The Cahn-Hilliard equation with dynamic boundary conditions

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Abstract

This paper is concerned with the following Cahn-Hilliard equation

$$\psi_t = \Delta\mu$$

where

$$\mu = -\Delta\psi - \psi + \psi^3,$$

subject on the boundary Γ to the following dynamic boundary condition

$$\sigma_s \Delta_{||}\psi - \partial_\nu \psi + h_s - g_s\psi = \frac{1}{\Gamma_s}\psi_t$$

and

$$\partial_\nu \mu = 0,$$

and the initial condition

$$\psi|_{t=0} = \psi_0.$$

This problem was recently proposed by physicists to describe spinodal decomposition of binary mixtures where the effective interaction between the wall (i.e., the boundary Γ) and two mixture components are short-ranged. The global existence and uniqueness of solutions to this initial boundary value problem with highest-order boundary conditions is proved.

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1 Introduction

It is well known that the following Cahn-Hilliard equation describes spinodal decomposition of binary mixtures that appears, for example, in cooling processes of alloys, glasses or polymer mixtures (see Cahn & Hilliard [4], Novick-Cohen & Segel [19], Kenzler et al. [16], and the references cited therein):

$$\psi_t = \Delta\mu \quad \text{in } [0,T]\times\Omega, \tag{1.1}$$

$$\mu = -\Delta\psi + a\psi + b\psi^3, \tag{1.2}$$

where $0 < T \leq \infty$, Ω is a bounded domain in $I\!R^n$, $n = 1, 2, 3$ with smooth boundary Γ and μ is called the chemical potential with a, b being constants, $b > 0, a < 0$. Without loss of generality, one can assume that $b = 1, a = -1$. It is clear that the equations (1.1), (1.2) can be written as a single nonlinear parabolic equation for ψ:

$$\psi_t = \Delta(-\Delta\psi + \psi^3 - \psi). \tag{1.3}$$

Equations (1.1), (1.2) have to be supplemented by the initial condition

$$\psi(0,\cdot) = \psi_0 \quad \text{in } \Omega \tag{1.4}$$

and two boundary conditions. In the literature, the usual boundary conditions considered are the following:

$$\partial_\nu\mu|_\Gamma = 0, \tag{1.5}$$

and

$$\partial_\nu\psi|_\Gamma = 0. \tag{1.6}$$

In the above $\partial_\nu = \nu\cdot\nabla$ denotes the exterior normal derivative at the boundary, and $\nu = \nu(x)$ denotes the exterior normal in $x \in \Gamma$. The boundary condition (1.5) has a clear physical meaning: There cannot be any exchange of the mixture constituents though the boundary Γ; it is easy to see from (1.3) and (1.5) that the total mass $\int_\Omega \psi dx$ is conserved for all time. The boundary condition (1.6) is usually called the variational boundary condition, which together with (1.5) results in decreasing of the following bulk free energy

$$F_b[\psi] := \int\limits_\Omega \left(\frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}\psi^2 + \frac{1}{4}\psi^4\right)(x)dx. \tag{1.7}$$

For the initial boundary value problem for the equations (1.1), (1.2) or for the equation (1.3) with boundary conditions (1.5), (1.6), the results on global existence, uniqueness and

large time behavior of solution have been established in the literature (see, for example, Elliott & Zheng [6], Zheng [26] and Temam [24], see also Kenmochi, Niezgódka & Pawłow [15] for the variational approach for the problem with constraint).
However, it was proposed by physicists in recent years that, when the effective interaction between the wall (i.e., the boundary Γ) and both mixture components is short-ranged, the following surface free energy functional

$$F_s[\psi] = \int_\Gamma \left(\frac{\sigma_s}{2}|\nabla_{||}\psi|^2 + f_s(\psi)\right) d\sigma \tag{1.8}$$

with

$$f_s(\psi) := -h_s\psi + \frac{g_s}{2}\psi^2 \tag{1.9}$$

and $\nabla_{||}$ being the tangential gradient operator on Γ, should be added to the above bulk free energy functional $F_b[\psi]$ to form a total free energy functional

$$F[\psi] = F_b[\psi] + F_s[\psi]. \tag{1.10}$$

In the above $\sigma_s > 0, g_s > 0, h_s$ are given constants where g_s accounts for a modification of the effective interaction between the components at the wall, and $h_s \neq 0$ describes the possible preferential attraction of one of the two components by the wall. It turns out that instead of the boundary condition (1.6), the following dynamic boundary condition

$$\frac{1}{\Gamma_s}\psi_t = \sigma_s\Delta_{||}\psi - \partial_\nu\psi + h_s - g_s\psi \quad \text{ on } \Gamma \tag{1.11}$$

is posed in order that the total free energy functional $F[\psi]$ will decrease with respect to time, i.e.,

$$\frac{d}{dt}F[\psi(t,\cdot)] = -\int\limits_\Omega |\nabla\mu|^2(t,x)dx - \frac{1}{\Gamma_s}\int\limits_\Gamma |\psi_t|^2(t,\sigma)d\sigma \leq 0. \tag{1.12}$$

Here Γ_s is a positive constant, and $\Delta_{||}$ denotes the tangential Laplace operator on the surface. A similar boundary condition can be derived by taking the continuum limit of simple lattice models within a direct mean-field approximation or by applying density functional theory. We refer to Fischer, Maass & Dieterich [10], [11], Binder & Frisch [5], Fischer et al. [12] and the references cited therein for details.

Remark 1.1 *In the one-dimensional case ($n = 1$) it is assumed that the term with the tangential Laplacian $\Delta_{||}$ simply does not appear in (1.11), or, in other words, the tangential gradient term in (1.8) vanishes.*

This paper is concerned with global existence and uniqueness of solutions to the initial boundary value problem (1.3), (1.5), (1.11), (1.4). While numerical experiments were carried out in a very recent paper [16], theoretical analysis for this kind of dynamic boundary condition is not available in the literature.

The new features of the present problem are: the boundary condition (1.11) involves ψ_t, which is called the dynamic boundary condition in literature, and the term with the tangential Laplacian $\Delta_{||}$. Before going into the detail of our results and proofs, let us first recall some related papers. As far as parabolic equations with dynamic boundary condition is concerned, we refer to Escher [8], [9], Hintermann [13] in which they establish existence and uniqueness of maximal solutions for quasilinear parabolic equations with dynamic boundary conditions in the framework of $W^{m,p}$. It should be mentioned that in their study the term with the tangential Laplacian $\Delta_{||}$ does not appear. It is not clear at all whether and how their approach can be extended to deal with the problem with boundary condition also involving the term with the tangential Laplacian $\Delta_{||}$. Besides, it seems to us that for the problem we studied, it would be very difficult to get uniform estimates in $W^{m,p}$ for $p \neq 2$ in order to obtain global existence, if we directly use their setting. It is also worthwhile noticing that in [13] the Agmon, Douglis, Nirenberg framework is used, as we shall do below.

We also refer to Sato and Aiki [22] for the results on global existence and uniqueness of *weak* solution for the phase filed equations. Again the term with the tangential Laplacian $\Delta_{||}$ does not appear in their boundary conditions. Besides, what they established is the existence and uniqueness of weak solution by variational approach. However, what we are concerned in this paper is existence and uniqueness of strong solution. The number of a priori estimates obtained in weak solution setting is much less than what needed for global existence and uniqueness of strong solution of the original nonlinear problem.

On the other hand, in literature there are some other papers dealing with problems for partial differential equations involving second partial derivatives on the boundary, see, e.g., [23]. But to our knowledge, there are no papers in literature dealing with problems with both dynamic boundary condition and the term with the tangential Laplacian $\Delta_{||}$ on the boundary .

To overcome the mathematical difficulties due to the presence of the dynamic boundary condition as well as the term with the tangential Laplacian $\Delta_{||}$ on the boundary, our strategy is the following. We shall first introduce an approximate problem (P_ε) depending on a small positive parameter ε; the equations of this problem will, for fixed ε, formally be the phase-field equations of Caginalp type (see [3, 27]), but now with a non-homogeneous dynamic boundary condition also involving the highest-order derivative in x with the tangential Laplacian. To solve this approximate problem, we study the corresponding linearized problem again with the same kind of boundary conditions:

$$\varepsilon\phi_t - \Delta\phi = f, \tag{1.13}$$

$$(\sigma_s \Delta_{||}\phi - \partial_\nu \phi - g_s\phi - \frac{1}{\Gamma_s}\phi_t)|_\Gamma = 0, \tag{1.14}$$

$$\phi(0, \cdot) = \phi_0. \tag{1.15}$$

After completion of the first version of our paper, a very recent paper by Prüss [21] attracts our attention. He discusses (1.13)–(1.15) as an example of general abstract parabolic systems and proves optimal regularity results. Since we also need a priori estimates in spaces that are suitable for the original nonlinear system, we give an alternative approach in different function spaces (refer to the remark in Section 2 for the detailed discussions).

Namely, in our approach, it will be necessary to use the results on elliptic operators with highest-order derivatives in the boundary condition which still form an elliptic boundary value problem in the sense of Hörmander [14] (or is elliptic with the complementary condition according to Agmon, Douglis & Nirenberg [1, 2], or is a normal elliptic system in the sense of Peetre [20]). In particular we use a result of Višik [25], and the correponding parabolic theory, respectively, compare Lions & Magenes [17, 18] or Temam [24].

Once the solvability of the linearized problem is proved, we proceed to use the contraction mapping theorem to prove the local existence and uniqueness of solutions to the approximate problem. The possibility of treating the nonlinear problem now naturally depends on the fact that the special nonlinearity in the differential equation is of semilinear power-type. Then, based on the uniform a priori estimates, we draw the conclusions on the global existence and uniqueness of the approximate problem. Furthermore, we show that some uniform a priori estimates we get for the approximate problems do not depend on ε. Therefore, they allow us to pass to the limit, as $\varepsilon \to 0$, to get the global existence of strong solutions for the original problem (1.3), (1.5), (1.11), (1.4). The uniqueness of solutions in the same class of functions can be proved by the standard energy method.

Remark 1.2 *In the one-dimensional case, because the tangential Laplacian in the boundary condition does not appear, some minor modifications in the proof are needed.*

This paper is organized as follows: In Section 2 we introduce the approximate problem (P_ε), and the solvability of the auxiliary linear problem with highest-order boundary conditions is extensively studied. In Section 3 we shall prove a local existence and uniqueness result for problem (P_ε). Uniform a priori estimates will be obtained in Section 4 to prove the global existence for the approximate problem. In Section 5 we shall show that sufficiently strong *a priori* estimates obtained in Section 4, which do not depend on ε, will allow us to pass to the limit, as $\varepsilon \to 0$, in certain Sobolev spaces as $n = 2, 3$. The proof of the uniqueness for the original problem (1.3), (1.5), (1.11), (1.4) is also given in that section. In Section 6 the case $n = 1$ is treated. Finally we shortly comment on the limiting case $\Gamma_s = \infty$ in Section 7.

We use standard notation for Sobolev spaces, e.g. for $H^s \equiv H^s(\Omega)$ or for $L^r(H^s) \equiv L^r([0,T]; H^s)$, cp. e.g. [17]. $\|\cdot\|_Y$ and $\|\cdot\|$ will denote the norms in the Banach space Y and in $L^2 \equiv L^2(\Omega)$, respectively.

2 The approximate problem (P_ε)

In order to find a solution ψ to the original problem (1.3), (1.5), (1.11), (1.4), we now introduce an approximate problem (P_ε) for some fixed, but arbitrary parameter ε, $0 < \varepsilon \leq 1$. Observe that the transformation

$$\hat{\psi} := \psi - \frac{h_s}{g_s} \tag{2.1}$$

turns the Cahn-Hilliard equation into one with the new chemical potential

$$\hat{\mu} := -\Delta\hat{\psi} - \hat{\psi} + \hat{\psi}^3 + 3\frac{h_s}{g_s}\hat{\psi}^2 + \frac{h_s^2}{g_s^2}\hat{\psi} + \frac{h_s^3}{g_s^3} + \frac{h_s}{g_s} \equiv -\Delta\hat{\psi} + N(\hat{\psi}), \tag{2.2}$$

where

$$N(\hat{\psi}) = -\hat{\psi} + \hat{\psi}^3 + 3\frac{h_s}{g_s}\hat{\psi}^2 + \frac{h_s^2}{g_s^2}\hat{\psi} + \frac{h_s^3}{g_s^3} + \frac{h_s}{g_s}, \tag{2.3}$$

and it has the effect that the second boundary condition reads

$$\frac{1}{\Gamma_s}\hat{\psi}_t = \sigma_s \Delta_{||}\hat{\psi} - \partial_\nu\hat{\psi} - g_s\hat{\psi} \quad \text{ on } \Gamma,$$

having thus removed the nonhomogeneous term h_s.

The approximate problem (P_ε) now consists in finding a solution (ϕ, u) satisfying

$$\varepsilon u_t - \Delta u = -\phi_t, \tag{2.4}$$

$$(\partial_\nu u)|_\Gamma = 0, \tag{2.5}$$

$$u(0,\cdot) = \mu_0 + \varepsilon\mu_1 =: u_0 \tag{2.6}$$

and

$$\varepsilon\phi_t - \Delta\phi = u - N(\phi), \tag{2.7}$$

$$(\sigma_s\Delta_{||}\phi - \partial_\nu\phi - g_s\phi - \frac{1}{\Gamma_s}\phi_t)|_\Gamma = 0, \tag{2.8}$$

$$\phi(0,\cdot) = \psi_0 - \frac{h_s}{g_s} =: \phi_0, \tag{2.9}$$

where

$$\mu_0 := -\Delta\phi_0 + N(\phi_0),$$

and μ_1 is an element in $H^1(\Omega)$ specified later (see (3.1), (3.2)). The system of differential equations in (2.4), (2.7) corresponds, up to scaling, to the phase-field equation of Caginalp type, as proposed and studied in [3], (see also [7], [27]). However, our boundary conditions, especially (2.8), are much more complicated than those studied there.

We now discuss the following auxiliary linear problem

$$\varepsilon\phi_t - \Delta\phi = f, \tag{2.10}$$

$$(\sigma_s\Delta_{||}\phi - \partial_\nu\phi - g_s\phi - \frac{1}{\Gamma_s}\phi_t)|_\Gamma = 0, \tag{2.11}$$

$$\phi(0,\cdot) = \phi_0. \tag{2.12}$$

This linear problem (2.10)–(2.12) already reflects many of the inherent difficulties of our original problem such as the dynamic boundary condition as well as the term with the tangential Laplacian $\Delta_{||}$ on the boundary. The tools that we use to solve it are variational methods combined with variations of Duhamel's formula from semigroup theory, and variational evolution equations techniques.

As already mentioned, problem (1.13)–(1.15) has also been studied recently by Prüss [21]. He discusses it as an example of general abstract parabolic systems and proves the L_p-maximal regularity results. More precisely, what he proves for the case $p = 2$ is the following (see [21], p.2): problem (1.13)–(1.15) admits a unique solution $\phi \in H^1([0,T];L^2(\Omega)) \bigcap L^2([0,T];H^2(\Omega))$ such that $\phi|_\Gamma \in H^{5/4}([0,T];L^2(\Gamma)) \bigcap L^2([0,T];H^{5/2}(\Gamma))$ if and only if $f \in L^2([0,T]\times\Omega), \phi_0 \in H^{3/2}(\Omega)$, and the compatibility condition $\phi_0|_\Gamma \in H^{3/2}(\Gamma)$ is satisfied. As one can see from Theorems 2.2–2.4 in Section 2 that we use a quite different approach and obtain existence, uniqueness and a priori estimate results in different function space setting such as V, H spaces, as introduced below in Subsection 2.1, to fit them into our original nonlinear problem. Roughly speaking, in comparison with [21], p.2, in our results less requirements on compatibility of initial data and more assumptions on the right-hand term f are needed. Besides, the constants appearing in our a priori estimates do not depend on t, T. This is important for the study of our original nonlinear problem as well as for the further study of the asymptotic behavior of solution to the original nonlinear problem as time tends to infinity.

2.1 The homogenous problem (2.10)–(2.12): $f = 0$

We first look at the homogeneous problem, i.e., we assume in this subsection that

$$f = 0.$$

To solve the homogeneous problem

$$\varepsilon\phi_t - \Delta\phi = 0, \tag{2.13}$$

$$(\sigma_s\Delta_{||}\phi - \partial_\nu\phi - g_s\phi - \frac{1}{\Gamma_s}\phi_t)|_\Gamma = 0, \tag{2.14}$$

$$\phi(0,\cdot) = \phi_0, \tag{2.15}$$

we *formally* use the method of separation of variables. The *ansatz*

$$\phi(t,x) \equiv \Phi(x)h(t) \tag{2.16}$$

leads to

$$\varepsilon\Phi h_t - (\Delta\Phi)h = 0, \tag{2.17}$$

$$((\sigma_s\Delta_{||}\Phi - \partial_\nu\Phi - g_s\Phi)h - \frac{1}{\Gamma_s}\Phi h_t)|_\Gamma = 0. \tag{2.18}$$

Therefore,

$$\frac{\Delta\Phi}{\Phi} = \frac{\varepsilon h_t}{h} \equiv -\lambda \in I\!R,$$

and then

$$(\frac{\sigma_s\Delta_{||}\Phi - \partial_\nu\Phi - g_s\Phi}{\Phi})|_\Gamma = \frac{h_t}{\Gamma_s h} = \frac{-\lambda}{\varepsilon\Gamma_s}.$$

Thus h should satisfy

$$h(t) = h(0)\mathrm{e}^{-\frac{\lambda}{\varepsilon}t} \tag{2.19}$$

and the pair (Φ, λ) should satisfy the eigenvalue problem

$$-\Delta\Phi = \lambda\Phi \ \text{ in } \ \Omega, \tag{2.20}$$

$$(\sigma_s\Delta_{||}\Phi - \partial_\nu\Phi - g_s\Phi)|_\Gamma = \frac{-\lambda}{\varepsilon\Gamma_s}\Phi|_\Gamma. \tag{2.21}$$

Multiplying (2.20) by a smooth function Ψ and using (2.21) we obtain that

$$\int_{\Omega} \nabla\Phi\nabla\Psi dx + \sigma_s \int_{\Gamma} \nabla_{||}\Phi\nabla_{||}\Psi d\sigma + g_s \int_{\Gamma} \Phi\Psi d\sigma = \lambda \left(\int_{\Omega} \Phi\Psi dx + \frac{1}{\varepsilon\Gamma_s} \int_{\Gamma} \Phi\Psi d\sigma \right). \quad (2.22)$$

The observations above suggest to establish the following abstract framework (we refer, e.g., to [24] for the basic knowledge on spectral analysis on Hilbert spaces and linear evolution equations):
Let V be a Hilbert space with the inner product defined by the left-hand side of equation (2.22), i.e., for any $\Psi, \Phi \in V$,

$$(\Psi, \Phi)_V = \int_{\Omega} \nabla\Phi\nabla\Psi dx + \sigma_s \int_{\Gamma} \nabla_{||}\Phi\nabla_{||}\Psi d\sigma + g_s \int_{\Gamma} \Phi\Psi d\sigma.$$

In other words,

$$V := \text{completion of } C^1(\bar{\Omega}) \text{ under } \|\cdot\|_V.$$

Clearly [17, Thm 9.4], we have

$$H^{3/2}(\Omega) \subset V \subset H^1(\Omega).$$

Let H be another Hilbert space with the inner product defined by the right-hand side of equation (2.22), i.e., for any $\Psi, \Phi \in H$,

$$(\Psi, \Phi)_H = \left(\int_{\Omega} \Phi\Psi dx + \frac{1}{\varepsilon\Gamma_s} \int_{\Gamma} \Phi\Psi d\sigma \right).$$

In other words,

$$H := \text{completion of } C^0(\bar{\Omega}) \text{ under } \|\cdot\|_H.$$

Clearly, H is a subset of $L^2(\Omega)$,

$$H^{1/2+\alpha}(\Omega) \subset H, \qquad \text{for } \alpha > 0,$$

and the injection from V into H is continuous and compact. Then it is well-known (e.g., see [24]) that the bilinear form

$$a(u, v) := (u, v)_V \quad (2.23)$$

defines a strictly positive self-adjoint unbounded operator A from

$$D(A) = \{u \in V, Au \in H\} \quad (2.24)$$

into H, and for any $v \in V$,

$$(Au, v)_H = (u, v)_V. \quad (2.25)$$

Moreover, the standard spectral theory allows us to define the power A^s of A for $s \in I\!R$, and we infer that there exists a complete orthonormal family of H, $\{w_j\}, j \in I\!N$, with $w_j \in D(A^s)$ for $s \in I\!R$, and a sequence λ_j, $0 < \lambda_1 \leq \lambda_2, \cdots, \lambda_j \to \infty$ as $j \to \infty$ such that

$$Aw_j = \lambda_j w_j. \tag{2.26}$$

Remark 2.1 *We have from (2.23) that $V = D(A^{1/2})$. Furthermore, it follows from (2.25) and the definition of the inner product in V and H that for any $u \in D(A)$, $v \in V$,*

$$\int_\Omega Auvdx + \frac{1}{\varepsilon\Gamma_s}\int_\Gamma Auvd\sigma = \int_\Omega \nabla u \nabla v dx + \sigma_s \int_\Gamma \nabla_{||}u\nabla_{||}vd\sigma + g_s\int_\Gamma uvd\sigma. \tag{2.27}$$

Taking $v \in C_0^\infty(\Omega)$ in (2.27) and taking integration by parts yields that $Au = -\Delta u \in H$. Furthermore, taking integration by parts in (2.27) yields that for $v \in V$,

$$-\int_\Gamma \partial_\nu uvd\sigma - \frac{1}{\varepsilon\Gamma_s}\int_\Gamma \Delta uvd\sigma = \sigma_s\int_\Gamma \nabla_{||}u\nabla_{||}vd\sigma + g_s\int_\Gamma uvd\sigma. \tag{2.28}$$

Thus, the following boundary condition on Γ holds in $H^{-1}(\Gamma)$:

$$(\sigma_s\Delta_{||}u - \partial_\nu u - g_s u - \frac{1}{\varepsilon\Gamma_s}\Delta u)|_\Gamma = 0. \tag{2.29}$$

Similarly, it follows from (2.26) that w_j satisfies

$$-\Delta w_j = \lambda_j w_j \quad in \;\; \Omega, \tag{2.30}$$

and

$$(\sigma_s\Delta_{||}w_j - \partial_\nu w_j - g_s w_j - \frac{1}{\varepsilon\Gamma_s}\Delta w_j)|_\Gamma = 0. \tag{2.31}$$

Having established this framework, we now state and prove the following theorem on solvability of the problem (2.13)–(2.15):

Theorem 2.2

For any $\phi_0 \in H$, there exists a unique solution $\phi \in C^0([0,\infty), H)\bigcap C^\infty((0,\infty), C^\infty(\bar{\Omega}))$ in the following sense that, for any $t > 0$, ϕ satisfies the equation (2.13) and the boundary condition (2.14) in the classical sense; as t tends to zero, $\|\phi(t,\cdot) - \phi_0\|_H \to 0$. Moreover, the solutuion ϕ can be explicitly expressed by

$$\phi(t,x) = \sum_{k=1}^\infty a_k e^{\frac{-\lambda_k}{\varepsilon}t} w_k(x) \tag{2.32}$$

where

$$a_k = (\phi_0, w_k)_H. \tag{2.33}$$

The family of operators $\{S(t)\}_{t\geq 0}$ on H given by (2.32),

$$\phi = S(t)\phi_0,$$

define a C_0-semigroup on H.

Proof: We first prove that w_k belongs to $C^\infty(\bar{\Omega})$. Indeed, taking the inner product of (2.26) with any element $v \in V$ in H yields

$$(w_j, v)_V = \lambda_j (w_j, v)_H,$$

i.e.,

$$\int_\Omega \nabla w_j \nabla v dx + \sigma_s \int_\Gamma \nabla_{||} w_j \nabla_{||} v d\sigma + g_s \int_\Gamma w_j v d\sigma = \lambda_j \left(\int_\Omega w_j v dx + \frac{1}{\varepsilon \Gamma_s} \int_\Gamma w_j v d\sigma \right). \quad (2.34)$$

In other words, w_j is a weak solution in V for the boundary value problem (2.20), (2.21). Notice that the following boundary value problem

$$-\Delta u = f, \quad (2.35)$$

$$(\sigma_s \Delta_{||} u - \partial_\nu u - g_s u)|_\Gamma = g \quad (2.36)$$

is an elliptic boundary value problem in the sense of Hörmander [14], or in the sense of Agmon, Douglis & Nirenberg [1, 2], or in the sense of Peetre [20], as mentioned before. For any $f \in H^m(\Omega), g \in H^{m-1/2}(\Gamma), m \in I\!N$, the problem (2.35), (2.36) admits a unique solution $u \in H^{m+2}(\Omega)$. Moreover, the following estimate holds:

$$\|u\|_{H^{m+2}(\Omega)} \leq C(\|f\|_{H^m(\Omega)} + \|g\|_{H^{m-1/2}(\Gamma)}). \quad (2.37)$$

Therefore, it follows that w_j is a solution to the problem (2.35), (2.36) with $f = \lambda_j w_j$, $g = \lambda_j w_j$. Then a bootstrap argument yields that w_j belongs to $H^{m+2}(\Omega)$ for any $m \in I\!N$, thus belongs to $C^\infty(\bar{\Omega})$. Moreover, it follows from (2.37) that

$$\|w_j\|_{H^3(\Omega)} \leq C(\|\lambda_j w_j\|_{H^1(\Omega)} + \|\lambda_j w_j\|_{H^{1/2}(\Gamma)}) \leq C(\lambda_j + \lambda_j^2) \quad (2.38)$$

By induction, it easily follows that for any $m \in I\!N$,

$$\|w_j\|_{H^{2m+1}(\Omega)} \leq C_m(\lambda_j^m + \lambda_j^{m+1}) \quad (2.39)$$

with C_m being a positive constant depending on m. Then it follows that $\phi \in C^0([0,\infty), H)$ $\bigcap C^\infty((0,\infty), C^\infty(\bar{\Omega}))$ and, for $t > 0$, ϕ explicitly expressed by (2.32) satisfies (2.13), (2.14), and the initial condition (2.15) is satisfied in the sense that as t tends to zero, $\|\phi(t,\cdot)-\phi_0\|_H \to 0$. The uniqueness can be easily seen from the following identity which is derived by multiplying (2.13) by ϕ, then integrating with respect to x, and using the boundary condition (2.14):

$$\frac{\varepsilon}{2}\frac{d}{dt}\|\phi\|^2 + \frac{1}{2\Gamma_s}\frac{d}{dt}\int_\Gamma \phi^2 d\sigma + \|\nabla\phi\|^2 + \int_\Gamma (\sigma_s|\nabla_{||}\phi|^2 + g_s\phi^2)d\sigma = 0. \tag{2.40}$$

It is a routine procedure to verify that the operators $S(t)$ from ϕ_0 to ϕ defined by (2.32) form a C_0-semigroup on H, and we can omit the details here. The proof of the theorem is completed. □

2.2 The problem (2.10)–(2.12) with general f

Now we return to the nonhomogeneous auxiliary problem (2.10)–(2.12). Since ε is a fixed positive constant, without loss of generality, we can assume in this section that $\varepsilon = 1$.

Notice that, because the boundary condition (2.11) involves ϕ_t, we cannot directly apply the Duhamel principle. For the time being, we assume that

$$\phi_0 \in V, \quad f \in H^2([0,T];L^2)$$

with some arbitrary, but fixed $T > 0$. We now introduce z as a solution to

$$\left.\begin{array}{rcll} -\Delta z &=& f, & \\ \mathcal{B}z := \sigma_s\Delta_{||}z - \partial_\nu z - g_s z &=& 0 & \text{on } \Gamma. \end{array}\right\} \tag{2.41}$$

It follows from the results on general elliptic equations as given in Hörmander [14], in particular a result of Višik [25] that applies to (2.41), cp. [14, page 264], that we may conclude that the problem (2.41) admits a unique solution z with

$$z \in H^2([0,T];H^2).$$

Now define

$$w := \phi - z.$$

Then w satisfies

$$\begin{array}{rcl} w_t - \Delta w &=& \phi_t - \Delta\phi - z_t + \Delta z \\ &=& -z_t =: \hat{f}, \end{array}$$

and on Γ we have

$$\begin{aligned}\mathcal{B}w - \frac{1}{\Gamma_s}\Delta w &= \mathcal{B}\phi - \frac{1}{\Gamma_s}\Delta\phi - \mathcal{B}z + \frac{1}{\Gamma_s}\Delta z \\ &= \mathcal{B}\phi - \frac{1}{\Gamma_s}(\phi_t - f) - \frac{1}{\Gamma_s}f \\ &= \mathcal{B}\phi - \frac{1}{\Gamma_s}\phi_t \\ &= 0.\end{aligned}$$

That is, w satisfies

$$\left.\begin{aligned} w_t - \Delta w &= \hat{f} \in H^1([0,T];V), \\ (\sigma_s\Delta_{||}w - \partial_\nu w - g_s w - \tfrac{1}{\Gamma_s}\Delta w)|_\Gamma &= 0, \\ w(0,\cdot) &= \phi_0 - z(0,\cdot) \in V = D(A^{\frac{1}{2}}). \end{aligned}\right\} \tag{2.42}$$

But this last problem can be solved since the corresponding homogeneous problem ($\hat{f} = 0$) is equivalent to the problem (2.13)–(2.15), not involving time derivatives on the boundary in this formulation. In other words, it can be written as an initial value problem for an abstract evolution equation:

$$\frac{dw}{dt} + Aw = \hat{f}, \tag{2.43}$$

$$w(0) = \phi_0 - z(0). \tag{2.44}$$

Hence the usual Duhamel principle applies to (2.42), and we can solve for w according to Subsection 2.1. Then the desired solution to (2.10)–(2.12) is of course given by

$$\phi := w + z.$$

A representation of the solution describing a modified Duhamel's principle can be given as follows. Let $\{S(t)\}_{t\geq 0}$ be the semigroup defined in Subsection 2.1, and let $\Delta_{\mathcal{B}}^{-1}f(t,\cdot)$ denote the solution $z(t,\cdot)$ to (2.41). Then the solution ϕ to (2.10)–(2.12) is given by

$$\begin{aligned}\phi(t,x) &= S(t)\left(\phi_0 - \Delta_{\mathcal{B}}^{-1}f(0,\cdot)\right) - \int_0^t S(t-\tau)\partial_\tau\Delta_{\mathcal{B}}^{-1}(f(\tau,\cdot))d\tau \\ &\quad + \Delta_{\mathcal{B}}^{-1}f(t,\cdot).\end{aligned} \tag{2.45}$$

Now we have the following result on solvability of the nonhomogeneous problem (2.10)–(2.12).

Theorem 2.3 *Let*

$$\phi_0 \in V, \quad f \in C^0([0,T];H^1), \quad f_t \in H^1([0,T];L^2).$$

Then there is a unique solution $\phi \in C^0([0,T];V)\bigcap C^1((0,T];V)\bigcap C^0((0,T];H^3)$ to the problem (2.10)–(2.12) such that $\phi = w + z$ with $z \in C^0([0,T];H^3), z_t \in H^1([0,T];H^2)$ being the unique solution to the problem (2.41) and w being the unique solution to the problem (2.42) with $w \in C^0([0,T];V)\bigcap C^1((0,T];V)\bigcap C^0((0,T];D(A^{\frac{3}{2}}))$.

Proof: It is clear from the previous constructions of z and w that the sum $\phi = w + z$ is a solution to the problem (2.10)–(2.12). The uniqueness of the solutions in the indicated class can be easily seen from the following identity which can be derived by multiplying (2.10) by ϕ, and integrating over Ω:

$$\frac{d}{dt}\left(\frac{1}{2}\|\phi\|^2 + \frac{1}{2\Gamma_s} + \int\limits_\Gamma \phi^2 d\sigma\right) + \|\nabla\phi\|^2 + \int\limits_\Gamma (\sigma_s|\nabla_{||}\phi|^2 + g_s\phi^2)d\sigma = \int\limits_\Omega f\phi dx.$$

The proof is completed. □

The following regularity result will be needed in the next section:

Theorem 2.4 *Let*

$$\phi_0 \in H^3,$$

$$f \in C^0([0,T];H^1), \quad f_t \in L^2([0,T];L^2),$$

and

$$w(0) = w_0 := \phi_0 - z(0) \in D(A^{3/2}) \tag{2.46}$$

which implies that

$$\phi_1 := \Delta\phi_0 + f(0) \in V. \tag{2.47}$$

Then the problem (2.10)–(2.12) admits a unique solution ϕ such that $\phi \in C^0([0,T];H^3), \phi_t \in C^0([0,T];V)\bigcap L^2([0,T];H^2), \phi_{tt} \in L^2([0,T];H)$. Moreover, for $t \in [0,T]$ the following estimates hold:

$$\|\phi(t)\|_V^2 + \int\limits_0^t \|\phi_t\|_H^2 d\tau \le C\left(\|\phi_0\|_V^2 + \int\limits_0^t \|f\|^2 d\tau\right), \tag{2.48}$$

$$\|\phi_t(t)\|_V^2 + \int\limits_0^t \|\phi_{tt}\|_H^2 d\tau \le C\left(\|\phi_1\|_V^2 + \int\limits_0^t \|f_t\|^2 d\tau\right), \tag{2.49}$$

$$\|\phi(t)\|_{H^3}^2 \le C\left(\|\phi_1\|_V^2 + \|f(t)\|_{H^1}^2 + \int\limits_0^t \|f_t\|^2 d\tau\right), \tag{2.50}$$

where C is a positive constant being independent of the solution ϕ, t, ϕ_0 and f.

Proof: We use a density argument. For the time being, we assume that $f \in C^0([0,T]; H^1)$, $f_t \in H^1([0,T]; L^2)$. As proved previously, the unique solution ϕ can be decomposed into $\phi = w + z$. By the elliptic theory previously mentioned, we now have

$$z \in C^0([0,T]; H^3),\ z_t \in H^1([0,T]; H^2).$$

Therefore,

$$\hat{f} = -z_t \in C^0([0,T]; V), \quad \hat{f}_t \in L^2([0,T]; V).$$

Since w is the unique solution to the initial value problem for the abstract evolution equation (2.43), (2.44), and now $w(0) = \phi_0 - z(0) \in D(A^{3/2})$, i.e., $\Delta\phi_0 + f(0) \in V$, and we infer from semigroup theory that $w \in C^0([0,T]; D(A^{3/2})), w_t \in C^0([0,T]; V)$, $w_t \in L^2([0,T]; D(A))$, $w_{tt} \in L^2([0,T]; H)$. On the other hand, it follows from (2.10), (2.11) that

$$-\Delta\phi = f - \phi_t \in C^0([0,T]; H^1), \tag{2.51}$$

and on the boundary Γ

$$\sigma_s \Delta_{||}\phi - \partial_\nu \phi - g_s \phi = \frac{1}{\Gamma_s}\phi_t \in H^1(\Gamma). \tag{2.52}$$

Therefore, by the elliptic theory, we have

$$\|\phi(t)\|_{H^3} \leq C\left(\|f(t) - \phi_t(t)\|_{H^1} + \|\phi_t(t)\|_{H^{1/2}(\Gamma)}\right), \tag{2.53}$$

hence

$$\|\phi(t)\|_{H^3} \leq C(\|f(t)\|_{H^1} + \|\phi_t(t)\|_{H^1}). \tag{2.54}$$

Thus, the solution ϕ belongs to the desired function class as stated. To obtain the estimate (2.48), we multiply the equation (2.10) by ϕ and ϕ_t, respectively, then integrate over Ω and use the boundary condition (2.11) to get

$$\frac{1}{2}\frac{d}{dt}\left(\|\phi(t)\|^2 + \frac{1}{\Gamma_s}\int_\Gamma \phi^2 d\sigma\right) + \|\nabla\phi(t)\|^2 + \int_\Gamma (\sigma_s|\nabla_{||}\phi|^2 + g_s\phi^2)d\sigma = \int_\Omega f\phi dx,$$

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Gamma (\sigma_s|\nabla_{||}\phi|^2 + g_s\phi^2)d\sigma + \|\nabla\phi(t)\|^2\right) + \|\phi_t\|^2 + \frac{1}{\Gamma_s}\int \phi_t^2 d\sigma = \int f\phi_t dx$$

Thus we easily deduce the estimate (2.48) by integrating with respect to t and using the Hölder inequality.

To obtain the estimate (2.49), we use the density argument again. For the time being, we assume that the solution ϕ is more regular with more regular f and initial data. Similarly, we differentiate the equation (2.10) with respect to t, then multiply it by ϕ_{tt}, integrate with respect to x and t, and use the boundary condition (2.11) to get

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Gamma (\sigma_s|\nabla_{||}\phi_t|^2 + g_s\phi_t^2)d\sigma + \|\nabla\phi_t(t)\|^2\right) + \|\phi_{tt}\|^2 + \frac{1}{\Gamma_s}\int \phi_{tt}^2 d\sigma = \int f_t\phi_{tt}dx. \quad (2.55)$$

We then easily deduce that

$$\|\phi_t(t)\|_V^2 + \int_0^t \|\phi_{tt}\|_H^2 d\tau \leq C\left(\|\phi_1\|_V^2 + \int_0^t \|f_t\|^2 d\tau\right). \quad (2.56)$$

Thus the estimate (2.49) follows. For the initial data and f as regular as stated in the present theorem, we use a sequence of more regular initial data and f to approximate, then pass to the limit. Combining (2.53) with (2.49) yields (2.50). Notice that in the estimates (2.48)–(2.50), it does not involve f_{tt}. Therefore, we can conclude the proof by the density argument. □

Lemma 2.5 *Suppose that the following compatibility condition*

$$(\sigma_s\Delta_{||}\phi_0 - \partial_\nu\phi_0 - g_s\phi_0 - \frac{1}{\varepsilon\Gamma_s}\Delta\phi_0 - \frac{1}{\varepsilon\Gamma_s}f(0,\cdot))|_\Gamma = 0 \quad (2.57)$$

is satisfied and

$$\Delta\phi_0 + f(0,\cdot) \in V, \quad (2.58)$$

then the condition (2.46) in the statement of Theorem 2.4 holds.

Proof: It follows from (2.44) and (2.41) that

$$\sigma_s\Delta_{||}w(0) - \partial_\nu w(0) - g_s w(0) - \frac{1}{\varepsilon\Gamma_s}\Delta w(0))|_\Gamma = 0, \quad (2.59)$$

i.e., $w(0) \in D(A)$, and $Aw(0) = -\Delta w(0) = -\Delta\phi_0 - f(0,\cdot) \in V$. Thus the proof is completed. □

3 Local existence for (P_ε) for $n = 2, 3$

Having solved the auxiliary problem (2.10)–(2.12), we are now able to solve the approximate problem (P_ε) given by (2.4)–(2.6), (2.7)–(2.9) by using the contraction mapping principle in an appropriate Banach space.

From now on we always assume that the following conditions (3.1), (3.2) on the initial data $\phi_0 = \psi_0 - \frac{h_s}{g_s}$, $u_0 = \mu_0 + \varepsilon\mu_1 = N(\phi_0) - \Delta\phi_0 + \varepsilon\mu_1$ are satisfied:

$$\phi_0 \in H^3, \;\; \mathcal{B}\phi_0 \in H^1(\Gamma), \;\; \mu_1 \in V, \tag{3.1}$$

which implies that $u_0 \in H^1$,

$$\Big(\mathcal{B}\phi_0 - \frac{1}{\varepsilon\Gamma_s}(\Delta\phi_0 + u_0 - N(\phi_0))\Big)\,|_\Gamma = \Big(\mathcal{B}\phi_0 - \frac{1}{\Gamma_s}\mu_1\Big)\,|_\Gamma = 0. \tag{3.2}$$

It turns out that

$$\phi_1 := \frac{\Delta\phi_0 + (u_0 - N(\phi_0))}{\varepsilon} = \mu_1 \in V. \tag{3.3}$$

Then we we have the following result on local existence and uniqueness.

Theorem 3.1 *Suppose that the initial data ϕ_0, u_0 satisfy the conditions above. Then there is a positive constant δ, which may depend on the initial data and on ε such that the problem (P_ε) admits a unique local solution ϕ, u such that $\phi \in C^0([0,\delta]; H^3), \phi_t \in C^0([0,\delta]; V), \phi_t \in L^2([0,\delta]; H^2), \phi_{tt} \in L^2([0,\delta]; H)$, $u \in C^0([0,\delta]; H^1) \cap L^2([0,\delta]; H^2)$, $u_t \in L^2([0,\delta]; L^2)$.*

Proof: Let

$$Y_1 := C^0([0,\delta], H^3) \bigcap C^1([0,\delta], V) \bigcap H^2([0,\delta], L^2),$$

$$Y_2 := C^0([0,\delta], H^1) \bigcap H^1([0,\delta]; L^2) \bigcap L^2([0,\delta]; H^2),$$

and

$$X_\delta = Y_1 \times Y_2.$$

For $(\chi, v) \in X_\delta$, we define (ϕ, u) to be the solution to

$$\left.\begin{array}{rcl} \varepsilon u_t - \Delta u &=& -\chi_t =: g, \\ (\partial_\nu u)|_\Gamma &=& 0, \\ u(0,\cdot) &=& u_0, \end{array}\right\} \tag{3.4}$$

and

$$\left.\begin{array}{rcl} \varepsilon\phi_t - \Delta\phi &=& v - N(\chi) =: f, \\ (\sigma_s\Delta_{||}\phi - \partial_\nu\phi - g_s\phi - \frac{1}{\Gamma_s}\phi_t)|_\Gamma &=& 0, \\ \phi(0,\cdot) &=& \phi_0. \end{array}\right\} \tag{3.5}$$

For $(\chi, v) \in X_\delta$, thus $g \in H^1([0,\delta]; L^2)$, by the well-known results for the heat equation, we have a unique solution $u \in Y_2, u \in C^0((0,\delta]; H^3), u_t \in C^0((0,T]; H^1)$ to the problem

(3.4). On the other hand, by the Sobolev imbedding theorem the right-hand side f in (3.5) satisfies

$$f \in C^0([0,\delta]; H^1), \quad f_t \in L^2([0,\delta]; L^2).$$

Moreover, it is easy to see from (3.1)–(3.3) that the conditions on initial data in Theorem 2.4 are satisfied. Therefore, by Theorem 2.4, the problem (3.5) admits a unique solution $\phi \in Y_1$. Thus, the mapping

$$S \,:\, (\chi, v) \;\mapsto\; (\phi, u)$$

is well defined as a mapping from X_δ into itself. Let

$$\phi_1 := \frac{1}{\varepsilon}(\Delta\phi_0 - N(\phi_0) + u_0)$$

and

$$\begin{aligned} Z_\delta := \Bigg\{ &(\phi, u) \in X_\delta \mid \max_{0\le t\le\delta} \|u(t)\|^2_{H^1} + \int_0^\delta \|u_t(\tau)\|^2 + \|u(\tau)\|^2_{H^2} d\tau \le M_1, \\ &\max_{0\le t\le\delta} \|\phi(t)\|^2_V \le M_2,\ \max_{0\le t\le\delta}(\|\phi_t(t)\|^2_V + \|\phi(t)\|^2_{H^3}) + \int_0^\delta \|\phi_{tt}(\tau)\|^2 d\tau \le 2M_3, \\ &\phi(0,\cdot) = \phi_0,\ \phi_t(0,\cdot) = \phi_1,\ u(0,\cdot) = u_0 \Big\}, \end{aligned}$$

where the positive constants M_1, M_2, M_3 will be determined below. We shall prove that S maps Z_δ into itself as a contraction in a suitable norm, if $0 < \delta \le 1$ is sufficiently small. For this purpose let $(\phi, u) = S((\chi, v))$ again, now for $(\chi, v) \in Z_\delta$. Then we conclude as usual from (3.4) that for $0 \le t \le \delta$,

$$\begin{aligned} \|u(t)\|^2_{H^1} + \int_0^t \|u_t(\tau)\|^2 + \|u(\tau)\|^2_{H^2} d\tau &\le C_1(\|u_0\|^2_{H^1} + \int_0^t \|g(\tau)\|^2 d\tau) \\ &\le C_1(\|u_0\|^2_{H^1} + \delta M_3), \end{aligned} \tag{3.6}$$

where C_1 (and in the sequel $C_2, \ldots$) denotes a positive constant depending at most on ε. We choose

$$M_1 := 2C_1 \|u_0\|^2_{H^1} \tag{3.7}$$

and δ such that

$$\delta M_3 \le \|u_0\|^2_{H^1}, \tag{3.8}$$

then we get from (3.7)

$$\|u(t)\|^2_{H^1} + \int_0^t \|u_t(\tau)\|^2 + \|u(\tau)\|^2_{H^2} d\tau \le M_1. \tag{3.9}$$

The estimate (2.48) from Theorem 2.4 yields that for $0 \le t \le \delta$,

$$\|\phi(t)\|_V^2 \le C_2 \left(\|\phi_0\|_V^2 + \delta M_1 + \delta(1 + M_2 + M_2^3)\right). \tag{3.10}$$

Choosing

$$M_2 := 2C_2\|\phi_0\|_V^2 \tag{3.11}$$

and δ such that

$$2C_2\delta(M_1 + 1 + M_2 + M_2^3) \le M_2, \tag{3.12}$$

we conclude from (3.10)

$$\|\phi(t)\|_V^2 \le M_2. \tag{3.13}$$

The estimate (2.49) from Theorem 2.4 yields that for $0 \le t \le \delta$,

$$\|\phi_t(t)\|_V^2 + \int_0^t \|\phi_{tt}(\tau)\|_H^2 \le C_3 \left(\|\phi_1\|_V^2 + M_1 + \delta(M_3 + M_3^3)\right). \tag{3.14}$$

Choosing

$$M_3 \le 2C_3(\|\phi_1\|_V^2 + M_1) \tag{3.15}$$

and δ such that

$$2C_3\delta(M_3 + M_3^3) \le M_3, \tag{3.16}$$

we obtain from (3.14)

$$\|\phi_t(t)\|_V^2 + \int_0^t \|\phi_{tt}(\tau)\|_H^2 \le M_3. \tag{3.17}$$

We use (2.50) from Theorem 2.4 to estimate $\|\phi(t)\|_{H^3}^2$:

$$\|\phi(t)\|_{H^3}^2 \le C_4(\|\phi_1\|_V^2 + M_1 + \delta(M_3 + M_3^3) + 1 + M_2 + M_2^3 + \|\nabla\chi^3\|^2). \tag{3.18}$$

The last term can be estimated as follows:

$$\begin{aligned} \|\nabla\chi^3\|^2 &= \|\nabla(\phi_0^3 + 3\int_0^t (\chi^2\chi_t)(\tau)d\tau)\|^2 \\ &\le C_5(\|\phi_0\|_{H^2}^6 + \delta^2 M_3^3). \end{aligned} \tag{3.19}$$

Combining (3.18) and (3.19) we obtain

$$\|\phi(t)\|_{H^3}^2 \le C_6(\|\phi_1\|_V^2 + M_1 + \delta(M_3 + M_3^3) + 1 + M_2 + M_2^3 + \|\phi_0\|_{H^2}^6 + \delta^2 M_3^3). \tag{3.20}$$

Choosing

$$M_3 \ge 2C_6(\|\phi_1\|_V^2 + M_1 + 1 + M_2 + M_2^3 + \|\phi_0\|_{H^2}^6). \tag{3.21}$$

and δ such that

$$2C_6(\delta(M_3+M_3^3)+\delta^2 M_3^3) \le M_3 \tag{3.22}$$

we obtain from (3.20)

$$\|\phi(t)\|_{H^3}^2 \le M_3. \tag{3.23}$$

Therefore, choosing M_1, M_2, M_3 and δ this way, we conclude from (3.9), (3.13), (3.17) and (3.23) that $(u,\phi) \in Z_\delta$, that is, S maps Z_δ into itself.

We notice that M_1, M_2, M_3 and δ depend at most on $(\|u_0\|_{H^1}, \|\phi_1\|_V, \|\phi_0\|_{H^3})$ and ε.

Finally, we show that $S : Z_\delta \longrightarrow Z_\delta$ is a contraction mapping with respect to the following norm

$$\|(\phi,u)\|_{Z_\delta} := \Bigg(\max_{0\le t\le\delta} \|u(t)\|_{H^1}^2 + \int_0^\delta \|u_t(\tau)\|^2 + \|u(\tau)\|_{H^2}^2 d\tau +$$
$$\max_{0\le t\le\delta} \|\phi(t)\|_V^2 + K_1(\max_{0\le t\le\delta}(\|\phi_t(t)\|_V^2 + \int_0^\delta \|\phi_{tt}(\tau)\|^2 d\tau) + K_2\|\phi(t)\|_{H^3}^2 \Bigg),$$

where the positive constants K_1, K_2 will be defined below. Let $(\chi_j, v_j) \in Z_\delta$, $j=1,2$, and $(\phi_j,u_j) := S((\chi_j,v_j)$. Then $\phi := \phi_1-\phi_2$, $u := u_1-u_2$, $\chi := \chi_1-\chi_2$ and $v := v_1-v_2$ satisfy

$$\left.\begin{array}{rcl} \varepsilon u_t - \Delta u &=& -\chi_t =: g, \\ (\partial_\nu u)|_\Gamma &=& 0, \\ u(0,\cdot) &=& 0, \end{array}\right\} \tag{3.24}$$

and

$$\left.\begin{array}{rcl} \varepsilon\phi_t - \Delta\phi &=& v + N(\chi_2) - N(\chi_1) =: f, \\ (\sigma_s\Delta_{||}\phi - \partial_\nu\phi - g_s\phi - \frac{1}{\Gamma_s}\phi_t)|_\Gamma &=& 0, \\ \phi(0,\cdot) &=& 0. \end{array}\right\} \tag{3.25}$$

As in (3.6) we now obtain

$$\|u(t)\|_{H^1}^2 + \int_0^t \|u_t(\tau)\|^2 + \|u(\tau)\|_{H^2}^2 d\tau \le C_7\delta\|(\chi,v)\|_{Z_\delta}^2 \le \frac{1}{24}\|(\chi,v)\|_{Z_\delta}^2, \tag{3.26}$$

if

$$\delta \le \frac{1}{24C_7}. \tag{3.27}$$

In analogy to (3.10) we get

$$\|\phi(t)\|_V^2 \le C_8\delta(2+M_2^2)\|(\chi,v)\|_{Z_\delta}^2 \le \frac{1}{24}\|(\chi,v)\|_{Z_\delta}^2, \tag{3.28}$$

if

$$\delta \leq \frac{1}{24C_8(2+M_2^2)}. \tag{3.29}$$

As in (3.14) we now conclude

$$\|\phi_t(t)\|_V^2 + \int_0^t \|\phi_{tt}(\tau)\|_H^2 \leq C_9 \left(\|(\chi, v)\|_{Z_\delta}^2 + \delta(1+M_3^2)\|(\chi, v)\|_{Z_\delta}^2 \right). \tag{3.30}$$

With

$$K_1 := \frac{1}{24C_9}$$

we obtain from (3.30)

$$K_1 \left(\|\phi_t(t)\|_V^2 + \int_0^t \|\phi_{tt}(\tau)\|_H^2 \right) \leq \frac{2}{24}\|(\chi, v)\|_{Z_\delta}^2 \tag{3.31}$$

if

$$\delta \leq \frac{1}{24C_9(1+M_3^2)}. \tag{3.32}$$

The estimate (3.20) now carries over to

$$\|\phi(t)\|_{H^3}^2 \leq C_{10}(1+M_2^2)(\|(\chi, v)\|_{Z_\delta}^2 + \delta(1+M_3^2+\delta M_3^2)\|(\chi, v)\|_{Z_\delta}^2. \tag{3.33}$$

Defining

$$K_2 := \frac{1}{24C_{10}(1+M_2^2)}$$

we get from (3.33)

$$K_2\|\phi(t)\|_{H^3}^2 \leq \frac{2}{24}\|(\chi, v)\|_{Z_\delta}^2 \tag{3.34}$$

if

$$\delta \leq \frac{1}{24C_{10}(1+2M_3^2)}. \tag{3.35}$$

Combining (3.26), (3.28) and (3.34), we conclude

$$\|(\phi, u)\|_{Z_\delta} \leq \frac{1}{2}\|(\chi, v)\|_{Z_\delta}.$$

Thus S is a contraction mapping and the proof of the local existence and uniqueness theorem is completed. □

Remark 3.2 *As the proof demonstrated, the parameter δ in Theorem 3.1 depends at most on $(\|u_0\|_{H^1}, \|\phi_1\|_V, \|\phi_0\|_{H^3})$ and ε.*

4 Global existence for (P_ε) for $n = 2, 3$

Denoting by $(\phi^\varepsilon, u^\varepsilon)$ the local solution of (P_ε) given by Theorem 3.1 we shall now prove further *a priori* estimates to justify the global existence. To do so, we reverse the transformation (2.1) and observe that (ϕ, u) then satisfies (dropping the index ε in most places)

$$\left.\begin{aligned} \varepsilon u_t - \Delta u &= -\phi_t, \\ (\partial_\nu u)|_\Gamma &= 0, \\ u(0,\cdot) &= u_0, \end{aligned}\right\} \tag{4.1}$$

and

$$\left.\begin{aligned} \varepsilon\phi_t - \Delta\phi - \phi + \phi^3 &= u, \\ (\sigma_s\Delta_{||}\phi - \partial_\nu\phi)|_\Gamma &= (\tfrac{1}{\Gamma_s}\phi_t - h_s + g_s\phi)|_\Gamma, \\ \phi(0,\cdot) &= \psi_0. \end{aligned}\right\} \tag{4.2}$$

Theorem 4.1 *Suppose that the initial data ϕ_0, u_0 satisfy the conditions (3.1)–(3.3) in Section 3. Then the problem (4.1), (4.2) admits a unique global solution such that for any $T > 0$, $\phi \in C^0([0,T]; H^3), \phi_t \in C([0,T]; V), \phi_t \in L^2([0,T]; H^2), \phi_{tt} \in L^2([0,T]; H), u \in C^0([0,T]; H^1) \bigcap L^2([0,T]; H^2), u_t \in L^2([0,T]; L^2)$.*

Proof: By the local existence and uniqueness result in the previous section it suffices to obtain uniform a priori estimates on $\|\phi(t)\|_{H^3}, \|u(t)\|_{H^1}, \|\phi_t(t)\|_V$.

Multiplying the equations (4.1) and (4.2) by u and ϕ_t, respectively, and integrating with respect to x, we obtain that

$$\frac{d}{dt}(\frac{1}{2}\varepsilon\|u\|^2 + F[\phi]) + \|\nabla u\|^2 + \varepsilon\|\phi_t\|^2 + \frac{1}{\Gamma_s}\int_\Gamma \phi_t^2 d\sigma = 0,$$

where $F[\phi]$ denotes the total energy functional defined in (1.10), implying after integration with respect to t and using the elementary estimate $|a| \le a^2 + 1/4$ for $a \in I\!R$,

$$\left.\begin{aligned} \varepsilon\|u\|^2 \le C, \quad \|\phi\|_V \le C, \quad \int_0^t \|\nabla u\|^2 d\tau \le C, \\ \varepsilon\int_0^t \|\phi_t\|^2 d\tau \le C, \quad \int_0^t\int_\Gamma \phi_t^2 d\sigma d\tau \le C, \end{aligned}\right\} \tag{4.3}$$

where C denotes — also in the sequel in different places with possibly different values — a positive constant that depends at most on ϕ_0, u_0, ϕ_1 but is independent of t, δ and ε.

Differentiating (4.2) with respect to t, multiplying the result by ϕ_t, integrating over Ω, and using (4.1), we get

$$\int_\Omega (-\varepsilon\phi_{tt} + \Delta\phi_t - 3\phi^2\phi_t + \phi_t)\phi_t dx = -\int_\Omega u_t\phi_t dx = \varepsilon\int_\Omega u_t^2 dx - \int_\Omega \Delta u u_t dx$$

which implies that

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u\|^2+\varepsilon\|\phi_t\|^2+\frac{1}{\Gamma_s}\int\limits_\Gamma\phi_t^2d\sigma)+\|\nabla\phi_t\|^2+\varepsilon\|u_t\|^2+3\int\limits_\Omega\phi^2\phi_t^2dx$$

$$+\int\limits_\Gamma(|\nabla_{||}\phi_t|^2+g_s\phi_t^2)d\sigma=\|\phi_t\|^2.$$

Multiplying (4.1) by ϕ_t and integrating by parts leads to

$$\|\phi_t\|^2=-\varepsilon\int\limits_\Omega u_t\phi_tdx-\int\limits_\Omega\nabla u\nabla\phi_tdx\le\varepsilon\|u_t\|\|\phi_t\|+\|\nabla u\|\|\nabla\phi_t\|.$$

By the inequality $ab\le\frac{a^2}{2}+\frac{b^2}{2}$, we deduce that

$$\|\phi_t\|^2\le\varepsilon^2\|u_t\|^2+2\|\nabla u\|\|\nabla\phi_t\|.$$

Hence, using (4.3),

$$\begin{aligned}\int\limits_0^t\|\phi_t\|^2d\tau &\le \varepsilon^2\int\limits_0^t\|u_t\|^2d\tau+2(\int\limits_0^t\|\nabla u\|^2d\tau)^{1/2}(\int\limits_0^t\|\nabla\phi_t\|^2d\tau)^{1/2}\\ &\le \varepsilon^2\int\limits_0^t\|u_t\|^2d\tau+C(\int\limits_0^t\|\nabla\phi_t\|^2d\tau)^{1/2}.\end{aligned}$$

Thus we deduce, observing $0<\varepsilon\le1$,

$$\left.\begin{array}{c}\|\nabla u\|\le C,\quad \varepsilon\|\phi_t\|^2\le C,\quad \int\limits_\Gamma\phi_t^2d\sigma\le C,\quad \int\limits_0^t\|\nabla\phi_t\|^2d\tau\le C,\\ \varepsilon\int\limits_0^t\|u_t\|^2d\tau\le C,\quad \int\limits_0^t\int\limits_\Gamma(|\nabla_{||}\phi_t|^2+\phi_t^2)d\sigma d\tau\le C,\quad \int\limits_0^t\|\phi_t\|^2d\tau\le C.\end{array}\right\}\tag{4.4}$$

It follows from equation (4.1) that

$$\int\limits_0^t\|\Delta u\|^2d\tau\le C.\tag{4.5}$$

As (2.55) in the proof in Theorem 2.4, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\int\limits_\Gamma(\sigma_s|\nabla_{||}\phi_t|^2+g_s\phi_t^2)d\sigma+\|\nabla\phi_t(t)\|^2\right)+\varepsilon\|\phi_{tt}\|^2+\frac{1}{\Gamma_s}\int\limits_\Gamma\phi_{tt}^2d\sigma\\ &\qquad=\int\limits_\Omega(u_t-3\phi^2\phi_t+\phi_t)\phi_{tt}dx,\end{aligned}\tag{4.6}$$

hence

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\int\limits_{\Gamma}(\sigma_s|\nabla_{||}\phi_t|^2+g_s\phi_t^2)d\sigma+\|\nabla\phi_t(t)\|^2-\|\phi_t\|^2+\int\limits_{\Omega}\frac{3}{2}\phi^2\phi_t^2dx\right)\\
&\qquad+\varepsilon\|\phi_{tt}\|^2+\frac{1}{\Gamma_s}\int\phi_{tt}^2d\sigma\\
=&\qquad\int\limits_{\Omega}(u_t\phi_{tt}+3\phi\phi_t^3)dx. \qquad (4.7)
\end{aligned}
$$

Since $n\le 3$, by the Young inequality and Sobolev's imbedding theorem, we have

$$
|3\int\limits_{\Omega}\phi\phi_t^3dx|\le C\|\phi\|_{L^6}\|\phi_t\|^3_{L^{\frac{18}{5}}}\le C\|\phi_t\|^3_{L^{\frac{18}{5}}}.
$$

By the Gagliardo–Nirenberg interpolation inequality we get

$$
\|\phi_t\|^3_{L^{\frac{18}{5}}}\le C_1\|\nabla\phi_t\|^{\frac{2n}{3}}\|\phi_t\|^{\frac{9-2n}{3}}+C_2\|\phi_t\|^3\le C\|\phi_t\|^2_{H^1}\|\phi_t\|.
$$

Therefore it follows from (4.4) that

$$
\int\limits_0^t\|\phi_t\|^3_{L^{\frac{18}{5}}}\le C\max_{0\le\tau\le t}\|\phi_t(\tau)\|\le\eta\max_{0\le\tau\le t}\|\phi_t(\tau)\|^2+C_\eta
$$

with η being a small positive constant. Integrating (4.7) with respect to t, then taking the maximum with respect to time over $[0,t]$, and choosing η small enough yields

$$
\begin{aligned}
&\int\limits_{\Gamma}(\sigma_s|\nabla_{||}\phi_t|^2+g_s\phi_t^2)d\sigma+\|\nabla\phi_t(t)\|^2+\int\limits_0^t\varepsilon\|\phi_{tt}\|^2d\tau+\int\limits_0^t\frac{2}{\Gamma_s}\int\limits_{\Gamma}\phi_{tt}^2d\sigma d\tau\\
&\qquad\le\frac{C}{\varepsilon}+\frac{C}{\varepsilon}\int\limits_0^t\|u_t\|^2d\tau\le\frac{C}{\varepsilon}+\frac{C}{\varepsilon^2}. \qquad (4.8)
\end{aligned}
$$

Thus,

$$
\varepsilon\|\phi_t\|_V\le C. \qquad (4.9)
$$

Integrating (4.2) over Ω yields

$$
-\varepsilon\int\limits_{\Omega}\phi_tdx+\int\limits_{\Gamma}\partial_\nu\phi d\sigma-\int\limits_{\Omega}(\phi^3-\phi)dx=\int\limits_{\Omega}udx.
$$

¿From the boundary conditions in (4.2) we conclude that

$$
\begin{aligned}
\int\limits_{\Gamma}\partial_\nu\phi d\sigma &= \sigma_s\int\limits_{\Gamma}\Delta_{||}\phi d\sigma-\frac{1}{\Gamma_s}\int\limits_{\Gamma}\phi_td\sigma+\int\limits_{\Gamma}(h_s-g_s\phi)d\sigma\\
&= -\frac{1}{\Gamma_s}\int\limits_{\Gamma}\phi_td\sigma+\int\limits_{\Gamma}(h_s-g_s\phi)d\sigma.
\end{aligned}
$$

On the other hand we deduce from (4.3) and (4.4) that

$$|\int_\Gamma (h_s - g_s\phi)d\sigma| \le C, \quad |\int_\Omega (\phi^3 - \phi)dx| \le C, \quad |\varepsilon \int_\Omega \phi_t dx| \le C, \quad |\int_\Gamma \phi_t d\sigma| \le C.$$

Therefore we have

$$|\int_\Omega u dx| \le C,$$

and, in combination with (4.4),

$$\|u\|_{H^1} \le C. \tag{4.10}$$

Finally we have the elliptic estimate (cp. Section 2 and the references [14, 1, 2, 20])

$$\|\phi\|_{H^3} \le C(\|f\|_{H^1} + \|g\|_{H^{1/2}(\Gamma)})$$

where

$$f := \Delta\phi = -u + \varepsilon\phi_t + \phi^3 - \phi, \quad g := \frac{1}{\Gamma_s}\phi_t - h_s + g_s\phi.$$

Since by (4.3)–(4.10)

$$\|f\|_{H^1} \le C, \quad \|g\|_{H^{1/2}(\Gamma)} \le C/\varepsilon,$$

we get

$$\|\phi(t)\|_{H^3} \le C/\varepsilon. \tag{4.11}$$

By these estimates, the global existence and uniqueness of strong solutions for problem (P_ε) with $\varepsilon > 0$ being fixed follows. □

5 Global existence for the original Cahn-Hilliard equation

We now turn to the original problem (1.3), (1.5), (1.11), (1.4) for the Cahn-Hilliard equation, i.e.,

$$\psi_t = \Delta\mu \quad \text{in } [0,T] \times \Omega, \tag{5.1}$$

$$\mu = -\Delta\psi - \psi + \psi^3, \tag{5.2}$$

$$\partial_\nu \mu|_\Gamma = 0, \tag{5.3}$$

$$\frac{1}{\Gamma_s}\psi_t = \sigma_s \Delta_{||}\psi - \partial_\nu\psi + h_s - g_s\psi \quad \text{on } \Gamma, \tag{5.4}$$

$$\psi(0,\cdot) = \psi_0 \quad \text{in } \Omega. \tag{5.5}$$

Then we have the following result.

Theorem 5.1 *Suppose that $n=2$ or $n=3$ and that the initial data ψ_0 satisfies $\psi_0 \in H^3$, $\mathcal{B}\phi_0 \in H^1(\Gamma)$. Then the initial boundary value problem (5.1)–(5.5) admits a unique global solution ψ such that for any $T>0$, $\psi \in C^0([0,T];H^1)\bigcap L^2([0,T];H^3), \psi_t \in L^2([0,T];V), \Delta\psi \in L^2([0,T];H^3)$, $\mu := -\Delta\psi - \psi + \psi^3 \in L^2([0,T];H^3)$.*

Proof: As in Section 3, let μ_1 be an element in V such that

$$\left(\mathcal{B}\psi_0 - \frac{1}{\Gamma_s}\mu_1\right)|_\Gamma = 0. \tag{5.6}$$

Consider the following (P_ε) problem:

$$\left.\begin{array}{rcl} \varepsilon u_t - \Delta u &=& -\phi_t, \\ (\partial_\nu u)|_\Gamma &=& 0, \\ u(0,\cdot) &=& u_0, \end{array}\right\} \tag{5.7}$$

and

$$\left.\begin{array}{rcl} \varepsilon\phi_t - \Delta\phi - \phi + \phi^3 &=& u, \\ (\sigma_s\Delta_{||}\phi - \partial_\nu\phi)|_\Gamma &=& (\frac{1}{\Gamma_s}\phi_t - h_s + g_s\phi)|_\Gamma, \\ \phi(0,\cdot) &=& \psi_0, \end{array}\right\} \tag{5.8}$$

where

$$u_0 = \mu_0 + \varepsilon\mu_1, \tag{5.9}$$

and

$$\mu_0 = -\Delta\psi_0 - \psi_0 + \psi_0^3.$$

Then it is easy to see from (5.6) and (5.9) that the conditions (3.1)–(3.2) in Section 3 are satisfied. Thus, by Theorem 4.1, the problem (5.1)–(5.5) admits a unique global solution $\phi^{(\varepsilon)}, u^{(\varepsilon)}$. Moreover, by the estimates (4.3)–(4.11) in the previous section, for any $T>0$ we have

$$\phi^{(\varepsilon)} \quad \text{uniformly bounded} \quad \text{in} \quad C^0([0,T];H^1)\bigcap L^2([0,T];H^3), \tag{5.10}$$

$$\phi_t^{(\varepsilon)} \quad \text{uniformly bounded} \quad \text{in} \quad L^2([0,T];V), \tag{5.11}$$

$$u^{(\varepsilon)} \quad \text{uniformly bounded} \quad \text{in} \quad C^0([0,T];H^1), \tag{5.12}$$

$$\sqrt{\varepsilon}u_t^{(\varepsilon)} \quad \text{uniformly bounded} \quad \text{in} \quad L^2([0,T];L^2), \tag{5.13}$$

$$\Delta u^{(\varepsilon)} \quad \text{uniformly bounded} \quad \text{in} \quad L^2([0,T];L^2). \tag{5.14}$$

Therefore, we have a subsequence in ε and $\phi^{(\varepsilon)}, u^{(\varepsilon)}$, still denoted by ε and $\phi^{(\varepsilon)}, u^{(\varepsilon)}$, and ψ, μ such that, as $\varepsilon \to 0$,

$$\phi^{(\varepsilon)} \to \psi, \quad \text{weak-*} \quad \text{in} \quad L^\infty([0,T];H^1), \tag{5.15}$$

$$\phi^{(\varepsilon)} \to \psi, \quad \text{weakly} \quad \text{in} \quad L^2([0,T];H^3), \tag{5.16}$$

$$\phi_t^{(\varepsilon)} \to \psi_t, \quad \text{weakly} \quad \text{in} \quad L^2([0,T];V), \tag{5.17}$$

$$u^{(\varepsilon)} \to \mu, \quad \text{weak-*} \quad \text{in} \quad L^\infty([0,T];H^1), \tag{5.18}$$

$$\varepsilon u_t^{(\varepsilon)} \to 0 \quad \text{strongly} \quad \text{in} \quad L^2([0,T];L^2), \tag{5.19}$$

and

$$\Delta u^{(\varepsilon)} \to \Delta\mu \quad \text{weakly} \quad \text{in} \quad L^2([0,T];L^2). \tag{5.20}$$

It follows from (5.15), (5.17) that ψ also belongs to $C^0([0,T];H^1)$ and $\psi|_{t=0} = \psi_0$. By the well-known Aubin compactness theorem, we deduce from (5.16), (5.17) that

$$\phi^{(\varepsilon)} \to \psi \quad \text{strongly} \quad \text{in} \quad L^2([0,T];H^{3-\eta}) \tag{5.21}$$

with η being a sufficiently small positive constant. Then it follows from the Sobolev imbedding theorem that

$$(\phi^{(\varepsilon)})^3 \to \psi^3 \quad \text{strongly} \quad \text{in} \quad L^2([0,T];L^2). \tag{5.22}$$

Taking the weak limit in both (5.7), (5.8) yields that (5.1) holds in $L^2([0,T];L^2)$, and (5.2) holds in $L^2([0,T];H^1)$. The boundary conditions (5.3) and (5.4) are satisfied in $L^2([0,T];H^{-1/2}(\Gamma))$ and in $L^2([0,T];H^{1/2}(\Gamma))$, respectively. Since $\psi_t \in L^2([0,T];V)$, by the regularity theorem for the elliptic equation, we deduce that $\mu \in L^2([0,T];H^3)$. Therefore, the proof for the global existence part of the present theorem is completed.

To prove the uniqueness, let ψ_1, ψ_2 be two solutions, and let $\psi = \psi_1 - \psi_2$. Then ψ and the corresponding μ satisfy

$$\Delta\mu = \psi_t, \tag{5.23}$$

$$-\Delta\psi + \psi_1^3 - \psi_2^3 - \psi = \mu, \tag{5.24}$$

$$\partial_\nu \mu|_\Gamma = 0, \tag{5.25}$$

$$\frac{1}{\Gamma_s}\psi_t = \sigma_s \Delta_{||}\psi - \partial_\nu\psi - g_s\psi \quad \text{on } \Gamma, \tag{5.26}$$

$$\psi(0,\cdot) = 0 \quad \text{in } \Omega. \tag{5.27}$$

Multiplying (5.24) by ψ_t and integrating over Ω, then using (5.23) and the boundary conditions, we have for $n \le 3$

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (|\nabla\psi|^2 + (\psi_1^2 + \psi_1\psi_2 + \psi_2^2)\psi^2)dx + \int_\Gamma (\sigma_s|\nabla_{||}\psi|^2 + g_s\psi^2)d\sigma\right)$$

$$
\begin{aligned}
&+\|\nabla\mu\|^2+\frac{1}{\Gamma_s}\int\limits_\Gamma|\psi_t|^2d\sigma\\
=\;& -\int\limits_\Omega\nabla\mu\nabla\psi dx+\frac{1}{2}\int\limits_\Omega(\psi_{1t}(2\psi_1+\psi_2)+\psi_{2t}(\psi_1+2\psi_2))\psi^2dx\\
\le\;& \frac{1}{2}(\|\nabla\mu\|^2+\|\nabla\psi\|^2)+2(\|\psi_{1t}\|+\|\psi_{2t}\|)(\|\psi_1\|_{L^6}+\|\psi_2\|_{L^6})\|\psi\|^2_{L^6}\\
\le\;& \frac{1}{2}(\|\nabla\mu\|^2+\|\nabla\psi\|^2)+C(\|\psi_1\|_{H^1}+\|\psi_2\|_{H^1})(\|\psi_{1t}\|+\|\psi_{2t}\|)\|\psi\|^2_{H^1}. \qquad (5.28)
\end{aligned}
$$

Let $y(t)$ be the nonnegative function defined as follows:

$$
y(t)=\int\limits_\Omega(|\nabla\psi|^2+(\psi_1^2+\psi_1\psi_2+\psi_2^2)\psi^2)dx+\int\limits_\Gamma\left(\sigma_s|\nabla_{||}\psi|^2+g_s\psi^2\right)d\sigma.
$$

Then it follows from (5.28) that

$$
\frac{dy}{dt}\le\alpha(t)y(t) \qquad (5.29)
$$

where

$$
\alpha(t)=C(\|\psi_1\|_{H^1}+\|\psi_2\|_{H^1})(\|\psi_{1t}\|+\|\psi_{2t}\|). \qquad (5.30)
$$

It is easy to see from (5.16), (5.17) that $\alpha\in L^1([0,T];I\!R)$. Thus we deduce from (5.29), the Gronwall inequality and (5.27) that $\psi(t)=0$ for all $t\in[0,T]$. The proof is completed. □

Remark 5.2 *We could also conclude the C^∞ smoothness of the solution as $t>0$, since the system is now a nonlinear parabolic one, cp. e.g. [27] for this general aspect.*

6 The case $n=1$

In the one-dimensional case the boundary condition (1.11) turns out to be

$$
\frac{1}{\Gamma_s}\psi_t=-\partial_\nu\phi+h_s-g_s\phi \quad \text{on } \Gamma, \qquad (6.1)
$$

where $\partial_\nu=\partial_x$ now. Therefore, we should make some changes in the previous sections. For instance, in Section 2, we should replace V by the usual H^1 in which the inner product is defined as follows

$$
(u,v)_{H^1}=\int\limits_\Omega\nabla u\nabla vdx+g_s\int\limits_\Gamma uvd\sigma.
$$

Accordingly, instead of using the results on elliptic equations with the boundary condition involving the operator $\Delta_{||}-\partial_\nu-g_s$, we should use the elliptic theory for the usual third kind of boundary condition, i.e., Robin's boundary condition. For this reason some estimates, e.g. (2.37) and (2.53) in Section 2 no longer hold the same way. However, for $n = 1$ we have $\Delta\phi = \phi_{xx}$, hence (2.54) still follows, and it finally turns out that Theorem 2.4 still holds. Keeping these minor changes in mind, we now have the corresponding result for problem (5.1)–(5.5) for $n = 1$.

Theorem 6.1 *Suppose that $n = 1$ and that the initial data ψ_0 satisfies $\psi_0 \in H^3$. Then the initial boundary value problem (5.1)–(5.5) (without the operator $\Delta_{||}$) admits a unique global solution ψ such that for any $T > 0$, $\psi \in C^0([0,T]; H^1) \bigcap L^2([0,T]; H^5), \psi_t \in L^2([0,T]; V)$, $\mu := -\Delta\psi - \psi + \psi^3 \in L^2([0,T]; H^3)$.*

7 The limiting case $\Gamma_s = \infty$

In the limiting case $\Gamma_s = \infty$ the nonlinear boundary condition (1.11) turns into a boundary condition of lower-order since the highest-order term vanishes now. This simpler case can be dealt with in an analogous way to the previous sections. For example, the eigenvalue problem (2.20), (2.21) becomes just an ordinary eigenvalue problem for the Laplace operator with boundary condition $\mathcal{B}\Phi = 0$, which in turn is reflected in the fact that H equals L^2. Also because the decomposition technique in Section 2 is no longer needed, the auxiliary function z in that section should be the trivial zero now, and so on. Keeping these remarks in mind, we obtain the corresponding results to Theorem 5.1, Theorem 6.1 by making minor changes in the proofs in the previous sections.

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References

[1] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I, Comm. Pure Appl. Math. **12** (1959), 623–727.

[2] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II, Comm. Pure Appl. Math. **17** (1964), 35–92.

[3] G. Caginalp: An analysis of a phase field model of a free boundary, Arch. Rational Mech. Anal. **92** (1986), 205–245.

[4] J.W. Cahn, E. Hilliard, Free energy of a nonuniform system. I. Interfacial free energy, J. Chem. Phys. **28** (1958), 258–367.

[5] K. Binder, H.L. Frisch, Dynamics of surface enrichment: a theory based on the Kawasaki spin-exchange model in the presence of a wall, Z. Phys. B **84** (1991), 403–418.

[6] C.M. Elliott, S. Zheng, On the Cahn-Hilliard equation, Arch. Rational Mech. Anal. **96** (1986), 339–357.

[7] C.M. Elliott, S. Zheng, Global existence and stability of solutions to the phase field equations, Free Boundary Problems, edited by K.H. Hoffmann and J. Sprekels, International Series of Numerical Mathematics **95**, pp. 46–58. Birkhäuser Verlag, Basel (1990).

[8] J. Escher, Quasilinear parabolic systems with dynamical boundary conditions, Comm. PDE **18** (1993), 1309–1364.

[9] J. Escher, On quasilinear fully parabolic boundary value problems, Differential Integral Equations **7** (1994), 1325–1343.

[10] H.P. Fischer, P. Maass, W. Dieterich, Novel surface modes in spinodal decomposition, Phys. Rev. Lett. **79** (1997), 893–896.

[11] H.P. Fischer, P. Maass, W. Dieterich, Diverging time and length scales of spinodal decomposition modes in thin films, Europhy. Lett. **42** (1998), 49–54.

[12] H. P. Fischer, J. Reinhard, W. Dieterich, J.-F. Gouyet, P. Maass, A. Majhofer, D. Reinel, Time-dependent density functional theory and the kinetics of lattice gas systems in contact with a wall, J. Chem. Phys. **108** (1998), 3028–3037.

[13] T. Hintermann, Evolution equations with dynamic boundary conditions, Proc. Roy. Soc. Edinburgh Sect A **113** (1989), 43–60.

[14] L. Hörmander, Linear partial differential operators, Grundlehren math. Wiss. **116**, Springer-Verlag, Berlin (1976).

[15] N. Kenmochi, M. Niezgódka, I. Pawłow, Subdifferential operator approach to the Cahn-Hilliard equation with constraint, J. Differential Equations **117** (1995), 320–356.

[16] R. Kenzler, F. Eurich, P. Maass, B. Rinn, J. Schropp, E. Bohl, W. Dieterich, Phase separation in confined geometries: Solving the Cahn-Hilliard equation with generic boundary conditions, Computer Phys. Comm. **133** (2001), 139–157.

[17] J.L. Lions, E. Magenes, Non-homogeneous boundary value problems and applications. I, Grundlehren math. Wiss. **181**, Springer-Verlag, New York (1972).

[18] J.L. Lions, E. Magenes, Non-homogeneous boundary value problems and applications. II, Grundlehren math. Wiss. **182**, Springer-Verlag, New York (1972).

[19] A. Novick-Cohen, L.A. Segel, Nonlinear aspects of the Cahn-Hilliard equation, Physica D **10** (1984), 277–298.

[20] J. Peetre, Another approach to elliptic boundary value problems, Comm. Pure Appl. Math. **14** (1961), 711–731.

[21] J. Prüss, Maximal regularity for abstract parabolic problems with inhomogeneous boundary data in L_p-spaces, Preprint (2001), to appear in: Proceedings "Equadiff 10, Prague, August 27–31, 2001".

[22] N. Sato and T. Aiki, Phase field equations with constraints under nonlinear dynamic boundary conditions, Comm. Appl. Anal. **5** (2001), 215–234.

[23] M. Shillor, Existence and continuity of a weak solution to the problem of a free boundary in a concentrated capacity, Proc. Roy. Soc. Edinburgh, Sect. A, **100** (1985), 271–280.

[24] R. Temam, Infinite-dimensional dynamical systems in mechanics and physics, Appl. Math. Sci. **68**, Springer-Verlag, New York (1988).

[25] M.I. Višik, On general boundary problems for elliptic differential equations (in Russian), Trudy Moskov. Mat. Obsč. **1** (1952), 187–246.

[26] S. Zheng, Asymptotic behavior of solution to the Cahn-Hilliard equation, Applicable Anal. **3** (1986), 165–184.

[27] S. Zheng, Nonlinear parabolic equations and hyperbolic-parabolic coupled systems, Pitman Monographs Surv. Pure Appl. Math. **76**, Longman; John Wiley & Sons, New York (1995).