THE CAUCHY PROBLEM FOR THERMOELASTIC PLATES WITH TWO TEMPERATURES

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ABSTRACT. We consider the decay rates of solutions to thermoelastic systems in materials where, in contrast to classical thermoelastic models for Kirchhoff type plates, two temperatures are involved, related by an elliptic equation. The arising initial value problems deal with systems of partial differential equations involving Schrödinger like equations, hyperbolic and elliptic equations. Depending on the model – with Fourier or with Cattaneo type heat conduction – we obtain polynomial decay rates without or with regularity loss. This way we obtain another example where the loss of regularity in the Cauchy problem corresponds to the loss of exponential stability in bounded domains. The well-posedness is done using semigroup theory in appropriate space reflecting the different regularity compared to the classical single temperature case, and the (optimal) decay estimates are obtained with sophisticated pointwise estimates in Fourier space.

Keywords: thermoelastic plate, Cauchy problem, Fourier and Cattaneo law, asymptotic behavior

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1. INTRODUCTION

Classical thermoelastic plates of Kirchhoff type modeled by

$$
u_{tt} + b\Delta^2 u + d\Delta\theta = 0,
$$
$$
(1.1) \qquad \theta_t + \operatorname{div} q - d\Delta u_t = 0,
$$
$$
\tau q_t + q + \kappa\nabla\theta = 0,
$$

for $(u, \theta, q) = (u, \theta, q)(t, x)$ denoting the displacement, the temperature and heat flux for $x \in \mathbb{R}^n$, $t \geq 0$, with $b, d, \kappa > 0, \tau \geq 0$, have been discussed in recent years with respect to well-posedness and asymptotic behavior in time (also for bounded and unbounded domains with boundaries, and both for $\tau = 0$ and for $\tau > 0$.

So-called non-simple materials are modeled by two temperatures, the thermodynamic temperature θ and the conductive temperature ψ, related to each other in the following way, see [2, 3, 4, 36],

$$
(1.2) \qquad \theta = \psi - a\Delta\psi
$$

with a constant $a \geq 0$. The corresponding extension of the classical thermoelastic plate model (1.1) then reads as

$$
u_{tt} + b\Delta^2 u + d\Delta\theta = 0,
$$
$$
(1.3) \qquad \theta_t + \operatorname{div} q - d\,\Delta u_t = 0,
$$
$$
\tau q_t + q + \kappa\nabla\psi = 0,
$$

with (1.2). Especially, for $\tau = 0$, (1.3) is rewritten as

(1.4)
$$u_{tt} + b\Delta^2 u + d\Delta\theta = 0,$$
$$\theta_t - \kappa\Delta\psi - d\,\Delta u_t = 0,$$

which represents Fourier type heat conduction.

For $a = 0$ in (1.2) we recover (1.1). This case was investigated in [31] cf. Said-Houari for the one-dimensional case [33]. A regularity loss in the description of polynomial decay of soultions was proved reflecting the loss of exponential stability in bounded domains known. In this case it is known that one has, for appropriate boundary conditions, exponential stability for the Fourier type heat conduction given by $\tau = 0$, while it is not exponentially stable for the Cattaneo (Maxwell) type given for a positive relaxation constant $\tau > 0$, see [26, 27, 8].

Here, we shall investigate initial value problems for the case $a > 0$. We are first interested in the well-posedness both for $\tau = 0$ and for $\tau > 0$, which is more delicate in comparison to the case $a = 0$, since there will be no regularity gain in the temperature triggered by the differential equations. The main topic will be to investigate the asymptotic behavior as time tends to infinity in terms of polynomial decay rates. The rates will be shown to be optimal, and a loss of regularity will be observed while turning from the Fourier to the Cattaneo model. This way we also contribute a further example where the different heat conduction models, one by Fourier ($\tau = 0$), one by Cattaneo ($\tau > 0$) predict different qualitative behavior. Similar effects are known for the thermoelastic Timoshenko system in one space dimensions. Here, we also have that the system with the Fourier model for heat conduction may show exponential stability in bounded domains (in the case of equal wave speeds of the two wave equations involved), while this property is lost with the Cattaneo model, see Fernańdez Sare & Racke [9]. Moreover, for the Cauchy problem in $\mathbb{R}^1$, one has the same effect, i.e., a regularity-loss phenomenon when changing from Fourier's to Cattaneo's law, see Ide & Kawashima [12], Ide & Haramoto & Kawashima [11], Ueda & Duan & Kawashima [35], Said-Houari & Kasimov [34].

Some further related papers are given as follows:

Case $a = 0$: For bounded domains and for $\tau = 0$, there are many results in particular on exponential stability, see for example [1, 13, 15, 16, 17, 18, 21, 22, 23]. For results for the Cauchy problem or in general exterior domains see for example [5, 6, 7, 22, 23, 31]. For $\tau > 0$, exponential stability in bounded domains is lost [26, 27, 8], for the Cauchy problem we encounter a regularity loss [31].

Case $a > 0$: As mentioned above, in [28], the bounded domain case was studied, and the exponential stablity was proved for $\tau = 0$, while it was shown not to be exponentially stable for $\tau > 0$.

The *second-order* system

$$u_{tt} - bu_{xx} + d\theta_x = 0,$$
$$\theta_t + q_x + du_{tx} = 0,$$
$$\tau q_t + q + \kappa\psi_x = 0,$$
$$\theta - \psi + a\psi_{xx} = 0,$$

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in one space dimension in a bounded interval has been studied with respect to exponential stability for $a = 0$ [25], the well-posedness was obtained in any space dimension [24]. The non-exponential stability for $\tau > 0$ was proved in in [20].

We remark that related *nonlinear* problems have been discussed in [10, 14, 19, 32]. Our main new contributions are

- First discussion of the fourth-order thermoelastic plate system with two temperatures in all of $\mathbb{R}^n$.
- Proof of well-posedness for rather weak regular solutions, both for $\tau = 0$ and for $\tau > 0$.
- Proof of optimal decay rates for both cases $(\tau = 0)$ and $(\tau > 0)$
- Demonstration of the loss of regularity while turning from the Fourier model to the Cattaneo one, this way yielding another example for the general picture of the correspondence between "loss of exponential stability in bounded domains" and "loss of regularity for the Cauchy problem".
- Clarification of the role of the heat conduction parameter κ in the decay estimates.

The methods used will be essentially sophisticated pointwise estimates of the solutions in Fourier space, and semigroup theory for the well-posedness.

The paper is organized as follows: We start in section 2 proving the well-posedness based on semigroup theory both for $\tau = 0$ and for $\tau > 0$. In section 3, we treat the case $\tau = 0$ and derive the decay estimates using the Fourier transform. Section 4 discusses the decay estimates for the case $\tau > 0$.

Throughout the paper, we use standard notation, in particular the Sobolev spaces $L^p = L^p(\mathbb{R}^n)$, $p \geq 1$, and $H^s = W^{s,2}(\mathbb{R}^n)$, $s \in \mathbb{N}_0$, with their associated norms $\| \cdot \|_{L^p}$, with abbreviation $\| \cdot \| := \| \cdot \|_{L^2}$, respectively $\| \cdot \|_{H^s}$. For the inner product in L^2 we use the notation $\langle \cdot, \cdot \rangle$. Furthermore, we use the Sobolev norm $\| \cdot \|_{H_a^s}$ with parameters τ and a as follows. Let $\tau \geq 0$ and $a \geq 0$ be real numbers, then

$$\|u\|_{H_a^1} := \|(1 + a|\xi|^2)^{1/2}\hat{u}\|_{L^2},$$

$$\|u\|_{H_{\tau,a}^{3+s}} := \|(1 + \tau|\xi|^2)(1 + a|\xi|^2)^{(1+s)/2}\hat{u}\|_{L^2},$$

with $s = 0, 1$, where $\hat{u}$ denotes the Fourier transform of u and $\xi \in \mathbb{R}^n$ is the Fourier variable. We observe that $\| \cdot \|_{H_0^s} = \| \cdot \|_{H_{0,0}^{3+s}} = \| \cdot \|_{L^2}$.

2. Well-posedness for $\tau \geq 0$

To discuss the decay estimate of the global solutions in time, we first consider the well-posedness of the systems, both for $\tau = 0$ and for $\tau > 0$, based on semigroup arguments. Especially, this argument is useful for the system with $\tau > 0$ to construct the global solution in time. Regarding the well-posedness of the systems under consideration, we point out, that due to the effects of the two temperatures in the model, the regularity for the temperature(s) is different from that for the case with a single temperature. In the heat equation instead of the Laplace operator, now, a bounded operator appears not triggering the regularity seemingly needed for the main elastic equation for the displacement. As a consequence, a connected regularity is considered.

We modify the work on the bounded domain case given in our recent paper [28] for the Cauchy problem considered here, and we present the details for the reader's convenience.

2.1. **Well-posedness for $\tau = 0$.** We start proving the well-posedness of the system (1.4) with $\tau = 0$, i.e. for (1.4) with the initial data

(2.1) $$u(0, x) = u_0(x), \qquad u_t(0, x) = u_1(x), \qquad \theta(0, x) = \theta_0(x)$$

for $x \in \mathbb{R}^n$. We rewrite (1.2) as

$$\psi = (\mathrm{Id} - a\Delta)^{-1} \theta,$$

where Id denotes the identity operator. Here $(\mathrm{Id} - a\Delta)^{-1}$ denotes the homeomorphism from L^2 onto H^2. Then (1.4) with (1.2) can be written as

(2.2)
$$u_{tt} + \Delta(b\Delta u + d\theta) = 0,$$
$$\theta_t - B\theta - d\Delta u_t = 0,$$

where $B : L^2 \to L^2$ is a bounded operator defined by $B := \kappa\Delta (\mathrm{Id} - a\Delta)^{-1}$.

Remark 2.1. *The second equation of (2.2) for, essentially, θ does not trigger any regularity for θ, in contrast to the situation where $a = 0$ (only one temperature $\theta = \psi$). For $a = 0$ we would have the classical operator $B = \kappa\Delta$ on its usual domain. On the other hand, in the first equation of (2.2) one needs, yet formally, $\Delta\theta$. This lack of regularity will be reflected in a lack of separate for regularity for u and θ. We shall have a connected regularity, see below.*

The operator B satisfies for (1.2)

$$\langle B\theta, \theta \rangle = -\kappa\|\nabla\psi\|^2 - \kappa a\|\Delta\psi\|^2 \leq 0.$$

We transform the system (2.2) into a system of first order in time for $U := (v, w, \theta)^T$ with $v := u_t$, $w := \Delta u$, where T denotes the transposed matrix:

$$U_t = \begin{pmatrix} 0 & -b\Delta & -d\Delta \\ \Delta & 0 & 0 \\ d\Delta & 0 & B \end{pmatrix} U =: A_f U, \qquad U(0, \cdot) = U^0 := (u_1, \Delta u_0, \theta_0)^T.$$

This formal system with the formal differential symbol A_f will be considered as an evolution equation in the associated Hilbert space $\mathcal{H} := L^2 \times L^2 \times L^2$ with inner product

$$\langle U, W \rangle_{\mathcal{H}} := \langle U^1, W^1 \rangle + b \langle U^2, W^2 \rangle + \langle U^3, W^3 \rangle,$$

where $U = (U^1, U^2, U^3)^T$ and $W = (W^1, W^2, W^3)^T$. Then our problem is associated with

(2.3) $$U_t = AU, \qquad U(t = 0) = U^0,$$

where

$$A : D(A) \subset \mathcal{H} \longrightarrow \mathcal{H}, \qquad AU := A_f U,$$

for $U \in D(A)$ with

$$D(A) := \{ U = (v, w, \theta)^T \in \mathcal{H} \mid v \in H^2, \ \Delta(bw + d\theta) \in L^2 \}.$$

In the definition of $D(A)$, the problem of the missing (separate) regularity for θ is reflected. cp. Remark 2.1. One just has the combined regularity $\Delta(bw + d\theta) \in L^2$, not writing Δw, $\Delta\theta \in L^2$, and this way $A_f U$ has to be interpreted.

As usual, $\Delta(bw + d\theta) \in L^2$ means

$$(2.4) \qquad \exists\, h \in L^2,\ \forall\, \phi \in C_0^\infty : \langle bw + d\theta, \Delta\phi\rangle = \langle h, \phi\rangle.$$

We remark that the difference to the bounded domain case considered in [28] lies in the choice of the transformation by U and the choice of the space $\mathcal{H}$. This modification is necessary essentially because in all of $\mathbb{R}^n$, the set H^2 is no longer a Banach space under the norm $\|\Delta \cdot \|$, in contrast to the situation in bounded domains considered in [28].

We will show that A generates a contraction semigroup.

Lemma 2.2. $D(A)$ is dense in $\mathcal{H}$, and for $U \in D(A)$ with (1.2), we have the dissipativity of A that

$$\operatorname{Re}\langle AU, U\rangle_{\mathcal{H}} = -\kappa\|\nabla\psi\|^2 - \kappa a\|\Delta\psi\|^2 \le 0.$$

Proof. $(C_0^\infty)^3 \subset D(A)$ is dense in $\mathcal{H}$. Furthermore, the computation

$$\operatorname{Re}\langle AU, U\rangle_{\mathcal{H}} = \operatorname{Re}\left(\langle -\Delta(bw + d\theta), v\rangle + b\langle \Delta v, w\rangle + \langle d\Delta v + B\theta, \theta\rangle\right)$$
$$= \langle B\theta, \theta\rangle = -\kappa\|\nabla\psi\|^2 - \kappa a\|\Delta\psi\|^2 \le 0$$

gives the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 2.3. *The range of* $\operatorname{Id} - A$ *equals* $\mathcal{H}$.

Proof. $(\operatorname{Id} - A)U = F$ is, for given $F = (F^1, F^2, F^3)^T \in \mathcal{H}$, equivalent to finding $U \in D(A)$ solving

$$\begin{cases} U^1 + \Delta(bU^2 + dU^3) = F^1, \\ \qquad\quad U^2 - \Delta U^1 = F^2, \\ U^3 - d\Delta U^1 - BU^3 = F^3. \end{cases}$$

Here, $U^2 := \Delta U^1 + F^2$ will be given if we find (U^1, U^3) satisfying

$$(2.5) \qquad \begin{cases} U^1 + \Delta(b\Delta U^1 + bF^2 + dU^3) = F^1, \\ \qquad\quad U^3 - d\Delta U^1 - BU^3 = F^3, \end{cases}$$

with $U \in D(A)$. For this purpose we consider the sesquilinear form

$$\beta : \mathcal{K} := H^2 \times L^2 \longrightarrow \mathbb{C},$$

where H^2 is equipped with the usual H^2-norm, and

$$\beta\left((U^1, U^3), (W^1, W^3)\right) := \langle U^1, W^1\rangle + \langle b\Delta U^1 + dU^3, \Delta W^1\rangle + \langle U^3, W^3\rangle$$
$$- \langle d\Delta U^1, W^3\rangle - \langle BU^3, W^3\rangle.$$

The variational problem associated to (2.5) is to find a (unique) $(U^1, U^3) \in \mathcal{K}$ satisfying for all $(W^1, W^3) \in \mathcal{K}$

$$\beta\left((U^1, U^3), (W^1, W^3)\right) = \langle F^1, W^1\rangle + \langle F^3, W^3\rangle - \langle bF^2, \Delta W^1\rangle.$$

The solvability of the variational problem follows from the theorem of Lax and Milgram, observing

$$\left|\beta\left((U^1,U^3),(W^1,W^3)\right)\right| \leq C\|(U^1,U^3)\|_{\mathcal{K}}\|(W^1,W^3)\|_{\mathcal{K}}$$

with some positive constant $C > 0$, and

$$\operatorname{Re}\beta\left((U^1,U^3),(U^1,U^3)\right) = \|U^1\|^2 + b\|\Delta U^1\|^2 + \|U^3\|^2 - \langle BU^3, U^3\rangle$$
$$\geq c\|(U^1,U^3)\|_{\mathcal{K}}^2$$

with some positive constant $c > 0$, by elliptic regularity. $\qquad\square$

By the Lumer-Phillips theorem we conclude the well-posedness:

Theorem 2.4. *A generates a contraction semigroup, and, for any $U^0 \in D(A)$, there is a unique solution U to (2.3) satisfying*

$$U \in C^1\left([0,\infty),\mathcal{H}\right) \cap C^0\left([0,\infty), D(A)\right).$$

2.2. **Well-posedness for $\tau > 0$.** The model (1.3), (1.2) for thermoelastic plates of Kirchhoff type with two temperatures under the Cattaneo law will now be shown to be well-posed. The well-posedness requires the choice of suitable representations of the solutions and corresponding phase spaces. The regularity issue is even more complicated due to the fact that the heat flux is not immediately of the same regularity as the gradient of the temperature ψ, as it was in the case of the Fourier model discussed in the previous section. The issue of only combined regularity for (u,θ,q) only, in contrast to separate regularity for each of u, θ, q, comes up again requiring the right spaces and domains of operators.

We consider the Cauchy problem (1.3), (1.2) with initial data

$$(2.6)\qquad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad \theta(0,x) = \theta_0(x), \quad q(0,x) = q_0(x)$$

for $x \in \mathbb{R}^n$. Defining $v := u_t$, $w := \sqrt{b}\Delta u$ we obtain from (1.3), (1.2)

$$v_t + \sqrt{b}\Delta w + d\Delta\theta = 0,$$
$$w_t - \sqrt{b}\Delta v = 0,$$
$$\theta_t + \operatorname{div} q - d\Delta v = 0,$$
$$\tau q_t + \nabla B_1\theta + q = 0,$$

where B_1 denotes the bounded operator $B_1 : L^2 \to H^2$, $B_1 := \kappa(\operatorname{Id} - a\Delta)^{-1}$. Let $U := (v,w,\theta,q)^T$. Then

$$(2.7)\qquad U_t = \begin{pmatrix} 0 & -\sqrt{b}\Delta & -d\Delta & 0 \\ \sqrt{b}\Delta & 0 & 0 & 0 \\ d\Delta & 0 & 0 & -\operatorname{div} \\ 0 & 0 & -\frac{1}{\tau}\nabla B_1 & -\frac{1}{\tau} \end{pmatrix} U =: A_{1,f}U,$$

$$(2.8)\qquad U(0,\cdot) = U^0 := (u_1, \sqrt{b}\Delta u_0, \theta_0, q_0)^T.$$

System (2.7), (2.8) will be considered as an evolution equation in the associated Hilbert space $\mathcal{H}_1 := L^2 \times L^2 \times L^2 \times \mathcal{D}$, where $\mathcal{D} := \{q \in L^2 \,|\, \mathrm{div}\, q \in L^2\}$, with inner product

$$\langle U, W \rangle_{\mathcal{H}_1} := \langle U^1, W^1 \rangle + \langle U^2, W^2 \rangle + \frac{1}{\tau}\langle U^3, B_1 W^3 \rangle + \langle U^4, W^4 \rangle + \langle \mathrm{div}\, U^4, \mathrm{div}\, W^4 \rangle,$$

where $U = (U^1, U^2, U^3, U^4)^T$ and $W = (W^1, W^2, W^3, W^4)^T$. Then we solve

$$(2.9) \qquad\qquad U_t = A_1 U, \qquad U(t = 0) = U^0,$$

where

$$A_1 : D(A_1) \subset \mathcal{H}_1 \longrightarrow \mathcal{H}_1, \qquad A_1 U := A_{1,f} U,$$

for $U \in D(A_1)$ with

$$D(A_1) := \{\, U = (v, w, \theta, q)^T \in \mathcal{H}_1 \,|\, v \in H^2, \, \Delta(\sqrt{b}w + d\theta) \in L^2 \,\}.$$

In the definition of $D(A_1)$, the problem of the missing (separate) regularity is reflected again, cf. the previous subsection. Again we show that A_1 generates a C_0-semigroup. For this purpose we write

$$A_1 = \underbrace{\begin{pmatrix} 0 & -\sqrt{b}\Delta & -d\Delta & 0 \\ \sqrt{b}\Delta & 0 & 0 & 0 \\ d\Delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{=:A_{11}} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{div} \\ 0 & 0 & -\frac{1}{\tau}\nabla B_1 & -\frac{1}{\tau} \end{pmatrix}}_{=:A_{12}}.$$

The operator $A_{12} : \mathcal{H}_1 \to \mathcal{H}_1$ is bounded, and for

$$A_{11} : D(A_{11}) := D(A_1) \subset \mathcal{H}_1 \longrightarrow \mathcal{H}_1$$

we have the following fact.

Lemma 2.5. (i) $D(A_{11})$ *is dense in* $\mathcal{H}_1$, *and* A_{11} *is dissipative,*

$$\mathrm{Re}\,\langle A_{11} U, U \rangle_{\mathcal{H}_1} = 0.$$

(ii) *The range of* $\mathrm{Id} - A_{11}$ *equals* $\mathcal{H}_1$.

Proof. (i) is easy again, and to find $U = (U^1, U^2, U^3, U^4)^T \in D(A_1)$ satisfying $(\mathrm{Id} - A_{11})U = F$, for given $F = (F^1, F^2, F^3, F^4)^T \in \mathcal{H}_1$, we may argue as in the proof of Lemma 2.3, now eliminating first U^2, U^3 and U^4 as follows. $(\mathrm{Id} - A_{11})U = F$ is equivalent to

$$\begin{cases} U^1 + \Delta(\sqrt{b}U^2 + dU^3) = F^1, \\ U^2 - \sqrt{b}\Delta U^1 = F^2, \\ U^3 - d\Delta U^1 = F^3, \\ U^4 = F^4. \end{cases}$$

After having found the appropriate U^1, we may take $U^2 := \sqrt{b}\Delta U^1 + F^2$, $U^3 := d\Delta U^1 + F^3$ and $U^4 := F^4$. For U^1 we have to solve

$$U^1 + \Delta(b\Delta U^1 + \sqrt{b}F^2 + dF^3 + d^2\Delta U^1) = F^1.$$

This is equivalent to finding $U^1 \in H^2$ satisfying for all $W^1 \in H^2$

$$(2.10) \qquad \langle U^1, W^1 \rangle + \langle (b + d^2)\Delta U^1 + \sqrt{b}F^2 + dF^3, \Delta W^1 \rangle = \langle F^1, W^1 \rangle.$$

Inspired by this equation, we define the bilinear form $\beta_{11} : \mathcal{K}_1 := H^2 \times H^2 \to \mathbb{C}$ with

$$\beta_{11}(U^1, W^1) := \langle U^1, W^1 \rangle + \langle (b + d^2)\Delta U^1, \Delta W^1 \rangle$$

and $\mathcal{F} : H^2 \to \mathbb{C}$ with

$$\mathcal{F}(W^1) := \langle F_1, W^1 \rangle - \langle \sqrt{b}F^2 + dF^3, \Delta W^1 \rangle.$$

Then (2.10) is described as $\beta_{11}(U^1, W^1) = \mathcal{F}(W^1)$. Here, $\mathcal{F}$ is a bounded linear functional on H^2. Furthermore, the bilinear form β_{11} satisfies

$$|\beta_{11}(U^1, W^1)| \leq C_1 \|U^1\|_{H^2} \|W^1\|_{H^2},$$

$$\beta_{11}(U^1, U^1) = \langle U^1, U^1 \rangle + \langle (b + d^2)\Delta U^1, \Delta U^1 \rangle \geq c_1 \|U^1\|_{H^2}^2,$$

for some positive constants and C_1 and c_1, so we may use the theorem of Lax and Milgram to find the solution $U^1 \in H^2$. $\qquad\square$

As a consequence we obtain the well-posedness of (2.9) in the following result.

Theorem 2.6. A_1 *generates a C_0-semigroup, and, for any $U^0 \in D(A_1)$, there is a unique solution U to* (2.9) *satisfying*

$$U \in C^1\left([0, \infty), \mathcal{H}_1\right) \cap C^0\left([0, \infty), D(A_1)\right).$$

3. Decay estimates for Fourier type heat conduction

In this section, we consider the case $\tau = 0$, i.e. system (1.4), (1.2), with initial data (2.1). The purpose of this section is to derive the optimal decay estimates for the global solutions in time .

3.1. **Decay estimates ($\tau = 0$).** To derive a representation of the solution and to get the decay estimates for the solutions, we introduce new functions $v := u_t$, $w := \sqrt{b}\Delta u$ for (1.4). Then our problem reads as

$$(3.1) \qquad \begin{aligned} v_t + \sqrt{b}\Delta w + d\Delta\theta &= 0, \\ w_t - \sqrt{b}\Delta v &= 0, \\ \theta_t - \kappa\Delta\psi - d\Delta v &= 0. \end{aligned}$$

Furthermore, (3.1) with (1.2) leads to

$$(3.2) \qquad (I - a\Delta)U_t - \Delta B_1 U - (I - a\Delta)\Delta B_2 U = 0,$$

where $U := (v, w, \theta)^T$ and

$$B_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \kappa \end{pmatrix}, \qquad B_2 := \begin{pmatrix} 0 & -\sqrt{b} & -d \\ \sqrt{b} & 0 & 0 \\ d & 0 & 0 \end{pmatrix}.$$

Applying the Fourier transform to (3.2), we have

$$(3.3) \qquad \hat{U}_t + \frac{|\xi|^2}{1 + a|\xi|^2}B_1\hat{U} + |\xi|^2 B_2\hat{U} = 0.$$

The is can be solved, and the solution of (3.3) can be written as

$$(3.4) \qquad \hat{U}(t, \xi) = e^{t\hat{\Phi}(i\xi)}\hat{U}_0(\xi),$$

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where $U_0 := (v_0, w_0, \theta_0)^T$ with $v_0 := u_1$ and $w_0 := \Delta u_0$, and

$$(3.5) \quad \hat{\Phi}(i\xi) := -\left(\frac{|\xi|^2}{1 + a|\xi|^2} B_1 + |\xi|^2 B_2 \right) = -\begin{pmatrix} 0 & -\sqrt{b}|\xi|^2 & -d|\xi|^2 \\ \sqrt{b}|\xi|^2 & 0 & 0 \\ d|\xi|^2 & 0 & \kappa|\xi|^2/(1 + a|\xi|^2) \end{pmatrix}.$$

Therefore, the semigroup $e^{t\Phi}$ associated with the system (3.2) is given by the formula

$$(3.6) \quad e^{t\Phi}\varphi := \mathcal{F}^{-1}[e^{t\hat{\Phi}(i\xi)}\hat{\varphi}(\xi)].$$

Now, our purpose is to derive the property of the solution operator $e^{t\Phi}$. The result on the decay estimates for (3.2) is stated as follows.

Theorem 3.1. *Let $e^{t\Phi}$ be the semigroup associated with the system (3.2) defined by (3.6). Then the following decay estimates hold for $1 \leq p \leq 2$ and $k \geq 0$.*

$$(3.7) \quad \|\partial_x^k e^{t\Phi}\varphi\|_{L^2} \leq C_0(1 + \nu_a t)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k}{2}}\|\varphi\|_{L^p} + \sqrt{3}e^{-\frac{\nu_a}{2}t}\|\partial_x^k \varphi\|_{L^2},$$

where

$$\nu_a := \frac{\nu}{1 + a}, \qquad \nu := \frac{bd^2\kappa}{12(8(b + d^2)^2 + b\kappa^2)},$$

and C_0 is a certain positive constant depends on only p and k.

The key of the proof of Theorem 3.1 is to get the pointwise estimate of the operator $e^{t\Phi}$ in Fourier space, which is stated as follows.

Proposition 3.2. *Let $\hat{\Phi}(i\xi)$ be the matrix defined in (3.5). Then the corresponding matrix exponential $e^{t\hat{\Phi}(i\xi)}$ satisfies the following pointwise estimate*

$$(3.8) \quad |e^{t\hat{\Phi}(i\xi)}| \leq \sqrt{3}e^{-\frac{1}{2}\rho(\xi)t}$$

for $t \geq 0$ and $\xi \in \mathbb{R}^n$, where

$$(3.9) \quad \rho(\xi) := \frac{\nu|\xi|^2}{1 + a|\xi|^2},$$

and ν is defined in Theorem 3.1.

Proof. We first derive the basic energy equation for the system (3.2). Taking the inner product (3.2) with $\hat{U}$, and taking real parts in the resulting equality, we have

$$(3.10) \quad \frac{1}{2}(1 + a|\xi|^2)\frac{\partial}{\partial t}|\hat{U}|^2 + \kappa|\xi|^2|\hat{\theta}|^2 = 0.$$

Next, we construct the dissipation terms. System (3.2) gives us

$$(3.11) \quad \begin{aligned} \hat{v}_t - \sqrt{b}|\xi|^2\hat{w} - d|\xi|^2\hat{\theta} &= 0, \\ \hat{w}_t + \sqrt{b}|\xi|^2\hat{v} &= 0, \\ \hat{\theta}_t + \kappa|\xi|^2\hat{\psi} + d|\xi|^2\hat{v} &= 0, \end{aligned}$$

with $\hat{\theta} = (1 + a|\xi|^2)\hat{\psi}$. We multiply the first and second equations in (3.11) by $-\bar{\hat{w}}$ and $-\bar{\hat{v}}$, respectively. Then, combining the resulting equations and taking real parts, we obtain

$$-\frac{\partial}{\partial t}\text{Re}(\hat{v}\bar{\hat{w}}) + \sqrt{b}|\xi|^2(|\hat{w}|^2 - |\hat{v}|^2) + d|\xi|^2\text{Re}(\hat{w}\bar{\hat{\theta}}) = 0.$$

Especially, using the Hölder inequality, we get

$$(3.12) \qquad -\sqrt{b}\frac{\partial}{\partial t}\mathrm{Re}(\hat{v}\bar{\hat{w}}) + \frac{b}{2}|\xi|^2|\hat{w}|^2 - b|\xi|^2|\hat{v}|^2 - \frac{d^2}{2}|\xi|^2|\hat{\theta}|^2 \le 0.$$

Furthermore, we multiply the first and third equations in (3.11) by $\bar{\hat{\theta}}$ and $\bar{\hat{v}}$, respectively. Then we have

$$\frac{\partial}{\partial t}\mathrm{Re}(\hat{v}\bar{\hat{\theta}}) + d|\xi|^2(|\hat{v}|^2 - |\hat{\theta}|^2) + \kappa|\xi|^2\mathrm{Re}(\hat{v}\bar{\hat{\psi}}) - \sqrt{b}|\xi|^2\mathrm{Re}(\hat{w}\bar{\hat{\theta}}) = 0.$$

Similarly as before, we obtain

$$(3.13) \qquad d\frac{\partial}{\partial t}\mathrm{Re}(\hat{v}\bar{\hat{\theta}}) + \frac{d^2}{2}|\xi|^2|\hat{v}|^2 - d^2|\xi|^2|\hat{\theta}|^2 - \frac{\kappa^2}{2}|\xi|^2|\hat{\psi}|^2 - d\sqrt{b}|\xi|^2\mathrm{Re}(\hat{w}\bar{\hat{\theta}}) \le 0.$$

Our next step is to combine the above inequalities to construct the energy estimate. First, computing $d^2/4 \times (3.12) + b \times (3.13)$, we get

$$\frac{\partial}{\partial t}\Big\{bd\mathrm{Re}(\hat{v}\bar{\hat{\theta}}) - \frac{d^2\sqrt{b}}{4}\mathrm{Re}(\hat{v}\bar{\hat{w}})\Big\} + \frac{bd^2}{4}|\xi|^2\Big(|\hat{v}|^2 + \frac{1}{2}|\hat{w}|^2\Big)$$
$$- d^2(b + \frac{d^2}{8})|\xi|^2|\hat{\theta}|^2 - \frac{b\kappa^2}{2}|\xi|^2|\hat{\psi}|^2 + bd\sqrt{b}|\xi|^2\mathrm{Re}(\hat{w}\bar{\hat{\theta}}) \le 0.$$

Here, using

$$bd\sqrt{b}|\hat{w}||\hat{\theta}| \le \frac{bd^2}{16}|\hat{w}|^2 + 4b^2|\hat{\theta}|^2,$$

we obtain

$$\frac{\partial}{\partial t}\Big\{bd\mathrm{Re}(\hat{v}\bar{\hat{\theta}}) - \frac{d^2\sqrt{b}}{4}\mathrm{Re}(\hat{v}\bar{\hat{w}})\Big\} + \frac{bd^2}{4}|\xi|^2\Big(|\hat{v}|^2 + \frac{1}{4}|\hat{w}|^2\Big)$$
$$- d^2\Big(b + \frac{d^2}{8}\Big)|\xi|^2|\hat{\theta}|^2 - \frac{b\kappa^2}{2}|\xi|^2|\hat{\psi}|^2 - 4b^2|\xi|^2|\hat{\theta}|^2 \le 0.$$

Especially, $\hat{\theta} = (1 + a|\xi|^2)\hat{\psi}$ gives us

$$(3.14) \qquad \frac{\partial}{\partial t}\Big\{bd\mathrm{Re}(\hat{v}\bar{\hat{\theta}}) - \frac{d^2\sqrt{b}}{4}\mathrm{Re}(\hat{v}\bar{\hat{w}})\Big\} + \frac{bd^2}{4}|\xi|^2\Big(|\hat{v}|^2 + \frac{1}{4}|\hat{w}|^2\Big) - 4g_0|\xi|^2|\hat{\theta}|^2 \le 0,$$

where

$$g_0 := (b + d^2)^2 + \frac{b\kappa^2}{8}.$$

Therefore, calculating $g_0 \times (3.10) + \kappa/8 \times (3.14)$ yields

$$(3.15) \qquad \frac{\partial}{\partial t}E_0(t, \xi) + \frac{bd^2\kappa}{32}|\xi|^2\Big(|\hat{v}|^2 + \frac{1}{4}|\hat{w}|^2\Big) + \frac{\kappa}{2}g_0|\xi|^2|\hat{\theta}|^2 \le 0,$$

where we define

$$E_0(t, \xi) := \frac{1}{2}g_0(1 + a|\xi|^2)|\hat{U}|^2 + \frac{\kappa}{8}\Big(bd\mathrm{Re}(\hat{v}\bar{\hat{\theta}}) - \frac{d^2\sqrt{b}}{4}\mathrm{Re}(\hat{v}\bar{\hat{w}})\Big).$$

Here, we easily obtain from

$$d^2\kappa\sqrt{b}|\hat{v}||\hat{w}| \le \frac{1}{2}(b\kappa^2|\hat{v}|^2 + d^4|\hat{w}|^2), \qquad d\kappa|\hat{v}||\hat{\theta}| \le d^2|\hat{v}|^2 + \frac{\kappa^2}{4}|\hat{\theta}|^2$$

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the estimate

$$(3.16) \qquad \frac{1}{4}g_0(1+a|\xi|^2)|\hat{U}|^2 \leq E_0(t,\xi) \leq \frac{3}{4}g_0(1+a|\xi|^2)|\hat{U}|^2.$$

Therefore, integrating (3.15) over t and applying (3.16) to the resulting estimate, we get

$$|\hat{U}(t,\xi)|^2 + \int_0^t \frac{\kappa|\xi|^2}{1+a|\xi|^2}\left\{\frac{bd^2}{8g_0}\left(|\hat{v}(t',\xi)|^2 + \frac{1}{4}|\hat{w}(t',\xi)|^2\right) + 2|\hat{\theta}(t',\xi)|^2\right\}dt' \leq 3|\hat{U}_0(\xi)|^2,$$

hence

$$\|\partial_x^k U(t)\|_{H_a^1}^2 + \frac{bd^2\kappa}{32g_0}\int_0^t \|\partial_x^{k+1} U(t')\|_{L^2}^2 dt' \leq 3\|\partial_x^k U_0\|_{H_a^1}^2,$$

for $k \geq 0$. Furthermore, (3.15) and

$$\frac{bd^2\kappa}{32}|\xi|^2\left(|\hat{v}|^2 + \frac{1}{4}|\hat{w}|^2\right) + \frac{\kappa}{2}g_0|\xi|^2|\hat{\theta}|^2 \geq \frac{bd^2\kappa}{128}|\xi|^2|\hat{U}|^2 \geq \frac{bd^2\kappa}{96g_0}\frac{|\xi|^2}{1+a|\xi|^2}E_0(t,\xi),$$

which comes from (3.16), gives us

$$\frac{\partial}{\partial t}E_0(t,\xi) + \frac{bd^2\kappa}{96g_0}\frac{|\xi|^2}{1+a|\xi|^2}E_0(t,\xi) \leq 0.$$

Thus, we obtain $E_0(t,\xi) \leq e^{-\rho(\xi)t}E_0(0,\xi)$, where $\rho(\xi)$ is defined in Proposition 3.2. This means $|\hat{U}(t,\xi)| \leq \sqrt{3}e^{-\rho(\xi)t/2}|\hat{U}_0(\xi)|$, and (3.4) gives the the desired pointwise estimate (3.8). $\qquad\square$

Proof of Theorem 3.1. We use the Hausdorff-Young inequality, $\|\hat{f}\|_{L^{p'}} \leq (2\pi)^{n/p'}\|f\|_{L^p}$ for $1 \leq p \leq 2$ and $1/p + 1/p' = 1$.

By (3.8), we have

$$\|\partial_x^k e^{t\Phi}\varphi\|^2 \leq 3\int_{|\xi|\leq 1}|\xi|^{2k}e^{-\rho(\xi)t}|\hat{\varphi}(\xi)|^2 d\xi + 3\int_{|\xi|\geq 1}|\xi|^{2k}e^{-\rho(\xi)t}|\hat{\varphi}(\xi)|^2 d\xi =: I_1 + I_2.$$

For the case $|\xi| \leq 1$, we compute $\rho(\xi) \geq \nu_a|\xi|^2$, and then

$$I_1 \leq 3\int_{|\xi|\leq 1}|\xi|^{2k}e^{-\nu_a|\xi|^2 t}|\hat{\varphi}(\xi)|^2 d\xi$$

$$\leq 3\||\xi|^{2k}e^{-\nu_a|\xi|^2 t}\|_{L^\sigma(|\xi|\leq 1)}\|\hat{\varphi}\|_{L^{p'}(|\xi|\leq 1)}^2 \leq C_0^2(1+\nu_a t)^{-n/(2\sigma)-k}\|\varphi\|_{L^p}^2$$

for $1/\sigma + 2/p' = 1$, $1/p + 1/p' = 1$ and $1 \leq p \leq 2$, where we used the Hölder inequality and the Hausdorff-Young inequality. Here we note that $1/\sigma = 2/p - 1$, and we then arrive at

$$I_1 \leq C_0^2(1+\nu_a t)^{-n(\frac{1}{p}-\frac{1}{2})-k}\|\varphi\|_{L^p}^2$$

for $1 \leq p \leq 2$, where C_0 is a positive constant depends on only p and k. For the case $|\xi| \geq 1$, we calculate $\rho(\xi) \geq \nu_a$ and

$$I_2 \leq 3e^{-\nu_a t}\int_{|\xi|\geq 1}|\xi|^{2k}|\hat{\varphi}(\xi)|^2 d\xi \leq 3e^{-\nu_a t}\|\partial_x^k \varphi\|_{L^2}^2.$$

Consequently, we get

$$\|\partial_x^k e^{t\Phi}\varphi\|^2 \leq C_0^2(1+\nu_a t)^{-n(\frac{1}{p}-\frac{1}{2})-k}\|\varphi\|_{L^p}^2 + 3e^{-\nu_a t}\|\partial_x^k \varphi\|_{L^2}^2.$$

and arrive at the desired decay estimate (3.7). $\qquad\square$

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3.2. **Optimality of the decay estimates ($\tau = 0$).** At the end of this section, we investigate the optimality of the pointwise estimates in Theorem 3.1. To this end, we consider the characteristic equation $\det(\lambda I - \hat{\Phi}(i\xi)) = 0$ for the system (3.2), which is equivalent to

$$(3.17) \qquad (1 + a|\xi|^2)\lambda^3 + \kappa|\xi|^2\lambda^2 + (b + d^2)(1 + a|\xi|^2)|\xi|^4\lambda + b\kappa|\xi|^6 = 0.$$

We consider the asymptotic expansion of $\lambda = \lambda(i\xi)$ for $|\xi| \to 0$ and for $|\xi| \to \infty$. These expansions essentially determine the asymptotic behavior of solutions.

We first consider the asymptotic expansion for $|\xi| \to 0$:

$$(3.18) \qquad \lambda(i\xi) = \sum_{k=0}^{\infty} \alpha_k|\xi|^k.$$

Substituting (3.18) into (3.17), we have identities with respect to the order of $|\xi|$. From the zeroth order and third order identities, we obtain $\alpha_0 = 0$ and $\alpha_1 = 0$, respectively. Furthermore, the sixth order identity gives $f(\alpha_2) = 0$, where

$$(3.19) \qquad f(z) := z^3 + \kappa z^2 + (b + d^2)z + b\kappa.$$

Let z_j with $j = 1, 2, 3$ be solutions for $f(z) = 0$. Then these solutions satisfy $z_1 + z_2 + z_3 = -\kappa$. Since $f(0) = b\kappa > 0$ and $f(-\kappa) = -d^2\kappa < 0$, we get $\mathrm{Re}(z_j) < 0$ for $j = 1, 2, 3$. Thus, we conclude that the solutions for (3.17) satisfy

$$(3.20) \qquad \lambda_j(i\xi) = z_j|\xi|^2 + O(|\xi|^4)$$

for $j = 1, 2, 3$.

Next, we consider the asymptotic expansion for $|\xi| \to \infty$. To this end, we take $\lambda = |\xi|^2\nu$ in (3.17) and obtain

$$(3.21) \qquad (a + |\xi|^{-2})\nu^3 + \kappa|\xi|^{-2}\nu^2 + (b + d^2)(a + |\xi|^{-2})\nu + b\kappa|\xi|^{-2} = 0.$$

We substitute

$$\nu(i\xi) = \sum_{k=0}^{\infty} \beta_k|\xi|^{-k}$$

into (3.21). Then the zeroth order identity gives $\beta_0 = 0, \pm\sqrt{b + d^2}i$. Furthermore, this yields $\beta_1 = 0$, $\beta_3 = 0$ and

$$\beta_2 = -\frac{\kappa(\beta_0^2 + b)}{a(3\beta_0^2 + b + d^2)}.$$

Therefore, we derive

$$(3.22) \qquad \begin{aligned} \lambda_j(i\xi) &= \pm\sqrt{b + d^2}i|\xi|^2 - \frac{d^2\kappa}{2a(b + d^2)} + O(|\xi|^{-2}), \\ \lambda_3(i\xi) &= -\frac{b\kappa}{a(b + d^2)} + O(|\xi|^{-2}) \end{aligned}$$

for $j = 1, 2$.

Consequently, the asymptotic expansions (3.20) and (3.22) tell us that the pointwise estimate (3.8) is optimal.

4. Decay estimates for Cattaneo type heat conduction

Now we turn to the Cattaneo case $\tau > 0$, i.e we look at (1.3), (1.2) with initial data (2.6). Our purpose is also to derive optimal decay estimates for global solutions in time.

4.1. Decay estimates for $\tau > 0$.

Note that we have already studied (1.3), (1.2) with $a = 0$ in [31]. Here we try to use similar arguments as in section 2. Introducing again $v := u_t$, $w := \sqrt{b}\Delta u$, our problem (1.3), (1.2) is rewritten as in the previous section as

$$
\begin{aligned}
v_t + \sqrt{b}\Delta w + d\Delta\theta &= 0, \\
w_t - \sqrt{b}\Delta v &= 0, \\
\theta_t + \operatorname{div} q - d\Delta v &= 0, \\
\tau q_t + \kappa\nabla\psi + q &= 0.
\end{aligned}
$$

(4.1)

Furthermore, (4.1) with (1.2) leads

$$
(4.2) \qquad \left(A_1^0 + (1 - a\Delta)L\right)U_t + \sum_{j=1}^n A^j\partial_{x_j}U - \Delta BU + \frac{1}{\tau}(1 - a\Delta)LU = 0,
$$

where $U := (\sqrt{\kappa}v, \sqrt{\kappa}w, \sqrt{\kappa}\theta, \sqrt{\tau}q)^T$ and

$$
A_1^0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \sum_{j=1}^n A^j\xi_j := \sqrt{\frac{\kappa}{\tau}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi \\ 0 & 0 & \xi^T & 0 \end{pmatrix},
$$

$$
B := \begin{pmatrix} 0 & -\sqrt{b} & -d & 0 \\ \sqrt{b} & 0 & 0 & 0 \\ d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad L := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \operatorname{Id} \end{pmatrix},
$$

where $\xi = (\xi_1, \cdots, \xi_n)$. Applying the Fourier transform to (4.2), we obtain

$$
(4.3) \qquad \left(A_1^0 + (1 + a|\xi|^2)L\right)\hat{U}_t + i|\xi|A(\omega)\hat{U} + |\xi|^2 B\hat{U} + \frac{1}{\tau}(1 + a|\xi|^2)L\hat{U} = 0,
$$

where $A(\omega) := \sum_{j=1}^n A^j\omega_j$ for $\omega := \xi/|\xi| \in S^{n-1}$ and $\omega = (\omega_1, \cdots, \omega_n)$. Furthermore, we introduce the new unknown function $\hat{V} := (\sqrt{\kappa}\hat{v}, \sqrt{\kappa}\hat{w}, \sqrt{\kappa}\hat{\theta}, \sqrt{\tau(1 + a|\xi|^2)}\hat{q})^T$. Then (4.3) can be rewritten as

$$
(4.4) \qquad \hat{V}_t + \frac{i|\xi|}{\sqrt{1 + a|\xi|^2}}A(\omega)\hat{V} + |\xi|^2 B\hat{V} + \frac{1}{\tau}L\hat{V} = 0.
$$

We observe that $A_1^0 + L$ is the identity matrix. The solution of (4.4) can be written as

$$
(4.5) \qquad \hat{V}(t, \xi) = e^{t\hat{\Phi}(i\xi)}\hat{V}_0(\xi),
$$

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where $\hat{V}_0 := (\sqrt{\kappa}\hat{v}_0, \sqrt{\kappa}\hat{w}_0, \sqrt{\kappa}\hat{\theta}_0, \sqrt{\tau(1 + a|\xi|^2)}\hat{q}_0)^T$ with $\hat{v}_0 = \hat{u}_1$ and $\hat{w}_0 = -\sqrt{b}|\xi|^2\hat{u}_0$, and

$$(4.6) \qquad \hat{\Phi}(i\xi) := -\Big(\frac{i|\xi|}{\sqrt{1 + a|\xi|^2}}A(\omega) + |\xi|^2 B + \frac{1}{\tau}L\Big).$$

Then we define the semigroup $e^{t\Phi}$ by the formula

$$(4.7) \qquad e^{t\Phi}\varphi := \mathcal{F}^{-1}[e^{t\hat{\Phi}(i\xi)}\hat{\varphi}(\xi)].$$

Our main purpose is now to derive the property of the solution operator $e^{t\Phi}$.

Theorem 4.1. *Let $e^{t\Phi}$ be the semigroup defined by (4.7). Then the following decay estimates hold for $1 \leq p \leq 2$ and $k \geq 0$.*

$$(4.8) \quad \|\partial_x^k e^{t\Phi}\varphi\|_{L^2} \leq C_0(1 + \nu_{\tau,a}t)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k}{2}}\|\varphi\|_{L^p} + C_1(1 + \nu_{\tau,a}t)^{-\frac{\ell}{4}}\|\partial_x^{k+\ell}\varphi\|_{L^2},$$

where

$$\nu_{\tau,a} := \frac{\tilde{\nu}}{(1 + \tau)^2(1 + a)},$$

$\tilde{\nu}$ is a positive constant depending on only b, d and κ, C_0 is a positive constant depending on only p and k, and C_1 is a positive constant depending only on ℓ.

From the property of the semigroup operator $e^{t\Phi}$, we derive the result on the following decay estimates for (1.3), (1.2).

Corollary 4.2. *Let (v, w, θ, q) be the solution for (1.3), (1.2) with the initial data (2.6). Then the following decay estimates hold for $1 \leq p \leq 2$ and $k, \ell \geq 0$.*

$$(4.9) \quad \begin{aligned} &\sqrt{\kappa}\|\partial_x^k(u_t, \Delta u, \theta)\|_{L^2} + \sqrt{\tau}\|\partial_x^k q\|_{H_a^1} \\ \leq{}& \sqrt{2}C_0(1 + \nu_{\tau,a}t)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k}{2}}\big(\sqrt{\kappa}\|(u_1, \Delta u_0, \theta_0)\|_{L^p} + \sqrt{\tau(1 + a)}\|q_0\|_{L^p}\big) \\ &+ \sqrt{2}C_1(1 + \nu_{\tau,a}t)^{-\frac{\ell}{4}}\big(\sqrt{\kappa}\|\partial_x^{k+\ell}(u_1, \Delta u_0, \theta_0)\|_{L^2} + \sqrt{\tau(1 + a)}\|\partial_x^{k+\ell+1}q_0\|_{L^2}\big), \end{aligned}$$

where $\nu_{\tau,a}$, C_0 and C_1 appeared in Theorem 4.1.

In Collorary 4.2 (and Theorem 4.1), the loss of regularity is clearly visible: to get certain decay rates one needs more regularity of the initial data on the right-hand side than can be estimated for the solutions on the left-hand side. This is in contrast to the situation for $\tau = 0$ stated in Theorem 3.1. The main estimate is the following pointwise one that will allow as conclude the rates of decay and to describe the effect of regularity loss.

The key of the proof of Theorem 3.1 is to get the pointwise estimate of the operator $e^{t\Phi}$ in Fourier space, which is stated as follows.

Proposition 4.3. *Let $\hat{\Phi}(i\xi)$ be the matrix defined in (4.6). Then the corresponding matrix exponential $e^{t\hat{\Phi}(i\xi)}$ satisfies the following pointwise estimate*

$$(4.10) \qquad |e^{t\hat{\Phi}(i\xi)}| \leq \sqrt{3}e^{-\frac{1}{2}\eta(\xi)t}$$

for $t \geq 0$ and $\xi \in \mathbb{R}^n$, where

$$(4.11) \qquad \eta(\xi) := \frac{\tilde{\nu}|\xi|^2}{(1 + \tau|\xi|^2)^2(1 + a|\xi|^2)},$$

and $\tilde{\nu}$ depends only on b, d and κ.

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The precise estimate clearly describes the role of the parameters τ and a, leading to regularity loss whenever $\tau > 0$. Formally, we recover the estimate (3.9) from section 3.

Remark 4.4. (i) *The estimate* (4.10) *tells us that it might be better to consider not only q but also the derivative of q to construct the solution operator. This corresponds to the fact that we introduced* $\operatorname{div} q$ *in the discussion of the well-posednes in subsection 3.2.*
(ii) *We have the explicit value of $\tilde{\nu}$ in Proposition 4.3. The details are given in the proof of Proposition 4.3.*

Proof of Proposition 4.3. We first derive the basic energy equality for the system (4.4) in the Fourier space. Taking the inner product (4.4) with $\hat{V}$, and taking the real parts for the resulting equality, we arrive at the basic energy equality

$$(4.12) \qquad \frac{1}{2}\frac{\partial}{\partial t}|\hat{V}|^2 + (1 + a|\xi|^2)|\hat{q}|^2 = 0.$$

Here we used the symmetry properties of $A(\omega)$ and L, and the skew-symmetry of B.

Now we construct the dissipation terms. The system (4.4) is equivalent to

$$(4.13) \qquad \begin{aligned} \hat{v}_t - \sqrt{b}|\xi|^2\hat{w} - d|\xi|^2\hat{\theta} &= 0, \\ \hat{w}_t + \sqrt{b}|\xi|^2\hat{v} &= 0, \\ \hat{\theta}_t - i\xi \cdot \hat{q} + d|\xi|^2\hat{v} &= 0, \\ \tau\hat{q}_t + \kappa i\xi\hat{\psi} + \hat{q} &= 0. \end{aligned}$$

with $\hat{\theta} = (1 + a|\xi|^2)\hat{\psi}$. We multiply the first and second equations in (4.13) by $-\bar{\hat{w}}$ and $-\bar{\hat{v}}$, respectively. Then, combining this two equations and taking real parts, we obtain

$$(4.14) \qquad -\frac{\partial}{\partial t}\operatorname{Re}(\hat{v}\bar{\hat{w}}) + \sqrt{b}|\xi|^2(|\hat{w}|^2 - |\hat{v}|^2) + d|\xi|^2\operatorname{Re}(\hat{w}\bar{\hat{\theta}}) = 0.$$

Furthermore, we multiply the first and the third equation in (4.13) by $d\bar{\hat{\theta}}$ and $d\bar{\hat{v}}$, respectively. Then we have

$$(4.15) \qquad d\frac{\partial}{\partial t}\operatorname{Re}(\hat{v}\bar{\hat{\theta}}) + d^2|\xi|^2(|\hat{v}|^2 - |\hat{\theta}|^2) + d\xi \cdot \operatorname{Re}(i\hat{v}\bar{\hat{q}}) - \sqrt{b}d|\xi|^2\operatorname{Re}(\hat{w}\bar{\hat{\theta}}) = 0.$$

Similarly we multiply the third equation in (4.13) by $\tau i\xi \cdot \bar{\hat{q}}$ and we take the inner product of the third equation in (4.13) with $-i\xi\bar{\hat{\theta}}$. This yields

$$\tau\xi \cdot \frac{\partial}{\partial t}\operatorname{Re}(i\hat{\theta}\bar{\hat{q}}) + \kappa(1 + a|\xi|^2)|\xi|^2|\hat{\psi}|^2 - \tau|\xi \cdot \hat{q}|^2 + \xi \cdot \operatorname{Re}(i\hat{\theta}\bar{\hat{q}}) + d\tau|\xi|^2\xi \cdot \operatorname{Re}(i\hat{v}\bar{\hat{q}}) = 0.$$

Especially, we obtain

$$(4.16) \qquad \begin{aligned} &\tau(1 + a|\xi|^2)\xi \cdot \frac{\partial}{\partial t}\operatorname{Re}(i\hat{\theta}\bar{\hat{q}}) + \kappa|\xi|^2|\hat{\theta}|^2 - \tau(1 + a|\xi|^2)|\xi \cdot \hat{q}|^2 \\ &+ (1 + a|\xi|^2)\xi \cdot \operatorname{Re}(i\hat{\theta}\bar{\hat{q}}) + d\tau(1 + a|\xi|^2)|\xi|^2\xi \cdot \operatorname{Re}(i\hat{v}\bar{\hat{q}}) = 0. \end{aligned}$$

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Our next step is to combine the above equalities to construct the energy estimate. First, computing $d^2/2 \times (4.14) + \sqrt{b} \times (4.15)$, we have

$$d\frac{\partial}{\partial t}\Big\{ -\frac{d}{2}\mathrm{Re}(\hat{v}\bar{\hat{w}}) + \sqrt{b}\mathrm{Re}(\hat{v}\bar{\hat{\theta}}) \Big\} + \frac{\sqrt{b}d^2}{2}|\xi|^2(|\hat{v}|^2 + |\hat{w}|^2) - \sqrt{b}d^2|\xi|^2|\hat{\theta}|^2$$

$$+ \sqrt{b}d\xi \cdot \mathrm{Re}(i\hat{v}\bar{\hat{q}}) + d\Big(\frac{d^2}{2} - b\Big)|\xi|^2\mathrm{Re}(\hat{w}\bar{\hat{\theta}}) = 0.$$

Here, using the Hölder inequality, we get

(4.17)
$$d\frac{\partial}{\partial t}\Big\{ -\frac{d}{2\sqrt{b}}\mathrm{Re}(\hat{v}\bar{\hat{w}}) + \mathrm{Re}(\hat{v}\bar{\hat{\theta}}) \Big\} + \frac{d^2}{2}|\xi|^2|\hat{v}|^2 + \frac{d^2}{4}|\xi|^2|\hat{w}|^2$$

$$- \Big\{ d^2 + \frac{1}{b}\Big(\frac{d^2}{2} - b\Big)^2 \Big\}|\xi|^2|\hat{\theta}|^2 + d\xi \cdot \mathrm{Re}(i\hat{v}\bar{\hat{q}}) \le 0.$$

On the other hand, from (4.16) we have

(4.18)
$$\tau(1 + a|\xi|^2)\xi \cdot \frac{\partial}{\partial t}\mathrm{Re}(i\hat{\theta}\bar{\hat{q}}) + \frac{\kappa}{2}|\xi|^2|\hat{\theta}|^2$$

$$- \Big\{ \tau|\xi|^2 + \frac{1}{2\kappa}(1 + a|\xi|^2) \Big\}(1 + a|\xi|^2)|\hat{q}|^2 + d\tau(1 + a|\xi|^2)|\xi|^2\xi \cdot \mathrm{Re}(i\hat{v}\bar{\hat{q}}) \le 0.$$

Then we calculate $\varepsilon_1 \times (4.17) + (4.18)$ and obtain

$$\frac{\partial}{\partial t}\Big\{ -\varepsilon_1\frac{d^2}{2\sqrt{b}}\mathrm{Re}(\hat{v}\bar{\hat{w}}) + \varepsilon_1 d\mathrm{Re}(\hat{v}\bar{\hat{\theta}}) + \tau(1 + a|\xi|^2)\xi \cdot \mathrm{Re}(i\hat{\theta}\bar{\hat{q}}) \Big\}$$

$$+ \varepsilon_1\frac{d^2}{2}|\xi|^2|\hat{v}|^2 + \varepsilon_1\frac{d^2}{4}|\xi|^2|\hat{w}|^2 + \Big\{ \frac{\kappa}{2} - \varepsilon_1\Big(d^2 + \frac{1}{b}\Big(b - \frac{d^2}{2}\Big)^2\Big) \Big\}|\xi|^2|\hat{\theta}|^2$$

$$- \Big\{ \tau|\xi|^2 + \frac{1}{2\kappa}(1 + a|\xi|^2) \Big\}(1 + a|\xi|^2)|\hat{q}|^2 + d\{\tau(1 + a|\xi|^2)|\xi|^2 + 1\}\xi \cdot \mathrm{Re}(i\hat{v}\bar{\hat{q}}) \le 0,$$

where ε_1 is a small positive parameter. Then we put $\varepsilon_1 = \kappa/(4g)$, where

$$g := d^2 + \frac{1}{b}\Big(b - \frac{d^2}{2}\Big)^2,$$

and obtain

$$\frac{\partial}{\partial t}\Big\{ -\frac{d^2\kappa}{8\sqrt{b}g}\mathrm{Re}(\hat{v}\bar{\hat{w}}) + \frac{d\kappa}{4g}\mathrm{Re}(\hat{v}\bar{\hat{\theta}}) + \tau(1 + a|\xi|^2)\xi \cdot \mathrm{Re}(i\hat{\theta}\bar{\hat{q}}) \Big\}$$

$$+ \frac{d^2\kappa}{8g}|\xi|^2|\hat{v}|^2 + \frac{d^2\kappa}{16g}|\xi|^2|\hat{w}|^2 + \frac{\kappa}{4}|\xi|^2|\hat{\theta}|^2$$

$$- \Big(\tau|\xi|^2 + \frac{1}{2\kappa}(1 + a|\xi|^2)\Big)(1 + a|\xi|^2)|\hat{q}|^2 + d\{\tau(1 + a|\xi|^2)|\xi|^2 + 1\}\xi \cdot \mathrm{Re}(i\hat{v}\bar{\hat{q}}) \le 0.$$

This estimate gives

$$\frac{\partial}{\partial t}\Big\{ -\frac{d^2\kappa}{8\sqrt{b}g}\mathrm{Re}(\hat{v}\bar{\hat{w}}) + \frac{d\kappa}{4g}\mathrm{Re}(\hat{v}\bar{\hat{\theta}}) + \tau(1 + a|\xi|^2)\xi \cdot \mathrm{Re}(i\hat{\theta}\bar{\hat{q}}) \Big\}$$

$$+ \frac{d^2\kappa}{16g}|\xi|^2(|\hat{v}|^2 + |\hat{w}|^2) + \frac{\kappa}{4}|\xi|^2|\hat{\theta}|^2 - G(|\xi|)|\hat{q}|^2 \le 0.$$

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Here, we calculate

$$G(|\xi|) := \left(\tau|\xi|^2 + \frac{1}{2\kappa}(1 + a|\xi|^2)\right)(1 + a|\xi|^2) + \frac{4g}{\kappa}(\tau(1 + a|\xi|^2)|\xi|^2 + 1)^2$$

$$= \frac{4g}{\kappa} + \frac{1}{\kappa}(\kappa + 8g)\tau|\xi|^2(1 + a|\xi|^2) + \frac{1}{2\kappa}(1 + 8g\tau^2|\xi|^4)(1 + a|\xi|^2)^2$$

$$\leq \frac{1}{2\kappa}(1 + 8g + 2(\kappa + 8g)\tau|\xi|^2 + 8g\tau^2|\xi|^4)(1 + a|\xi|^2)^2$$

$$\leq \frac{1 + \kappa + 8g}{2\kappa}(1 + \tau|\xi|^2)^2(1 + a|\xi|^2)^2.$$

This yields

(4.19)
$$\frac{\partial}{\partial t}\left\{ -\frac{d^2\kappa}{8\sqrt{bg}}\mathrm{Re}(\hat{v}\bar{\hat{w}}) + \frac{d\kappa}{4g}\mathrm{Re}(\hat{v}\bar{\hat{\theta}}) + \tau(1 + a|\xi|^2)\xi \cdot \mathrm{Re}(i\hat{\theta}\bar{\hat{q}}) \right\} + \frac{\kappa}{4}|\xi|^2|\hat{\theta}|^2$$

$$+ \frac{d^2\kappa}{16g}|\xi|^2(|\hat{v}|^2 + |\hat{w}|^2) - \frac{1 + \kappa + 8g}{2\kappa}(1 + \tau|\xi|^2)^2(1 + a|\xi|^2)^2|\hat{q}|^2 \leq 0.$$

Therefore, calculating $(1 + \tau|\xi|^2)^2(1 + a|\xi|^2) \times (4.12) + \varepsilon_0 \times (4.19)$ leads to

(4.20)
$$\frac{\partial}{\partial t}E(t, \xi) + \varepsilon_0 \frac{d^2\kappa}{16g}|\xi|^2(|\hat{v}|^2 + |\hat{w}|^2) + \varepsilon_0 \frac{\kappa}{4}|\xi|^2|\hat{\theta}|^2$$

$$+ \left(1 - \varepsilon_0 \frac{1 + \kappa + 8g}{2\kappa}\right)(1 + \tau|\xi|^2)^2(1 + a|\xi|^2)^2|\hat{q}|^2 \leq 0.$$

where the energy $E(t, \xi)$ is defined by

$$E(t, \xi) := \frac{1}{2}(1 + \tau|\xi|^2)^2(1 + a|\xi|^2)\big(\kappa(|\hat{v}|^2 + |\hat{w}|^2 + |\hat{\theta}|^2) + \tau(1 + a|\xi|^2)|\hat{q}|^2\big)$$

$$+ \varepsilon_0\Big(-\frac{d^2\kappa}{8\sqrt{bg}}\mathrm{Re}(\hat{v}\bar{\hat{w}}) + \frac{d\kappa}{4g}\mathrm{Re}(\hat{v}\bar{\hat{\theta}}) + \tau(1 + a|\xi|^2)\xi \cdot \mathrm{Re}(i\hat{\theta}\bar{\hat{q}})\Big).$$

Finally, we choose the positive parameter ε_0 suitably. We compute

$$E(t, \xi) \geq \frac{1}{2}(1 + \tau|\xi|^2)^2(1 + a|\xi|^2)\big(\kappa(|\hat{v}|^2 + |\hat{w}|^2 + |\hat{\theta}|^2) + \tau(1 + a|\xi|^2)|\hat{q}|^2\big)$$

$$- \frac{\varepsilon_0}{2}\Big(\frac{d\kappa}{4g}\Big(1 + \frac{d}{2\sqrt{b}}\Big)|\hat{v}|^2 + \frac{d^2\kappa}{8\sqrt{bg}}|\hat{w}|^2 + \Big(1 + \frac{d\kappa}{4g}\Big)|\hat{\theta}|^2 + \tau^2|\xi|^2(1 + a|\xi|^2)^2|\hat{q}|^2\Big)$$

$$\geq \frac{1}{2}(1 + \tau|\xi|^2)^2(1 + a|\xi|^2)\big(\kappa(|\hat{v}|^2 + |\hat{w}|^2 + |\hat{\theta}|^2) + \tau(1 + a|\xi|^2)|\hat{q}|^2\big)$$

$$- \frac{\varepsilon_0}{2}\Big(1 + \frac{d\kappa}{4g}\Big(1 + \frac{d}{2\sqrt{b}}\Big)\Big)\big(|\hat{v}|^2 + |\hat{w}|^2 + |\hat{\theta}|^2 + \tau^2|\xi|^2(1 + a|\xi|^2)^2|\hat{q}|^2\big).$$

Thus, choosing ε_0 such that

$$\kappa - \varepsilon_0\Big(1 + \frac{d\kappa}{4g}\Big(1 + \frac{d}{2\sqrt{b}}\Big)\Big) \geq \frac{\kappa}{2}, \qquad 1 - \varepsilon_0\Big(1 + \frac{d\kappa}{4g}\Big(1 + \frac{d}{2\sqrt{b}}\Big)\Big) \geq \frac{1}{2},$$

we can obtain

$$E(t, \xi) \geq \frac{1}{4}(1 + \tau|\xi|^2)^2(1 + a|\xi|^2)\big(\kappa(|\hat{v}|^2 + |\hat{w}|^2 + |\hat{\theta}|^2) + \tau(1 + a|\xi|^2)|\hat{q}|^2\big).$$

Here, ε_0 should satisfy

$$(4.21) \qquad \varepsilon_0 \leq \frac{\kappa}{2}\left(1 + \frac{d\kappa}{4g}\left(1 + \frac{d}{2\sqrt{b}}\right)\right)^{-1}, \qquad \varepsilon_0 \leq \frac{1}{2}\left(1 + \frac{d\kappa}{4g}\left(1 + \frac{d}{2\sqrt{b}}\right)\right)^{-1}.$$

Similarly, we also compute

$$\begin{aligned}
E(t,\xi) &\leq \frac{1}{2}(1 + \tau|\xi|^2)^2(1 + a|\xi|^2)\big(\kappa(|\hat{v}|^2 + |\hat{w}|^2 + |\hat{\theta}|^2) + \tau(1 + a|\xi|^2)|\hat{q}|^2\big) \\
&\quad + \frac{\varepsilon_0}{2}\left(1 + \frac{d\kappa}{4g}\left(1 + \frac{d}{2\sqrt{b}}\right)\right)\big(|\hat{v}|^2 + |\hat{w}|^2 + |\hat{\theta}|^2 + \tau^2|\xi|^2(1 + a|\xi|^2)^2|\hat{q}|^2\big) \\
&\leq \frac{3}{4}(1 + \tau|\xi|^2)^2(1 + a|\xi|^2)\big(\kappa(|\hat{v}|^2 + |\hat{w}|^2 + |\hat{\theta}|^2) + \tau(1 + a|\xi|^2)|\hat{q}|^2\big)
\end{aligned}$$

for suitably small ε_0 which satisfy (4.21). Consequently, because of (4.20), we set $\varepsilon_0 = c_*$ with

$$c_* := \min\left\{\frac{\kappa}{1 + \kappa + 8g}, \frac{\kappa}{2}\left(1 + \frac{d\kappa}{4g}\left(1 + \frac{d}{2\sqrt{b}}\right)\right)^{-1}, \frac{1}{2}\left(1 + \frac{d\kappa}{4g}\left(1 + \frac{d}{2\sqrt{b}}\right)\right)^{-1}\right\}$$

to obtain

$$(4.22) \qquad \begin{aligned}
&\frac{\partial}{\partial t}E(t,\xi) + c_*\frac{d^2\kappa}{16g}|\xi|^2(|\hat{v}|^2 + |\hat{w}|^2) + c_*\frac{\kappa}{4}|\xi|^2|\hat{\theta}|^2 \\
&\quad + \frac{1}{2}(1 + \tau|\xi|^2)^2(1 + a|\xi|^2)^2|\hat{q}|^2 \leq 0
\end{aligned}$$

and

$$(4.23) \qquad \begin{aligned}
&\frac{1}{4}(1 + \tau|\xi|^2)^2(1 + a|\xi|^2)\big(\kappa(|\hat{v}|^2 + |\hat{w}|^2 + |\hat{\theta}|^2) + \tau(1 + a|\xi|^2)|\hat{q}|^2\big) \\
&\leq E(t,\xi) \leq \frac{3}{4}(1 + \tau|\xi|^2)^2(1 + a|\xi|^2)\big(\kappa(|\hat{v}|^2 + |\hat{w}|^2 + |\hat{\theta}|^2) + \tau(1 + a|\xi|^2)|\hat{q}|^2\big).
\end{aligned}$$

By using (4.22) and (4.23), we arrive at (4.10) and the energy estimate. Indeed, integrating (4.22) over t and applying (4.23) to the resulting estimate, we obtain

$$\begin{aligned}
&|\hat{V}(t,\xi)|^2 + \int_0^t \frac{c_*\kappa|\xi|^2}{(1 + \tau|\xi|^2)^2(1 + a|\xi|^2)}\left\{\frac{d^2}{4g}(|\hat{v}(t',\xi)|^2 + |\hat{w}(t',\xi)|^2) + |\hat{\theta}(t',\xi)|^2\right\}dt' \\
&\quad + \int_0^t 2(1 + a|\xi|^2)|\hat{q}(t',\xi)|^2 dt' \leq 3|\hat{V}_0(\xi)|^2.
\end{aligned}$$

This gives

$$\begin{aligned}
&\kappa\|\partial_x^k(v,w,\theta)(t)\|_{H^3_{\tau,a}}^2 + \tau\|\partial_x^k q(t)\|_{H^4_{\tau,a}}^2 + \frac{c_*d^2\kappa}{4g}\int_0^t\|\partial_x^{k+1}(v,w,\theta)(t')\|_{L^2}^2 dt' \\
&\quad + 2\int_0^t\|\partial_x^k q(t')\|_{H^4_{\tau,a}}^2 dt' \leq 3(\kappa\|\partial_x^k(v_0,w_0,\theta_0)\|_{H^3_{\tau,a}}^2 + \tau\|\partial_x^k q_0\|_{H^4_{\tau,a}}^2)
\end{aligned}$$

for $k \geq 0$. On the other hand, (4.22) gives

$$\frac{\partial}{\partial t}E(t,\xi) + \frac{c_*d^2}{16g}|\xi|^2\{\kappa(|\hat{v}|^2 + |\hat{w}|^2 + |\hat{\theta}|^2) + \tau(1 + a|\xi|^2)|\hat{q}|^2\} \leq 0.$$

Moreover, (4.23) and the above estimate lead

$$\frac{\partial}{\partial t}E(t,\xi) + \frac{c_*d^2}{12g}\frac{|\xi|^2}{(1+\tau|\xi|^2)^2(1+a|\xi|^2)}E(t,\xi) \le 0.$$

This yields $E(t,\xi) \le e^{-\eta(\xi)t}E(0,\xi)$, and hence $|\hat{V}(t,\xi)| \le \sqrt{3}e^{-\eta(\xi)t/2}|\hat{V}_0(\xi)|$, where $\eta(\xi)$ is defined in (4.11) with $\tilde{\nu} := c_*d^2/(12g)$. Consequently, this estimate together with (4.5) gives (4.10). $\qquad\square$

As an easy consequence of Proposition 4.3, we can obtain the decay estimate for the solutions in Theorem 4.1 and Collorary 4.2.

Proof of Theorem 4.1 and Collorary 4.2. First, we prove Theorem 4.1. Because of (4.10) we have

$$(4.24)\qquad \begin{aligned}\|\partial_x^k e^{t\Phi}\varphi\|_{L^2}^2 &\le 3\int_{|\xi|\le 1}|\xi|^{2k}e^{-2c_0\eta(\xi)t}|\hat{\varphi}(\xi)|^2 d\xi + 3\int_{|\xi|\ge 1}|\xi|^{2k}e^{-2c_0\eta(\xi)t}|\hat{\varphi}(\xi)|^2 d\xi\\ &=: I_1 + I_2.\end{aligned}$$

For the case $|\xi| \le 1$, we employ the same argument as in the proof of Theorem 3.1. More precisely, because of $\eta(\xi) \ge \nu_{\tau,a}|\xi|^2$, we get

$$I_1 \le 3\int_{|\xi|\le 1}|\xi|^{2k}e^{-\nu_{\tau,a}|\xi|^2 t}|\hat{\varphi}(\xi)|^2 d\xi \le C_0^2(1+\nu_{\tau,a}t)^{-n(\frac{1}{p}-\frac{1}{2})-k}\|\varphi\|_{L^p}^2$$

for $1 \le p \le 2$, where C_0 is also a positive constant depending only on p and k. On the other hand, for the case $|\xi| \ge 1$, we calculate $\eta(\xi) \ge \nu_{\tau,a}|\xi|^{-4}$ and

$$\begin{aligned}I_2 &\le 3\int_{|\xi|\ge 1}|\xi|^{2k}e^{-\nu_{\tau,a}|\xi|^{-4}t}|\hat{\varphi}(\xi)|^2 d\xi\\ &\le 3\sup_{|\xi|\ge 1}\{|\xi|^{-2\ell}e^{-\nu_{\tau,a}|\xi|^{-4}t}\}\int_{|\xi|\ge 1}|\xi|^{2(k+\ell)}|\hat{\varphi}(\xi)|^2 d\xi\\ &\le C_1^2(1+\nu_{\tau,a}t)^{-\ell/2}\|\partial_x^{k+\ell}\varphi\|_{L^2}^2\end{aligned}$$

for $\ell \ge 0$, where C_1 is also positive constant depending only on ℓ. Consequently, substituting the estimates of I_1 and I_2 into (4.24), we get (4.8) and Theorem 4.1 is proved.

To prove Corollary 4.2, we modify the proof of Theorem 4.1. The left-hand side of (4.24) is estimated by

$$\begin{aligned}\|\partial_x^k V(t)\|_{L^2}^2 &\le 3\int_{|\xi|\le 1}|\xi|^{2k}e^{-\eta(\xi)t}|\hat{V}_0(\xi)|^2 d\xi + 3\int_{|\xi|\ge 1}|\xi|^{2k}e^{-\eta(\xi)t}|\hat{V}_0(\xi)|^2 d\xi\\ &=: J_1 + J_2.\end{aligned}$$

For the low frequency part, we estimate

$$\begin{aligned}J_1 &\le 3\kappa\int_{|\xi|\le 1}|\xi|^{2k}e^{-\eta(\xi)t}|(\hat{v}_0,\hat{w}_0,\hat{\theta}_0)(\xi)|^2 d\xi + 3\tau\int_{|\xi|\le 1}(1+a|\xi|^2)|\xi|^{2k}e^{-\eta(\xi)t}|\hat{q}_0(\xi)|^2 d\xi\\ &\le 3\int_{|\xi|\le 1}|\xi|^{2k}e^{-\nu_{\tau,a}|\xi|^2 t}\big(\kappa|(\hat{v}_0,\hat{w}_0,\hat{\theta}_0)(\xi)|^2 + \tau(1+a)|\hat{q}_0(\xi)|^2\big)d\xi\\ &\le C_0^2(1+\nu_{\tau,a}t)^{-n(\frac{1}{p}-\frac{1}{2})-k}\big(\kappa\|(v_0,w_0,\theta_0)\|_{L^q}^2 + \tau(1+a)\|q_0\|_{L^q}^2\big)\end{aligned}$$

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for $1 \leq q \leq 2$, where C_0 is same as before. Similarly, for the high frequency part, we compute

$$J_2 \leq 3\kappa \int_{|\xi| \geq 1} |\xi|^{2k} e^{-\eta(\xi)t} |(\hat{v}_0, \hat{w}_0, \hat{\theta}_0)(\xi)|^2 d\xi + 3\tau \int_{|\xi| \geq 1} (1 + a|\xi|^2) |\xi|^{2k} e^{-\eta(\xi)t} |\hat{q}_0(\xi)|^2 d\xi$$

$$\leq 3 \int_{|\xi| \geq 1} |\xi|^{2k} e^{-\eta(\xi)t} \left(\kappa |(\hat{v}_0, \hat{w}_0, \hat{\theta}_0)(\xi)|^2 + \tau(1 + a) |\xi|^2 |\hat{q}_0(\xi)|^2 \right) d\xi$$

$$\leq C_1^2 (1 + \nu_{\tau,a} t)^{-\ell/2} \left(\kappa \|\partial_x^{k+\ell}(v_0, w_0, \theta_0)\|_{L^2}^2 + \tau(1 + a) \|\partial_x^{k+\ell+1} q_0\|_{L^2}^2 \right)$$

for $\ell \geq 0$, where C_1 is also same as before. Therefore, using

$$\|\partial_x^k V(t)\|_{L^2}^2 = \kappa \|\partial_x^k (v, w, \theta)(t)\|_{L^2}^2 + \tau \|\partial_x^k q(t)\|_{H_a^1}^2,$$

we arrive at the desired decay estimate (4.9), and the proof is completed. $\qquad\square$

4.2. Optimality of the decay estimates ($\tau > 0$).

At the rest of this section, we investigate the optimality of the estimates in Theorem 4.1 and Corollary 4.2. For this purpose we consider the characteristic equation for the system (4.1) given by

$$(4.25) \quad \begin{aligned} \tau(1 + a|\xi|^2)\lambda^4 + (1 + a|\xi|^2)\lambda^3 + \{\tau(b + d^2)(1 + a|\xi|^2)|\xi|^2 + \kappa\}|\xi|^2\lambda^2 \\ + (b + d^2)(1 + a|\xi|^2)|\xi|^4\lambda + b\kappa|\xi|^6 = 0. \end{aligned}$$

Similarly as before, we consider the asymptotic expansion of $\lambda = \lambda(|\xi|)$ for $|\xi| \to 0$ and for $|\xi| \to \infty$. In the low frequency parts, the solutions for (4.25) are expanded as

$$\lambda_j(|\xi|) = z_j|\xi|^2 + O(|\xi|^4), \qquad \lambda_4(|\xi|) = -\frac{1}{\tau} + \kappa|\xi|^2 + O(|\xi|^4)$$

for $j = 1, 2, 3$, where the z_j are solutions for $f(z) = 0$ with (3.19). Observe that $\mathrm{Re}(z_j) < 0$ for $j = 1, 2, 3$. On the other hand, in the high frequency parts, this yields

$$(4.26) \quad \lambda_j(|\xi|) = \pm\sigma i|\xi|^2 \pm \frac{d^2\kappa i}{2\sigma^3 a\tau}|\xi|^{-2} - \frac{d^2\kappa}{2\sigma^4 a\tau}\left(\frac{1}{\tau} \pm \frac{\sigma i}{a}\right)|\xi|^{-4} + O(|\xi|^{-6})$$

for $j = 1, 2$, and

$$\lambda_j(|\xi|) = -\frac{1}{2\tau}\left(1 \pm \sqrt{1 - \frac{4b\kappa\tau}{\sigma^2 a}}\right) + O(|\xi|^{-2})$$

for $j = 3, 4$, where $\sigma := \sqrt{b + d^2}$. Especially, the expansion (4.26) gives us

$$\mathrm{Re}(\lambda_j)(|\xi|) = -\frac{d^2\kappa}{2(b + d^2)^2 a\tau^2}|\xi|^{-4} + O(|\xi|^{-6})$$

for $j = 1, 2$, and leads to the regularity-loss phenomena. Consequently, these asymptotic expansions tell us that the pointwise estimate (4.10) is optimal.

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