta_t\|^2_{\partial\Omega} + \frac{\beta}{2\delta}\langle (\operatorname{div}\sigma)\theta_t, \theta_k\rangle - \frac{\beta\gamma}{\delta}\frac{d}{dt}\langle \operatorname{div} q,\, \sigma_k\partial_k\theta\rangle + \frac{\beta\gamma}{\delta}\langle \operatorname{div} q_t, \sigma_k\partial_k\theta\rangle \\
&\quad -\beta\langle \operatorname{div} u_t,\, \sigma_k\partial_k\theta_t\rangle \\
&= -\frac{\beta}{\delta}\|\theta_t\|^2_{\partial\Omega} + \frac{\beta}{2\delta}\langle (\operatorname{div}\sigma)\theta_t, \theta_t\rangle - \frac{\beta\gamma}{\delta}\frac{d}{dt}\langle \operatorname{div} q, \sigma_k\partial_k\theta\rangle \\
&\quad -\frac{\beta\gamma}{\tau_0\delta}\langle \operatorname{div} q, \sigma_k\partial_k\theta\rangle - \frac{\beta\gamma\kappa}{\tau_0\delta}\langle \Delta\theta, \sigma_k\partial_k\theta\rangle - \beta\langle \operatorname{div} u_t, \sigma_k\partial_k\theta_t\rangle.
\end{aligned} \tag{3.15}$$

Since

$$\begin{aligned}
\langle \Delta\theta, \sigma_k\partial_k\theta\rangle &= \langle \nu\nabla\theta,\, \nu_k\partial_k\theta\rangle_{\partial\Omega} - \langle \nabla\theta, (\nabla\sigma_k)\partial_k\theta\rangle - \langle \nabla\theta, \sigma_k\partial_k\nabla\theta\rangle \\
&= \left\|\frac{\partial\theta}{\partial\nu}\right\|^2_{\partial\Omega} - \langle \nabla\theta, (\nabla\sigma_k)\partial_k\theta\rangle - \frac{1}{2}\|\nabla\theta\|^2_{\partial\Omega} + \frac{1}{2}\langle (\operatorname{div}\sigma)\nabla\theta, \nabla\theta\rangle \\
&= \frac{1}{2}\left\|\frac{\partial\theta}{\partial\nu}\right\|^2_{\partial\Omega} - \langle \nabla\theta, (\nabla\sigma_k)\partial_k\theta\rangle + \frac{1}{2}\langle (\operatorname{div}\sigma)\nabla\theta, \nabla\theta\rangle,
\end{aligned}$$

where we used the boundary condition $\theta_{|\partial\Omega} = 0$ to conclude that $\nabla\theta = \frac{\partial\theta}{\partial\nu}\cdot\nu$, we get from (3.15)

$$\begin{aligned}
0 &= \frac{2\beta}{\delta}\|\theta_t\|^2_{\partial\Omega} - \frac{\beta}{\delta}\langle (\operatorname{div}\sigma)\theta_t, \theta_t\rangle + \frac{2\beta\gamma}{\delta}\frac{d}{dt}\langle \operatorname{div} q,\, \sigma_k\partial_k\theta\rangle \\
&\quad + \frac{2\beta\gamma}{\tau_0\delta}\langle \operatorname{div} q, \sigma_k\partial_k\theta\rangle + \frac{\beta\gamma\kappa}{\tau_0\delta}\left\|\frac{\partial\theta}{\partial\nu}\right\|^2_{\partial\Omega} - \frac{2\beta\gamma\kappa}{\tau_0\delta}\langle \nabla\theta, (\nabla\sigma_k)\partial_k\theta\rangle \\
&\quad - \frac{\beta\gamma\kappa}{\tau_0\delta}\langle (\operatorname{div}\sigma)\nabla\theta, \nabla\theta\rangle + 2\beta\langle \operatorname{div} u_t,\, \sigma_k\partial_k\theta_t\rangle,
\end{aligned} \tag{3.16}$$

On the other hand we obtain in the same way, differentiating (1.1) with respect to t and multiplying by $\sigma_k\partial_k u_t$ — actually an application of Lemma 2.3 from [3] —,

$$0 = \left\|\frac{\partial u_t}{\partial\nu}\right\|^2_{\partial\Omega} + (\mu+\lambda)\left\|\operatorname{div} u_t\right\|^2_{\partial\Omega} - 2\frac{d}{dt}\langle u_{tt}, \sigma_k\partial_k u_t\rangle$$

$$
\begin{aligned}
&-\langle(\operatorname{div}\sigma)u_{tt},u_{tt}\rangle-2\mu\langle\partial_j u_t^i,(\partial_j\sigma_k)\partial_k u_t^i\rangle\\
&+\mu\langle(\operatorname{div}\sigma)\nabla u_t,\nabla u_t\rangle-2(\mu+\lambda)\langle\operatorname{div}u_t,(\nabla\sigma_k)\partial_k u_t\rangle\\
&+(\mu+\lambda)\langle(\operatorname{div}\sigma)\operatorname{div}u_t,\,\operatorname{div}u_t\rangle-2\beta\langle\nabla\theta_t,\sigma_k\partial_k u_t\rangle. \qquad (3.17)
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
\operatorname{Re}\langle\operatorname{div}u_t,\sigma_k\partial_k\theta_k\rangle &= -\operatorname{Re}\langle\partial_k\partial_m u_t^m,\sigma_k\theta_t\rangle-\operatorname{Re}\langle\partial_m u_t^m,(\partial_k\sigma_k)\theta_t\rangle\\
&= -\operatorname{Re}\langle\theta_t,\sigma_k\partial_k\partial_m u_t^m\rangle-k\langle\theta_t(\partial_m\sigma_k)\partial_k u_t^m\rangle\\
&\quad+\operatorname{Re}\langle\theta_t,(\partial_m\sigma_k)\partial_k u_t^m-(\partial_k\sigma_k)\partial_m u_t^m\rangle\\
&= \operatorname{Re}\langle\partial_m\theta_t,\sigma_k\partial_k u_t^m\rangle+\operatorname{Re}\langle\theta_t,(\partial_m\sigma_k)\partial_k u_t^m-(\partial_t\sigma_k)\partial_m u_t^m\rangle\\
&= \operatorname{Re}\langle\nabla\theta_t,\sigma_k\partial_k u_t\rangle+\operatorname{Re}\langle\theta_t,(\nabla\sigma_k)\partial_k u_t-(\operatorname{div}\sigma)\operatorname{div}u_t\rangle. \qquad (3.18)
\end{aligned}
$$

Adding (3.16) and (3.17), taking real parts and observing (3.18), we obtain

$$
\begin{aligned}
&\frac{2\beta}{\delta}\|\theta_t\|^2_{\partial\Omega}+\frac{\beta\gamma\kappa}{\tau_0\delta}\Big\|\frac{\partial\theta}{\partial\nu}\Big\|^2_{\partial\Omega}+\Big\|\frac{\partial u_t}{\partial\nu}\Big\|^2_{\partial\Omega}+(\mu+\lambda)\Big\|\operatorname{div}u_t\Big\|^2_{\partial\Omega}\\
&\quad+\frac{d}{dt}\operatorname{Re}\left(\frac{2\beta\gamma}{\delta}\langle\operatorname{div}q,\sigma_k\partial_k\theta\rangle-2\langle u_{tt},\sigma_k\partial_k u_t\rangle\right)\\
&\leq\; c_2(\|\theta_t\|^2+\|\nabla\theta\|^2+\|\nabla u_t\|^2+\|\Delta u\|^2). \qquad (3.19)
\end{aligned}
$$

Until now we only used $\operatorname{rot}u=0$ from the assumptions (3.1), (3.2). In the next step we also exploit $\operatorname{rot}q=0$ and $\nu\times q=0$ on $\partial\Omega$.

Multiplying (1.2) by σq yields

$$
\tau_0\langle\theta_t,\sigma q\rangle+\gamma\langle\operatorname{div}q,\sigma q\rangle+\delta\langle\operatorname{div}u_t,\sigma q\rangle=0 \qquad (3.20)
$$

with

$$
|\tau_0\langle\theta_t,\sigma q\rangle|\leq\tilde\varepsilon\|\theta_t\|^2+\frac{c_3}{\tilde\varepsilon}\|q\|^2 \qquad (3.21)
$$

for some $\tilde\varepsilon>0$ to be determined later.

$$
|\delta\langle\operatorname{div}u_t,\sigma q\rangle|\leq\tilde\varepsilon\|\operatorname{div}u_t\|^2+\frac{c_4}{\tilde\varepsilon}|q\|^2, \qquad (3.22)
$$

$$
\begin{aligned}
\langle\operatorname{div}q,\sigma q\rangle &= \langle\partial_j q_j,\sigma_k q_k\rangle=\langle\nu_j q_j,\sigma_k q_k\rangle_{\partial\Omega}-\langle q_j,(\partial_j\sigma_k)g_k\rangle-\langle q_j,\sigma_k\partial_j g_k\rangle\\
&= \|\nu q\|^2_{\partial\Omega}-\langle q_j,(\partial_j\sigma_k)q_k\rangle-\langle q_j,\sigma_k\partial_k q_j\rangle\\
&= \|\nu q\|^2-\langle q_j,(\partial_j\sigma_k)q_k\rangle-\frac12\|q\|^2_{\partial\Omega}+\frac12\langle q,(\operatorname{div}\sigma)q\rangle\\
&= \frac12\|\nu q\|^2-\langle q_j,(\partial_j\sigma_k)q_k\rangle+\frac12\langle q,(\operatorname{div}\sigma)q\rangle. \qquad (3.23)
\end{aligned}
$$

Combining (3.20)–(3.23) we conclude ($\tilde{\varepsilon}$ small)

$$\|\nu q\|^2 \le \tilde{\varepsilon}(\|\theta_t\|^2 + \|\operatorname{div} u_t\|^2) + \frac{c_5}{\tilde{\varepsilon}}\|q\|^2. \tag{3.24}$$

From (3.9), (3.13), (3.14), (3.19) (multiplied by $\frac{\hat{\varepsilon}}{\mu+\lambda}$), (3.24) follows for sufficiently small $\tilde{\mu}, \hat{\varepsilon}, \tilde{\varepsilon}$:

$$\frac{1}{\alpha}\|\nabla u_t\|^2 + \frac{1}{4}\|\Delta u\|^2 + c_6\|\theta_t\|^2 + \frac{d}{dt}H(t) \le c_7(\|q_t\|^2 + \|q\|^2), \tag{3.25}$$

where (cp. (3.10) for $G_1(t)$)

$$H(t) := G_1(t) + \frac{\hat{\varepsilon}}{\mu+\lambda}\mathrm{Re}\left(\frac{2\beta\gamma}{\delta}\langle \operatorname{div} q, \sigma_k\partial_k\theta\rangle - 2\langle u_{tt}, \sigma_k\partial_k u_t\rangle\right) - 2\tilde{\mu}\langle q, \nabla\theta\rangle. \tag{3.26}$$

A suitable Lyapunov function F is defined by

$$F(t) := \frac{1}{\varepsilon}(E_1(t) + E_2(t)) + H(t),$$

where $\varepsilon > 0$.

Combining (3.3), (3.5), (3.11), (3.12) and (3.25) we see that F satisfies, for sufficiently small ε,

$$\frac{d}{dt}F(t) \le -d_1(E_1(t) + E_2(t)). \tag{3.27}$$

for some $d_1 > 0$. On the other hand (ε small enough)

$$\exists C_1, C_2 > 0 \;\; \forall t \ge 0: \;\; C_1 E(t) \le F(t) \le C_2 E(t),$$

with

$$E(t) = E_1(t) + E_2(t). \tag{3.28}$$

Hence we conclude from (3.27)

$$\frac{d}{dt}F(t) \le -d_0 F(t),$$

with

$$d_0 := \frac{d_1}{C_2},$$

whence

$$F(t) \le e^{-d_0 t}F(0)$$

follows. Using (3.28) again we conclude

$$E(t) \le C_0 e^{-d_0 t}E(0)$$

with

$$C_0 := \frac{C_2}{C_1}.$$

Q.E.D.

Remarks:

1. The conditions (3.1), (3.2) will be satisfied in the radially symmetric case, see section 4. The conditions on q, i.e. $\operatorname{rot} q = 0$, and $\nu \times q = 0$ on $\partial\Omega$ are not very restrictive, since for q_0 with $\operatorname{rot} q_0 = 0$ and $\nu \times q_0 = 0$ on $\partial\Omega$, it follows from the differential equation (1.3).

2. The constants d_0 and C_0 in Theorem 3.1 depend on $\Gamma := (\alpha, \beta, \gamma, \delta, \tau_0)$ and on Ω, where the latter dependence is given through the constant C_p in the first Poincaré estimate for v in $H_0^1(\Omega)$,

$$\|v\| \leq C_p \|\nabla v\|,$$

   and through the estimates on $\sigma, \nabla\sigma$. The dependence on Γ could be described explicitly. Bounds for d_0 calculated in [12] in one dimension indicate a small decay rate.

As in one dimension [12] it can be shown that the energy of the difference of the solution $(u^{\tau_0}, \theta^{\tau_0}, q^{\tau_0})$ to (1.1)–(1.5) and the solution $(\tilde{u}, \tilde{\theta}, \tilde{q})$ to the corresponding system with $\tau_0 = 0$ vanishes of order τ_0^2, provided the compatibility condition

$$q(0,\cdot) = -\kappa \nabla \tilde{\theta}(0,\cdot)$$

holds. If $E_1^{\tau_0}$ denotes the first-order energy for the difference $(u^{\tau_0} - \tilde{u},\ \theta^{\tau_0} - \tilde{\theta},\ q^{\tau_0} - \tilde{q}) \equiv (v, \varrho, z)$, then

$$\frac{d}{dt} E_1^{\tau_0}(t) = -\beta\gamma \|z\|^2 - \tau_0 \beta\gamma\kappa \langle \nabla\tilde{\theta}_t, z \rangle$$

implying

$$E_1^{\tau_0}(t) \leq \tau_0^2 \frac{\beta\gamma\kappa^2}{2} \int_0^t \|\nabla\tilde{\theta}_t(s,\cdot)\|^2 ds \tag{3.29}$$

which holds in general, i.e. without assuming (3.1), (3.2).

Under the assumptions (3.1), (3.2) we have from [3]

$$\int_0^\infty \|\nabla\tilde{\theta}_t(s,\cdot)\|^2 ds < \infty$$

implying

$$\exists C > 0 \ \forall t \geq 0 : \quad E_1^{\tau_0}(t) \leq C\tau_0^2.$$

4 Radial symmetry

Having proved Theorem 3.1 it follows now as in classical thermoelasticity ($\tau_0 = 0$), see [3], the exponential stability for radially symmetric domains Ω *and* radially symmetric data. We recall that in a ball Ω there are — non radially symmetric — data for which the energy E(t) does not decay at all, cp. section 1. We summarize the considerations from [3] carried over to our case $\tau_0 \neq 0$:

Let Ω be a smoothly bounded, radially symmetrical domain, i.e.

$$x \in \Omega \Rightarrow \forall R \in O(2) \quad (\text{if } n = 2) \text{ resp. } SO(3) \ (\text{if } n = 3) : \ Rx \in \Omega,$$

where $(S)O$ denotes the (special) orthogonal group. A vector field v and a function ψ from Ω to $\mathbb{R}^n$ and $\mathbb{R}$, respectively, are called radially symmetric if

$$\forall R \in O(2) \quad (\text{if } n=2) \text{ or } SO(3) \quad (\text{if } n=3)\ \forall x \in \Omega : v(Rx) = Rv(x) \text{ and } \psi(Rx) = \psi(x), \text{ resp.}$$

v and ψ are radially symmetric if and only if there are $\chi : [0,n) \to \mathbb{R}$ and $\zeta : [0,\infty)$ such that for $x \in \Omega$

$$v(x) = x\chi(|x|), \qquad \psi(x) = \zeta(|x|).$$

Any radially symmetric vector field v satisfies

$$\operatorname{rot} v = 0 \text{ in } \Omega, \quad \nu \times v = 0 \text{ on } \partial\Omega. \tag{4.1}$$

If the data u_0, u_1, θ_0, q_0 in (1.4) are radially symmetric then the solution (u, θ, q) to (1.1)–(1.5) is radially symmetric for all times. This can be seen taking for $R \in 0(2)$ resp. $SO(3)$ (R^t: transposed matrix)

$$v(t,x) := R^t u(t,Rx),\ \varrho(t,x) := \theta(t,Rx),\ h(t,x) := R^t q(t,Rx).$$

Then (v, ϱ, h) also satisfies (1.1)–(1.5), hence $(v, \varrho, h) = (u, \theta, q)$ implying the radial symmetry. In particular, u and q satisfy (4.1), hence (3.1), (3.2) is satisfied, and we can apply Theorem 3.1.

Theorem 4.1 *Let Ω be radially symmetric and let the initial data u_0, u_1, θ_0, q_0 be radially symmetric. Then the energy $E(t) = E_1(t) + E_2(t)$ associated to the solution (u, θ, q) if (1.1)–(1.4) decays exponentially, i.e.*

$$\exists d_0, C_0 > 0 \quad \forall t \geq 0, \qquad E(t) \leq C_0 e^{-d_0 t} E(0).$$

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