Global stability for damped Timoshenko systems*

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Abstract: We consider a nonlinear Timoshenko system as an initial-boundary value problem in a one-dimensional bounded domain. The system has a dissipative mechanism through frictional damping being present only in the equation for the rotation angle. We first give an alternative proof for a sufficient and necessary condition for exponential stability for the linear case. Polynomial stability is proved in general. The global existence of small, smooth solutions and the exponential stability is investigated for the nonlinear case.

1 Introduction

We consider the following nonlinear Timoshenko system with a frictional damping in one equation:

$$\rho_1\varphi_{tt} - \sigma_1(\varphi_x,\psi)_x = 0, \tag{1.1}$$

$$\rho_2\psi_{tt} - \chi(\psi_x)_x + \sigma_2(\varphi_x,\psi) + d\psi_t = 0, \tag{1.2}$$

where the functions φ,ψ depending on $(t,x)\in(0,\infty)\times(0,L)$ model the transverse displacement of a beam with reference configuration $(0,L)\subset\mathbb{R}$, and the rotation angle of a filament, respectively.
By ρ_1,ρ_2,d we are denoting positive constants, and the given non-linear functions σ_1, σ_2 will be assumed to satisfy for $j=1,2$:

$$\sigma_{j,\varphi_x}(0,0) = \sigma_{j,\psi}(0,0) = k, \tag{1.3}$$

$$\chi_{\psi_x}(0) = b, \tag{1.4}$$

with positive constants k and b. A simple example for σ_1 with essential nonlinearity in the first variable is given by

$$\sigma(r,s) = k\frac{r}{\sqrt{1+r^2}} + ks,$$

the nonlinear part corresponding to a vibrating string. The linearized system then consists of

$$\rho_1\varphi_{tt} - k(\varphi_x+\psi)_x = 0, \tag{1.5}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x+\psi) + d\psi_t = 0, \tag{1.6}$$

the common linear Timoshenko system, cp. [1], [8].
Boundary conditions both for the linear and the nonlinear system will be given for $t\geq 0$ by

$$\varphi(t,0) = \varphi(t,L) = \psi_x(t,0) = \psi_x(t,L) = \theta(t,0) = \theta(t,L) = 0. \tag{1.7}$$

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Additionally one has initial conditions

$$\varphi(0,\cdot)=\varphi_0,\ \ \varphi_t(0,\cdot)=\varphi_1,\ \ \psi(0,\cdot)=\psi_0,\ \ \psi_t(0,\cdot)=\psi_1 \text{ in } (0,L). \tag{1.8}$$

If $d=0$ then (1.5), (1.6) build a purely hyperbolic system where the energy is conserved and a solution, respectively the energy, does not decay at all, of course. Moreover, the system (1.1), (1.2) is expected to develop singularities in finite time because of its typical nonlinear hyperbolic character.
Soufyane [8] proved for the boundary conditions $\phi=\psi=0$, also for positive $d=d(x)$, that the linearized system is exponentially stable if and only if

$$\frac{\rho_1}{k}=\frac{\rho_2}{b} \tag{1.9}$$

holds, that is, if and only if the wave speeds associated to (1.5), (1.6), respectively, are equal. Here we present an alternative proof in the Sections 2 and 3, which in particular for the only-if-part in Section 3 is simpler than using the approach from [6] as done by Soufyane.
A weaker type of dissipation for the linearized system, also being present only in the equation (1.2) for ψ, was considered in [1] replacing $d\psi_t$ by a memory term $\int\limits_0^t g(t-s)\psi_{xx}(s,x)ds$. For exponential type kernels g the exponential stability follows again if and only if (1.9) holds.

In [5] we investigated Timoshenko systems where the dissipation arises not through a fricitional damping but through the impact of heat conduction being coupled to the differential equation (1.2) for ψ.

Here, we also prove in Section 4 the polynomial stability of the linearized system when the condition (1.9) does not hold, which means that the dissipation is not strong enough to produce exponential stability, but even in this case we get $1/t$-decay of the solution. In turn, we prove in Section 5 that the nonlinear system admits global, small, smooth solutions that decay exponentially, if the wave speeds are equal.

Energy methods and spectral analysis arguments will be used that will have to combine methods previously used for Timoshenko systems as in [1], for systems with Kelvin-Voigt damping [2], and for nonlinear systems as described for Cauchy problems in [7].

Standard notations for function spaces are used.

2 Exponential stability, linear case

We rewrite the linearized initial-boundary value problem (1.5)–(1.8) as first-order system for $V:=(\varphi,\varphi_t,\psi,\psi_t)'$. Then V formally satisfies

$$V_t=AV, \qquad V(t=0)=V_0 \tag{2.1}$$

where $V_0:=(\varphi_0,\varphi_1,\psi_0,\psi_1)'$ and A is the (formal) differential operator

$$A=\begin{pmatrix} 0 & 1 & 0 & 0 \\ k/\rho_1\partial_x^2 & 0 & k/\rho_1\partial_x & 0 \\ 0 & 0 & 0 & 1 \\ -k/\rho_2\partial_x & 0 & b/\rho_2\partial_x^2-k/\rho_2 & -d \end{pmatrix}. \tag{2.2}$$

Let

$$\mathcal{H} := H_0^1((0,L)) \times L^2((0,L)) \times H^1((0,L)) \times L^2((0,L))$$

be the Hilbert space with norm given by

$$\begin{aligned} \|V\|_{\mathcal{H}}^2 &= \|(V^1,V^2,V^3,V^4)'\|_{\mathcal{H}}^2 \\ &\equiv \rho_1\|V^2\|_{L^2}^2 + b\|V_x^3\|_{L^2}^2 + k\|V_x^1 + V^3\|_{L^2}^2 + \rho_2\|V^4\|_{L^2}^2. \end{aligned}$$

Then A, formally given in (2.2), with domain

$$\begin{aligned} D(A) \;:=\; &\{V \in \mathcal{H} \mid V^1 \in H^2((0,L)),\;\; V^2 \in H_0^1((0,L)),\;\; V^3 \in H^2((0,L)), \\ &V_x^3 \in H_0^1((0,L)),\;\; V^4 \in L^2((0,L))\}, \end{aligned}$$

generates a contraction semigroup $\{e^{tA}\}_{t\geq 0}$. We observe that for a solution (φ,ψ) to (1.5)–(1.8), and the corresponding V, the norm $\|V(t)\|_{\mathcal{H}}^2$ equals twice the energy $E(t)$ of (φ,ψ,θ) defined by

$$E(t) := \frac{1}{2}\int_0^L (\rho_1\varphi_t^2 + \rho_2\psi_t^2 + b\psi_x^2 + k|\varphi_x + \psi|^2)(t,x)dx. \tag{2.3}$$

Hence proving the exponential stability of the semigroup $\{e^{tA}\}_{t\geq 0}$ will prove the following theorem on the exponential decay of the energy.

Theorem 2.1 *Let the initial data satisfy*

$$\varphi_0,\, \psi_{0,x} \in H_0^1((0,L)), \quad \varphi_1, \psi_1 \in L^2((0,L)).$$

If the equality (1.9) holds then there are constants $C > 0$, $\kappa > 0$ being independent of the initial data such that for all $t \geq 0$

$$E(t) \leq C \quad E(0)e^{-2\kappa t}.$$

PROOF: By well-known characterizations (see e.g. [4, Theorem 1.3.2]) of the exponential stability of contraction semigroups it is sufficient and necessary to prove

$$i\mathbb{R} \subset \varrho(A) \qquad \text{(resolvent set)} \tag{2.4}$$

and

$$\exists C > 0 \forall V \in D(A) \forall w \in \mathbb{R} : C\|(A - iw)V\|_{\mathcal{H}}^2 \geq \|V\|_{\mathcal{H}}^2 \tag{2.5}$$

While (2.4) is not difficult to prove we shall show (2.5) now. Let $(iw - A)V = F$, $V = (v^1, v^2, v^3, v^4)'$, i.e.

$$iwv^1 - v^2 = f^1, \tag{2.6}$$

$$iwv^2 - \alpha(v_{xx}^1 + v_x^3) = f^2, \tag{2.7}$$

$$iwv^3 - v^4 = f^3, \tag{2.8}$$

$$iwv^4 - \beta v^3_{xx} + \gamma(v^1_x + v^3) + dv^4 = f^4, \tag{2.9}$$

where

$$\alpha := \frac{k}{\varrho_1}, \ \ \beta := \frac{b}{\varrho_2}, \ \ \gamma := \frac{k}{\varrho_2}. \tag{2.10}$$

Multiplying (2.9) by $\overline{v^4}$ we get

$$iw\int_0^L |v^4|^2 dx + \beta \int_0^L v^3_x \overline{v^4_x} dx + \gamma \int_0^L (v^1_x + v^3)\overline{v^4} dx + \int_0^L d|v^4|^2 dx = \int_0^L f^4 \overline{v^4} dx, \tag{2.11}$$

while (2.8) yields

$$\int_0^L v^3_x \overline{v^4_x} dx = -iw \int_0^L |v^3_x|^2 dx - \int_0^L v^3_x \overline{f^3_x} dx$$

implying

$$\begin{aligned} iw\int_0^L |v^4|^2 dx - i\beta w \int_0^L |v^3_x|^2 dx + \gamma \int_0^L (v^1_x + v^3)\overline{v^4} dx + \int_0^L d|v^4|^2 dx \\ = \int_0^L f^4 \overline{v^4} dx + \int_0^L v^3_x \overline{f^3_x} dx. \end{aligned} \tag{2.12}$$

Using (2.6) and (2.8) we see

$$\alpha \int_0^L (v^1_x + v^3)\overline{v^2_x} dx = -i\alpha w \int_0^L (v^1_x + v^3)\overline{v^1_x} dx - \alpha \int_0^L (v^1_x + v^3)\overline{f^1_x} dx, \tag{2.13}$$

$$\gamma \int_0^L (v^1_x + v^3)\overline{v^4} dx = -i\gamma w \int_0^L (v^1_x + v^3)\overline{v^3} dx - \gamma \int_0^L (v^1_x + v^3)\overline{f^3} dx. \tag{2.14}$$

We conclude from (2.11)–(2.14)

$$\varrho_2 \int_0^L d|v^4|^2 dx = \mathrm{Re}\left\{ \int_0^L \varrho_2 f^4 \overline{v^4} + \varrho_2 v^3_x \overline{f^3_x} + k(v^1_x + v^3)\overline{f^3} + \varrho_1 f^2 \overline{v^2} + \varrho_1 (v^1_x + v^3)\overline{f^1_x} dx \right\} \tag{2.15}$$

A multiplication of (2.9) by $\overline{v^3}$ yields, using (2.8),

$$-\int_0^L |v^4|^2 dx + \beta \int_0^L |v^3_x|^2 dx + \gamma \int_0^L v^1_x \overline{v^3} dx + \gamma \int_0^L |v^3|^2 dx + \int_0^L dv^4 \overline{v^3} dx = \int_0^L f^4 \overline{v^3} + v^4 \overline{f^3} dx \tag{2.16}$$

Let w denote the solution to

$$-w_{xx} = v^3_x, \qquad w(0) = w(L) = 0,$$

i.e.

$$w(x) = -\int_0^x v^3(y) dx + \frac{x}{L}\int_0^L v^3(y) dy \equiv G(v^3_x)(x).$$

A multiplication of (2.7) by $\overline{w}$ gives

$$\alpha \int_0^L v_x^1 \overline{w}_x dx = \alpha \int_0^L |w_x|^2 dx + \int_0^L v^2 \left(\overline{G(v_x^4)} + \overline{G(f_x^3)} \right) dx + \int_0^L f^3 w \, dx. \tag{2.17}$$

Observing

$$\int_0^L v_x^1 \overline{w_x} \, dx = - \int_0^L v_x^1 \overline{v^3} dx$$

we conclude from (2.16), (2.17)

$$\begin{aligned} -\varrho_2 \int_0^L |v^4|^2 dx + b \int_0^L |v_x^3| dx - k \left(\int_0^L |w_x|^2 dx - \int_0^L |v^3|^2 dx \right) & \\ -\varrho_1 \int_0^L v^2 (\overline{G(v_x^4)} + \overline{G(f_x^3)}) dx - \varrho_1 \int_0^L f^2 \overline{w} \, dx + \varrho_2 \int_0^L d v^4 \overline{v^3} dx & \\ = \quad \varrho_2 \int_0^L \varrho^4 \overline{v^3} dx + \varrho_2 \int_0^L v^4 \overline{\varrho^3} dx. & \end{aligned}$$

Since

$$\int_0^L |w_x|^2 \, dx \leq \int_0^L |v^3|^2 dx$$

we obtain

$$\begin{aligned} b \int_0^L |v_x^3|^2 dx \quad \leq \quad & \varrho_2 \int_0^L |v^4|^2 dx + \mathrm{Re} \left\{ \int_0^L \varrho_1 v^2 (\overline{G(v_x^4)} + \overline{G(f_x^3)}) + \varrho_1 f^2 \overline{w} \right. \\ & \left. -d \varrho_2 v^4 \overline{v^3} + \varrho_2 f^4 \overline{v^3} + \varrho_2 v^4 \overline{f^3} dx \right\}. \end{aligned} \tag{2.18}$$

A multiplication of (2.9) by $\overline{v_x^1 + v^3}$ yields

$$\begin{aligned} iw \int_0^L v^4 \overline{v_x^1} dx + iw \int_0^L v^4 \overline{v^3} dx + \beta \int_0^L v_x^3 \overline{(v_{xx}^1 + v_x^3)} dx & \\ + \gamma \int_0^L |v_x^1 + v^3|^2 dx + \int_0^L d v^4 \overline{(v_x^1 + v^3)} \, dx = \int_0^L f^4 \overline{(v_x^1 + v^3)} dx, & \end{aligned}$$

and, using (2.7),

$$\begin{aligned} iw \int_0^L v^4 \overline{v_x^4} dx + iw \int_0^L v^4 \overline{v^3} dx - i \frac{\beta w}{\alpha} \int_0^L v_x^3 \overline{v^2} dx + \beta \int_0^L v_x^3 \overline{f^2} dx & \\ + \gamma \int_0^L |v_x^1 + v^3|^2 dx + d \int_0^L v^4 \overline{(v_x^1 + v^3)} dx = \int_0^L f^4 \overline{(v_x^1 + v^3)} dx. & \end{aligned}$$

Since we get from (2.6), (2.8)

$$-iw\int_0^L v^4\overline{v_x^1}dx = -iw\int_0^L v_x^3\overline{v^2}dx + iw\int_0^L v^3\overline{f_x^1} + \overline{v^2}f_x^3 - f^3\overline{f_x^1}dx$$

we obtain — here the role of the assumption (1.9) appears —

$$\begin{aligned}
iw\underbrace{(1-\beta/\alpha)}_{=0}\int_0^L v_x^3\overline{v^2}dx + \gamma\int_0^L |v_x^1+v^3|^2dx + \int_0^L dv^4\overline{(v_x^1+v^3)}\,dx \\
-\int_0^L |v^4|^2dx - \int_0^L v^4\overline{f^3}dx + \int_0^L dv^4\overline{(v_x^1+v^3)}dx - \beta\int_0^L v^3\overline{f_x^2} \\
-iw\int_0^L v^3\overline{f_x^1}dx - \int_0^L \overline{v^2}f_x^3 + f^3\overline{f_x^1}dx - \int_0^L f^4\overline{(v_x^1+v^3)}dx = 0
\end{aligned} \tag{2.19}$$

Finally, we multiply (2.7) by $\overline{v^1}$ and get

$$\int_0^L |v^2|^2dx = \alpha\int_0^L (v_x^1+v^3)\overline{v_x^1} - \int_0^L f^3\overline{v^1}dx - \int_0^L v^2\overline{f^1}dx. \tag{2.20}$$

Taking the real parts of

$$N_1\cdot(2.15) + N_2\cdot(2.18) + N_3\cdot(2.19) + (2.20) \tag{2.21}$$

we obtain for suitable numbers $N_1 \gg N_2 \gg N_3 \gg 1$.

$$\|V\|_{\mathcal{H}}^2 \le C\|F\|_{\mathcal{H}}^2,$$

where C is a positive constant being independent of w (and V). This proves (2.5) and hence Theorem 2.1.

Q.E.D.

Theorem 2.1 now implies the exponential stability of the semigroup as usual:

$$\exists c_1 > 0 \quad \forall t \ge 0 \quad \forall V_0 \in \mathcal{H} : \|e^{tA}V_0\|_{\mathcal{H}} \le c_1 e^{-\kappa t}\|V_0\|_{\mathcal{H}}. \tag{2.22}$$

For the nonlinear part we shall need estimates for higher derivatives of (φ,ψ) or V, respectively. Observe that if $V_0 \in D(A)$ then we can estimate $AV(t)$ in the same way as $V(t)$ is estimated in (2.22), which in turn implies that $(V_x^1, V_x^2, V_x^3, V_x^4)'$ can be estimated in the $\|\cdot\|_{\mathcal{H}}$-norm, hence we may estimate $((\varphi_x)_x, (\varphi_t)_x, (\psi_x)_x, (\psi_t)_x)'$ in L^2.

Let for $s \in \mathbb{N}$

$$\mathcal{H}_s := (H^s \times H^{s-1} \times H^s \times H^{s-1})((0,L)).$$

with natural norms $\|\cdot\|_{H^s}$ for the components.

For $V_0 \in D(A^{s-1})$, we thus can estimate

$$\|V(t)\|_{\mathcal{H}_s} \le c_s\|V_0\|_{\mathcal{H}_s}e^{-\kappa t}. \tag{2.23}$$

with c_s being a positive constant, independent of V_0 and t.

3 Non-uniform stability, linear case

Having proved the sufficiency of the condition (1.9) for exponential stability, we now prove its necessity with the criteria (2.4), (2.5) as already done similarly in [2] or [5].

Theorem 3.1 *If*

$$\frac{\varrho_1}{k} \neq \frac{\varrho_2}{b}$$

then the system associated to (1.5)–(1.8) is not exponentially stable.

PROOF: It suffices to show the existence of sequences $(\lambda_n)_n \subset \mathbb{R}$ with $\lim_{n\to\infty} |\lambda_n| = \infty$ and $(V_n)_n \subset D(A), (F_n)_n \subset \mathcal{H}$ such that $(i\lambda_n - A)V_n = F_n$ is bounded and

$$\lim_{n\to\infty} \|V_n\|_{\mathcal{H}} = \infty.$$

We choose $F \equiv F_n$ with

$$F = (0, f^2, 0, f^4)',$$

where

$$\begin{aligned} f^2(x) &:= \sin\frac{n\pi x}{L} = \sin(\delta\lambda x), \\ f^4(x) &:= \cos\frac{n\pi x}{L} = \cos(\delta\lambda x), \end{aligned}$$

with

$$\delta := \frac{1}{\sqrt{\alpha}}, \quad \lambda \equiv \lambda_n = \frac{\sqrt{\alpha}\, n\pi}{L}.$$

Then

$$\|F_n\|^2_{\mathcal{H}} = L$$

and the solution $V = (v^1, v^2, v^3, v^4)'$ to $(i\lambda - A) = F$ has to satisfy

$$i\lambda v^1 - v^2 = 0, \tag{3.1}$$

$$i\lambda v^2 - \alpha(v^1_{xx} + v^3_x) = f^2, \tag{3.2}$$

$$i\lambda v^3 - v^4 = 0, \tag{3.3}$$

$$i\lambda v^4 - \beta v^3_{xx} + \gamma(v^1_x + v^3) + dv^4 = f^4. \tag{3.4}$$

This will determine v^2, v^4 and we obtain for v^1, v^3:

$$-\lambda^2 v^1 - \alpha v^1_{xx} - \alpha v^3_x = f^2, \tag{3.5}$$

$$-\lambda^2 v^3 - \beta v^3_{xx} + \gamma(v^1_x + v^3) + id\lambda v^3 = f^4. \tag{3.6}$$

The ansatz

$$v^1(x) = A\sin(\delta\lambda x), \quad v^3(x) = B\cos(\delta\lambda x) \tag{3.7}$$

will solve (3.5), (3.6) for appropriate $A = A(\lambda)$, $B = B(\lambda)$ which are determined below. Plugging the ansatz (3.7) into (3.5), (3.6) A and B have to satisfy

$$(\alpha\delta^2 - 1)\lambda^2 A + \alpha\delta\lambda B = 1, \tag{3.8}$$

$$(\beta\delta^2 - 1)\lambda^2 B + \gamma\delta\lambda A + \gamma B + id\lambda B = 1. \tag{3.9}$$

Since $\alpha\delta^2 = 1$ we conclude from (3.8)

$$B = \frac{1}{\sqrt{\alpha}}\frac{1}{\lambda} \tag{3.10}$$

and hence from (3.9)

$$A = -\frac{1}{\lambda^2} + \frac{1}{\gamma}(\sqrt{\alpha} - id)\frac{1}{\lambda} - (\beta/\alpha - 1)\,\frac{1}{\gamma}. \tag{3.11}$$

Then

$$\begin{aligned}
v^1(x) &= \left(-\frac{1}{\lambda^2} + \frac{1}{\gamma}(\sqrt{\alpha} - id)\frac{1}{\lambda} - \frac{1}{\gamma}(\beta/\alpha - 1)\right)\sin(\delta\lambda x),\\
v^2(x) &= (-\frac{i}{\lambda} + \frac{1}{\gamma}(i\sqrt{\alpha} + d) - \frac{i}{\gamma}(\beta/\alpha - 1)\lambda)\sin(\delta\lambda x),\\
v^3(x) &= \frac{1}{\sqrt{\alpha}}\frac{1}{\lambda}\cos(\delta\lambda x),\\
v^4(x) &= \frac{i}{\sqrt{\alpha}}\cos(\delta\lambda x).
\end{aligned}$$

Remember that α and β are given in (2.10). Noting

$$\int_0^L |v^2|^2\,dx = \frac{L}{2}|(-\frac{i}{\lambda}+\frac{1}{\gamma}(i\sqrt{\alpha}+d)-\frac{i}{\gamma}(\beta/\alpha-1)\lambda)|^2 \geq -\frac{L}{2}|-\frac{i}{\lambda}+\frac{1}{\gamma}(i\sqrt{\alpha}+d)|^2+\frac{L}{4}|\frac{1}{\gamma}(\beta/\alpha-1)|^2\lambda^2$$

it follows that

$$\lim_{\lambda\to\infty}\|V_n\|_{\mathcal{H}} \geq \lim_{\lambda\to\infty}\int_0^L |v^2|^2\,dx = \infty$$

which completes the proof.

Q.E.D.

Remark: We could treat the case of arbitrary $d \in L^\infty((0,L))$ in this section in the same way. The proof above also works for $d = 0$. Let the corresponding operator and the sequence constructed above be denoted by A_0, $V_{n,0}$, respectively. Then we have

$$(i\lambda_n - A)V_{n,0} = (i\lambda_n - A_0)V_{n,0} + (0,0,0,dV^4_{n,0})'$$

which implies

$$\begin{aligned}
\|(i\lambda_n - A)V_{n,0}\|_{\mathcal{H}} &\leq \|(i\lambda_n - A_0)V_{n,0}\|_{\mathcal{H}} + \|d(\cdot)V^4_{n,0}\|\\
&\leq \|(i\lambda_n - A_0)V_{n,0}\|_{\mathcal{H}} + \|d\|_{L^\infty}\|V^4_{n,0}\|,
\end{aligned}$$

hence

$$\sup_n \|(i\lambda_n - A)V_{n,0}\|_{\mathcal{H}} < \infty$$

while still

$$\lim_{n\to\infty}\|V_{n,0}\|_{\mathcal{H}} = \infty.$$

4 Polynomial stability, linear case

Even for the case that there is no exponential stability we can prove the following polynomial decay in general. Let

$$E(t) \equiv E(t, \varphi, \psi) \equiv E_1(t)$$

denote the energy defined in (2.3), and let

$$E_2(t) := E(t, \varphi_t, \psi_t)$$

denote the energy of second order, for a suitability smooth solution, of course. Then we shall prove

Theorem 4.1 *Let the initial data satisfy*

$$\varphi_0, \psi_0 \in H^2((0,L)), \varphi_0, \psi_{0,x}, \psi_1, \psi_{1,x} \in H_0^1((0,L)).$$

Then there is $C > 0$ such that for all $t > 0$:

$$E(t) \leq C(E_2(0) + E_1(0))t^{-1}.$$

PROOF: We have

$$\frac{d}{dt} E_1(t) = -d \int_0^L |\psi_t|^2 dx, \qquad \frac{d}{dt} E_2(t) = -d \int_0^L |\psi_{tt}|^2 dx \tag{4.1}$$

and

$$\frac{d}{dt} \int_0^L \varrho_2 \psi \overline{\psi_t} dx = \varrho_2 \int_0^L |\psi_t|^2 dx + b \int_0^L \psi_{xx} \overline{\psi} dx - k \int_0^L (\varphi_x + \psi) \overline{\psi} dx - d \int_0^L \psi_t \overline{\psi} dx. \tag{4.2}$$

Let w denote the solution to

$$-w_{xx} = \psi_x, \qquad w(0) = w(L) = 0,$$

then

$$\frac{d}{dt} \int_0^L \varrho_1 \varphi_t \overline{w}\, dx = -k \int_0^L \varphi \overline{\psi_x}\, dx + k \int_0^L |w_x|^2\, dx + \varrho_1 \int_0^L \varphi_t \overline{w_t}\, dx. \tag{4.3}$$

Summing up (4.2), (4.3) we get

$$\underbrace{\frac{d}{dt} \left\{ \int_0^L \varrho_2 \psi \overline{\psi_t} + \varrho_1 \varphi_t \overline{w} + \frac{d}{2} |\psi|^2 dx \right\}}_{=: F_1(t)} \leq \varrho_2 \int_0^L |\psi_t|^2 dx - b \int_0^L |\psi_x|^2 dx + \varrho_1 \int_0^L \varphi_t \overline{w_t} dx,$$

hence, using (4.1), for $N_1 > 0$

$$\frac{d}{dt} \{N_1 E(t) + F_1(t)\} \leq -(Nd - \varrho_2) \int_0^L |\psi_t|^2 dx - b \int_0^L |\varphi_x|^2\, dx + \varrho_1 \int_0^L \varphi_t \overline{w_t}\, dx. \tag{4.4}$$

Moreover,

$$\frac{d}{dt}\int_0^L \varrho_2\psi_t\overline{(\varphi_x+\psi)}\,dx = -b\int_0^L \psi_x\overline{(\varphi_x+\psi)_x} - k\int_0^L |\varphi_x+\psi|^2dx - d\int_0^L \psi_t\overline{(\varphi_x+\psi)}dx$$

$$+\varrho_2\,\frac{d}{dt}\int_0^L \psi_t\overline{\varphi_x}dx - \varrho_2\int_0^L \psi_{tt}\overline{\varphi_x}\,dx + \varrho_2\int_0^L |\varphi_t|^2dx$$

$$= \quad -\frac{b\varrho_1}{k}\int_0^L \psi_x\overline{\varphi_{tt}}\,dx - k\int_0^L |\varphi_x+\psi|^2dx - d\int_0^L \psi_t\overline{(\varphi_x+\psi)}dx$$

$$+\varrho_2\,\frac{d}{dt}\int_0^L \psi_t\overline{\varphi_x}\,dx - \varrho_2\int_0^L \psi_{tt}\overline{\varphi_x}\,dx + \varrho_2\int_0^L |\varphi_t|^2\,dx. \tag{4.5}$$

Observing

$$\int_0^L \psi_x\overline{\varphi_{tt}}\,dx = \frac{d}{dt}\left\{\int_0^L \psi_x\overline{\varphi_t} - \psi_{xt}\overline{\varphi}\,dx\right\} - \int_0^L \psi_{tt}\overline{\varphi_x}\,dx$$

we conclude for

$$\begin{aligned} F_2(t) \quad &:= \quad \int_0^L \varrho_2\psi_t\overline{(\varphi_x+\psi)} - \varrho_2\psi_t\overline{\varphi_x} + \frac{b\varrho_1}{k}(\psi_x\overline{\varphi_t} - \psi_{xt}\overline{\varphi})dx \\ \frac{d}{dt}F_2(t) \quad &= \quad \left(\frac{b\varrho_1}{k} - \varrho_2\right)\int_0^L \psi_{tt}\overline{\varphi_x}\,dx - k\int_0^L |\varphi_x+\psi|^2dx \\ &\quad -d\int_0^L \psi_t\overline{(\varphi_x+\psi)}dx + \varrho_2\int_0^L |\psi_t|^2dx. \end{aligned} \tag{4.6}$$

Let, for $N_2 > 0$,

$$F_3(t) := N_2(N_1E(t) + F_1(t)) + F_2(t).$$

Then (4.4), (4.6) imply

$$\begin{aligned} \frac{d}{dt}F_3(t) \quad &\leq \quad -\left(N_2(Nd-\varrho_2) - \left(\varrho_2 + \frac{d^2}{2k}\right)\right)\int_0^L |\psi_t|^2dx \\ &\quad -\left(N_2b - \frac{kL^2}{\pi^2}\right)\int_0^L |\psi_x|^2dx + \varrho_1N_2\int_0^L \varphi_t\overline{w_t}dx \\ &\quad +\left(\frac{b\varrho_1}{k} - \varrho_2\right)\int_0^L \psi_{tt}\overline{\varphi_x}dx - \frac{k}{4}\int_0^L |\varphi_x+\psi|^2dx - \frac{k}{8}\int_0^L |\varphi_x|^2dx. \end{aligned} \tag{4.7}$$

Moreover,

$$\frac{d}{dt}\left\{-\int_0^L \varrho_1\varphi_t\,\overline{\varphi}dx\right\} = -\varrho_1\int_0^L |\varphi_t|^2dx + k\int_0^L |\varphi_x+\psi|^2dx - k\int_0^L (\varphi_x+\psi)\overline{\psi}\,dx$$

$$\leq \quad -\varrho_1 \int_0^L |\varphi_t|^2 dx + k \int_0^L |\varphi_x + \psi|^2 dx + \frac{k}{2} \int_0^L |\varphi_x|^2 dx + \frac{3kL^2}{2\pi^2} \int_0^L |\psi_x|^2 dx. \tag{4.8}$$

Combining (4.7) and (4.8) we obtain

$$\begin{aligned}
\frac{d}{dt}\left\{F_3(t) - \frac{\varrho_1}{8}\int_0^L \varphi_t\overline{\varphi}\, dx\right\} \quad \leq \quad & -\left(N_2(Nd-\varrho_2) - \left(\varrho_2 + \frac{d^2}{2k}\right)\right)\int_0^L |\psi_t|^2 dx \\
& -\left(N_2 b - \frac{19}{16}\frac{kL^2}{\pi^2}\right)\int_0^L |\psi_x|^2 dx + N_2\varrho_1 \int_0^L \varphi_t \overline{w_t}\, dx \\
& + \left(\frac{b\varrho_1}{k} - \varrho_2\right)\int_0^L \psi_{tt}\overline{\varphi_x}\, dx - \frac{k}{8}\int_0^L |\varphi_x+\psi|^2 dx \\
& -\frac{k}{16}\int_0^L |\varphi_x|^2 dx - \frac{\varrho_1}{8}\int_0^L |\varphi_t|^2 dx.
\end{aligned}$$

Since

$$\left(\frac{b\varrho_1}{k} - \varrho_2\right)\left|\int_0^L \psi_{tt}\overline{\varphi_x}\, dx\right| \leq c_1 \int_0^L |\psi_{tt}|^2\, dx + \frac{k}{32}\int_0^L |\varphi_x|^2 dx$$

and

$$\varrho_1 N_0 \left|\int_0^L \varphi_t \overline{w_t} dx\right| \leq \frac{\varrho_1}{16}\int_0^L |\varphi_t|^2 dx + c_1 N_2^2 \int_0^L |\psi_t|^2 dx$$

for some constant $c_1 > 0$ we conclude

$$\begin{aligned}
\frac{d}{dt}\left\{F_3(t) - \frac{\varrho_1}{8}\int_0^L \varphi_t\overline{\varphi}\, dx\right\} \quad \leq \quad & -\left[N_2(Nd-\varrho_2) - (\varrho_2 + \frac{d^2}{2k} + c_1 N_2^2)\right]\int_0^t |\psi_t|^2 dx \\
& -\left[N_2 b - \frac{19}{16}\frac{kL^2}{\pi^2}\right]\int_0^L |\psi_x|^2 dx - \frac{k}{8}\int_0^L |\varphi_x+\psi|^2 dx \\
& -\frac{k}{32}\int_0^L |\varphi_x|^2 dx - \frac{\varrho_1}{16}\int_0^L |\varphi_t|^2 dx + c_1 \int_0^L |\psi_{tt}|^2 dx.
\end{aligned}$$

Choosing N_2 and N sufficiently large the coefficients in brackets $[\ldots]$ become positive, hence

$$\frac{d}{dt}\left\{F_3(t) - \frac{\varrho_1}{8}\int_0^L \varphi_t\overline{\varphi}\, dx\right\} \leq c_1 \int_0^L |\psi_{tt}|^2 dx - c_2 E_1(t), \tag{4.9}$$

with some constant $c_2 > 0$.

Let the final functional $\mathcal{L}$ be defined by

$$\mathcal{L}(t) := N_1 E_2(t) + F_3(t) - \frac{\varrho_1}{8}\int_0^L \varphi_t \overline{\varphi}\, dx.$$

Then (4.1) and (4.9) imply

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) &\leq -(N_1 d - c_1)\int_0^L |\varphi_{tt}|^2 dx - c_2 E_1(t)\\ &\leq -c_2 E_1(t) \quad \text{for } N_1 \geq \frac{c_1}{d}.\end{aligned}$$

This implies

$$\int_0^t E_1(r)dr \leq \frac{1}{c_2}(\mathcal{L}(0) - \mathcal{L}(t)).$$

Observing, for N_1 large enough,

$$\exists c_3, c_4 > 0\, \forall t \geq 0 : c_3(E_1(t) + E_2(t)) \leq \mathcal{L}(t) \leq c_4(E_1(t) + E_2(t))$$

we get for $t \geq 0$

$$\int_0^t E_1(r)dr \leq \frac{c_4}{c_2}(E_1(0) + E_2(0).$$

A standard conclusion using

$$\frac{d}{dt}(tE_1(t)) \leq E_1(t)$$

gives

$$E_1(t) \leq \frac{c_4}{c_2}(E_1(0) + E_2(0))t^{-1}.$$

Q.E.D.

5 Global stability, non-linear case

Now we return to the nonlinear system (1.1)–(1.4), (1.7), (1.8). The local well-posedness is standard, cp. for example [3]:
Let for $m \geq 2$, $j \geq 1$

$$\varphi_m(\cdot) := (\partial_t^m \varphi)(0,\cdot), \quad \psi_m(\cdot) := (\partial_t^m \psi)(0,\cdot), \quad \theta_j(\cdot) := (\partial_t^j \theta)(0,\cdot)$$

be defined through the differential equations (1.1), (1.2) and the initial conditions (1.7). For $s \geq 3$ assume

$$\varphi_m \in H^{s-m}((0,L)) \cap H_0^1((0,L)), \quad 2 \leq m \leq s-1, \quad \varphi_s, \psi_s \in L^2((0,L)), \tag{5.1}$$

$$\psi_{m,x} \in H^{s-m-1}((0,L)) \cap H_0^1((0,L)), \quad 2 \leq m \leq s-1, \quad \psi_s \in L^2((0,L)). \tag{5.2}$$

Theorem 5.1 *Let $s = 3$ and assume the compatibility conditions (5.1), (5.2). Then there is $T = T\left(\left\|(\varphi_0, \varphi_1, \psi_0, \psi_1)\right\|_{\mathcal{H}_3}\right) > 0$ such that (1.1), (1.2), (1.7), (1.8)has a unique local solution (φ, ψ) satisfying*

$$(\varphi, \psi) \in \bigcap_{k=0}^{3} C^k([0,T], H^{3-k}((0,L)).$$

and (φ, ψ) satisfy the boundary conditions (1.7).

As in the previous sections we rewrite everything as a first-order system for $V = (\varphi, \varphi_t, \psi, \psi_t)$ and obtain as in

$$V_t = AV + F(V, V_x), \quad V(t=0) = V_0 \tag{5.3}$$

where A is defined in Section 2 and

$$\begin{aligned} F(V, V_x) &\equiv (0, \sigma_1(\varphi_x, \psi)_x - k(\varphi_x + \psi)_x, 0, \chi(\psi_x)_x - b\psi_{xx} + \sigma_2(\varphi_x, \psi)_x - k(\varphi_x + \psi), 0)' = \\ &(0, \sigma_1(V_x^1, V^3)_x - k(V_x^1 + V^3)_x, 0, \chi(V_x^3)_x - bV_{xx}^3 + \sigma_2(V_x^1, V^3)_x - k(V_x^1 + V^3), 0)'. \end{aligned} \tag{5.4}$$

The (local) solution satisfies

$$V(t) = e^{tA}V_0 + \int_0^t e^{(t-r)A}F(V, V_x)(r)dr. \tag{5.5}$$

The technique that we use is an adaption of one known suitable for Cauchy problems, see [7]. We point out that the energy method as for the linearized problem, see e.g. [3], seems not to work here because it does not seem to be possible to exploit the condition $\rho_1/k = \rho_2/b$ for the nonlinear system. Therefore the perturbation arguments of the method used here are more appropriate.

Without loss of generality we assume φ_x, ψ, ψ_x to be small enough a priori.

We follow the steps described in [7], similarly as in [5]. The first step is to prove the following high energy estimate:

Lemma 5.2 *There are constants $c_1, c_2 > 0$, neither depending on V_0 nor on T, such that the local solution given in Theorem 5.1 satisfies for $t \leq [0, T]$:*

$$\|V(t)\|_{\mathcal{H}_3}^2 \leq c_1 \|V_0\|_{\mathcal{H}_3}^2 e^{c_2 \int_0^t \|V(r)\|_{\mathcal{H}_2}} dr.$$

For the proof which uses multiplicative techniques as well as Gagliardo-Nirenberg and Moser type inequalities, we refer to the analogous considerations in [5].

Next we want to prove a weighted a priori estimate for $\|V(t)\|_{\mathcal{H}_2}$. Using the representation (5.5) and (2.23) — if $\rho_1/k = \rho_2/b$ is satisfied! — we can estimate

$$\begin{aligned} \|V(t)\|_{\mathcal{H}_2} &\leq \|e^{At}V_0\|_{\mathcal{H}_2} + \int_0^t \|e^{(t-r)A}\mathcal{F}(V, V_x)(r)\|_{\mathcal{H}_3} dr \\ &\leq c_1 e^{-\kappa t}\|V_0\|_{\mathcal{H}_2} + c_1 \int_0^t e^{-\kappa(t-r)}\|F(V, V_x)\|_{\mathcal{H}_2} dr. \end{aligned} \tag{5.6}$$

Here we exploited the boundary condition

$$\varphi(t, 0) = \varphi(t, L) = \psi_x(t, 0) = \psi_x(t, L) = 0, \quad t \geq 0,$$

which implies using the differential equation (1.1) that also

$$\varphi_{xx}(t, 0) = \varphi_{xx}(t, L) = 0, \quad t \geq 0.$$

Looking at $\mathcal{F}$ this implies that

$$\mathcal{F}(V, V_x)(r) \in D(A), \quad r \geq 0,$$

hence (2.23) can be applied for $s = 2$.

Lemma 5.3 $\exists c > 0\ \forall W \in \mathcal{H}_3: \quad \|F(W, W_x)\|_{\mathcal{H}_2} \leq c\|W\|_{\mathcal{H}_2}\|W\|_{\mathcal{H}_3}$.

For the proof cp. [5] or [7] (in $\mathbb{R}^n$).
Using Lemma 5.4 we conclude from (5.6)

$$\|V(t)\|_{\mathcal{H}_2} \leq ce^{-\kappa t}\|V_0\|_{\mathcal{H}_2} + c\int_0^t e^{-\kappa(t-r)}\|V(r)\|_{\mathcal{H}_2}\|V(r)\|_{\mathcal{H}_3}dr, \tag{5.7}$$

which is the starting point to prove the following weighted a priori estimate.

Lemma 5.4 *For $0 \leq t \leq T$ let*

$$M_2(t) := \sup_{0 \leq r \leq t}\left(e^{\kappa r}\|V(r)\|_{\mathcal{H}_2}\right),$$

and let

$$\frac{\rho_1}{k} = \frac{\rho_2}{b}.$$

Then there are $M_0 > 0$ and $\delta > 0$ such that if $\|V_0\|_{\mathcal{H}_3} < \delta$ we have for all $0 \leq t \leq T$:

$$M_2(t) \leq M_0 < \infty$$

M_0 is independent of T (and of V_0).

Proof: From (5.7) and the energy estimate in Lemma 5.2 we conclude

$$\|V(t)\|_{\mathcal{H}_2} \leq c\|V_0\|_{\mathcal{H}_2}e^{-\kappa_t} + c\int_0^t e^{-\kappa(t-r)}\|V(r)\|_{\mathcal{H}_2}\|V_0\|_{\mathcal{H}_3}e^{c\int_0^r \|V(\tau)\|_{\mathcal{H}_2}dr}.$$

If $\|V_0\|_{\mathcal{H}_3} \leq \delta \quad$ (δ to be determined) we get

$$\begin{aligned}\|V(t)\|_{\mathcal{H}_2} &\leq c\delta e^{-\kappa t} + c\delta e^{c\int_0^t \|V(\tau)\|_{\mathcal{H}_2}d\tau}\int_0^t e^{-\kappa(t-r)}\|V(r)\|_{\mathcal{H}_2}dr \\ &\leq c\delta e^{-\kappa t} + c\delta e^{cM_2(t)}M_2(t)\int_0^t e^{-\kappa(t-r)}e^{-\kappa r}dr\end{aligned}$$

which implies

$$M_2(t) \leq c\delta + c\delta e^{cM_2(t)}M_2(t)\sup_{0 \leq t < \infty} e^{\kappa t}\int_0^t e^{-\kappa(t-r)}e^{-\kappa r}dr. \tag{5.8}$$

Since

$$\sup_{0\le t<\infty} e^{\kappa t}\int_0^t e^{-\kappa(t-r)}e^{-\kappa r}dr \le c < \infty,$$

we obtain from (5.8) for $0 \le t \le T$:

$$M_2(t) \le c\delta + c\delta M_2(t)e^{cM_2(t)}. \tag{5.9}$$

By standard arguments (cp. e.g. [7]), considering the function

$$f(x) := c\delta(1 + cxe^{cx}) - x$$

it follows that $M_2(t)$ is uniformly bounded by the first zero M_0 of f if δ and $M_2(0)$ are sufficiently small.
This proves Lemma 5.5.

Q.E.D.
Now we can formulate and prove the main theorem on global existence and exponential decay.

Theorem 5.5 *Let $s = 3$ and assume the conditions (5.1), (5.2) on the initial data. If*

$$\frac{\rho_1}{k} = \frac{\rho_2}{b}$$

there is a $\delta > 0$ such that if $\|V_0\|_{\mathcal{H}_3} < \delta$ there exists a unique global solution (φ, ψ) to (1.1), (1.2), (1.7), (1.8) satisfying

$$(\varphi,\psi) \in \bigcap_{k=0}^{3} C^k([0,\infty), H^{3-k}((0,L)),$$

and $(\partial_t^s\varphi, \partial_j^s\psi)$ satisfy the boundary conditions (1.7) for $0 \le j \le 2$.
Moreover, there is a constant $C_0(V_0) > 0$ such that for $t \ge 0$:

$$\|V(t)\|_{\mathcal{H}_2} \le C_0 e^{-\kappa t},$$

with κ from Theorem 2.1.

PROOF: From Lemma 5.2 and Lemma 5.5 we conclude for the local solution

$$\|V(t)\|_{\mathcal{H}_3} \le c\|V_0\|_{\mathcal{H}_3} e^{c\int_0^t \|V(r)\|_{\mathcal{H}_2}dr} \le c\|V_0\|_{\mathcal{H}_3}e^{cM_0} \le c\|V_0\|_{\mathcal{H}_4},$$

c being independent of t or V_0, from where the global existence follows by the usual continuation argument. The claim on exponential decay of $\|V(t)\|_{\mathcal{H}_2}$ now is a consequence of Lemma 5.5.

Q.E.D.
Remarks: 1. Imposing further conditions on the nonlinearity — essentially those that make the nonlinear function F in (5.4) cubic near zero — one can also prove that it suffices to have only small $\mathcal{H}^2$-norm and bounded $\mathcal{H}^3$-norm, see [5].
2. All results hold, with easy modifications of the proofs, for the case of arbitrary smooth, strictly positive $d = d(x)$, cp. also the remarks in Section 3 for arbitrary bounded functions d.

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