Global Well-Posedness and Polynomial Decay for a Nonlinear Timoshenko-Cattaneo System under Minimal Sobolev Regularity

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Abstract

We consider the nonlinear stability of the Timoshenko-Cattaneo system in the one-dimensional whole space. The Timoshenko system consists of two coupled wave equations with non-symmetric relaxation, and describes vibrations of the beam with shear deformation and rotational inertia effect. Generally, if the relaxation is not symmetric, the dissipation is produced through the complicated interaction of the components of the system, and their decay estimates and the energy estimates are of regularity-loss type. In this paper, we introduce the mathematical method to control such a weak dissipativity by investigating the Timoshenko system with Cattaneo's law, which is the first order approximation of Fourier's law with its time-delay effect. Racke & Said-Houari (2012), showed the global existence and the decay estimate of solutions by assuming high regularity $H^8 \cap L^1$ on the small initial data to control their weak dissipativity. In contrast, we prove the global existence in H^2 by energy methods without any negative weights. Our regularity assumption is the same as that needed to show the local existence. That is, we do not need to assume the extra higher regularity on the initial data. Besides, the optimal decay estimate in $H^2 \cap L^1$ is shown by using the time decay inequality of L^p-L^q-L^r type.

Keywords: Timoshenko systems; Cattaneo's law; Global existence; Decay estimate; Regularity-loss

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1 Introduction

In this paper, we consider the Cauchy problem for a nonlinear version of the dissipative Timoshenko system with heat conduction following Cattaneo's law in one-dimensional whole space. This problem was first considered by Racke & Said-Houari in [15] in the form:

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ \psi_{tt} - \sigma(\psi_x)_x - (\varphi_x - \psi) + \gamma\psi_t + b\theta_x = 0, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ \theta_t + \tilde{q}_x + b\psi_{tx} = 0, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ \tau_0 \tilde{q}_t + \tilde{q} + \kappa\theta_x = 0, & (t,x) \in (0,\infty) \times \mathbb{R} \end{cases} \tag{1.1}$$

with the initial data

$$(\varphi_x, \psi_x, \varphi_t, \psi_t, \theta, \tilde{q})(0,x) = (\varphi_{0,x}, \psi_{0,x}, \varphi_1, \psi_1, \theta_0, \tilde{q}_0). \tag{1.2}$$

The original Timoshenko system consists of the first two equations with $\gamma = b = 0$, and $\theta = \tilde{q} \equiv 0$, which was first introduced by S.P. Timoshenko ([22, 23]) to describe the vibration of the so-called Timoshenko beams: the model takes into account not only transversal movement but also shear deformation and rotational bending effects. On the other hand, the last two equations with $b = 0$ represent the heat conduction described by Cattaneo's law, which is the first-order approximation of Fourier's law ($\tilde{q}(t) + \kappa\theta_x = 0$) with a time-delay effect $\tilde{q}(t+\tau_0) + \kappa\theta_x = 0$. Therefore, we regard τ_0 as a small parameter satisfying $\tau_0 \in (0,1]$.

Here, $t \geq 0$ is the time variable, $x \in \mathbb{R}$ is the spacial variable which denotes the point on the center line of the beam. φ and ψ are the unknown functions of $t \geq 0$ and $x \in \mathbb{R}$, which denote the transversal displacement, the negative rotation angle of linear filaments perpendicular to the mid-line in the reference configuration. And θ and $\tilde{q}$ are the unknown functions of $t \geq 0$ and $x \in \mathbb{R}$, which denote appropriately weighted (first-order) thermal and heat flux moments. $\sigma(\eta)$ of the nonlinear term associated with the nonlinear elastic response function (and not the geometric nonlinearity) is assumed to be a smooth function of η such that $\sigma'(\eta) > 0$ for any η under considerations. The coefficiences a, b, γ, κ are positive constants: here we note that some of the constants (such as the density, the beam thickness, the heat capacity, Timoshenko's correction factor, etc.) are normalized.

When we formally let $\tau_0 \to 0$ in (1.1), we have Fourier's law $\tilde{q} = -\kappa\theta_x$ from the last two equations in (1.1). This together with the first two equations in (1.1) yields the Timoshenko-Fourier system with parabolic heat conduction:

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ \psi_{tt} - \sigma(\psi_x)_x - (\varphi_x - \psi) + \gamma\psi_t + b\theta_x = 0, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ \theta_t + b\psi_{tx} = \kappa\theta_{xx}, & (t,x) \in (0,\infty) \times \mathbb{R}. \end{cases} \tag{1.3}$$

1.1 Formulation of the problem

We introduce the change of variables $v = \varphi_x - \psi$, $u = \varphi_t$, $z = a\psi_x$, $y = \psi_t$ as in [6], and $q = \tilde{q}/\sqrt{\kappa}$ as in [13]. Then we can rewrite the system (1.1) into the first-order system as follows:

$$\begin{cases} v_t - u_x + y = 0, \\ y_t - \sigma(z/a)_x - v + \gamma y + b\theta_x = 0, \\ u_t - v_x = 0, \\ z_t - ay_x = 0, \\ \theta_t + by_x + \sqrt{\kappa}q_x = 0, \\ \tau_0 q_t + \sqrt{\kappa}\theta_x + q = 0. \end{cases} \tag{1.4}$$

Equivalently, let $U = (v, y, u, z, \theta, q)^T$, we have

$$A^0 U_t + F(U)_x + LU = 0, \tag{1.5}$$

where $A^0 = \mathrm{diag}\,(1, 1, 1, 1, 1, \tau_0)^T$, $F(U) = (-u, -\sigma(z/a) + b\theta, -v, -ay, by + \sqrt{\kappa}q, \sqrt{\kappa}\theta)^T$ and

$$L = \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & \gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

Note that the relaxation matrix L is not symmetric such that $\ker L \neq \ker L_1$, where L_1 denotes the symmetric part of L. Thus, it is concluded that the general theory of the dissipative structure called Shizuta-Kawashima's condition developed in [21, 24] is not applicable to our system (1.5). Generally, when the relaxation is not symmetric, the dissipativity is produced through the complicated interaction of the components of the system, and therefore even optimal decay estimates or energy estimates are of regularity-loss (See the next subsection for details).

1.2 Known results

In [19], the linear system of (1.1)

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, & (t, x) \in (0, \infty) \times \mathbb{R}, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + \gamma\psi_t + b\theta_x = 0, & (t, x) \in (0, \infty) \times \mathbb{R}, \\ \theta_t + \tilde{q}_x + b\psi_{tx} = 0, & (t, x) \in (0, \infty) \times \mathbb{R}, \\ \tau_0\tilde{q}_t + \tilde{q} + \kappa\theta_x = 0, & (t, x) \in (0, \infty) \times \mathbb{R} \end{cases} \tag{1.6}$$

is considered. It is shown that the solution $\tilde{U} = (v, y, u, z, \theta, q)^T$ to (1.6) satisfies the following decay estimate:

$$\|\partial_x^k \tilde{U}(t)\|_{L^2} \leq C(1+t)^{-1/4-k/2}\|\tilde{U}_0\|_{L^1} + C(1+t)^{-\ell/2}\|\partial_x^{k+\ell}\tilde{U}_0\|_{L^2},$$

where $\tilde{U}_0$ is the corresponding initial data, k and ℓ are nonnegative integers, and C and c are positive constants. We observe that in order to obtain the decay rate of $t^{-1/4-k/2}$, we have to assume the additional ℓ-th order regularity on the initial data to make the decay rate $t^{-\ell/2}$ faster than $t^{-1/4-k/2}$. Therefore the decay estimate can not avoid regularity-loss.

For the nonlinear system (1.1), in order to control the weak dissipativity caused by the regularity-loss property, Racke & Said-Houari in [15] introduce the following time-weighted norms $\tilde{E}(t)$ and $\tilde{D}(t)$

$$
\begin{aligned}
\tilde{E}(t)^2 &:= \sup_{0\le\tau\le t}(1+\tau)^{-\frac{1}{2}}\|U(\tau)\|_{H^s}^2,\\
\tilde{D}(t)^2 &:= \int_0^t \Big\{(1+\tau)^{-\frac{3}{2}}\|U(\tau)\|_{H^s} + (1+\tau)^{-\frac{1}{2}}\|v(\tau)\|_{H^{s-1}}^2\\
&\qquad + (1+\tau)^{-\frac{1}{2}}\Big(\|y(\tau)\|_{H^s}^2 + \|\partial_x\theta(\tau)\|_{H^{s-1}}^2 + \|q(\tau)\|_{H^s}^2\Big)\Big\}\,d\tau.
\end{aligned}
$$

for the solution $U=(v,y,u,z,\theta,q)^T$ to (1.5), and by using the energy method they show the global existence of small U for $s\ge 8$. Besides, the decay estimate for lower-order derivatives of the solution is obtained.

Proposition 1.1 ([15]). *Assume that the initial data satisfy $U_0\in H^s\cap L^1$ for $s\ge 8$ and put $\tilde{E}_1 := \|U_0\|_{H^s}+\|U_0\|_{L^1}$. Then there exists a positive constant $\tilde{\delta}_1$ such that if $\tilde{E}_1\le\tilde{\delta}_1$, the Cauchy problem (1.5) with the initial data U_0 has a unique global solution $U(t)$ with $U\in C([0,\infty);H^s)\cap C^1([0,\infty);H^{s-1})$. Moreover the solution $U(t)$ satisfies the energy estimate*

$$\tilde{E}(t)^2+\tilde{D}(t)^2\le C\tilde{E}_1^2$$

and the following decay estimate for lower-order derivatives

$$\|\partial_x^k U(t)\|_{L^2}\le C\tilde{E}_1(1+t)^{-1/4-k/2},$$

where $0\le k\le [s/2]-1$, and $C>0$ is a constant.

Remark. The result in Proposition 1.1 requires the regularity $s\ge 8$ and absolute integrability on the small initial data. Also, the norms $\tilde{E}(t)$ and $\tilde{D}(t)$ contains the time weights with negative exponents. These were crucial in [15] to overcome the difficulty caused by the regularity-loss property. Moreover, we note that the time decay rate of the solution $t^{-1/4-k/2}$ is the same as that of the corresponding linear system (1.6). Therefore, it seems that their decay rate is optimal.

1.3 Aim

The Timoshenko system is very important as a prototype of symmetric hyperbolic systems (the Timoshenko-Cattaneo system, etc.) or symmetric hyperbolic-parabolic systems (the Timoshenko-Fourier system, etc.) because the system has weaker dissipative structure than the one characterized by the general theory established by S. Kawashima and his collaborators in [21, 24].

In this paper, we demonstrated a mathematical method to control such weak dissipativity. We investigate the nonlinear stability of the system by introducing frictional damping and Cattaneo's type heat conduction as the dissipative mechanism, and prove the global existence and uniqueness of solutions under smallness assumption on the initial data in the Sobolev space H^2. Also, for small initial data in $H^2 \cap L^1$, we show that the solutions in L^2 decay at the the optimal rate $t^{-1/4}$ as $t \to \infty$. Racke & Said-Houari (2012) showed the same results in $H^8 \cap L^1$ in [15]. Therefore, our results can be regarded as an improvement over their regularity assumptions on the initial data from H^8 to H^2. First, we prove the global existence in H^2 by using the improved energy method without any negative weights. Besides, the optimal decay in $H^2 \cap L^1$ is also shown by using the alternative method, based on the energy method in the Fourier space and the refined time decay inequarity of L^p-L^q-L^r type. We expect that our methods should contribute not only to overcoming the difficulties caused by non-symmetric relaxations but also to application of beam structures in the field of Material Engineering.

Finally, we would like to mention the other works on the Timoshenko system with different effects, see, e.g., [5, 14, 16, 25] for frictional dissipation case, [3, 12, 18, 20] for thermal dissipation case, and [1, 2, 9, 10, 11, 17] for memory-type dissipation case. Especially, for the L^p-L^q-L^r-type decay estimate, which is the key to show the nonlinear stability for the Timoshenko systems, see [26]. For the physical derivation, see, e.g., [4].

Notations. Let $\hat{f} = \mathcal{F}[f]$ be the Fourier transform of f:

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) := \int_{\mathbb{R}} f(x)e^{-i\xi x}dx.$$

For $1 \le p \le \infty$, we denote by $L^p = L^p(\mathbb{R})$ the usual Lebesgue space on $\mathbb{R}$ enclosed with the norm $\|\cdot\|_{L^p}$. Also, for nonnegative integer s, we denote by $H^s = H^s(\mathbb{R})$ the Sobolev space of L^2 functions, equipped with the norm $\|\cdot\|_{H^s}$. In this paper, every positive constant is denoted by the same symbol C or c if no confusion arises.

2 Global existence

The main purpose of this paper is to improve the regularity assumptions in Proposition 1.1.

First, we show the global in time existence result of (1.4) under the lowest regularity assumptions on the initial data. To state the results, for the given solution $U = (v, y, u, z, \theta, q)^T$ to the Cauchy problem (1.4) with the initial data

$$U_0 = (v, y, u, z, \theta, q)(x, 0) = (\varphi_{0,x} - \psi_0, \psi_1, \varphi_1, a\psi_{0,x}, \theta_0, q_0),$$

we define the vector function V by $V = (v, y, u, z, \theta, \sqrt{\tau_0}\, q)^T$ and write $|V|^2 = |(v, y, u, z, \theta)|^2 + \tau_0|q|^2$, so that

$$\|V\|_{L^2}^2 = \|(v, y, u, z, \theta)\|_{L^2}^2 + \tau_0\|q\|_{L^2}^2.$$

Here we remark that in the case of $\tau_0 \to 0$, the function V can be regarded as the solution to the Cauchy problem of the Timoshenko-Fourier system (1.3). Moreover, we introduce $E(t)^2$ and $\mathcal{D}(t)$ by

$$E(t)^2 := \sup_{0\le\tau\le t} \|V(\tau)\|_{H^s}^2,$$

$$\mathcal{D}(t) := \int_0^t \|v(\tau)\|_{H^{s-1}}^2 + \|y(\tau)\|_{H^s}^2 + \|\partial_x u(\tau)\|_{H^{s-2}}^2 + \|\partial_x z(\tau)\|_{H^{s-1}}^2 + \|\partial_x \theta(\tau)\|_{H^{s-1}}^2 + \tau_0\|q(\tau)\|_{H^s}^2 \, d\tau.$$

We note that $\mathcal{D}(t)$ has one order regularity-loss for (v, u) but no regularity-loss for (y, z, θ, q).

Theorem 2.1 (Global existence)**.** *Assume $\sigma'(\eta) > 0$ and $V_0 \in H^s$ for $s \geq 2$. Put $E_0 = \|V_0\|_{H^s}$. Then there exists a positive constant δ_0 such that if $E_0 \leq \delta_0$, the Cauchy problem* (1.4) *with the initial data V_0 has a unique global solution $V(t)$ satifying*

$$V \in C([0, \infty); H^s) \cap C^1([0, \infty); H^{s-1}).$$

This global existence result can be shown by the combination of a local existence result and a priori estimate. Since our system (1.4) is a symmetric hyperbolic system, the local existence is already obtained in [7] by the standard method based on the successive approximation sequence, which needs H^s for $s \geq 2$ in the case of one dimension. Therefore, the key is to show the desired a priori estimate stated as follows:

Proposition 2.2 (A priori estimate)**.** *Assume $\sigma'(\eta) > 0$ and $V_0 \in H^s$ for $s \geq 2$. Suppose $T > 0$ and $\delta > 0$. Let $V(t)$ be the function corresponding to the solution U to the problem* (1.4) *with the initial data V_0 satisfying $V \in C([0, \infty); H^s) \cap C^1([0, \infty); H^{s-1})$ and*

$$\sup_{0 \leq t \leq T} \|V(t)\|_{L^\infty} \leq \delta. \tag{2.1}$$

Then there exists a positive constant δ_0 independent of T such that if $E_0 = \|V_0\|_{H^s} \leq \delta_0$, we have

$$E(t)^2 + \mathcal{D}(t) \leq C E_0^2 \tag{2.2}$$

for $t \in [0, T]$, where $C > 0$ is a constant independent of T.

To prove the above a priori estimate, we build the following energy inequality by using the improved energy method shown later.

Proposition 2.3 (Energy inequality)**.** *Assume $\sigma'(\eta) > 0$ and $V_0 \in H^s$ for $s \geq 2$. Put $T > 0$. Let $V(t)$ be the function corresponding to the solution $U(t)$ to the problem* (1.4) *with the initial data V_0 satisfying* (2.1)*. Then we have the energy inequality*

$$E(t)^2 + \mathcal{D}(t) \leq C E_0^2 + C E(t) \mathcal{D}(t) \tag{2.3}$$

for $t \in [0, T]$, where $C > 0$ is a constant independent of T.

The desired a priori estimate (2.2) easily follows from the energy inequality (2.3), provided $E_0 = \|V_0\|_{H^s}$ is suitably small. Therefore, it is sufficient to prove (2.3).

Proof of Proposition 2.3. Again,

$$v_t - u_x + y = 0, \tag{2.4a}$$

$$y_t - \sigma(z/a)_x - v + \gamma y + b\theta_x = 0, \tag{2.4b}$$

$$u_t - v_x = 0, \tag{2.4c}$$

$$z_t - a y_x = 0, \tag{2.4d}$$

$$\theta_t + b y_x + \sqrt{\kappa} q_x = 0, \tag{2.4e}$$

$$\tau_0 q_t + \sqrt{\kappa}\theta_x + q = 0. \tag{2.4f}$$

Our proof is divided into 4 steps.

Step 1. (**Basic energy and dissipation of y & q**)
We compute (2.4a) $\times$ v + (2.4b) $\times$ y + (2.4c) $\times$ u + (2.4d) $\times$ $\{\sigma(z/a)-\sigma(0)\}/a$ + (2.4e) $\times$ θ + (2.4f) $\times$ q, and integrate with respect to x. This yields

$$\frac{1}{2}\frac{d}{dt}E_0^{(0)}+\gamma\|y\|_{L^2}^2+\|q\|_{L^2}^2=0, \tag{2.5}$$

where

$$E_0^{(0)}:=\|(v,y,u,\theta,\sqrt{\tau_0}q)\|_{L^2}^2+\int_{\mathbb{R}}S(z)dx.$$

Here

$$S(z):=\int_0^{z/a}(\sigma(\eta)-\sigma(0))d\eta,$$

which behaves like z^2 as $z\to 0$. Since $E_0^{(0)}$ is equivalent to $\|V\|_{L^2}^2$, by integrating (2.5) with respect to t, we obtain

$$\|V(t)\|_{L^2}^2+\int_0^t\|y(\tau)\|_{L^2}^2d\tau+\tau_0\int_0^t\|q(\tau)\|_{L^2}^2d\tau\le CE_0^2, \tag{2.6}$$

where we used $\tau_0\le 1$. Next, we apply ∂_x^k to (1.4). Then we have

$$\partial_x^k v_t-\partial_x^k u_x+\partial_x^k y=0, \tag{2.7a}$$

$$\partial_x^k y_t-\sigma'(z/a)(\partial_x^k z/a)_x-\partial_x^k v+\gamma\partial_x^k y+b\partial_x^k\theta_x=[\partial_x^k,\sigma'(z/a)](z/a)_x], \tag{2.7b}$$

$$\partial_x^k u_t-\partial_x^k v_x=0, \tag{2.7c}$$

$$\partial_x^k z_t-a\partial_x^k y_x=0, \tag{2.7d}$$

$$\partial_x^k\theta_t+b\partial_x^k y_x+\sqrt{\kappa}\partial_x^k q_x=0, \tag{2.7e}$$

$$\tau_0\partial_x^k q_t+\sqrt{\kappa}\partial_x^k\theta_x+\partial_x^k q=0, \tag{2.7f}$$

where $[A,B]:=AB-BA$. We compute (2.7a) $\times$ $\partial_x^k v$ + (2.7b) $\times$ $\partial_x^k y$ + (2.7c) $\times$ $\partial_x^k u$ + (2.7d) $\times$ $\sigma'(z/a)\partial_x^k z/a^2$ + (2.7e) $\times$ $\partial_x^k\theta$ + (2.7f) $\times$ $\partial_x^k q$, and integrate with respect to x. This gives

$$\frac{1}{2}\frac{d}{dt}E_0^{(k)}+\gamma\|\partial_x^k y\|_{L^2}^2+\|\partial_x^k q\|_{L^2}^2\le CR_0^{(k)} \tag{2.8}$$

for $1\le k\le s$, where

$$E_0^{(k)}:=\|\partial_x^k(v,y,u,\theta,\sqrt{\tau_0}q)\|_{L^2}^2+\int_{\mathbb{R}}\sigma'(z/a)|\partial_x^k(z/a)|^2dx,$$
$$R_0^{(k)}:=\int_{\mathbb{R}}|y_x||\partial_x^k z|^2+|z_x||\partial_x^k z||\partial_x^k y|+|[\partial_x^k,\sigma'(z/a)]z_x||\partial_x^k y|dx.$$

Here, in the term $R_0^{(k)}$, we used the relation $z_t=ay_x$ from (2.4d). Now we integrate (2.8) with respect to t and add up for k with $1\le k\le s$. Since $E_0^{(k)}$ is equivalent to $\|\partial_x^k V\|_{L^2}^2$, we obtain

$$\|\partial_x V(t)\|_{H^{s-1}}^2+\int_0^t\|\partial_x y(\tau)\|_{H^{s-1}}^2d\tau+\tau_0\int_0^t\|\partial_x q(\tau)\|_{H^{s-1}}^2d\tau\le CE_0^2+CE(t)\mathcal{D}(t). \tag{2.9}$$

Here, we have used the following estimates for $R_0^{(k)}$:

$$R_0^{(k)} \le C\|\partial_x(y,z)\|_{L^\infty}\|\partial_x^k(y,z)\|_{L^2}^2, \qquad \sum_{k=1}^{s}\int_0^t R_0^{(k)}(\tau)d\tau \le CE(t)\mathcal{D}(t).$$

Consequently, adding (2.6) and (2.9), we arrive at

$$E(t)^2 + \int_0^t \|y(\tau)\|_{H^s}^2 d\tau + \tau_0\int_0^t \|q(\tau)\|_{H^s}^2 d\tau \le CE_0^2 + CE(t)\mathcal{D}(t). \tag{2.10}$$

Step 2. (**Dissipation of** v)
We rewrite the system (1.4) in the form

$$\begin{aligned}
&v_t - u_x + y = 0,\\
&y_t - az_x - v + \gamma y + b\theta_x = g(z)_x,\\
&u_t - v_x = 0,\\
&z_t - ay_x = 0,\\
&\theta_t + by_x + \sqrt{\kappa}q_x = 0,\\
&\tau_0 q_t + \sqrt{\kappa}\theta_x + q = 0,
\end{aligned} \tag{2.11}$$

where $g(z) := \sigma(z/a) - \sigma(0) - \sigma'(0)z/a = O(z^2)$ near $z = 0$. We apply ∂_x^k to (2.11). Then we have

$$\partial_x^k v_t - \partial_x^k u_x + \partial_x^k y = 0, \tag{2.12a}$$

$$\partial_x^k y_t - a\partial_x^k z_x - \partial_x^k v + \gamma\partial_x^k y + b\partial_x^k\theta_x = \partial_x^k g(z)_x, \tag{2.12b}$$

$$\partial_x^k u_t - \partial_x^k v_x = 0, \tag{2.12c}$$

$$\partial_x^k z_t - a\partial_x^k y_x = 0, \tag{2.12d}$$

$$\partial_x^k\theta_t + b\partial_x^k y_x + \sqrt{\kappa}\partial_x^k q_x = 0, \tag{2.12e}$$

$$\tau_0\partial_x^k q_t + \sqrt{\kappa}\partial_x^k\theta_x + \partial_x^k q = 0. \tag{2.12f}$$

To create the dissipation term $\|\partial_x^k v\|_{L^2}^2$, we compute $(2.12b) \times (-\partial_x^k v) + (2.12a) \times (-\partial_x^k y) + (2.12c) \times (-a\partial_x^k z) + (2.12d) \times (-a\partial_x^k u)$, and integrate with respect to x to obtain

$$\begin{aligned}
\frac{d}{dt}E_1^{(k)} + \|\partial_x^k v\|_{L^2}^2 \le \|\partial_x^k y\|_{L^2}^2 + \gamma\|\partial_x^k v\|_{L^2}\|\partial_x^k y\|_{L^2} + b\|\partial_x^k v\|_{L^2}\|\partial_x^k\theta_x\|_{L^2}\\
+ (a^2-1)\int_{\mathbb{R}} \partial_x^k y\,\partial_x^k u_x\,dx + R_1^{(k)}
\end{aligned}$$

for $0 \le k \le s-1$, where

$$\begin{aligned}
E_1^{(k)} &:= -\int_{\mathbb{R}} \partial_x^k v\,\partial_x^k y\,dx - a\int_{\mathbb{R}} \partial_x^k u\,\partial_x^k z\,dx,\\
R_1^{(k)} &:= \int_{\mathbb{R}} |\partial_x^k v|\,|\partial_x^k g(z)_x|\,dx.
\end{aligned}$$

By using the Young's inequality, we have

$$\frac{d}{dt}E_1^{(k)} + (1-\varepsilon)\|\partial_x^k v\|_{L^2}^2 \le C_\varepsilon(\|\partial_x^k y\|_{L^2}^2 + \|\partial_x^k \theta_x\|_{L^2}^2) + (a^2-1)\int_{\mathbb{R}} \partial_x^k y\, \partial_x^k u_x\, dx + R_1^{(k)} \tag{2.13}$$

for any small $\varepsilon > 0$, where C_ε is a constant depending on ε. Adding (2.13) up over k, with k and $k+1$ and integrating by parts, we have

$$\begin{aligned}\frac{d}{dt}\big(E_1^{(k)} + E_1^{(k+1)}\big) + (1-\varepsilon)\|\partial_x^k v\|_{H^1}^2 &\le C_\varepsilon(\|\partial_x^k y\|_{H^1}^2 + \|\partial_x^k \theta_x\|_{H^1}^2) \\ &\quad + (a^2-1)\int_{\mathbb{R}}(\partial_x^k y\, \partial_x^k u_x - \partial_x^{k+1} y_x\, \partial_x^{k+1} u)\, dx + R_1^{(k)} + R_1^{(k+1)} \\ &\le C_\varepsilon(\|\partial_x^k y\|_{H^1}^2 + \|\partial_x^k \theta_x\|_{H^1}^2) + |a^2-1|\|\partial_x^k y\|_{H^2}\|\partial_x^{k+1} u\|_{L^2} + R_1^{(k)} + R_1^{(k+1)}\end{aligned}$$

for $0 \le k \le s-2$. We integtate this inequality with respect to t and add up over k, with $0 \le k \le s-2$. Noting that $\sum_{k=0}^{s-1} |E_1^{(k)}| \le C\|V\|_{H^{s-1}}^2$ and using the Young's inequality, we obtain

$$\begin{aligned}\int_0^t \|v(\tau)\|_{H^{s-1}}^2 \le \varepsilon \int_0^t \|\partial_x u(\tau)\|_{H^{s-2}}^2 d\tau + C_\varepsilon \int_0^t \big(\|y(\tau)\|_{H^s}^2 + \|\partial_x \theta\|_{H^{s-1}}^2\big)\, d\tau \\ + CE_0^2 + CE(t)^2 + CE(t)\mathcal{D}(t)\end{aligned} \tag{2.14}$$

for any small $\varepsilon > 0$, where C_ε is a constant depending on ε. Here we also used the following estimates for $R_1^{(k)}$:

$$R_1^{(k)} \le C\|z\|_{L^\infty}\|\partial_x^k v\|_{L^2}\|\partial_x^{k+1} z\|_{L^2}, \qquad \sum_{k=0}^{s-1}\int_0^t R_1^{(k)}(\tau)d\tau \le CE(t)\mathcal{D}(t).$$

Step 3. (**Dissipation of u, z & θ**)
To get the dissipation term $\|\partial_x^{k+1} u\|_{L^2}^2$, we compute $(2.12a) \times (-\partial_x^k u_x) + (2.12c) \times \partial_x^k v_x$, and integrating with respect to x, we have

$$\frac{d}{dt}E_2^{(k)} + \|\partial_x^{k+1} u\|_{L^2}^2 \le \|\partial_x^{k+1} v\|_{L^2}^2 + \|\partial_x^k y\|_{L^2}\|\partial_x^{k+1} u\|_{L^2} \tag{2.15}$$

for $0 \le k \le s-2$, where $E_2^{(k)} := -\int_{\mathbb{R}} \partial_x^k v\, \partial_x^{k+1} u\, dx$. We integrate (2.15) with respect to t and sum over k with $0 \le k \le s-2$. Then we easily get

$$\int_0^t \|\partial_x u(\tau)\|_{H^{s-2}}^2 d\tau \le C\int_0^t \big(\|v(\tau)\|_{H^{s-1}}^2 + \|y(\tau)\|_{H^{s-2}}^2\big)\, d\tau + CE_0^2 + CE(t)^2. \tag{2.16}$$

In order to create the dissipation term $\|\partial_x^{k+1} z\|_{L^2}^2$, we compute $(2.12b)\times(-\partial_x^k z_x) + (2.12d) \times \partial_x^k y_x$, and integrating with respect to t, we obtain

$$\begin{aligned}\frac{d}{dt}E_3^{(k)} + a\|\partial_x^{k+1} z\|_{L^2}^2 \le a\|\partial_x^{k+1} y\|_{L^2}^2 + \|\partial_x^k v - \gamma\partial_x^k y\|_{L^2}\|\partial_x^k z_x\|_{L^2} \\ + \|\partial_x^k \theta_x\|_{L^2}\|\partial_x^k z_x\|_{L^2} + R_3^{(k)}\end{aligned}$$

for $0 \le k \le s-1$, where

$$E_3^{(k)} := -\int_{\mathbb{R}} \partial_x^k y \, \partial_x^{k+1} z \, dx, \qquad R_3^{(k)} := \int_{\mathbb{R}} |\partial_x^{k+1} z| \, |\partial_x^{k+1} g(z)| \, dx.$$

By using the Young's inequality, we obtain

$$\frac{d}{dt} E_3^{(k)} + a(1-\varepsilon)\|\partial_x^{k+1} z\|_{L^2}^2 \le C_\varepsilon (a\|\partial_x^k y\|_{H^1}^2 + \|\partial_x^k v\|_{L^2}^2 + \|\partial_x^k \theta_x\|_{L^2}^2) + R_3^{(k)} \tag{2.17}$$

for any small $\varepsilon > 0$, where C_ε is a constant depending on ε. We integrate (2.17) with respect to t and sum over k with $0 \le k \le s-1$. This yields

$$\begin{aligned} \int_0^t \|\partial_x z(\tau)\|_{H^{s-1}}^2 d\tau \le C \int_0^t \left(\|v(\tau)\|_{H^{s-1}}^2 + \|y(\tau)\|_{H^s}^2 + |\partial_x \theta(\tau)\|_{H^{s-1}}^2 \right) d\tau \\ + CE_0^2 + CE(t)^2 + CE(t)\mathcal{D}(t). \end{aligned} \tag{2.18}$$

Here we have used the estimates

$$R_3^{(k)} \le C\|z\|_{L^\infty} \|\partial_x^{k+1} z\|_{L^2}^2, \qquad \sum_{k=0}^{s-1} \int_0^t R_3^{(k)}(\tau) d\tau \le CE(t)\mathcal{D}(t).$$

In order to create the dissipation term $\|\partial_x^{k+1}\theta\|_{L^2}^2$, we compute $(2.12f) \times \partial_x^k \theta_x + (2.12e) \times (-\tau_0 \partial_x^k q_x)$, and integrating with respect to t, we obtain

$$\begin{aligned} \tau_0 \frac{d}{dt} E_4^{(k)} + \sqrt{\kappa}\|\partial_x^{k+1}\theta\|_{L^2}^2 \le \tau_0 \sqrt{\kappa} \|\partial_x^{k+1} q\|_{L^2}^2 \\ + \tau_0 b \|\partial_x^k y_x\|_{L^2} \|\partial_x^k q_x\|_{L^2} - \|\partial_x^k \theta_x\|_{L^2} \|\partial_x^k q\|_{L^2} \end{aligned} \tag{2.19}$$

for $0 \le k \le s-1$, where $E_4^{(k)} := -\int_{\mathbb{R}} \partial_x^k \theta \, \partial_x^k q \, dx$. We integrate (2.19) with respect to t and sum over k with $0 \le k \le s-1$. Then we easily get

$$\int_0^t \|\partial_x \theta(\tau)\|_{H^{s-1}}^2 d\tau \le C \int_0^t \left(\|y(\tau)\|_{H^s}^2 + \tau_0 \|q(\tau)\|_{H^s}^2 \right) d\tau + CE_0^2 + CE(t)^2. \tag{2.20}$$

Step 4. (**Build the energy inequality**)
Finally, combining (2.14), (2.16), (2.18) and (2.20), and then taking $\varepsilon > 0$ suitably small, we arrive at the estimate

$$\begin{aligned} &\int_0^t \left(\|v(\tau)\|_{H^{s-1}}^2 + \|\partial_x u(\tau)\|_{H^{s-2}}^2 + \|\partial_x z(\tau)\|_{H^{s-1}}^2 + \|\partial_x \theta(\tau)\|_{H^{s-1}}^2 \right) d\tau \\ &\le C \int_0^t \left(\|y(\tau)\|_{H^s}^2 + \tau_0 \|q(\tau)\|_{H^s}^2 \right) d\tau + CE_0^2 + CE(t)^2 + CE(t)\mathcal{D}(t). \end{aligned}$$

This combined with the basic estimate (2.10) yields the desired inequality $E(t)^2 + D(t)^2 \le CE_0^2 + CE(t)\mathcal{D}(t)$. Thus the proof of Proposition 2.3 is comptlete. $\square$

Remark. We note that our proof of Proposition 2.3 also holds true in the case of $\tau_0 = 0$. In the case of $\tau_0 = 0$, the classification of the system changes: the Timoshenko-Cattaneo system is regarded as the symmetric hyperbolic system, whereas the Timoshenko-Fourier system (the Timoshenko-Cattaneo system with $\tau_0 = 0$) is regarded as the symmetric hyperbolic-parabolic system. However, the local existence to the symmetric hyperbolic-parabolic system is already obtained in [8]. Therefore, we can say that the global-in-time existence and uniqueness result of *Timoshenko-Fourier system* (1.3) has just been shown in the above mentioned proof.

3 Decay estimate

Next, we show the optimal decay of solutions with the initial data in $H^2 \cap L^1$.

Theorem 3.1 (Optimal L^2 decay estimate)**.** *Assume $\sigma'(\eta) > 0$ and $V_0 \in H^s \cap L^1$ for $s \geq 2$. Put $E_1 := \|V_0\|_{H^2} + \|V_0\|_{L^1}$. Then there is a positive constant δ_1 such that if $E_1 \leq \delta_1$, then the function $V(t)$ obtained in Theorem 2.1 satisfies the following optimal L^2 decay estimate:*

$$\|V(t)\|_{L^2} \leq CE_1(1+t)^{-1/4}, \tag{3.1}$$

where $C > 0$ is a constant.

To this end, we first derive the pointwise estimate of solutions in the Fourier space. We recall that the system (1.4) is written in the form of (2.11) or in the vector notation as

$$A^0 U_t + AU_x + LU = G_x, \tag{3.2}$$

where $G_x = (0, g(z)_x, 0, 0)^T$ with $g(z)_x = O(z)$ as $z \to 0$; the coefficient matrices A^0, A and L are given by

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau_0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & b & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & \sqrt{\kappa} \\ 0 & 0 & 0 & 0 & \sqrt{\kappa} & 0 \end{pmatrix},$$

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proposition 3.2 (Pointwise estimate)**.** *Let $V(t)$ be the function corresponding to the solution $U(t)$ to the problem* (3.2) *with the initial data V_0. Then the Fourier image $\hat{V}$ satisfies the pointwise estimate*

$$|\hat{V}(\xi,t)|^2 \leq Ce^{-c\rho(\xi)t}|\hat{V}_0(\xi)|^2 + C\int_0^t e^{-c\rho(\xi)(t-\tau)}\xi^2|\hat{g}(\xi,\tau)|^2 d\tau \tag{3.3}$$

for $\xi \in \mathbb{R}$ and $t \geq 0$, where $\rho(\xi) := \xi^2/(1+\xi^2)^2$, and C and c are positive constants.

Then we estimate both terms in the right-hand side of the inequality (3.3) sharply by applying the following decay estimate of L^2-L^q-L^r type.

Lemma 3.3 (Decay estimate of L^2-L^q-L^r-type [25])**.** *Let V be a function satisfying*

$$|\hat{V}(\xi,t)| \leq C|\xi|^m e^{-c\rho(\xi)t}|\hat{V}_0(\xi)| \tag{3.4}$$

for $\xi \in \mathbb{R}$ and $t \geq 0$, where $\rho(\xi) = \xi^2/(1+\xi^2)^2$, $m \geq 0$, and V_0 is a given function. Then we have

$$\begin{aligned} \|\partial_x^k V(t)\|_{L^2} \leq{}& C(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k+m}{2}}\|V_0\|_{L^q} \\ &+ C(1+t)^{-\frac{\ell}{2}+\frac{1}{2}(\frac{1}{r}-\frac{1}{2})}\|\partial_x^{k+m+\ell}V_0\|_{L^r}, \end{aligned} \tag{3.5}$$

where $k \geq 0$, $1 \leq q, r \leq 2$, $\ell > \frac{1}{r} - \frac{1}{2}$ ($\ell \geq 0$ if $r = 2$).

Remark. The first (resp. the second) term on the right-hand side of (3.5) icorresponds to the low-frequency region $|\xi| \leq 1$ (resp. high-frequency region $|\xi| \geq 1$). When $m = 0$, $q = 1$ and $r = 2$, the estimate (3.5) is reduced to

$$\|\partial_x^k V(t)\|_{L^2} \leq C(1+t)^{-1/4-k/2}\|V_0\|_{L^1} + C(1+t)^{-\ell/2}\|\partial_x^{k+\ell} V_0\|_{L^2},$$

that is, the decay estimate of L^2-L^1-L^2-type, which is just the previous decay estimate first obtained in [6].

Thanks to the above L^2-L^q-L^r-type estimate, we get to have the sharp estimates of both terms on the right-hand side of the inequality (3.3).

This yields the decay estimate (3.1) with the same decay rate as shown in [13] to the linearized system (1.6). In [13], the decay rate in [13] is shown optimal based on the characterization of the dissipative structure by using the eigenvalue of the linearized system (1.6). Besides, we assume no extra higher regularity on the initial data more than we need to show the local existence. Therefore, we can say that we obtain the optimal decay estimate under the minimal regularity assumptions on the initial data. The outline of the proof of Lemma 3.3 is as follows. From the Plancherel theorem and (3.4), we have

$$\|\partial_x^k V(t)\|_{L^2}^2 = \int_{\mathbb{R}} \xi^{2k}|\hat{V}(\xi,t)|^2 d\xi \leq C\int_{\mathbb{R}} \xi^{2(k+m)} e^{-c\rho(\xi)t}|\hat{V}_0(\xi)|^2 d\xi$$

We divide the last integral into two parts corresponding to $|\xi| \leq 1$ and $|\xi| \geq 1$, respectively, and estimate each part by applying the Hölder's inequality and the Hausdorff-Young's inequality. See [25] for details.

3.1 Proof of Proposition 3.2

First, by taking the Fourier transform of (2.11), we have

$$\hat{v}_t - i\xi\hat{u} + \hat{y} = 0, \tag{3.6a}$$

$$\hat{y}_t - ai\xi\hat{z} - \hat{v} + \gamma\hat{y} + bi\xi\hat{\theta} = i\xi\hat{g}, \tag{3.6b}$$

$$\hat{u}_t - i\xi\hat{v} = 0, \tag{3.6c}$$

$$\hat{z}_t - ai\xi\hat{y} = 0, \tag{3.6d}$$

$$\hat{\theta}_t + bi\xi\hat{y} + \sqrt{\kappa}i\xi\hat{q} = 0, \tag{3.6e}$$

$$\tau_0\hat{q}_t + \sqrt{\kappa}i\xi\hat{\theta} + \hat{q} = 0, \tag{3.6f}$$

where $g = g(z)$. We construct the Lyapunov function of the system (3.6) in the Fourier space. The computations below are essentially same as the proof of Proposition 2.3.

Step 1. (**Basic energy and dissipation of y & q**)
We compute (3.6a) $\times \bar{\hat{v}}$ + (3.6b) $\times \bar{\hat{y}}$ + (3.6c) $\times \bar{\hat{u}}$ + (3.6d) $\times \bar{\hat{z}}$ and take the real part. This yields

$$\frac{1}{2}E_{0,t} + \gamma|\hat{y}|^2 + |\hat{q}|^2 = \mathrm{Re}\,\{i\xi\bar{\hat{y}}\hat{g}\},$$

where $E_0 := |\hat{V}|^2$. Applying the Young's inequality, we have

$$E_{0,t} + |\hat{y}|^2 + \tau_0|\hat{q}|^2 \le C\xi^2|\hat{g}|^2. \tag{3.7}$$

Step 2. (**Dissipation of** $\hat{v}$)
To create the dissipation term for $\hat{v}$, we compute (3.6b) $\times\,(-\bar{\hat{v}})$ + (3.6a) $\times\,(-\bar{\hat{y}})$ + (3.6c) $\times\,(-a\bar{\hat{z}})$ + (3.6d) $\times\,(-a\bar{\hat{u}})$ and take the real part. This gives

$$\begin{aligned} E_{1,t} + |\hat{v}|^2 - |\hat{y}|^2 &= \gamma\,\mathrm{Re}\,(\bar{\hat{v}}\hat{y}) - \,\mathrm{Re}\,\{i\xi(\bar{\hat{y}}\hat{u} + a^2\bar{\hat{u}}\hat{y})\} - \mathrm{Re}\,\{i\xi\bar{\hat{v}}\hat{g}\} \\ &= \gamma\,\mathrm{Re}\,(\bar{\hat{v}}\hat{y}) - (a^2-1)\xi\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{y}) - \xi\,\mathrm{Re}\,\{i\bar{\hat{v}}\hat{g}\}, \end{aligned}$$

where $E_1 := -\mathrm{Re}\,(\hat{v}\bar{\hat{y}} + a\hat{u}\bar{\hat{z}})$. We multiply this equality by $1+\xi^2$. Then, using the Young's inequality, we obtain

$$(1+\xi^2)E_{1,t} + c_1(1+\xi^2)|\hat{v}|^2 \le \varepsilon\xi^2|\hat{u}|^2 + C_\varepsilon(1+\xi^2)^2|\hat{y}|^2 + C(1+\xi^2)\xi^2|\hat{g}|^2 \tag{3.8}$$

for any small $\varepsilon > 0$, where c_1 is a positive constant with $c_1 < 1$ and C_ε is a constant depending on ε.

Step 3. (**Dissipation of** $\hat{u}$, $\hat{z}$ **&** $\hat{\theta}$)
To create the dissipation term $|\hat{u}|^2$, we compute (3.6a) $\times\, i\xi\bar{\hat{u}}$ − (3.6c) $\times\, i\xi\bar{\hat{v}}$ and take the real part. The result is

$$\xi E_{2,t} + \xi^2(|\hat{u}|^2 - |\hat{v}|^2) + \xi\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{y}) = 0, \tag{3.9}$$

where $E_2 := \mathrm{Re}\,(i\hat{v}\bar{\hat{u}})$. For the dissipation term $|\hat{z}|^2$, we compute (3.6b) $\times\, i\xi\bar{\hat{z}}$ − (3.6d) $\times\, i\xi\bar{\hat{y}}$ and take the real part. Then we have

$$\xi E_{3,t} + a\xi^2(|\hat{z}|^2 - |\hat{y}|^2) - \xi\,\mathrm{Re}\,\{i\bar{\hat{z}}(\hat{v} - \gamma\hat{y})\} = -\xi^2\,\mathrm{Re}\,\{\bar{\hat{z}}\hat{g}\}, \tag{3.10}$$

where $E_3 := \mathrm{Re}\,(i\hat{y}\bar{\hat{z}})$. Then, we create a dissipation for $|\hat{\theta}|^2$. To this end, we multiply (3.6e) and (3.6f) by $i\xi\tau_0\bar{\hat{q}}$ and $-i\xi\bar{\hat{\theta}}$, respectively, add the resulting equations, and take the real part. This yields

$$\tau_0\xi E_{4,t} + \sqrt{\kappa}\xi^2|\hat{\theta}|^2 = \tau_0\sqrt{\kappa}\xi^2|\hat{q}|^2 + \tau_0 b\xi^2\mathrm{Re}\,(\hat{y}\bar{\hat{q}}) + \xi\,\mathrm{Re}\,(i\bar{\hat{\theta}}\hat{q}), \tag{3.11}$$

where $E_4 = \mathrm{Re}\,(i\hat{\theta}\bar{\hat{q}})$. This equality becomes trivial when $\tau_0 = 0$, because we have $\hat{q} = -\sqrt{\kappa}i\xi\hat{\theta}$ for $\tau_0 \to 0$.

Now we combine (3.9), (3.10) and (3.11) such that (3.9) + (3.10) $\times\,(1+\xi^2)$ + (3.11) $\times\,(1+\xi^2)$. This gives

$$\begin{aligned} &\xi\{E_2 + (1+\xi^2)(E_3 + \tau_0 E_4)\}_t + \xi^2|\hat{u}|^2 + a(1+\xi^2)\xi^2|\hat{z}|^2 + \sqrt{\kappa}(1+\xi^2)\xi^2|\theta|^2 \\ &= \xi^2|\hat{v}|^2 + \tau_0\sqrt{\kappa}(1+\xi^2)\xi^2|\hat{q}|^2 + a(1+\xi^2)\xi^2|\hat{y}|^2 + (1+\xi^2)\xi\,\mathrm{Re}\,\{i\bar{\hat{z}}(\hat{v} - \gamma\hat{y})\} \\ &\quad - \xi\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{y}) - (1+\xi^2)\xi^2\,\mathrm{Re}\,\{\bar{\hat{z}}\hat{g}\} + \tau_0 b(1+\xi^2)\xi^2\mathrm{Re}\,(\hat{y}\bar{\hat{q}}) + (1+\xi^2)\xi\,\mathrm{Re}\,(i\bar{\hat{\theta}}\hat{q}). \end{aligned}$$

Using the Young's inequality, we get

$$\begin{aligned} &\xi\{E_2 + (1+\xi^2)(E_3 + \tau_0 E_4)\}_t + c_1\xi^2|\hat{u}|^2 + c_2(1+\xi^2)\xi^2|\hat{z}|^2 + c_3(1+\xi^2)\xi^2|\hat{\theta}|^2 \\ &\le C(1+\xi^2)|\hat{v}|^2 + C(1+\xi^2)^2|\hat{y}|^2 + \tau_0 C(1+\xi^2)^2|\hat{q}|^2 + C(1+\xi^2)\xi^2|\hat{g}|^2, \end{aligned} \tag{3.12}$$

where c_1, c_2 and c_3 are positive constants satisfying $c_1 < 1$, $c_2 < a$ and $c_3 < \sqrt{\kappa}$, respectively.

Step 4. (**Build the Lyapunov function**)
Letting $\alpha_1 > 0$, we combine (3.8) and (3.12) such that $(3.8) + (3.12) \times \alpha_1$. Then we have

$$
\begin{aligned}
&\big\{(1+\xi^2)E_1 + \alpha_1\xi\{E_2 + (1+\xi^2)(E_3+\tau_0 E_4)\}\big\}_t + (c_1 - \alpha_1 C)(1+\xi^2)|\hat{v}|^2 \\
&\qquad + (\alpha_1 c_1 - \varepsilon)\xi^2|\hat{u}|^2 + \alpha_1 c_2(1+\xi^2)\xi^2|\hat{z}|^2 + \alpha_1 c_3(1+\xi^2)\xi^2|\hat{\theta}|^2 \\
&\le \alpha_1 C_\varepsilon(1+\xi^2)^2|\hat{y}|^2 + \alpha_1\tau_0 C(1+\xi^2)^2|\hat{q}|^2 + C_{\alpha_1}(1+\xi^2)\xi^2|\hat{g}|^2,
\end{aligned}
\tag{3.13}
$$

where C_{ε,α_1} and C_{α_1} are constants depending on (ε, α_1) and α_1, respectively. Also, letting $\alpha_2 > 0$, we combine (3.7) and (3.13) such that $(3.7) + (3.13) \times \frac{\alpha_2}{(1+\xi^2)^2}$. Then, putting

$$
E := E_0 + \frac{\alpha_2}{1+\xi^2}\Big(E_1 + \frac{\alpha_1\xi}{1+\xi^2}\{E_2 + (1+\xi^2)(E_3+\tau_0 E_4)\}\Big), \tag{3.14}
$$

we obtain

$$
\begin{aligned}
&E_t + \alpha_2(c_1-\alpha_1 C)\frac{1}{1+\xi^2}|\hat{v}|^2 + (\gamma - \alpha_2\alpha_1 C_\varepsilon)|\hat{y}|^2 + \alpha_2(\alpha_1 c_1 - \varepsilon)\frac{\xi^2}{(1+\xi^2)^2}|\hat{u}|^2 \\
&+ \alpha_2\alpha_1 c_2\frac{\xi^2}{1+\xi^2}|\hat{z}|^2 + \alpha_2\alpha_1 c_3\frac{\xi^2}{1+\xi^2}|\hat{\theta}|^2 + \tau_0(1-\alpha_2\alpha_1 C_\varepsilon)|\hat{q}|^2 \\
&\le C_{\alpha_1,\alpha_2}\xi^2|\hat{g}|^2,
\end{aligned}
\tag{3.15}
$$

where C_{α_1,α_2} is a constant depending on (α_1, α_2). Here, we see that there is a small positive constant α_0 such that if α_1, $\alpha_2 \in (0, \alpha_0]$, then E in (3.14) is equivalent to $|\hat{V}|^2$, that is,

$$
c_0|\hat{V}|^2 \le E \le C_0|\hat{V}|^2, \tag{3.16}
$$

where c_0 and C_0 are positive constants. Futhermore, we choose $\alpha_1 \in (0, \alpha_0]$ such that $c_1 - \alpha_1 C > 0$ and take $\varepsilon > 0$ so small as $\alpha_1 c_1 - \varepsilon > 0$. Finally, we choose $\alpha_2 \in (0, \alpha_0]$ such that both $\gamma - \alpha_2\alpha_1 C_\varepsilon > 0$ and $1 - \alpha_2\alpha_1 C_\varepsilon > 0$ hold. Then (3.15) becomes

$$
E_t + cF \le C\xi^2|\hat{g}|^2, \tag{3.17}
$$

where

$$
F := \frac{1}{1+\xi^2}|\hat{v}|^2 + |\hat{y}|^2 + \frac{\xi^2}{(1+\xi^2)^2}|\hat{u}|^2 + \frac{\xi^2}{1+\xi^2}|\hat{z}|^2 + \frac{\xi^2}{1+\xi^2}|\hat{\theta}|^2 + \tau_0|\hat{q}|^2. \tag{3.18}
$$

This suggests that E in (3.14) is the desired Lyapunov function of the system (3.6). Noting (3.16), we find that $F \ge c\rho(\xi)E$, where $\rho(\xi) = \xi^2/(1+\xi^2)^2$. Therefore (3.17) becomes to $E_t + c\rho(\xi)E \le C\xi^2|\hat{g}|^2$. Solving this ordinary differential inequality for E and using (3.16), we arrive at the desired estimate (3.3) in the form

$$
|\hat{V}(\xi,t)|^2 \le Ce^{-c\rho(\xi)t}|\hat{V}_0(\xi)|^2 + C\int_0^t e^{-c\rho(\xi)(t-\tau)}\xi^2|\hat{g}(\xi,\tau)|^2 d\tau.
$$

This completes the proof of Proposition 3.2. $\square$

3.2 Proof of Theorem 3.1

Let V be the function corresponding to the solution U to the problem (1.4) obtained in Theorem 2.1. Then V satisfies (3.2). Therefore, we have the pointwise estimate (3.3). Now, we integrate (3.3) with respect to ξ. Applying the Plancherel's theorem, we obtain

$$\begin{aligned}\|V(t)\|_{L^2}^2 &= \int_{\mathbb{R}} |\hat{V}(\xi,t)|^2 d\xi \\ &\le C\int_{\mathbb{R}} e^{-c\rho(\xi)t}|\hat{V}_0(\xi)|^2 d\xi + C\int_0^t\int_{\mathbb{R}} e^{-c\rho(\xi)(t-\tau)}\xi^2|\hat{g}(\xi,\tau)|^2 d\xi d\tau =: I + J.\end{aligned} \tag{3.19}$$

We estimate the terms I and J by applying Lemma 3.3. For I, using (3.5) with $m = 0$, we have

$$\begin{aligned} I &= C\int_{\mathbb{R}} e^{-c\rho(\xi)t}|\hat{V}_0(\xi)|^2\, d\xi \\ &\le \underbrace{C(1+t)^{-\frac{1}{2}}\|V_0\|_{L^1}^2}_{k=0,\ q=1} + \underbrace{C(1+t)^{-1}\|\partial_x V_0\|_{L^2}^2}_{k=0,\ \ell=1,\ r=2} \\ &\le CE_1^2(1+t)^{-\frac{1}{2}}, \end{aligned} \tag{3.20}$$

where $E_1 = \|V_0\|_{H^2} + \|V_0\|_{L^1}$. On the other hand, for J we use (3.5) with $m = 1$. Then we obtain

$$\begin{aligned} J &= C\int_0^t\int_{\mathbb{R}} e^{-c\rho(\xi)(t-\tau)}\xi^2|\hat{g}(\xi,\tau)|^2\, d\tau d\xi \\ &\le C\int_0^t \underbrace{(1+t-\tau)^{-\frac{3}{2}}\|g(\tau)\|_{L^1}^2}_{k=0,\ q=1}\, d\tau + C\int_0^t \underbrace{(1+t-\tau)^{-\frac{1}{2}}\|\partial_x^2 g(\tau)\|_{L^1}^2}_{k=0,\ \ell=1,\ r=1}\, d\tau \\ &=: J_1 + J_2. \end{aligned}$$

Here, we introduce the norms $N(t)$ and $D(t)$ by

$$N(t) = \sup_{0\le\tau\le t}(1+\tau)^{\frac{1}{4}}\|V(\tau)\|_{L^2}, \qquad D(t)^2 = \int_0^t \|\partial_x z(\tau)\|_{H^1}^2\, d\tau.$$

We know from Theorem 2.1 that $D(t) \le CE_0 \le CE_1$. For the low-frequency part J_1, since $\|g\|_{L^1} \le C\|z\|_{L^2}^2$, we have

$$\begin{aligned} J_1 &\le C\int_0^t (1+t-\tau)^{-\frac{3}{2}}\|z(\tau)\|_{L^2}^4\, d\tau \\ &\le CN(t)^4\int_0^t (1+t-\tau)^{-\frac{3}{2}}(1+\tau)^{-1}\, d\tau \le CN(t)^4(1+t)^{-1}. \end{aligned} \tag{3.21}$$

For the high-frequency part J_2, using $\|\partial_x^2 g\|_{L^1} \le C\|z\|_{L^2}\|\partial_x^2 z\|_{L^2}$, we have

$$\begin{aligned} J_2 &\le C\int_0^t (1+t-\tau)^{-\frac{1}{2}}\|z(\tau)\|_{L^2}^2\|\partial_x^2 z(\tau)\|_{L^2}^2\,d\tau \\ &\le CN(t)^2\int_0^t (1+t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{1}{2}}\|\partial_x^2 z(\tau)\|_{L^2}^2\,d\tau \\ &\le CN(t)^2D(t)^2\sup_{0\le\tau\le t}\left\{(1+t-\tau)^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}}\right\} \\ &\le CN(t)^2D(t)^2(1+t)^{-\frac{1}{2}}. \end{aligned} \tag{3.22}$$

Combining (3.20), (3.21) and (3.22) and using $D(t) \le CE_1$, we obtain

$$(1+t)^{\frac{1}{2}}\|V(t)\|_{L^2}^2 \le CE_1^2 + CN(t)^4 + CE_1^2N(t)^2.$$

Thus, we have the inequality $N(t)^2 \le CE_1^2 + CN(t)^4 + CE_1^2N(t)^2$, and this yields

$$N(t) \le CE_1 + CN(t)^2 + CE_1N(t).$$

By considering the graphs of the left-hand side and the right-hand side, we know that if CE_1 is suitably small, for example, if we take such a small E_1 that

$$2CE_1 > CE_1 + C(2CE_1)^2 + CE_1(2CE_1), \tag{3.23}$$

therefore,

$$\frac{1}{2C(2C+1)} > E_1$$

holds, we have $N(t) \le N_1$ or $N_2 \le N(t)$, where N_1, N_2 $(N_1 < N_2)$ are two positive roots of the equarity

$$N = CE_1 + CN^2 + CE_1N.$$

Now, since

$$N(0) \le CE_1 < N_1 \quad (C \ge 1)$$

and $N(t)$ is continuous, we conclude that

$$N(t) \le N_1, \quad t \ge 0.$$

Besides, by (3.23), we have $N_1 \le 2CE_1$. Consequently, we arrive at $N(t) \le 2CE_1$, provided that E_1 is suitably small. Thus we have proved the desired decay estimate $\|V(t)\|_{L^2} \le 2CE_1(1+t)^{-1/4}$. This completes the proof of Theorem 3.1. □

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