Stability for thermoelasticity of type III

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Abstract: We consider initial-boundary value problems in a hyperbolic thermoelastic system, called thermoelasticity of type III. First, we prove the exponential stability in one space dimension for different boundary conditions with energy methods and spectral methods, respectively. Then the exponential stability in more two or three space dimensions is proved for radially symmetric situations. Finally, the equipartition of energy is investigated.

1 Introduction

The classical theory of thermoelasticity as exposed for example in Carlson's article [1] has found generalizations and modifications into various thermoelastic models that run under the label *hyperbolic* thermoelasticity, see the surveys of Chandrasekharaiah [2] or Hetnarski and Ignaczak [9]. The notion *hyperbolic* reflects the fact that thermal *waves* are modeled, avoiding the physical paradox of infinite propagation speed of the classical model. Of course, it is a natural question to ask whether the dissipation through the hyperbolic heat equation — mathematically represented through a system of wave equations with a kind of damping — is strong enough to produce a similar stability of the system as would be predicted in the classical hyperbolic-parabolic theory.

For some models like that of thermoelasticity with second sound, the exponential stability for bounded reference configurations in one space dimensions was recently established in [22], also the exponential stability for radially symmetric situations in two or three space dimensions, see [23].

In the decade of the 1990's Green and Naghdi proposed three new thermoelastic theories based on an entropy equality rather than the usual entropy inequality [5, 6, 7, 8]. The constitutive assumptions for the heat flux vector are different in each theory. Thus, they obtained three theories that they called thermoelasticity of type I, thermoelasticity of type II, and thermoelasticity of type III, respectively. When the theory of type I is linearized we obtain the classical system of thermoelasticity. The theory of type II does not allow the dissipation of the energy and it is usually known as thermoelasticity without energy dissipation. It proposes the coupling of two hyperbolic equations. We believe that the mathematical and physical analysis could help to clarify the applicability of these theories. Several previous contributions [15, 16, 17, 18, 19, 20, 21] and the present paper are adressed to this end. Here we shall investigate the system of thermoelasticity of type III. We shall first look at the one-dimensional situation. There the governing differential equations for the displacement (function) $u = u(t, x)$ and the temperature difference

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$\theta = \theta(t,x)$, $t \geq 0$, $x \in \Omega := (0,L)$, $L > 0$ fixed, are given by

$$\begin{aligned} u_{tt} - \alpha u_{xx} + \beta\theta_x &= 0 \quad \text{in } [0,\infty)\times(0,L), && (1.1)\\ \theta_{tt} - \delta\theta_{xx} + \gamma u_{ttx} - \kappa\theta_{txx} &= 0 \quad \text{in } [0,\infty)\times(0,L), && (1.2) \end{aligned}$$

where $\alpha, \beta, \gamma, \delta, \kappa$ are positive constants. Additionally, one has initial conditions

$$u(t=0) = u^0, \quad u_t(t=0) = u^1, \quad \theta(t=0) = \theta^0, \quad \theta_t(t=0) = \theta^1, \tag{1.3}$$

and boundary conditions. We shall consider either the boundary conditions

$$u(x=0) = u(x=L) = \theta(x=0) = \theta(x=L) = 0 \tag{1.4}$$

or

$$u(x=0) = u(x=L) = \theta_x(x=0) = \theta_x(x=L) = 0. \tag{1.5}$$

We shall show the exponential stability proving the exponential convergence of the associated energy for both boundary conditions.
Remark: Instead of the positivity of β and γ it is sufficient to require that $\beta\gamma > 0$ holds.

For simplicity we assumed a homogeneous medium being reflected in the constance of the coefficients in the differential equations, see below for a general non-homogeneous (and in more than one space dimension also possibly anisotropic) medium. The exponential stability results in one dimension for both types of boundary conditions also extend to the case of non-homogeneous media. The energy method used for boundary condition (1.4) — in contrast to the spectral method used for boundary condition (1.5) — can be carried over for both types of boundary conditions.

In more than one space dimension we do not expect more dissipation and decay, respectively, than in the hyperbolic-parabolic case of classical thermoelasticity. For the latter it is known, that one cannot hope for exponential stability in general, but radial symmetry (reference configuration Ω and data) suffices. Hence we shall prove here, that for thermoelasticity of type III still the radially symmetric case produces exponentially stable solutions. The differential equations — for the general non-homogeneous, anisotropic case — for the displacement (vector) u and the temperature difference θ are

$$\begin{aligned} u_{tt} - (2\mu+\lambda)\nabla\operatorname{div} u + \beta\nabla\theta &= 0 \quad \text{in } [0,\infty)\times\Omega, && (1.6)\\ \theta_{tt} - \delta\Delta\theta + \gamma\operatorname{div} u_{tt} - \kappa\Delta\theta_t &= 0 \quad \text{in } [0,\infty)\times\Omega, && (1.7) \end{aligned}$$

where μ and λ (the two Lamé constants) as well as β, δ, γ and κ are positive constants. Additionally, we have the initial conditions (1.3), and we consider the boundary conditions

$$u_{|\partial\Omega} = 0, \quad \theta_{|\partial\Omega} = 0. \tag{1.8}$$

Under the assumption that $\nabla \times u$ vanishes identically, which is satisfied in particular for the radially symmetric case, we shall obtain the exponential decay of the associated energy. In general, the behavior of the system is expected to be at least as "hyperbolic" as for classical thermoelasticity, that is, in general there will be no uniform decay, cp. [11] for classical thermoelasticity. Hence, the exponential stability in radially symmetric situations seems to be the optimal result one can hope for.

Finally, we shall prove the asymptotic equipartition of energy for the general non-homogeneous system

$$\begin{aligned} \rho\ddot{u}_i - (a_{ijkh}u_{h,k})_{,j} + (a_{ij}\theta)_{,j} &= 0 \quad \text{in } [0,\infty)\times\Omega, && (1.9)\\ \zeta\ddot{\theta} - (k_{ij}\theta_{,i})_{,j} + a_{ij}\ddot{u}_{i,j} - (b_{ij}\dot{\theta}_{,i})_{,j} &= 0 \quad \text{in } [0,\infty)\times\Omega, && (1.10)\end{aligned}$$

where a dot $\cdot$ and a subscript $,j$ denote differentiation with respect to t and x_j, respectively, and where the Einstein summation convention is used. The conditions on the coefficients will be made precise in Section 3. Again we have initial conditions (1.3) and the boundary conditions (1.8).

The paper is organized as follows: In Section 2 we prove the exponential stability first for one-dimensional models in Subsection 2.1, for the boundary conditions $u = \theta = 0$ using an energy method, and for the boundary conditions $u = \theta_x = 0$ using a spectral method. In Subsection 2.2 the exponential stability is proved for the radially symmetric case in two or three space dimensions. Section 3 presents the proof of the asymptotic equipartion of energy.

2 Exponential stability

The existence theory for the initial-boundary value problems considered in our paper is not too difficult (cp. [11, 15, 22, 23] for related approaches), but omitted here. Hence we shall assume in the sequel that all functions have the regularity needed for the manipulations performed.

2.1 One-dimensional models

In one space dimension we consider the equations (1.1), (1.2) together with the initial conditions (1.3) and either the boundary conditions (1.4) or (1.5). If $v := u_t$ denotes the velocity field, we obtain from (1.1)–(1.3)

$$\begin{aligned} v_{tt} - \alpha v_{xx} + \beta\theta_{xt} &= 0 \quad \text{in } [0,\infty)\times(0,L), && (2.1)\\ \theta_{tt} - \delta\theta_{xx} + \gamma v_{tx} - \kappa\theta_{txx} &= 0 \quad \text{in } [0,\infty)\times(0,L), && (2.2)\end{aligned}$$

$$v(t=0) = v^0 := u^1, \quad v_t(t=0) = v^1 := \alpha u^0_{xx} - \beta\theta^0_x, \quad \theta(t=0) = \theta^0, \quad \theta_t(t=0) = \theta^1. \quad (2.3)$$

The associated energy of first order is defined by

$$E_1(t) \equiv E_1(t; v, \theta) := \frac{1}{2}\int_0^L (\gamma v_t^2 + \alpha\gamma v_x^2 + \beta\theta_t^2 + \delta\beta\theta_x^2)\, dx.$$

We shall also need the second-order energy term

$$E_2(t) := E_1(t; v_t, \theta_t).$$

in the next subsection.

2.1.1 Energy methods for the boundary conditions $u = \theta = 0$

Using the energy method, i.e. appropriate multipliers to construct an appropriate Lyapunov functional, we shall prove the exponential decay of the energy term

$$F(t) := E_1(t) + E_2(t) + \frac{\gamma^2}{16\alpha\delta}\int_0^L \theta_{xx}^2(t,\cdot)dx.$$

The energy method was a useful tool for classical thermoelasticity, cp. [11], as well as for thermoelasticity with second sound, cp. [22, 23]. For the initial-boundary value problem (2.1)–(2.3) together with the boundary condition (1.4) we have

Theorem 2.1

$$\exists C_0 > 0 \qquad \exists d_0 > 0 \qquad \forall t \geq 0 : \qquad F(t) \leq C_0 e^{-d_0 t} F(0).$$

PROOF: Multiplying (2.1) by γv_t in L^2, and (2.2) by $\beta\theta_t$ in L^2, and summation yields

$$\frac{d}{dt}E_1 = -\beta\kappa\int_0^L \theta_{tx}^2 dx, \tag{2.4}$$

similarly,

$$\frac{d}{dt}E_2 = -\beta\kappa\int_0^L \theta_{ttx}^2. \tag{2.5}$$

Multiplying (2.2) by θ in L^2 we conclude

$$\begin{aligned}
\frac{d}{dt}\underbrace{\left\{\int_0^L (\theta_t\theta + \frac{\kappa}{2}\theta_x^2 + \gamma v_x\theta)\,dx\right\}}_{=:F_1(t)} &= \int_0^L \theta_t^2 dx - \delta\int_0^L \theta_x^2 dx + \gamma\int_0^L v_x\theta_t dx \\
&\leq \int_0^L \theta_t^2 dx - \delta\int_0^L \theta_x^2 dx + \varepsilon_1\int_0^L v_x^2 dx + \frac{c}{\varepsilon_1}\int_0^L \theta_{tx}^2 dx, \qquad (2.6)
\end{aligned}$$

where $c > 0$ (will) denote(s) various positive constants and $\varepsilon_1 > 0$ is still arbitrary.

Multiplying (2.1) by $-\frac{v_{xx}}{\alpha}$ in L^2 yields

$$\begin{aligned}
\frac{d}{dt}\left\{\frac{1}{\alpha}\int_0^L v_{tx}v_x dx\right\} &= \frac{1}{\alpha}\int_0^L v_{tx}^2 - \int_0^L v_{xx}^2 dx + \frac{\beta}{\alpha}\int_0^L \theta_{tx}v_{xx}dx \\
&\leq -\frac{2}{3}\int_0^L v_{xx}^2 dx + \frac{1}{\alpha}\int_0^L v_{tx}^2 dx + c\int_0^L \theta_{tx}^2 dx. \qquad (2.7)
\end{aligned}$$

Multiplying (2.2) by $\frac{3}{\alpha\gamma}v_{tx}$ in L^2 and partial integration leads to

$$\frac{3}{\alpha}\int_0^L v_{tx}^2 dx = -\frac{3\delta}{\alpha\gamma}\int_0^L \theta_x v_{txx}dx + \frac{3}{\alpha\gamma}\int_0^L \theta_{ttx}v_t dx + \frac{3\kappa}{\alpha\gamma}\int_0^L \theta_{txx}v_{tx}dx + \frac{3\delta}{\alpha\gamma}\Big[\theta_x v_{tx}\Big]_{x=0}^{x=L}$$

$$\leq \quad -\frac{3\delta}{\alpha\gamma}\frac{d}{dt}\left\{\int_0^L \theta_x v_{xx} dx\right\} + \frac{3\delta}{\alpha\gamma}\int_0^L \theta_{tx} v_{xx} dx + \frac{3}{\alpha\gamma}\int_0^L \theta_{ttx} v_t dx$$

$$+\frac{3\kappa}{\alpha\gamma}\int_0^L \theta_{txx} v_{tx} dx + c\Big\|\theta_x\Big\|_{L^\infty(\partial\Omega)}\Big\|v_{tx}\Big\|_{L^\infty(\partial\Omega)}.$$

Observing

$$\int_0^L \theta_{txx} v_{tx} dx = -\frac{d}{dt}\int_0^L \theta_{tx} v_{xx} dx + \int_0^L \theta_{ttx} v_{xx} dx + \Big[\theta_{tx} v_{tx}\Big]_{x=0}^{x=L},$$

and using the differential equation (2.1) again we obtain

$$\begin{aligned}
\frac{3}{\alpha}\int_0^L v_{tx}^2 dx \quad \leq \quad & -\frac{3}{\alpha\gamma}\frac{d}{dt}\left\{\frac{\delta}{\alpha}\int_0^L \theta_x v_{tt} dx + \kappa\int_0^L \theta_{tx} v_{xx} dx\right\} - \frac{3\delta\beta}{\alpha^2\gamma}\frac{d}{dt}\left\{\int_0^L \theta_x\theta_{tx} dx\right\} \\
& +\frac{1}{6}\int_0^L v_{xx}^2 dx + \frac{1}{\alpha}\int_0^L v_{tx}^2 dx + c\int_0^L \theta_{tx}^2 + \theta_{ttx}^2 dx \\
& +c\Big(\Big\|\theta_x\Big\|_{L^\infty(\partial\Omega)} + \Big\|\theta_{tx}\Big\|_{L^\infty(\partial\Omega)}\Big)\Big\|v_{tx}\Big\|_{L^\infty(\partial\Omega)}. \qquad (2.8)
\end{aligned}$$

Adding (2.8) to (2.7) we get

$$\begin{aligned}
& \underbrace{\frac{d}{dt}\left\{\int_0^L (\frac{1}{\alpha} v_{tx} v_x + \frac{3\delta}{\alpha^2\gamma}\theta_x v_{tt} + \frac{3\delta\beta}{\alpha^2\gamma}\theta_x\theta_{tx}) dx + \frac{3\kappa}{\alpha\gamma}\int_0^L \theta_{tx} v_{xx} dx\right\}}_{=:F_2(t)} \\
& \leq \quad -\frac{1}{2}\int_0^L v_{xx}^2 dx - \frac{1}{\alpha}\int_0^L v_{tx}^2 dx + c\left(\int_0^L \theta_{tx}^2 + \theta_{ttx}^2 dx\right) \\
& \qquad +c\Big(\Big\|\theta_x\Big\|_{L^\infty(\partial\Omega)} + \Big\|\theta_{tx}\Big\|_{L^\infty(\partial\Omega)}\Big)\Big\|v_{tx}\Big\|_{L^\infty(\partial\Omega)}. \qquad (2.9)
\end{aligned}$$

Multiplication of (2.2) by $\frac{\theta_{xx}}{\delta}$ in L^2 yields

$$\int_0^L \theta_{xx}^2 dx \leq -\frac{1}{\delta}\int_0^L \theta_{ttx}\theta_x dx + \frac{\gamma^2}{2\delta^2}\int_0^L v_{tx}^2 dx + \frac{1}{2}\int_0^L \theta_{xx}^2 dx - \frac{d}{dt}\left\{\frac{\kappa}{\delta}\int_0^L \theta_{xx}^2 dx\right\},$$

which implies

$$\frac{d}{dt}\left\{\frac{\kappa}{\delta}\int_0^L \theta_{xx}^2 dx\right\} \leq -\frac{1}{2}\int_0^L \theta_{xx}^2 dx + c\int_0^L \theta_{ttx}^2 + \theta_x^2 dx + \frac{\gamma^2}{2\delta^2}\int_0^L v_{tx}^2 dx. \qquad (2.10)$$

Applying the Sobolev imbedding $W^{1,1}((0,L)) \leq L^\infty((0,L))$ we get

$$\Big\|\theta_x\Big\|_{L^\infty(\partial\Omega)}^2 \leq \varepsilon_2^2\int_0^L \theta_{xx}^2 dx + \frac{c}{\varepsilon_2^2}\int_0^L \theta_x^2 dx, \qquad (2.11)$$

and, using (2.2) again,

$$\left\|\theta_{tx}\right\|^2_{L^\infty(\partial\Omega)} \leq c\varepsilon_2^2 \int_0^L (\theta_{tt}^2 + \theta_{xx}^2 + v_{tx}^2)dx + \frac{c}{\varepsilon_2^2} \int_0^L \theta_{tx}^2 dx, \tag{2.12}$$

where $\varepsilon_2 > 0$ is still arbitrary.
Combining (2.11) and (2.12) with (2.9) we obtain

$$\begin{aligned} \frac{d}{dt}F_2(t) \quad &\leq \quad -\frac{1}{2}\int_0^L v_{xx}^2 dx - \frac{1}{\alpha}\int_0^L v_{tx}^2 dx + c\int_0^L \theta_{tx}^2 + \theta_{ttx}^2 dx \\ &\quad + c\varepsilon_2 \int_0^L (\theta_{xx}^2 + \theta_{tt}^2)dx + \frac{c}{\varepsilon_2^3}\int_0^L \theta_x^2 dx + \varepsilon_2 \left\|v_{tx}\right\|_{L^\infty(\partial\Omega)}. \end{aligned} \tag{2.13}$$

The term $\left\|v_{tx}\right\|_{L^\infty(\partial\Omega)}$ is also dealt with as similar terms in classical thermoelasticity (cp. [11, Section 3.1.1]).
Let

$$\varphi(x) := 1 - \frac{2}{L}x.$$

Differentiating (2.1) with respect to t and multiplying the result by $\frac{2\varepsilon_2}{\alpha}\varphi v_{tx}$ in L^2 yields

$$\begin{aligned} \frac{d}{dt}\left\{\frac{2\varepsilon_2}{\alpha}\int_0^L v_{tt}\varphi v_{tx} dx\right\} \quad &= \quad -\frac{\varepsilon_2}{\alpha}\int_{\partial\Omega} v_{tt}^2 ds - \varepsilon_2 \int_{\partial\Omega} v_{tx}^2 ds + \frac{4\varepsilon_2}{\alpha L}\int_0^L v_{tt}^2 dx \\ &\quad + \frac{4\varepsilon_2}{L}\int_0^L v_{tx}^2 dx - \frac{2\varepsilon_2\beta}{\alpha}\int_0^L \theta_{ttx}\varphi v_{tx}\, dx. \end{aligned} \tag{2.14}$$

If

$$F_3(t) := F_2(t) + \frac{2\varepsilon_2}{\alpha}\int_0^L v_{tt}\varphi v_{tx} dx$$

then we obtain from (2.13), (2.14) choosing ε_2 small enough (only depending on L, α)

$$\begin{aligned} \frac{d}{dt}F_3(t) \quad &\leq \quad -\frac{1}{4}\int_0^L v_{xx}^2 dx - \frac{1}{4\alpha}\int_0^L v_{tx}^2 dx + c\int_0^L \theta_{tx}^2 + \theta_{ttx}^2 dx \\ &\quad + c\varepsilon_2 \int_0^L \theta_{xx}^2 dx + \frac{c}{\varepsilon_2^3}\int_0^L \theta_x^2 dx + \frac{4\varepsilon_2}{\alpha L}\int_0^L v_{tt}^2 dx. \end{aligned}$$

Observing from (2.1) that

$$v_{tt}^2 \leq 2\alpha^2 v_{xx}^2 + 2\beta^2\theta_{tx}^2,$$

we conclude

$$\begin{aligned} \frac{d}{dt}F_3(t) \quad &\leq \quad -\frac{1}{16\alpha^2}\int_0^L v_{tt}^2 dx - \frac{1}{8}\int_0^L v_{xx}^2 dx - \frac{1}{4\alpha}\int_0^L v_{tx}^2 \\ &\quad + c\int_0^L \theta_{tx}^2 + \theta_{ttx}^2 dx + c\varepsilon_2 \int_0^L \theta_{xx}^2 dx + \frac{c}{\varepsilon_2^3}\int_0^L \theta_x^2 dx. \end{aligned} \tag{2.15}$$

Now, multiplying (2.1) by v in L^2, we get

$$\frac{d}{dt}\int_0^L v_t v\,dx \le -\frac{\alpha}{2}\int_0^L v_x^2 dx + \int_0^L v_t^2 dx + c\int_0^L \theta_{tx}^2\,dx. \tag{2.16}$$

We have

$$\int_0^L v_t^2 dx \le c_p \int_0^L v_{tx}^2, \qquad \int_0^L \theta_t^2 + \theta_{tt}^2 dx \le c_p \int_0^L \theta_{tx}^2 + \theta_{ttx}^2 dx, \tag{2.17}$$

where $c_p := L^2/\pi^2$.
The desired Lyapunov functional $G = G(t)$ is now given by

$$G(t) := N(E_1(t) + E_2(t)) + MF_1(t) + F_3(t) + \frac{1}{4\alpha c_p}\int_0^L v_t(t,x)v(t,x)dx,$$

where $N, M > 1$ have to be chosen appropriately, namely:
Choosing ε_2 small enough (essentially for the θ_{xx}-terms), then M large enough (essentially for the θ_x-term), then ε_1 small enough (essentially for the v_x-term), finally choosing N large enough, we conclude from (2.4)–(2.6), (2.9), (2.10), (2.15)–(2.17)

$$\frac{d}{dt}G(t) \le -c_1 F(t) \tag{2.18}$$

with $c_1 > 0$ being constant. Since for sufficiently large N we have

$$\exists c_2, c_3 > 0 \qquad \forall t \ge 0: \qquad c_2 F(t) \le G(t) \le c_3 F(t) \tag{2.19}$$

we conclude from (2.18)

$$\frac{d}{dt}G(t) \le -d_0 G(t)$$

with some $d_0 > 0$, hence G decays exponentially,

$$G(t) \le e^{-d_0 t} G(0)$$

and the proof is completed using (2.19) once more.

Q.E.D.

Remark: For the boundary conditions studied in the next section, the energy method also applies in an even simpler manner since the boundary terms would disappear. Then one has to work only with the first-order energy $E_1(t)$ that can be shown to decay exponentially, without any need to work with higher-order energy terms as above.

2.1.2 Spectral methods for the boundary conditions $u = \theta_x = 0$

We now consider the system (2.1)–(2.3) together with the boundary condition (1.5). Without loss of generality we assume $L = \pi$. Now, $(v = 0, \theta = c_0 t)$, $c_0 \neq 0$ can be a solution for which the first-order energy $E_1(t) = \frac{\beta c_0^2 \pi}{2}$ does not decay at all. Observing from (2.2), (1.5) that $\frac{d}{dt}\int_0^\pi \theta(t,x)dx = 0$, we can define

$$\bar\theta(t,x) := \theta(t,x) - \frac{t}{\pi}\int_0^\pi \theta^1(x)dx - \frac{1}{\pi}\int_0^\pi \theta^0(x)dx,$$

and $(v,\bar\theta)$ satisfy the same differential equations and boundary conditions as (v,θ), but now we have

$$\int_0^\pi \bar\theta(t,x)dx = 0 \tag{2.20}$$

for all t. In the sequel, we work with $\bar\theta$ but write θ again for simplicity.

For the boundary conditions under consideration, the ansatz

$$v(t,x) = \sum_{n=1}^{\infty} a_n(t)\sin(nx), \quad \theta(t,x) = \sum_{n=1}^{\infty} b_n(t)\cos(nx)$$

is compatible with the differential equations and with the boundary conditions (as well as with (2.20). The coefficients (a_n, b_n) have to satisfy for all $n \in \mathbb{N}$

$$\begin{aligned} \ddot a_n + \alpha n^2 a_n - \beta n \dot b_n &= 0, \\ \ddot b_n + \delta n^2 b_n + \gamma n \dot a_n + \kappa n^2 \dot b_n &= 0, \end{aligned}$$

which implies that both a_n and b_n satisfy the following differential equation

$$\frac{d^4}{dt^4}w(t) + \kappa n^2 \frac{d^3}{dt^3}w(t) + (\alpha+\delta+\beta\gamma)n^2\frac{d^2}{dt^2}w(t) + \alpha\kappa n^4 \frac{d}{dt}w(t) + \alpha\delta n^4 = 0.$$

To assure the exponential stability of the first-order energy, we shall prove that the roots of the associated characteristic polynomial

$$\chi_n(x) := x^4 + \kappa n^2 x^3 + (\alpha+\delta+\beta\gamma)n^2x^2 + \alpha\kappa n^4 x + \alpha\delta n^4$$

have a strictly negative real part, i.e. we shall prove

Theorem 2.2

$$\exists\, \varepsilon_* > 0\; \forall n \in \mathbb{N}\; \forall\, x, \chi_n(x) = 0 : \; \Re x < -\varepsilon_*.$$

PROOF: It is equivalent to prove that for $y := x + \varepsilon$ any root of the polynomial

$$p_n(y) := (y-\varepsilon)^4 + \kappa n^2 (y-\varepsilon)^3 + (\alpha+\delta+\beta\gamma)n^2(y-\varepsilon)^2 + \alpha\kappa n^4(y-\varepsilon) + \alpha\delta n^4$$

has negative real part for some (sufficiently small) ε. Rewriting

$$\begin{aligned} p_n(y) = y^4 + [\underbrace{-4\varepsilon}_{=:P_3(\varepsilon)} + \kappa n^2]y^3 + [\underbrace{6\varepsilon^2 - 3\kappa n^2 \varepsilon}_{=:P_2(\varepsilon)} + (\alpha+\delta+\beta\gamma)n^2]y^2 + \\ [\underbrace{-4\varepsilon^3 + 3\kappa n^2\varepsilon^2 - 2(\alpha+\delta+\beta\gamma)n^2\varepsilon}_{=:P_1(\varepsilon)} + \alpha\kappa n^4]y + \end{aligned}$$

$$[\underbrace{\varepsilon^4 - \kappa n^2\varepsilon^3 + (\alpha+\delta+\beta\gamma)n^2\varepsilon^2 - \alpha\kappa n^4\varepsilon}_{=:P_0(\varepsilon)} + \alpha\delta n^4]$$
$$\equiv y^4 + \sum_{j=0}^{3} l_j y^j$$

we can apply the Hurwitz criterion which assures that the real parts of all the roots of p_n are negative if the following two conditions on the coefficients and the associated Hurwitz determinants are satisfied:

$$(i) \quad l_j > 0, \quad j = 0,1,2,3.$$
$$(ii) \quad H_2 := \begin{vmatrix} l_1 & l_0 \\ l_3 & l_2 \end{vmatrix} > 0, \quad H_3 := \begin{vmatrix} l_1 & l_0 & 0 \\ l_3 & l_2 & l_1 \\ 0 & 1 & l_3 \end{vmatrix} > 0.$$

It is easy to see that the first condition (i) is satisfied if

$$\varepsilon \leq \varepsilon_1$$

for some $\varepsilon_1 > 0$ depending at most on the coefficients. We compute

$$\begin{aligned} H_2 &= [\alpha\kappa n^4 + P_1(\varepsilon)][(\alpha+\delta+\beta\gamma)n^2 + P_2(\varepsilon)] - [\alpha\delta n^4 + P_0(\varepsilon)][\kappa n^2 + P_3(\varepsilon)] \\ &= (\alpha+\beta\gamma)\alpha\kappa n^6 + Q_1(\varepsilon) \end{aligned}$$

with

$$Q_1(\varepsilon) := \alpha\kappa n^4 P_2(\varepsilon) + (\alpha+\delta+\beta\gamma)n^2 P_1(\varepsilon) + P_1(\varepsilon)P_2(\varepsilon) - \alpha\delta n^4 P_3(\varepsilon) - P_0(\varepsilon)P_3(\varepsilon).$$

Since P_1, P_2, P_3 are at most quadratic in n, and P_0 is at most of order 4 in n, we conclude that Q_1 is at most of order 6 in n, with factor ε in each summand. Therefore, we have that H_2 is positive if

$$\varepsilon \leq \varepsilon_2$$

for some $\varepsilon_2 > 0$ depending at most on the coefficients.

Finally, we consider H_3 and observe that

$$\begin{aligned} H_3 &= l_3 H_2 - l_1^2 \\ &= [\kappa n^2 + P_3(\varepsilon)][(\alpha+\beta\gamma)\alpha\kappa n^6 + Q_1(\varepsilon)] - [\alpha\kappa n^4 + P_1(\varepsilon)]^2 \\ &= \beta\gamma\alpha\kappa^2 n^8 + Q_2(\varepsilon), \end{aligned}$$

where $Q_2(\varepsilon)$ is at most of order 8 in n, with factor ε in each summand. Therefore, we have that H_3 is positive if

$$\varepsilon \leq \varepsilon_3$$

for some $\varepsilon_3 > 0$ depending at most on the coefficients.

Choosing

$$\varepsilon_* := \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} > 0$$

the conditions (i) and (ii) of the Hurwitz criterion are satisfied and we have completed the proof of the Theorem.

Q.E.D.

2.2 Radial symmetry in $\mathbb{R}^n$, $n = 2, 3$

Already in classical thermoelasticity the dissipation through heat conduction is not strong enough to lead to exponential decay in general, for example if there exist reflecting rays in the reference configuration Ω, see the work of Koch [12] and Lebeau and Zuazua [13], or [11]. On the other hand it was possible to show also for thermoelasticity with second sound that it is not worse than classical thermoelasticity. That is, for radially symmetric situations — domain Ω *and* data —, u and θ decay uniformly to zero. We shall prove that still the same holds for thermoelasticity of type III considered here.

In the homogeneous isotropic case the differential equations are given by (1.6), (1.7) with initial conditions (1.3) and boundary conditions (1.8). Let $v := u_t$ denote the velocity field and assume

$$\nabla \times v = 0 \qquad \text{in} \qquad [0,\infty) \times \Omega. \tag{2.21}$$

If $\alpha := 2\mu + \lambda$ we obtain from (1.6)–(1.8)

$$v_{tt} - \alpha \nabla \operatorname{div} v + \beta \nabla \theta_t \;=\; 0 \quad \text{in } [0,\infty) \times \Omega, \tag{2.22}$$

$$\theta_{tt} - \delta\Delta\theta + \gamma \operatorname{div} v_t - \kappa \Delta \theta_t \;=\; 0 \quad \text{in } [0,\infty) \times \Omega, \tag{2.23}$$

$$v(t=0) = v^0 := u^1, \quad v_t(t=0) = v^1 := \alpha\nabla \operatorname{div} u^0 - \beta\nabla\theta^0, \quad \theta(t=0) = \theta^0, \quad \theta_t(t=0) = \theta^1. \tag{2.24}$$

Remark: We recall that the rotation in $\mathbb{R}^2$ is defined for a vector field $w : \mathbb{R}^2 \to \mathbb{R}^2$ to be the scalar

$$\operatorname{rot}\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} := \partial_1 w^2 - \partial_2 w^1, \qquad \left(\partial_j = \frac{\partial}{\partial x_j}\right),$$

and for the scalar $g : \mathbb{R}^2 \to \mathbb{R}$ to be the vector field

$$\operatorname{rot} g := \begin{pmatrix} \partial_2 g \\ -\partial_1 g \end{pmatrix}.$$

Then the formula

$$\Delta = \nabla \operatorname{div} - \operatorname{rot}\operatorname{rot}$$

holds in $\mathbb{R}^2$ as in $\mathbb{R}^3$. Moreover, we have for any $u \in (W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega))^n$ satisfying (2.21)

$$\int_\Omega |\nabla u|^2 dx = \int_\Omega |\operatorname{div} u|^2 dx. \tag{2.25}$$

The first-order energy is given by

$$E_1(t) := \frac{1}{2}\int_\Omega (\gamma |v_t|^2 + \alpha\gamma|\nabla v|^2 + \beta\theta_t^2 + \delta\beta|\nabla\theta|^2)dx \equiv E_1(t; v, \theta)$$

the second-order energy by

$$E_2(t) := E_1(t; v_t, \theta_t).$$

Let

$$E(t) := E_1(t) + E_2(t) + \frac{\gamma^2}{16\alpha\delta}\int_\Omega |\Delta\theta|^2 dx$$

We shall prove for the initial-boundary value-problem (2.22)– (2.24), (1.8):

Theorem 2.3 *If (2.21) holds, then we have*

$$\exists C_1 > 0 \quad \exists d_1 > 0 \quad \forall t \geq 0: \qquad E(t) \leq C_1 e^{d_1 t} E(0).$$

Corollary 2.4 *Let Ω be radially symmetric, and let the initial data $u_0, u_1, \theta_0, \theta_1$ (resp. $v_0, v_1, \theta_0, \theta_1$) be radially symmetric. Then we have*

$$\exists C_1 > 0 \quad \exists d_1 > 0 \quad \forall t \geq 0: \qquad E(t) \leq C_1 e^{-d_1 t} E(0).$$

PROOF of Corollary 2.4: For $R \in 0(2)$ (orthogonal group in $\mathbb{R}^2$) resp. $R \in SO(3)$ (special orthogonal group in $\mathbb{R}^3$) let

$$w(t,x) := R' v(t, Rx), \quad \psi(t,x) := \theta(t, Rx),$$

R' denoting the transposed matrix. Then (w, ψ) satisfies the same initial-boundary value problem (2.22)–(2.24), (1.8), hence, by uniqueness $(w,\psi) = (v, \theta)$. Thus (w, ψ) is radially symmetric, in particular $\nabla \times w = 0$. Now Theorem 2.3 applies.

Q.E.D.

Remark: For a characterization of radial symmetry cp. [23, 10] or [11].
The following proof is an adaption of those in [23, 10], in combination with the proof of Theorem 2.1.

PROOF of Theorem 2.3:
As in Section 2.1.1 — see (2.4)–(2.10) – we obtain using (2.25)

$$\frac{d}{dt} E_1 = -\beta\kappa \int_\Omega |\nabla \theta_t|^2 dx, \tag{2.26}$$

$$\frac{d}{dt} E_2 = -\beta\kappa \int_\Omega |\nabla \theta_{tt}|^2 dx, \tag{2.27}$$

$$\frac{d}{dt} \underbrace{\left\{ \int_\Omega (\theta_t \theta + \frac{\kappa}{2} |\nabla \theta|^2 + \gamma \operatorname{div} v\, \theta)\, dx \right\}}_{=:F_1(t)}$$
$$\leq \int_\Omega \theta_t^2 - \delta \int_\Omega |\nabla \theta|^2 + \varepsilon_1 \int_\Omega |\operatorname{div} v|^2 dx + \frac{c}{\varepsilon_1} \int_\Omega |\nabla \theta_t|^2 dx, \tag{2.28}$$

where $\varepsilon_1 > 0$ is still arbitrary,

$$\frac{d}{dt} \underbrace{\left\{ \int_\Omega (\frac{1}{\alpha} \nabla v_t \nabla v + \frac{3\delta}{\alpha^2 \gamma} \nabla \theta v_{tt} + \frac{3\delta\beta}{\alpha^2\gamma} \nabla\theta \nabla\theta_t + \frac{3\kappa}{\alpha\gamma} \nabla \theta_t \nabla \operatorname{div} v) dx \right\}}_{=:F_2(t)}$$
$$\leq -\frac{1}{2} \int_\Omega |\Delta v|^2 dx - \frac{1}{\alpha} \int_\Omega |\nabla v_t|^2 dx + c \int_\Omega |\nabla \theta_t|^2 + |\nabla \theta_{tt}|^2 dx$$
$$+ \varepsilon_2 \int_{\partial\Omega} |\operatorname{div} v_t|^2 ds + \frac{c}{\varepsilon_2} \int_{\partial\Omega} |\partial_\nu \theta|^2 + |\partial_\nu \theta_t|^2 ds, \tag{2.29}$$

where $\varepsilon_2 > 0$ is still arbitrary,

$$\frac{d}{dt}\left\{\int_\Omega \frac{\kappa}{\delta}|\Delta\theta|^2 dx\right\} \le -\frac{1}{2}\int_\Omega |\Delta\theta|^2 dx + c\int_\Omega |\nabla\theta_{tt}|^2 + |\nabla\theta|^2 dx + \frac{\gamma^2}{2\delta^2}\int_\Omega |\operatorname{div} v_t|^2 dx. \tag{2.30}$$

Since $\partial\Omega$ is assumed to be smooth, there is a vector field $\sigma \in (C^1(\overline{\Omega}))^3$ such that σ equals the exterior normal ν on $\partial\Omega$. Then

$$\int_{\partial\Omega} |\partial_\nu\theta|^2 ds = \int_{\partial\Omega} |\sigma\nabla\theta|^2 ds,$$

and we conclude from the imbedding $W^{1,1}(\Omega) \subset L^1(\partial\Omega)$ and elliptic regularity that

$$\int_{\partial\Omega} |\partial_\nu\theta|^2 ds \le \varepsilon_2^2 \int_\Omega |\Delta\theta|^2 dx + \frac{c}{\varepsilon_2^2}\int_\Omega |\nabla\theta|^2 dx, \tag{2.31}$$

and, using the differential equation (2.23) again,

$$\int_{\partial\Omega} |\partial_\nu\theta_t|^2 ds \le \varepsilon_2^2 \int_\Omega (|\theta_{tt}|^2 + |\Delta\theta|^2 + |\operatorname{div} v_t|^2) dx + \frac{c}{\varepsilon_2^2}\int_\Omega |\nabla\theta_t|^2 dx. \tag{2.32}$$

Combining (2.31) and (2.32) with (2.29) we obtain

$$\begin{aligned}\frac{d}{dt}F_2(t) \le\; & -\frac{1}{2}\int_\Omega |\Delta v|^2 dx - \frac{1}{\alpha}\int_\Omega |\nabla v_t|^2 dx + c\int_\Omega |\nabla\theta_t|^2 + |\nabla\theta_{tt}|^2 dx \\ & + c\varepsilon_2 \int_\Omega |\Delta\theta|^2 dx + \frac{c}{\varepsilon_2^3}\int_\Omega |\nabla\theta|^2 dx + \varepsilon_2 \int_{\partial\Omega} |\operatorname{div} v_t|^2 ds.\end{aligned} \tag{2.33}$$

The last boundary integral can be dealt with as in [23, 10] using [11, Lemma 4.1]. We get

$$\begin{aligned}\frac{d}{dt}\frac{2\varepsilon_2}{\mu+\lambda}\int_\Omega v_{tt}\sigma_k\partial_k v_t dx \le\; & -\varepsilon_2 \int_{\partial\Omega} |\operatorname{div} v_t|^2 ds + c\varepsilon_2 \int_\Omega |v_{tt}|^2 + |\nabla v_t|^2 dx \\ & - \frac{2\beta\varepsilon_2}{\mu+\lambda}\int_\Omega \nabla\theta_{tt}\sigma_k\partial_k v_t dx.\end{aligned} \tag{2.34}$$

Let

$$F_3(t) := F_2(t) + \frac{2\varepsilon_2}{\mu+\lambda}\int_\Omega v_{tt}\sigma_k\partial_k v_t dx.$$

Choosing ε_2 small enough it follows from (2.33), (2.34) that

$$\begin{aligned}\frac{d}{dt}F_3(t) \le\; & -\frac{1}{16\alpha^2}\int_\Omega |v_{tt}|^2 dx - \frac{1}{8}\int_\Omega |\Delta v|^2 dx - \frac{1}{4\alpha}\int_\Omega |\nabla v_t|^2 dx + c\int_\Omega |\nabla\theta_t|^2 + |\nabla\theta_{tt}|^2 dx \\ & + c\varepsilon_2 \int_\Omega |\Delta\theta|^2 dx + \frac{c}{\varepsilon_2^3}\int_\Omega |\nabla\theta|^2 dx.\end{aligned} \tag{2.35}$$

As in (2.16) we have

$$\frac{d}{dt}\int_\Omega v_t v\,dx \le -\frac{\alpha}{2}\int_\Omega |\nabla v|^2 dx + \int_\Omega |v_t|^2 + c\int_\Omega |\nabla\theta_t|^2 dx. \tag{2.36}$$

The Lyapunov functional $G = G(t)$ is defined in analogy to Section 2.1.1 by

$$G(t) := N(E_1(t) + E_2(t)) + MF_1(t) + F_3(t) + \frac{1}{4\alpha c_p}\int_\Omega v_t(t,x)v(t,x)dx,$$

c_p being now the Poincaré constant in $\int_\Omega |w|^2 dx \le c_p \int_\Omega |\nabla w|^2 dx$ for $w \in (W_0^{1,2}(\Omega))^n$. Choosing N, M large enough and $\varepsilon_1, \varepsilon_2$ small enough similar to Section 2.1.1, we obtain

$$\frac{d}{dt}G(t) \le -c_1 E(t)$$

with some constant $c > 0$, and it follows the assertion of the Theorem.

Q.E.D.

3 Equipartition of energy

The partition of the energy for several kinds of elastic and thermoelastic problems was developed by several authors [4, 14, 3]. We study the general, possibly non-homogeneous, anisotropic but with a center of symmetry case, where we have the differential equations (1.9),(1.10), i.e.

$$\rho\ddot{u}_i - (a_{ijkh}u_{h,k})_{,j} + (a_{ij}\theta)_{,j} = 0 \quad \text{in } [0,\infty)\times\Omega, \tag{3.1}$$
$$\zeta\ddot{\theta} - (k_{ij}\theta_{,i})_{,j} + a_{ij}\ddot{u}_{i,j} - (b_{ij}\dot{\theta}_{,i})_{,j} = 0 \quad \text{in } [0,\infty)\times\Omega, \tag{3.2}$$

together with the initial conditions (1.3) and the boundary conditions (1.8). Here the Einstein summation convention is used, and the indices run from 1 to n, n being the space dimension. (u_i) is the displacement vector ρ denotes the mass density; ζ denotes the specific heat; (a_{ijkh}) is the elasticity tensor; (a_{ij}) is the matrix of thermal expansion; (b_{ij}) is the matrix of thermal conductivity and (k_{ij}) is a matrix which is characteristic of this theory. All coefficients are assumed to depend smoothly on $x \in \Omega$ and to satisfy:
(i) There are positive constant ρ_0, ζ_0 such that $\rho \ge \rho_0$ and $\zeta \ge \zeta_0$.
(ii) The tensors a_{ijkl}, k_{ij} and b_{ij} have the following symmetries:

$$a_{ijrs} = a_{rsij}, \;\; k_{ij} = k_{ji}, \;\; b_{ij} = b_{ji}.$$

(iii) There exist positive constants a_0, k_0, b_0 such that

$$a_{ijrs}e_{ij}e_{sr} \ge a_0 e_{ij}e_{ij},$$

$$k_{ij}\tau_{,i}\tau_{,j} \ge k_0\tau_{,i}\tau_{,i},$$

$$b_{ij}\tau_{,i}\tau_{,j} \ge b_0\tau_{,i}\tau_{,i}.$$

for all symmetric tensors (e_{ij}) and all vectors (τ_i).
Introducing τ^0 as solution to

$$(k_{ij}\tau^0_{,i})_{,j} = \rho\theta^1 + a_{ij}u^1_{i,j}, \quad \tau^0_{|\partial\Omega} = 0,$$

and defining the common new thermal variable τ by

$$\tau(t,x) := \int_0^t \theta(s,x)ds + \tau^0(x) \tag{3.3}$$

we obtain from (3.1)–(3.3) the following system of differential equations

$$\rho\ddot{u}_i - (a_{ijkh}u_{h,k})_{,j} + (a_{ij}\theta)_{,j} = 0 \quad \text{in } [0,\infty)\times\Omega, \tag{3.4}$$

$$\zeta\ddot{\tau} - (k_{ij}\tau_{,i})_{,j} + a_{ij}\dot{u}_{i,j} - (b_{ij}\dot{\theta}_{,i})_{,j} = 0 \quad \text{in } [0,\infty)\times\Omega, \tag{3.5}$$

together with the initial conditions

$$u(t=0) = u^0, \quad u_t(t=0) = u^1, \quad \tau(t=0) = \tau^0, \quad \tau_t(t=0) = \tau^1 := \theta^0, \tag{3.6}$$

and the boundary conditions

$$u_{|\partial\Omega} = 0, \quad \tau_{|\partial\Omega} = 0. \tag{3.7}$$

We recall an energy equality which is well known, cp. the energy estimates in the previous sections. If (u,τ) is a solution of the initial-boundary value problem (3.4)–(3.7), then the equality

$$\begin{aligned} E(t) &:= \frac{1}{2}\int_\Omega (c\theta^2 + k_{ij}\tau_{,i}\tau_{,j} + \rho\dot{u}_i\dot{u}_i + a_{ijkl}u_{i,j}u_{l,k})(t,x)dx + 2\int_0^t\int_\Omega (b_{ij}\theta_{,i}\theta_{,j})(s,x)dxds \\ &= E(0), \quad t \geq 0, \end{aligned} \tag{3.8}$$

is satisfied.

Lemma 3.1 *Let (u,τ) be a solution of the initial-boundary value problem (3.4)–(3.7), then the equality*

$$\begin{aligned} &2\int_\Omega \rho\dot{u}_i u_i dx + \int_\Omega b_{ij}\tau_{,i}\tau_{,j}dx + 2\int_\Omega k_{ij}\eta_{,i}\tau_{,j}dx \\ &= 2\int_0^t\int_\Omega (\rho\dot{u}_i\dot{u}_i + k_{ij}\tau_{,i}\tau_{,j} - a_{ijkl}u_{i,j}u_{l,k} - c\theta^2)dxds + 2\int_\Omega \rho\dot{u}^0_i u^0_i dx \\ &\quad +2\int_0^t\int_\Omega \Phi\theta(s)dxds + \int_\Omega b_{ij}\tau^0_{,i}\tau^0_{,j}dx, \end{aligned}$$

is satisfied, where

$$\eta(t,\cdot) := \int_0^t \tau(s,\cdot)ds \text{ and } \Phi := a_{ij}u^0_{i,j} - (b_{ij}\tau^0_{,j})_{,i} + c\theta^0.$$

PROOF: Using the differential equation (3.4) we obtain

$$\int_\Omega \rho u_i\dot{u}_i dx = \int_\Omega \rho u^0_i\dot{u}^0_i dx + \int_0^t\int_\Omega (\rho\dot{u}_i(s)\dot{u}_i(s) + \rho u_i(s)\ddot{u}_i(s))dxds$$

$$= \int_\Omega \rho u^0_i\dot{u}^0_i dx + \int_0^t\int_\Omega (\rho\dot{u}_i(s)\dot{u}_i(s) - a_{ijhk}u_{i,j}u_{k,h} + a_{ij}u_{i,j}\theta)dxds.$$

After an integration of the equation (3.5), we see

$$a_{ij}u_{i,j} = a_{ij}u^0_{i,j} - c\theta + (k_{ij}\eta_{,i})_{,j} + (b_{ij}\tau_{,i})_{,j} + c\dot{\tau}^0 - (b_{ij}\tau^0_{,i})_{,j}.$$

Thus, we get

$$\int_\Omega \rho u_i\dot{u}_i dx + \int_0^t\int_\Omega (k_{ij}\eta_{,j}\theta_{,i} + b_{ij}\tau_{,j}\theta_{,i})dxds$$
$$= \int_0^t\int_\Omega (\rho\dot{u}_i\dot{u}_i - a_{ijhk}u_{i,j}u_{k,h} - c\theta^2)dxds + \int_\Omega \rho u^0_i\dot{u}^0_i dx + \int_0^t\int_\Omega \Phi\theta(s)dxds.$$

If we recall that

$$\frac{d}{ds}(k_{ij}\eta_{,j}\tau_{,i}) = k_{ij}\eta_{,j}\theta_{,i} + k_{ij}\tau_{,i}\tau_{,j},$$

and

$$\frac{d}{ds}(\frac{1}{2}b_{ij}\tau_{,j}\tau_{,i}) = b_{ij}\tau_{,j}\theta_{,i}.$$

the lemma is proved.

Q.E.D.

Lemma 3.2 *Let (u,τ) be a solution of the initial-boundary value problem (3.4)–(3.7), then the equality*

$$2\int_\Omega \rho\dot{u}_i u_i dx + 2\int_\Omega k_{ij}\eta_{,i}\tau_{,j}dx + \int_\Omega b_{ij}\tau_{,i}\tau_{,j}dx$$
$$= \int_\Omega \rho(\dot{u}^0_i u_i(2t) + u^0_i\dot{u}_i(2t))dx + \int_\Omega b_{ij}\tau^0_{,i}\tau_{,j}(2t)dx$$
$$+ \int_0^t\int_\Omega \Phi\theta(s)dxds - \int_t^{2t}\int_\Omega \Phi\theta(s)dxds$$

is satisfied.

Proof: First, we recall the equality

$$2\int_\Omega \rho u_i(t)\dot{u}_i(t)dx = \int_\Omega \rho(\dot{u}^0_i u_i(2t) + u^0_i\dot{u}_i(2t))dx$$
$$+ \int_0^t\int_\Omega (\rho u_i(2t-s)\ddot{u}_i(s) - \rho u_i(s)\ddot{u}_i(2t-s))dxds.$$

It follows that

$$2\int_\Omega \rho u_i(t)\dot{u}_i(t)dx = \int_\Omega \rho(\dot{u}^0_i u_i(2t) + u^0_i\dot{u}_i(2t))dx$$
$$+ \int_0^t\int_\Omega (a_{ij}\theta(s)u_{i,j}(2t-s) - a_{ij}\theta(2t-s)u_{i,j}(s))dxds.$$

But

$$\int_\Omega (a_{ij}\theta(s)u_{i,j}(2t-s) - a_{ij}\theta(2t-s)u_{i,j}(s))dx$$
$$= -\int_\Omega k_{ij}\eta_{,i}(2t-s)\theta_{,j}(s)dx - \int_\Omega b_{ij}\tau_{,i}(2t-s)\theta_{,j}(s)dx$$
$$+ \int_\Omega k_{ij}\eta_{,i}(s)\theta_{,j}(2t-s)dx + \int_\Omega b_{ij}\tau_{,i}(s)\theta_{,j}(2t-s)dx$$

$$+\int_\Omega (\Phi\theta(s)-\Phi\theta(2t-s))dx.$$

From the previous equalities, we obtain

$$2\int_\Omega \rho u_i\dot{u}_i dx = \int_\Omega \rho(\dot{u}_i^0 u_i(2t)+u_i^0\dot{u}_i(2t))dv$$

$$+\int_0^t\int_\Omega k_{ij}(\eta_{,i}\theta_{,j}(2t-s)-\eta_{,i}(2t-s)\theta_{,j}(s))dxds$$

$$+\int_0^t\int_\Omega b_{ij}(\tau_{,i}\theta_{,j}(2t-s)-\tau_{,i}(2t-s)\theta_{,j}(s))dxds$$

$$+\int_0^t\int_\Omega (\Phi\theta(s)-\Phi\theta(2t-s))dxds.$$

If we recall the equalities

$$\frac{d}{ds}(k_{ij}(-\eta_{,i}(s)\tau_{,j}(2t-s)-\eta_{,i}(2t-s)\tau_{,j}(s))) = k_{ij}(\eta_{,i}(s)\theta_{,j}(2t-s)-\eta_{,i}(2t-s)\theta_{,j}(s))$$

and

$$\frac{d}{ds}\frac{1}{2}b_{ij}(-\tau_{,i}(s)\tau_{,j}(2t-s)) = b_{ij}(\tau_{,i}(s)\theta_{,j}(2t-s)-\tau_{,i}(2t-s)\theta_{,j}(s)),$$

the lemma is proved.

Q.E.D.

In order to formulate the result on the equipartition of energy, it will be useful to introduce the functions:

$$K(t) = \frac{1}{2t}\int_0^t\int_\Omega (\rho\dot{u}_i\dot{u}_i + k_{ij}\tau_{,i}\tau_{,j})dxds,$$

$$V(t) = \frac{1}{2t}\int_0^t\int_\Omega (c\theta^2 + a_{ijkl}u_{i,j}u_{l,k})dxds.$$

$$D(t) = \frac{1}{2t}\int_0^t\int_\Omega c\theta^2(s)dxds,$$

$$F(t) = \frac{1}{2t}\int_0^t\int_\Omega a_{ijkl}u_{i,j}u_{l,k}dxds.$$

Theorem 3.3 *Let (u,τ) be a solution of the initial-boundary value problem (3.4)–(3.7), then we have*

$$\lim_{t\to\infty} D(t) = 0 \tag{3.9}$$

and

$$\lim_{t\to\infty} K(t) = \lim_{t\to\infty} V(t). \tag{3.10}$$

PROOF: The proof of (3.9) is a direct consequence of the energy equality (3.8).
To prove (3.10), we notice that Lemma 3.1 and Lemma 3.2 imply

$$\int_0^t\int_\Omega (\rho\dot{u}_i\dot{u}_i + k_{ij}\tau_{,i}\tau_{,j} - a_{ijkl}u_{i,j}u_{l,k} - c\theta^2)dxds$$

$$= \frac{1}{2}\int_\Omega \rho(\dot{u}_i^0 u_i(2t)+u_i^0\dot{u}_i(2t))dx - \frac{1}{2}\int_0^{2t}\int_\Omega \Phi\theta(s)dxds - \int_\Omega \rho\dot{u}_i^0 u_i^0 dx.$$

$$\frac{1}{2}\int_\Omega b_{ij}\tau^0_{,i}\tau_{,j}(2t)dx - \frac{1}{2}\int_\Omega b_{ij}\tau^0_{,i}\tau^0_{,j}dx.$$

Now, we can estimate

$$|K(t)-V(t)| = |-\frac{1}{2t}\int_\Omega \rho\dot{u}^0_i u^0_i + \frac{1}{4t}\int_\Omega \rho(\dot{u}^0_i u_i(2t) + u^0_i\dot{u}_i(2t))dx - \frac{1}{4t}\int_\Omega \Phi\Big(\int_0^{2t}\theta(s)ds\Big)dx$$

$$+\frac{1}{4t}\int_\Omega b_{ij}\tau^0_{,i}\tau_{,j}(2t)dx - \frac{1}{4t}\int_\Omega b_{ij}\tau^0_{,i}\tau^0_{,j}dx|.$$

$$\leq \frac{1}{4t}\int_\Omega \rho(\dot{u}^0_i\dot{u}^0_i + u^0_i u^0_i)dx + \frac{1}{8t}\int_\Omega \rho(\dot{u}^0_i\dot{u}^0_i + u^0_i u^0_i)dx$$

$$+\frac{1}{8t}\int_\Omega \rho(\dot{u}_i(2t)\dot{u}_i(2t) + u_i(2t)u_i(2t))dx + |\frac{1}{4t}\int_\Omega \Phi(\tau(2t)-\tau^0)dx|$$

$$\frac{1}{8t}\int_\Omega b_{ij}\tau_{,i}(2t)\tau_{,j}(2t)dx + \frac{3}{8t}\int_\Omega b_{ij}\tau^0_{,i}\tau^0_{,j}dx$$

$$\leq \frac{3}{8t}\int_\Omega \rho(\dot{u}^0_i\dot{u}^0_i + u^0_i u^0_i)dx + \frac{1}{8t}\int_\Omega \rho(\dot{u}_i(2t)\dot{u}_i(2t)dx + u_i(2t)u_i(2t))dx$$

$$+\frac{1}{4t}\Big(\int_\Omega \Phi^2 dx\Big)^{1/2}\Big(\int_\Omega (\tau^0)^2 dx\Big)^{1/2} + \frac{1}{4t}\Big(\int_\Omega \Phi^2 dx\Big)^{1/2} + \Big(\int_\Omega \tau^2(2t)dx\Big)^{1/2}$$

$$\frac{1}{8t}\int_\Omega b_{ij}\tau_{,i}(2t)\tau_{,j}(2t)dx + \frac{3}{8t}\int_\Omega b_{ij}\tau^0_{,i}\tau^0_{,j}dx$$

$$= \frac{3}{8t}\int_\Omega \rho(\dot{u}^0_i\dot{u}^0_i + u^0_i u^0_i)dx + \frac{1}{8t}\int_\Omega \rho(\dot{u}_i(2t)\dot{u}_i(2t) + \frac{1}{8t}u_i(2t)u_i(2t))dx$$

$$+\frac{1}{8t}\int_\Omega b_{ij}\tau_{,i}(2t)\tau_{,j}(2t)dx + \frac{3}{8t}\int_\Omega b_{ij}\tau^0_{,i}\tau^0_{,j}dx$$

$$+\frac{1}{4t}\Big(\int_\Omega \Phi^2 dx\Big)^{1/2}\Big[\Big(\int_\Omega (\tau^0)^2 dx\Big)^{1/2} + \Big(\int_\Omega \tau^2(2t)dx\Big)^{1/2}\Big].$$

From the energy equality (3.8), we deduce

$$\int_\Omega \rho\dot{u}_i\dot{u}_i dx \leq 2E(0), \tag{3.11}$$

$$\int_\Omega \rho u^2 dx \leq 2\frac{\rho}{a_0 m}E(0), \tag{3.12}$$

$$\int_\Omega \tau^2 dx \leq 2\frac{1}{k_0 m}E(0), \tag{3.13}$$

$$\int_\Omega b_{ij}\tau_{,i}(2t)\tau_{,j}(2t)dx \leq 2C^*E(0) \tag{3.14}$$

where m depends on Ω, and C^* depends on (b_{ij}) and k_0. In view of the estimates (3.11)–(3.14), it follows that

$$|K(t)-V(t)| \leq \frac{C}{t}, \tag{3.15}$$

where C is a constant that depends on the constitutive parameters, the geometry of Ω and on $E(0)$. From (3.15), it follows that

$$\lim_{t\to\infty} K(t) = \lim_{t\to\infty} F(t).$$

If we recall (3.9), the theorem is proved.

Q.E.D.

Remark: Similar results can be obtained for other boundary conditions.

References

[1] Carlson, D.E.: Linear thermoelasticity. *Handbuch der Physik* **VIa/2** (1972), 297–346.

[2] Chandrasekharaiah, DS.: Hyperbolic thermoelasticity: a review of recent literature. *Appl. Mech. Rev.* **51** (1998), 705–729.

[3] Chirita, S.: On the asymptotic partition of the energy in linear thermoelasticity. *Quart. Appl. Math.* **45** (1987), 327–340.

[4] Goldstein, J.A., Sandefur, J.T.: Asymptotic equipartition for differential equations in Hilbert space. *Trans. Amer. Math. Soc.* **21** (1976), 397–406.

[5] Green, A.E., Naghdi, P.M.: A re-examination of the basic postulates of themomechanics, *Proc. Royal Society London A* **432** (1991), 171–194.

[6] Green, A.E., Naghdi, P.M.: On undamped heat waves in an elastic solid, *J. Thermal Stresses* **15** (1992), 253–264.

[7] Green, A.E., Naghdi, P.M.: Thermoelasticity without energy dissipation, *J. Elasticity* **31** (1993), 189–208.

[8] Green, A.E., Naghdi, P. M.: A unified procedure for contruction of theories of deformable media.I. Classical continuum physics,II. Generalized continua, III. Mixtures of interacting continua, *Proc. Royal Society London A* **448** (1995), 335–356, 357–377, 379–388.

[9] Hetnarski, R.B., Ignazack, J.: Generalized thermoelasticity. *J. Thermal Stresses* **22** (1999), 451–476.

[10] Jiang, S., Muñoz Rivera, J.E., Racke, R.: Asymptotic stability and global existence in thermoelasticity with symmetry. *Quart. Appl. Math.* **56** (1998), 259–275.

[11] Jiang, S., Racke, R.: *Evolution equations in thermoelasticity.* π Monographs Surveys Pure Appl. Math. 112, Chapman & Hall/CRC, Boca Raton (2000).

[12] Koch, H.: Slow decay in linear thermoelasticity. *Quart. Appl. Math.* **58** (2000), 601–612.

[13] Lebeau, G., Zuazua, E.: Decay rates for the three-dimensional linear system of thermoelasticity. *Arch. Rational Mech. Anal.* **148** (1999), 179–231.

[14] Levine, H. A.: An equipartition of energy theorem for weak solutions of evoutionary equations in Hilbert spaces: The Lagrange identity method. *J. Differential Equations* **24** (1977), 197–210.

[15] Martín, S., Quintanilla, R.:. Existence of unbounded solutions in thermoelasticity. *Waves and Stability in Continuous Media, Proceedings of the 11th Conference on WASCOM 2001* (Eds. Roberto Monaco, Miriam Pandolfi Bianchi and Salvatore Rionero) World Scientific (2002).

[16] Quintanilla, R., Straughan, B.: Growth and uniqueness in thermoelasticity. *Proc. Royal Society London A* **456** (2000), 1419–1429.

[17] Quintanilla, R.: Damping of end effects in a thermoelastic theory. *Appl. Math. Letters* **14** (2001), 137–141.

[18] Quintanilla, R.: Instability and non-existence in the nonlinear theory of thermoelasticity without energy dissipation. *Continuum Mech. Thermodyn.* **13**, (2001), 121–129.

[19] Quintanilla, R.: Structural stability and continuous dependence of solutions in thermoelasticity of type III. *Discrete and Continuous Dynamical Systems, B.* **1** (2001), 463–470.

[20] Quintanilla, R.: On existence in thermoelasticity without energy dissipation. *J. Thermal Stresses*, **25** (2002), 195–202.

[21] Quintanilla, R., Straughan, B.: Discontinuity waves in type III thermoelasticity. *Manuscript.*

[22] Racke, R.: Thermoelasticity with second sound — exponential stability in linear and non-linear 1-d. *Math. Meth. Appl. Sci.* **25** (2002), 409–441.

[23] Racke, R.: Asymptotic behavior of solutions in linear 2- or 3-d thermoelasticity with second sound. To appear in: *Quart. Appl. Math.*

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