GLOBAL WELL–POSEDNESS OF THE CAUCHY PROBLEM FOR THE 3D JORDAN–MOORE–GIBSON–THOMPSON EQUATION

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ABSTRACT. We consider the Cauchy problem of a third-order in time nonlinear equation known as the Jordan–Moore–Gibson–Thompson (JMGT) equation arising in acoustics as an alternative model to the well-known Kuznetsov equation. We show a local existence result in appropriate function spaces, and, using the energy method together with a bootstrap argument, we prove a global existence result for small data, without using the linear decay. Finally, polynomial decay rates in time for a norm related to the solution will be obtained.

1. INTRODUCTION

In this paper, we consider the nonlinear Jordan–Moore–Gibson–Thompson equation:

$$\tau u_{ttt} + u_{tt} - c^2\Delta u - \beta\Delta u_t = \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{B}{2A}(u_t)^2 + |\nabla u|^2\right), \tag{1.1a}$$

where $x \in \mathbb{R}^3$ (Cauchy problem in 3D), and $t > 0$, and where $\tau > 0$ is a time relaxation parameter, the unknown $u = u(x,t)$ is the acoustic velocity potential, c is the speed of sound, β is the parameter of diffusivity and A and B are the constants of nonlinearity. We consider the initial conditions

$$u(t=0) = u_0, \qquad u_t(t=0) = u_1 \qquad u_{tt}(t=0) = u_2. \tag{1.1b}$$

Equation (1.1a) appears as a generalization of the Kuznetsov equation (see equation (1.3) below). Both equations are used as models in what is called nonlinear acoustics that deals with finite-amplitude wave propagation in fluids and solids and related phenomena, see the books of Beyer [1] or Rudenko and Soluyan [31]. In particular, the JMGT equation arises from modeling high-frequency ultra sound waves, see [24] for more details.

The derivation of equation (1.1a) (see [14] and [33]) can be obtained from the general equations of fluid mechanics by means of some asymptotic expansions in powers of small parameters, cf. Appendix B for the derivation.

In the derivation of (1.1a), the Cattaneo (or Maxwell–Cattaneo) law was used which accounts for finite speed of propagation of the heat transfer and eliminates the paradox

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of infinite speed of propagation for pure heat conduction associated with the Fourier law (i.e., $\tau = 0$). Here τ is a small relaxation parameter. If we use in (B.1) the Fourier law

$$q = -K\nabla\theta, \tag{1.2}$$

then we can derive the Kuznetsov equation

$$u_{tt} - c^2\Delta u - \beta\Delta u_t = \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{B}{2A}(u_t)^2 + |\nabla u|^2\right), \tag{1.3}$$

which is a well-known model and widely used in nonlinear acoustics, see the derivation of (1.3) in [17] and [10]. Hence equation (1.1a) can be seen as a "hyperbolic" version of (1.3). Equation (1.3) is written in terms of the acoustic velocity potential $v = -\nabla u$. It can be also expressed in terms of the acoustic pressure fluctuation $\tilde{p}$ as

$$\frac{1}{c^2}\tilde{p}_{tt} - \Delta\tilde{p} - \frac{\beta}{c^2}\Delta\tilde{p}_t = \partial_{tt}\left(\frac{1}{\varrho_0 c^4}\frac{B}{2A}\tilde{p}^2 + \frac{\varrho_0}{c^2}(v\cdot v)\right) \tag{1.4}$$

such that the identity

$$\varrho_0 v_t = -\Delta\tilde{p}$$

holds. Assuming that the local nonlinear effects can be neglected, that is making the replacement $v\cdot v = (\frac{1}{\varrho_0 c}\tilde{p})^2$ on the right-hand side of (1.4), we arrive at the so-called Westervelt equation:

$$\frac{1}{c^2}\tilde{p}_{tt} - \Delta\tilde{p} - \frac{\beta}{c^2}\Delta\tilde{p}_t = \partial_{tt}\left(\frac{1}{\varrho_0 c^4}\Big(1 + \frac{B}{2A}\Big)\tilde{p}^2\right) \tag{1.5}$$

or in terms of u through the relation $\varrho u_t = p$ as

$$u_{tt} - c^2\Delta u - \beta\Delta u_t = \frac{\partial}{\partial t}\left(\frac{1}{c^2}\Big(1 + \frac{B}{2A}\Big)(u_t)^2\right). \tag{1.6}$$

Analogously to the above reduction of the Kuznetsov equation to the Westervelt equation, we can reduce equation (1.1a) to

$$\tau u_{ttt} + u_{tt} - c^2\Delta u - \beta\Delta u_t = \frac{\partial}{\partial t}\left(\frac{1}{c^2}\Big(1 + \frac{B}{2A}\Big)(u_t)^2\right). \tag{1.7}$$

1.1. **Previous work.** The starting point of the nonlinear analysis lies in the results for the linearization, often referred to as the Moore–Gibson–Thompson (MGT) equation:

$$\tau u_{ttt} + \alpha u_{tt} - c^2\Delta u - \beta\Delta u_t = 0. \tag{1.8}$$

Equation (1.8) has been extensively studied lately; see, for example [4, 5, 6, 23, 28, 29] and the references therein. As we will see from the results of previous works, even at the linear level, the mathematical analysis raises nontrivial issues.

In [14] (see also [15]), the authors considered the linear equation in bounded domains

$$\tau u_{ttt} + \alpha u_{tt} + c^2\mathcal{A}u + \beta\mathcal{A}u_t = 0, \tag{1.9}$$

where $\mathcal{A}$ is a positive self-adjoint operator, and showed that by neglecting diffusivity of the sound coefficient ($\beta = 0$) there arises a lack of existence of a semigroup associated with the linear dynamics. On the other hand, they proved that when the diffusivity of the sound is strictly positive ($\beta > 0$), the linear dynamics is described by a strongly continuous semigroup, which is exponentially stable provided the dissipativity condition $\gamma := \alpha - \tau c^2/\beta > 0$ is fulfilled, which is, for our equation (1.1a), equivalent to (since without loss of generality, we are assuming the damping parameter $\alpha = 1$)

$$\beta - \tau c^2 > 0. \tag{1.10}$$

This condition (1.10) will be assumed throughout the paper.

For $\gamma = 0$ the energy is conserved (the same type of results are obtained in [3] using energy methods, or in [23] using the analysis of the spectrum of the operator). The exponential decay rate results in [23] are completed in [29], where an explicit scalar product when the operator is normal allows the authors to obtain the optimal exponential decay rate of the solutions. Finally, in [9], the authors showed the chaotic behavior of the system when $\gamma < 0$. Equation (1.9) with a viscoelastic damping of a memory type has been also considered in [20] and [19], where exponential stability results have been obtained.

The dissipativity condition (1.10) can also be understood in looking at the zeros z_j, $j = 1, 2, 3$, of the characteristic polynomial associated to our equation (1.1a) after having applied the Fourier transform $\mathcal{F}_{x\to\xi}$ to the linearized part:

$$\tau z^3 + z^2 + \beta|\xi|^2 z + c^2|\xi|^2 = 0. \tag{1.11}$$

Computing the associated Hurwitz matrix (see [21, p. 459])

$$\mathbb{H}_3 := \begin{pmatrix} 1 & \tau & 0 \\ c^2|\xi|^2 & \beta|\xi|^2 & 1 \\ 0 & 0 & c^2|\xi|^2 \end{pmatrix},$$

and the determinants d_j of the minors $D_j = ((\mathbb{H}_3)_{km})_{k,m=1,\dots,j}$, we have

$$d_1 = 1, \quad d_2 = |\xi|^2(\beta - \tau c^2), \quad d_3 = c^2|\xi|^2 d_2.$$

Thus, $\mathrm{Re}(z_j) < 0,\ j = 1, 2, 3$, holds if and only if the dissipativity condition (1.10) holds. Hence, the assumption (1.10) seems also a necessary condition for the stability of (1.8).

Equation (1.7) (which is called Jordan-Moore-Gibson-Thompson-Westervelt) has been investigated in [15] and its linear form in [14]. The authors in [15] used the estimates of the higher-level energies obtained for the linear model in [14] to establish global well-posedness and decay rates of solutions to the initial and boundary value problem associated to (1.7). Of course (1.7) is simpler compared to (1.1a), due to the absence of the gradient nonlinearity $\nabla u \nabla u_t$ in (1.7). Such a nonlinearity renders the mathematical analysis more difficult.

The MGT and JMGT equations have been studied recently from various points of view. The study of the controllability properties of the MGT type equations can be found for instance in [4, 22]. The MGT equation in $\mathbb{R}^N$ with a power source nonlinearity of the form $|u|^p$ has been considered in [7] where some blow up results have been shown for the critical case $\tau c^2 = \beta$. The MGT and JMGT equations with a memory term have been also investigated recently. For the MGT with memory, the reader is refereed to [2, 8, 11] and to [18, 25, 26] for the JMGT with memory. In particular in [11] (for bounded domain) and in [2] (in the whole space $\mathbb{R}^N$), and due to the presence of the memory damping term, the stability condition (1.10) has been pushed to the critical case $\tau c^2 = \beta$.

The singular limit problem when $\tau \to 0$ has been rigorously justified in [16]. The authors in [16] showed that the limit of (1.1a) as $\tau \to 0$ leads to the Kuznetsov equation (1.3). We also refer to [15, 16] for the analysis of (1.7) in smoothly bounded domains.

In this paper, we consider the Jordan–Moore–Gibson–Thompson equation in its full generality (i.e., (1.1a)) for the Cauchy problem $x \in \mathbb{R}^3$. Under the assumption $0 < \tau c^2 < \beta$: first, by using the contraction mapping theorem in appropriately chosen spaces, we show a local existence result in some appropriate functional spaces, second by using some energy-type estimates we prove a global existence result for small initial data by constructing an appropriate energy norm and show that this norm remains uniformly bounded with respect to time, without using the linear decay which is a standard way to proving small data existence for non-linear evolution equations, cf. [30]. If we want to use the linear decay, then we need to control a complicated time-weighted energy norms, which requires integrability in time to some norms of the solutions, which is not always the case. In addition, a good understanding of the linear problem is necessary. Here our method is based on the structure of the equation.

We rewrite the right-hand side of equation (1.1a) in the form

$$\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{B}{2A}(u_t)^2 + |\nabla u|^2\right) = \frac{1}{c^2}\frac{B}{A}u_t u_{tt} + 2\nabla u \nabla u_t,$$

and introduce the new variables

$$v = u_t \qquad \text{and} \qquad w = u_{tt},$$

Without loss of generality, we assume from now on

$$c = 1.$$

Then equation (1.1a) can be rewritten as the following first order system

$$\begin{cases} u_t = v, \\ v_t = w, \\ \tau w_t = \Delta u + \beta \Delta v - w + \dfrac{B}{A} v w + 2\nabla u \nabla v, \end{cases} \tag{1.12}$$

with the initial data

$$u(t=0) = u_0, \qquad v(t=0) = v_0, \qquad w(t=0) = w_0. \tag{1.13}$$

Understanding the asymptotic behavior of the linearized problem is critical for proving the decay rate of the nonlinear problem. The first result (for $x \in \mathbb{R}^3$) in this direction has been presented in the recent paper [28], where the authors used the energy method in the Fourier space to show that under the assumption $\beta > \tau$ the energy norm of the solution $\|V(t)\|_{L^2} = \|(\tau u_{tt} + u_t, \nabla(\tau u_t + u), \nabla u_t)(t)\|_{L^2}$ decays in $\mathbb{R}^n$ with the rate $(1+t)^{-n/4}$. They also proved that this decay rate is optimal, by using the eigenvalues expansion method. Some other decay rates for $\|u(t)\|_{L^2}$ were also presented in [28] by using the explicit formula of the Fourier image of the solution.

1.2. **Main results.** In this section, we state the main results of this paper. The global existence result is summarized in the following theorem, the proof of which is given in Section 2 and constitutes the major contribution of this work.

Theorem 1.1. *Assume that $0 < \tau < \beta$ and let $s > \frac{5}{2}$. Assume that $u_0, v_0, w_0 \in H^s(\mathbb{R}^3)$. Then there exists a small positive constant δ, such that if*

$$\begin{aligned}\mathcal{E}_s^2(0) &= \|(v_0 + \tau w_0)\|_{H^{s+1}}^2 + \|\Delta v_0\|_{H^s}^2 + \|\nabla v_0\|_{H^s}^2 \\ &\quad + \|\Delta(u_0 + \tau v_0)\|_{H^s}^2 + \|\nabla(u_0 + \tau v_0)\|_{H^s}^2 + \|w_0\|_{H^s}^2 \leq \delta,\end{aligned}$$

then the local solution u to (1.1) given in Theorem 1.2 exists globally in time.

The necessary local existence theorem is proved in Section 3 and given by

Theorem 1.2. *Assume that $0 < \tau < \beta$ and let $s > \frac{5}{2}$. Let $\mathbf{U}_0 = (u_0, v_0, w_0)^T$ be such that*

$$\begin{aligned}\mathcal{E}_s^2(0) &= \|(v_0 + \tau w_0)\|_{H^{s+1}}^2 + \|\Delta v_0\|_{H^s}^2 + \|\nabla v_0\|_{H^s}^2 \\ &\quad + \|\Delta(u_0 + \tau v_0)\|_{H^s}^2 + \|\nabla(u_0 + \tau v_0)\|_{H^s}^2 + \|w_0\|_{H^s}^2 \leq \tilde{\delta}_0\end{aligned} \tag{1.14}$$

for some $\tilde{\delta}_0 > 0$. Then, there exists a small time $T = T(\mathcal{E}_s(0)) > 0$ such that problem (1.1) has a unique solution u on $[0,T) \times \mathbb{R}^3$ satisfying

$$\mathcal{E}_s^2(T) + \mathcal{D}_s^2(T) \leq C_{\tilde{\delta}_0},$$

where $\mathcal{E}_s^2(T)$ and $\mathcal{D}_s^2(T)$ are given in (2.3), determining the regularity of u, and $C_{\tilde{\delta}_0}$ is a positive constant depending on $\tilde{\delta}_0$.

In the next theorem, we state the decay rate of the solution. Its proof is given in Section 4.

Theorem 1.3. *Assume that $0 < \tau < \beta$ and $s > 5/2$. Let u be the global solution of (1.1). Let $v_0 = u_t(t=0)$, $v_1 = u_{tt}(t=0)$ and $v_2 = u_{ttt}(t=0)$ satisfying $v_0, v_1, v_2 \in$*

$L^1(\mathbb{R}^3) \cap H^s(\mathbb{R}^3)$ and $(v_1, v_2) \in L^{1,1}(\mathbb{R}^3)$ with $\int_{\mathbb{R}^3} v_i(x)dx = 0$, $i = 1, 2$. Assume that $\|\mathbf{V}_0\|_{H^s \cap L^1}$ is small enough. Then, the following decay estimates hold:

$$\|\nabla^j \mathbf{V}(t)\|_{L^2} + \|\nabla^j u_t(t)\|_{L^2} \le C \left(\|\mathbf{V}_0\|_{L^1} + \|\nabla^j \mathbf{V}_0\|_{L^2}\right)(1+t)^{-3/4-j/2},$$

for all $0 \le j \le s$, where C is a constant independent of t and the initial data.

The remaining part of this paper is organized as follows: In Section 2, we prove the global existence of solutions for small data. We employ the energy method together with some commutator estimates to prove a global existence result for small initial data in appropriate Sobolev spaces. We should mention that the method we used to prove the global existence does not depend on decay estimates for the linearized equation. As a result, the global existence is proved under the same regularity assumption required for the local existence which is proved in Section 3, where we apply the contraction mapping theorem to show the local well-posedness of (1.1). Finally, Section 4 is devoted to the decay estimate for the norm $\|(u_t + \tau u_{tt}, \nabla(u + \tau u_t), \nabla u_t)\|_{L^2}$. In fact, based on the decay estimates obtained in [28], for the linearized problem, we prove that the same decay result holds for the nonlinear problem.

In Appendix A we collect some useful lemmas as well as results on the decay for the linearized problem that we will use in the proof of the main results. In Appendix B we present a derivation of equation (1.1a).

We introduce some notations that will be used throughout the paper. Let $\|.\|_{L^q}$ and $\|.\|_{H^\ell}$ stand for the $L^q(\mathbb{R}^3)$-norm $(2 \le q \le \infty)$ resp. the $H^\ell(\mathbb{R}^3)$-norm. We define the weighted function space, $L^{1,1}(\mathbb{R}^3)$ as follows: $u \in L^{1,1}(\mathbb{R}^3)$ iff $u \in L^1(\mathbb{R}^3)$ and

$$\|u\|_{L^{1,1}} := \int_{\mathbb{R}^3} (1+|x|)|u(x)|dx < \infty.$$

The symbol $[A, B] = AB - BA$ denotes the commutator. The constant C denotes a generic positive constant that appears in various inequalities and may change its value in different occurrences.

2. Global existence– Proof of Theorem 1.1

In this section we prove the global existence for the nonlinear problem (1.1) resp. its first-order version (1.12). The proof of Theorem 1.1 will be given through several lemmas. Our goal is to control the solution of (1.12) uniformly in a suitable norm as $t \to \infty$. In order to state our main result, we introduce the energy norm, $\mathcal{E}_k(t)$, and the corresponding dissipation norm, $\mathcal{D}_k(t)$, as follows:

$$\begin{aligned} \mathcal{E}_k^2(t) \;=\; \sup_{0\le\sigma\le t} \Big(& \left\|\nabla^k(v+\tau w)(\sigma)\right\|_{H^1}^2 + \left\|\Delta\nabla^k v(\sigma)\right\|_{L^2}^2 + \left\|\nabla^{k+1} v(\sigma)\right\|_{L^2}^2 \\ & + \left\|\Delta\nabla^k(u+\tau v)(\sigma)\right\|_{L^2}^2 + \left\|\nabla^{k+1}(u+\tau v)(\sigma)\right\|_{L^2}^2 + \|\nabla^k w(\sigma)\|_{L^2}^2 \Big), \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \mathcal{D}_k^2(t) &= \int_0^t \Big(\left\|\nabla^{k+1} v(\sigma)\right\|_{L^2}^2 + \left\|\Delta\nabla^k v(\sigma)\right\|_{L^2}^2 + \|\nabla^k w(\sigma)\|_{L^2}^2 \\ &\quad + \left\|\Delta\nabla^k (u+\tau v)(\sigma)\right\|_{L^2}^2 + \left\|\nabla^{k+1}(v+\tau w)(\sigma)\right\|_{L^2}^2 \Big) d\sigma. \end{aligned} \tag{2.2}$$

For some positive integer $s \geq 1$ that will be fixed later on, we define

$$\mathcal{E}_s^2(t) = \sum_{k=0}^{s} \mathcal{E}_k^2(t) \qquad \text{and} \qquad \mathcal{D}_s^2(t) = \sum_{k=0}^{s} \mathcal{D}_k^2(t). \tag{2.3}$$

We also define

$$Y_s(t) := \mathcal{E}_s^2(t) + \mathcal{D}_s^2(t).$$

The main goal is to prove by a continuity argument that for s large enough, $Y_s(t)$ is uniformly bounded for all time if the initial energy $\mathcal{E}_s^2(0) = Y_s(0)$ is sufficiently small. Due to the presence of the term $-\beta\Delta_t u$ in (1.1a) and the special nonlinearity, the global existence is proved without using the decay of the linearized problem.

Proposition 2.1. *Assume that $0 < \tau < \beta$ and let $s > \frac{5}{2}$, then the following estimate holds for t in an interval $[0,T]$ of local existence:*

$$Y_s(t) \leq CY_s(0) + CY_s^{3/2}(t), \tag{2.4}$$

where C is a positive constant that does not depend on t, T.

The main step towards the proof of (2.4) is to show the estimate (2.5) below. With this estimate in hand, the proof of Proposition 2.1 is a direct consequence of Proposition 2.2. We omit the details.

Proof of Theorem 1.1. From (2.4), we conclude in a standard way that there is $\alpha > 0$ small enough such that if $Y_s(0) = \mathcal{E}_s(0) \leq \alpha$, then there is $K > 0$, independent of T, such that

$$Y_s(t) \leq K,$$

for all $t \in [0,T]$. This uniform estimate allows to continue the local solution to $T = \infty$ as usual.

Now, it remains to prove Proposition 2.2. □

Proposition 2.2. *Assume that $0 < \tau < \beta$ and let $s > \frac{5}{2}$. Then, the following estimate holds:*

$$\mathcal{E}_s^2(t) + \mathcal{D}_s^2(t) \leq C\mathcal{E}_s^2(0) + C\mathcal{E}_s(t)\mathcal{D}_s^2(t). \tag{2.5}$$

The proof of Proposition 2.2 will be given in several steps and constitutes the majority of Section 2. The main idea is to use energy estimates. The most difficult part in the proof is to control "in a nice way" the nonlinear terms. This will be done by a repeated use of some functional inequalities such as: Gagliardo–Nirenberg interpolation inequality and Sobolev embedding theorems.

2.1. **First order energy estimates.**

Lemma 2.3. *The energy functional associated to system (1.12) is*

$$E_1(t) := \frac{1}{2}\int_{\mathbb{R}^3}\left(|v+\tau w|^2+\tau(\beta-\tau)|\nabla v|^2+|\nabla(u+\tau v)|^2\right)dx$$

and satisfies, for all $t\geq 0$, the identity

$$\frac{d}{dt}E_1(t)+(\beta-\tau)\left\|\nabla v\right\|_{L^2}^2=\mathbf{R}_1, \tag{2.6}$$

where

$$\mathbf{R}_1 := \int_{\mathbb{R}^3}\left(\frac{B}{A}vw+2\nabla u\nabla v\right)(v+\tau w)\,dx.$$

Proof. Summing up the second and the third equation in (1.12), we get

$$(v+\tau w)_t=\Delta u+\beta\Delta v+\frac{B}{A}vw+2\nabla u\nabla v. \tag{2.7}$$

Multiplying (2.7) by $v+\tau w$ and integrating by parts over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|v+\tau w|^2dx+\beta\int_{\mathbb{R}^3}|\nabla v|^2dx\\
=&\ -\int_{\mathbb{R}^3}\nabla u\,(\nabla v+\tau\nabla w)\,dx-\beta\tau\int_{\mathbb{R}^3}\nabla v\nabla w dx\\
&+\int_{\mathbb{R}^3}\left(\frac{B}{A}vw+2\nabla u\nabla v\right)(v+\tau w)\,dx.
\end{aligned} \tag{2.8}$$

We have

$$\frac{1}{2}\tau(\beta-\tau)\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla v|^2dx=\tau(\beta-\tau)\int_{\mathbb{R}^3}(\nabla w\nabla v)dx. \tag{2.9}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla(u+\tau v)|^2dx\\
=&\ \tau\int_{\mathbb{R}^3}\nabla w\nabla u dx+\tau^2\int_{\mathbb{R}^3}\nabla w\nabla v dx+\int_{\mathbb{R}^3}\nabla v\nabla u dx+\tau\int_{\mathbb{R}^3}|\nabla v|^2dx.
\end{aligned} \tag{2.10}$$

Summing up (2.8), (2.9) and (2.10), then (2.6) holds. This finishes the proof of Lemma 2.3. □

Next, we define the energy of second order

$$E_2(t) := \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla(v+\tau w)|^2+\tau(\beta-\tau)|\Delta v|^2+|\Delta(u+\tau v)|^2\right)dx.$$

The following lemma is proved analogously.

Lemma 2.4. *The energy functional $E_2(t)$ satisfies, for all $t \geq 0$, the identity*

$$\frac{d}{dt}E_2(t) + (\beta - \tau)\|\Delta v\|_{L^2}^2 = \mathbf{R}_2, \tag{2.11}$$

where

$$\mathbf{R}_2 := -\int_{\mathbb{R}^3}\left(\frac{B}{A}vw + 2\nabla u\nabla v\right)\Delta(v+\tau w)dx.$$

Now, we define

$$E(t) := E_1(t) + E_2(t).$$

Then, we have from (2.6) and (2.11)

$$\frac{d}{dt}E(t) + (\beta-\tau)(\|\nabla v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2) = \mathbf{R}_1 + \mathbf{R}_2. \tag{2.12}$$

Now, multiplying the third equation in (1.12) by w and integrating over $\mathbb{R}^3$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\tau|w|^2dx + \int_{\mathbb{R}^3}|w|^2dx &= \int_{\mathbb{R}^3}(\Delta u + \beta\Delta v)wdx \\ &\quad + \int_{\mathbb{R}^3}\left(\frac{B}{A}vw + 2\nabla u\nabla v\right)wdx \\ &\leq C(\|\Delta u\|_{L^2} + \|\Delta v\|_{L^2})\|w\|_{L^2} + |\tilde{\mathbf{R}}_1| \\ &\leq C(\|\Delta(u+\tau v)_{L^2} + \|\Delta v\|_{L^2})\|w\|_{L^2} + |\tilde{\mathbf{R}}_1|\end{aligned}$$

with

$$\tilde{\mathbf{R}}_1 := \int_{\mathbb{R}^3}\left(\frac{B}{A}vw + 2\nabla u\nabla v\right)wdx.$$

Applying Young's inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\tau|w|^2dx + \frac{1}{2}\int_{\mathbb{R}^3}|w|^2dx \leq C(\|\Delta(u+\tau v)_{L^2}^2 + \|\Delta v\|_{L^2}^2) + |\tilde{\mathbf{R}}_1|. \tag{2.13}$$

Collecting (2.12) $+ 2\varepsilon_0$(2.13), we get

$$\begin{aligned}&\frac{d}{dt}(E(t) + \varepsilon_0\tau\|w\|_{L^2}) + (\beta-\tau)(\|\nabla v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2) + \varepsilon_0\|w\|_{L^2}^2 \\ &\leq 2C\varepsilon_0(\|\Delta(u+\tau v)_{L^2}^2 + \|\Delta v\|_{L^2}^2) + |\mathbf{R}_1| + |\mathbf{R}_2| + 2\varepsilon_0|\tilde{\mathbf{R}}_1|.\end{aligned} \tag{2.14}$$

Now, we define the functional $F_1(t)$ as

$$F_1(t) := \int_{\mathbb{R}^3}\nabla(u+\tau v)\nabla(v+\tau w)dx.$$

Then, we have

Lemma 2.5. *For any $\epsilon_0 > 0$, we have*

$$\begin{aligned}&\frac{d}{dt}F_1(t) + (1-\epsilon_0)\int_{\mathbb{R}^3}|\Delta(u+\tau v)|^2dx \\ &\leq \int_{\mathbb{R}^3}|\nabla(v+\tau w)|^2dx + C(\epsilon_0)\int_{\mathbb{R}^3}|\Delta v|^2dx + |\tilde{\mathbf{R}}_2|\end{aligned} \tag{2.15}$$

with

$$\tilde{\mathbf{R}}_2 = -\int_{\mathbb{R}^3} \left(\frac{B}{A} vw + 2\nabla u \nabla v \right) \Delta\left(u + \tau v\right) dx.$$

Proof. Multiplying equation (2.7) by $-\Delta(u + \tau v)$ and $(u_t + \tau v_t)$ by $-\Delta(v + \tau w)$ we get, respectively,

$$\begin{aligned}
-\int_{\mathbb{R}^3} (v+\tau w)_t \Delta(u+\tau v) &= -\int_{\mathbb{R}^3} (\Delta u + \beta \Delta v)(\Delta u + \tau \Delta v) dx \\
&\quad - \int_{\mathbb{R}^3} \left(\frac{B}{A} vw + 2\nabla u \nabla v \right) \Delta\left(u + \tau v\right) dx \\
&= -\int_{\mathbb{R}^3} (\Delta u + \beta \Delta v + \tau \Delta v - \tau \Delta v)(\Delta u + \tau \Delta v) dx \\
&\quad - \int_{\mathbb{R}^3} \left(\frac{B}{A} vw + 2\nabla u \nabla v \right) \Delta\left(u + \tau v\right) dx
\end{aligned}$$

and

$$-\int_{\mathbb{R}^3} (u+\tau v)_t \Delta(v+\tau w) dx = -\int_{\mathbb{R}^3} (\tau w + v)\Delta(v+\tau w) dx.$$

Integrating by parts and summing up the above two equations, we obtain

$$\begin{aligned}
&\frac{d}{dt} F_1(t) + \int_{\mathbb{R}^3} |\Delta(u+\tau v)|^2 dx - \int_{\mathbb{R}^3} |\nabla(v+\tau w)|^2 dx \\
=\ & (\tau - \beta) \int_{\mathbb{R}^3} (\Delta v(\Delta u + \tau \Delta v)) dx - \int_{\mathbb{R}^3} \left(\frac{B}{A} vw + 2\nabla u \nabla v \right) \Delta\left(u + \tau v\right) dx.
\end{aligned}$$

Applying Young's inequality for any $\epsilon_0 > 0$, we obtain (2.15). This finishes the proof of Lemma 2.5. □

Next, we define the functional $F_2(t)$ as

$$F_2(t) := -\tau \int_{\mathbb{R}^3} \nabla v \nabla (v + \tau w) dx.$$

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$$F_2(t) := -\tau \int_{\mathbb{R}^3} \nabla v \nabla (v + \tau w) dx.$$

Lemma 2.6. *For any $\epsilon_1, \epsilon_2 > 0$, we have*

$$\begin{aligned}
&\frac{d}{dt} F_2(t) + (1-\epsilon_1) \int_{\mathbb{R}^3} |\nabla(v+\tau w)|^2 dx \\
\le\ & C(\epsilon_1, \epsilon_2) \int_{\mathbb{R}^3} (|\nabla v|^2 + |\Delta v|^2) dx + \epsilon_2 \int_{\mathbb{R}^3} |\Delta(u+\tau v)|^2 dx + |\mathbf{R}_3|,
\end{aligned} \tag{2.16}$$

where

$$\mathbf{R}_3 = \tau \int_{\mathbb{R}^3} \left(\frac{B}{A} vw + 2\nabla u \nabla v \right) \Delta v dx.$$

Proof. Multiplying the second equation in (1.12) by $\tau\Delta(v+\tau w)$ and (2.7) by $\tau\Delta v$, and integrating over $\mathbb{R}^3$ we obtain, respectively,

$$\tau\int_{\mathbb{R}^3} v_t\Delta(v+\tau w)dx = \tau\int_{\mathbb{R}^3} w\Delta(v+\tau w)dx$$

and

$$\begin{aligned}
&\tau\int_{\mathbb{R}^3}(v+\tau w)_t\Delta v dx\\
=\;&\tau\int_{\mathbb{R}^3}(\Delta u+\beta\Delta v)\Delta v dx+\tau\int_{\mathbb{R}^3}\left(\frac{B}{A}vw+2\nabla u\nabla v\right)\Delta v dx\\
=\;&\int_{\mathbb{R}^3}\Big(\tau\Delta u+\tau\beta\Delta v+\tau^2\Delta v-\tau^2\Delta v+(v+\tau w)-(v+\tau w)\Big)\Delta v dx\\
&+\tau\int_{\mathbb{R}^3}\left(\frac{B}{A}vw+2\nabla u\nabla v\right)\Delta v dx.
\end{aligned}$$

Using integration by parts, we obtain

$$\begin{aligned}
&\frac{d}{dt}F_2(t)+\int_{\mathbb{R}^3}|\nabla(v+\tau w)|^2dx-\tau(\beta-\tau)\int_{\mathbb{R}^3}|\Delta v|^2dx\\
=\;&\tau\int_{\mathbb{R}^3}\Delta(u+\tau v)\Delta v dx+\int_{\mathbb{R}^3}\nabla(v+\tau w)\nabla v dx\\
&+\tau\int_{\mathbb{R}^3}\left(\frac{B}{A}vw+2\nabla u\nabla v\right)\Delta v dx.
\end{aligned}$$

Thus we obtain the estimate (2.16) for any $\epsilon_1,\epsilon_2>0$. $\square$

Now, let

$$\mathcal{H}(t):=F_1(t)+\gamma_1F_2(t),$$

where $\gamma_1>0$ will be determined later. Hence, we have from (2.15) and (2.16) that

$$\begin{aligned}
&\frac{d}{dt}\mathcal{H}(t)+(1-\epsilon_0-\gamma_1\epsilon_2)\left\|\Delta\left(u+\tau v\right)\right\|_{L^2}^2+(\gamma_1(1-\epsilon_1)-1)\left\|\nabla(v+\tau w)\right\|_{L^2}^2\\
\leq\;&\gamma_1C(\epsilon_1,\epsilon_2)\left\|\nabla v\right\|_{L^2}^2+(C(\epsilon_0)+\gamma_1C(\epsilon_1,\epsilon_2))\left\|\Delta v\right\|_{L^2}^2+|\tilde{\mathbf{R}}_2|+\gamma_1|\mathbf{R}_3|.
\end{aligned}$$

In the above estimate, we can fix our constants in such a way that the coefficients in front of the norm terms are positive. This can be achieved as follows: we pick ϵ_0 and ϵ_1 small enough such that $\epsilon_0<1$ and $\epsilon_1<1$. After that, we take γ_1 large enough such that

$$\gamma_1>\frac{1}{1-\epsilon_1}.$$

Once γ_1 and ϵ_0 are fixed, we select ϵ_2 small enough such that

$$\epsilon_2<\frac{1-\epsilon_0}{\gamma_1}.$$

Consequently, we deduce that for all $t \geq 0$,

$$\begin{aligned} &\frac{d}{dt}\mathcal{H}(t) + \Big\{ \|\Delta (u+\tau v)\|_{L^2}^2 + \|\nabla(v+\tau w)\|_{L^2}^2 \Big\} \\ (2.17) \qquad \leq\ & C\|\nabla v\|_{L^2}^2 + C\|\Delta v\|_{L^2}^2 + C|\tilde{\mathbf{R}}_2| + C|\mathbf{R}_3|. \end{aligned}$$

where C here is a generic positive constant that depends on ϵ_0, ϵ_1 and γ_1.

We define the Lyapunov functional $L(t)$ as

$$L(t) := \gamma_0(E(t) + \varepsilon_0 \tau \|w(t)\|_{L^2}^2) + \mathcal{H}(t), \tag{2.18}$$

where γ_0 is a large positive constant.

Now, taking the derivative of (2.18) and using (2.14) and (2.17), we find

$$\begin{aligned} &\frac{d}{dt}L(t) + (\gamma_0(\beta-\tau) - 2C)\|\nabla v\|_{L^2}^2 + (\gamma_0(\beta-\tau) - 2C - 2C\gamma_0\varepsilon_0)\|\Delta v\|_{L^2}^2 \\ &+\varepsilon_0\gamma_0\|w\|_{L^2}^2 \\ &+(1-2C\gamma_0\varepsilon_0)\|\Delta(u+\tau v)\|_{L^2}^2 + \|\nabla(v+\tau w)\|_{L^2}^2 \\ (2.19)\quad \leq\ & C(|\mathbf{R}_1| + |\tilde{\mathbf{R}}_1| + |\mathbf{R}_2| + |\tilde{\mathbf{R}}_2| + |\mathbf{R}_3|). \end{aligned}$$

Next, we take γ_0 large enough such that $\gamma_0 > \frac{4C}{\beta-\tau}$ and then we fix ε_0 small enough such that $\varepsilon_0 \leq \frac{1}{2C\gamma_0}$, so we get from (2.19)

$$\begin{aligned} &\frac{d}{dt}L(t) + \|\nabla v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 + \|w\|_{L^2}^2 + \|\Delta(u+\tau v)\|_{L^2}^2 + \|\nabla(v+\tau w)\|_{L^2}^2 \\ (2.20)\quad &\leq C(|\mathbf{R}_1| + |\tilde{\mathbf{R}}_1| + |\mathbf{R}_2| + |\tilde{\mathbf{R}}_2| + |\mathbf{R}_3|). \end{aligned}$$

In the following lemma, we show the equivalence between the functional $L(t)$ and $E(t) + \|w\|_{L^2}^2$.

Lemma 2.7. *There exist two positive constants c_1 and c_2 such that for all $t \geq 0$*

$$c_1(E(t) + \|w\|_{L^2}^2) \leq L(t) \leq c_2(E(t) + \|w\|_{L^2}^2). \tag{2.21}$$

Proof. We have by using Hölder's inequality

$$\begin{aligned} |F_1(t) + \gamma_1 F_2(t)| &\leq C(\|\nabla v\|_{L^2} + \|\nabla(u+\tau v)\|_{L^2})\|\nabla(v+\tau w)\|_{L^2} \\ &\leq CE(t) \leq C(E(t) + \|w\|_{L^2}^2). \end{aligned}$$

This gives (2.21) for γ_0 large enough. □

Now, integrating (2.20) with respect to t and exploiting (2.21), we obtain

$$\begin{aligned} \mathcal{E}_0^2(t) + \mathcal{D}_0^2(t) \leq C\mathcal{E}_0^2(0) \ &+\ C\int_0^t \Big(|\mathbf{R}_1(\sigma)| + |\tilde{\mathbf{R}}_1(\sigma)| + |\mathbf{R}_2(\sigma)| \\ (2.22) \qquad &+ |\tilde{\mathbf{R}}_2(\sigma)| + |\mathbf{R}_3(\sigma)|\Big)d\sigma, \end{aligned}$$

where

$$\mathcal{E}_0^2(t) \equiv \sup_{0\leq\sigma\leq t} (E(\sigma) + \|w(\sigma)\|_{L^2}^2)$$

and

$$\mathcal{D}_0^2(t) = \int_0^t \Big(\|\nabla v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 + \|w\|_{L^2}^2 + \|\Delta (u+\tau v)\|_{L^2}^2 + \|\nabla(v+\tau w)\|_{L^2}^2 \Big) ds.$$

Our goal now is to estimate $|\mathbf{R}_1|$, $|\mathbf{R}_2|, \dots$ in the right-hand side of (2.22). First, we have

$$\begin{aligned} |\mathbf{R}_1| &= \left| \int_{\mathbb{R}^3} \left(\frac{B}{A} vw + 2\nabla u \nabla v \right) (v+\tau w)\, dx \right| \\ &\leq C\Big| \int_{\mathbb{R}^3} vw(v+\tau w) dx \Big| + C\Big| \int_{\mathbb{R}^3} \nabla u \nabla v\, (v+\tau w)\, dx \Big| \\ &\equiv I_1 + I_2. \end{aligned}$$

First, we estimate I_1 as follows:

$$\begin{aligned} I_1 &= C\Big| \int_{\mathbb{R}^3} vw(v+\tau w) dx \Big| \\ &\leq C\|w\|_{L^2}\|v\|_{L^4}^2 + C\|v\|_{L^2}\|w\|_{L^4}^2. \end{aligned}$$

Using the Ladyzhenskaya interpolation inequality in 3D (which is a particular case of (A.3))

$$\|f\|_{L^4} \leq c\|f\|_{L^2}^{1/4}\|\nabla f\|_{L^2}^{3/4} \tag{2.23}$$

we get

$$\begin{aligned} \|w\|_{L^2}\|v\|_{L^4}^2 &\leq C\|w\|_{L^2}\|v\|_{L^2}^{1/2}\|\nabla v\|_{L^2}^{3/2} \\ &= C\|v\|_{L^2}^{1/2}\|\nabla v\|_{L^2}^{1/2}\|\nabla v\|_{L^2}\|w\|_{L^2} \\ &\leq C(\|v\|_{L^2} + \|\nabla v\|_{L^2})\|\nabla v\|_{L^2}\|w\|_{L^2} \end{aligned} \tag{2.24}$$

and

$$\|v\|_{L^2}\|w\|_{L^4}^2 \leq C\|v\|_{L^2}\|w\|_{L^2}^{1/2}\|\nabla w\|_{L^2}^{3/2}. \tag{2.25}$$

We have

$$\begin{aligned} &\int_0^t \|v(\sigma)\|_{L^2}\|w(\sigma)\|_{L^2}^{1/2}\|\nabla w(\sigma)\|_{L^2}^{3/2} d\sigma \\ \leq\;& C \sup_{0\leq\sigma\leq t} \|v(\sigma)\|_{L^2}\Big(\int_0^t \|w(\sigma)\|_{L^2}^2 d\sigma\Big)^{1/4} \Big(\int_0^t \|\nabla w(\sigma)\|_{L^2}^2 d\sigma\Big)^{3/4} \\ \leq\;& C\mathcal{E}_0(t)\mathcal{D}_0^2(t). \end{aligned} \tag{2.26}$$

Similarly, we have for the term on the right-hand side of (2.24)

$$\int_0^t (\|v(\sigma)\|_{L^2} + \|\nabla v(\sigma)\|_{L^2})\|\nabla v(\sigma)\|_{L^2}\|w(\sigma)\|_{L^2} d\sigma \leq C\mathcal{E}_0(t)\mathcal{D}_0^2(t). \tag{2.27}$$

Consequently, collecting (2.26) and (2.27), we obtain, using

$$\|\nabla w(t)\|_{L^2}^2 \leq \|\nabla v(t)\|_{L^2}^2 + \|\nabla(v+\tau w)(t)\|_{L^2}^2,$$

that

$$\int_0^t I_1(\sigma)d\sigma \leq C\mathcal{E}_0(t)\mathcal{D}_0^2(t). \tag{2.28}$$

We can estimate I_2 as follows:

$$\begin{aligned} I_2 = \left|\int_{\mathbb{R}^3} \nabla u \nabla v\,(v+\tau w)\,dx\right| &\leq \left|\int_{\mathbb{R}^3} v\nabla u\nabla v dx\right| + \left|\int_{\mathbb{R}^3} \tau w \nabla u \nabla v dx\right| \\ &= J_1 + J_2. \end{aligned} \tag{2.29}$$

It is clear that

$$J_2 \leq C\|\nabla u\|_{L^\infty}\|\nabla v\|_{L^2}\|w\|_{L^2}.$$

Then, Hölder's inequality implies

$$\int_0^t J_2(\sigma)d\sigma \leq C \sup_{0\leq\sigma\leq t}\|\nabla u(\sigma)\|_{L^\infty}\mathcal{D}_0^2(t).$$

The difficulty is to estimate the term J_1. This is done in the following lemma.

Lemma 2.8. *We have the estimate*

$$\int_0^t J_1(\sigma)d\sigma \leq C\mathcal{E}_0(t)\mathcal{D}_0^2(t).$$

Proof. First, we have, by Hölder's inequality

$$J_1 \leq C\|v\|_{L^6}\|\nabla u\|_{L^3}\|\nabla v\|_{L^2}. \tag{2.30}$$

Now, applying the interpolation inequality, which holds for $n = 3$, (see (A.3))

$$\|f\|_{L^3} \leq C\|f\|_{L^2}^{1/2}\|\nabla f\|_{L^2}^{1/2} \tag{2.31}$$

we obtain

$$\|\nabla u\|_{L^3} \leq C\|\nabla u\|_{L^2}^{1/2}\|\nabla^2 u\|_{L^2}^{1/2}. \tag{2.32}$$

Consequently, using the above estimates, (2.30) becomes

$$\begin{aligned} J_1 &\leq C\|\nabla u\|_{L^2}^{1/2}\|\nabla^2 u\|_{L^2}^{1/2}\|\nabla v\|_{L^2}^2 \\ &\leq C(\|\nabla u\|_{L^2} + \|\nabla^2 u\|_{L^2})\|\nabla v\|_{L^2}^2. \end{aligned} \tag{2.33}$$

Now, using the fact that

$$\|\nabla^k u\|_{L^2} \leq C(\|\nabla^k(u+\tau v)\|_{L^2} + \|\nabla^k v\|_{L^2}), \qquad k \geq 1,$$

together with (2.33), we obtain

$$\begin{aligned}\int_0^t J_1(\sigma)d\sigma &\leq \sup_{0\leq\sigma\leq t}\left(\|\nabla u(\sigma)\|_{L^2}+\|\nabla^2 u(\sigma)\|_{L^2}\right)\int_0^t \|\nabla v(\sigma)\|_{L^2}^2 d\sigma\\ &\leq C\mathcal{E}_0(t)\mathcal{D}_0^2(t),\end{aligned}$$

where we have used the fact that $\|\nabla^2 u\|_{L^2}=\|\Delta u\|_{L^2}$ This completes the proof of Lemma 2.8. □

Consequently, we deduce from above that

$$\int_0^t |\mathbf{R}_1(\sigma)|d\sigma \leq C\mathcal{E}_0(t)\mathcal{D}_0^2(t). \tag{2.34}$$

Similarly, we have as in the estimate of $\mathbf{R}_1$,

$$\int_0^t |\tilde{\mathbf{R}}_1(\sigma)|d\sigma \leq C\sup_{0\leq\sigma\leq t}\|\nabla u(\sigma)\|_{L^\infty}\mathcal{D}_0^2(t)\leq C\mathcal{E}_0(t)\mathcal{D}_0^2(t). \tag{2.35}$$

Using integration by parts, we have

$$\begin{aligned}\mathbf{R}_2 &= -\int_{\mathbb{R}^3}\left(\frac{B}{A}vw+2\nabla u\nabla v\right)\Delta(v+\tau w)dx\\ &= \int_{\mathbb{R}^3}\nabla\left(\frac{B}{\tau A}v\left(v+\tau w-v\right)+2\nabla\left(u+\tau v-\tau v\right)\nabla v\right)\nabla(v+\tau w)dx\\ &= \int_{\mathbb{R}^3}\left(\frac{B}{\tau A}v\nabla\left(v+\tau w\right)+\frac{B}{\tau A}\nabla v\left(v+\tau w\right)-\nabla\left|v\right|^2\right)\nabla(v+\tau w)dx,\\ &\quad+\int_{\mathbb{R}^3}\left(2H(u+\tau v)\nabla v+2H(v)\nabla\left(u+\tau v\right)-4\tau H(v)\nabla v\right)\nabla(v+\tau w)dx,\end{aligned}$$

where $H(f)=(\partial_{x_i}\partial_{x_j}f)_{,1\leq i,j\leq 3}$ is the Hessian matrix of f. Using the fact that $\|H\left(f\right)\|_{L^2}=\|\Delta f\|_{L^2}$, together with Hölder's inequality, we get

$$\begin{aligned}|\mathbf{R}_2| &\leq C\left(\|v\|_{L^\infty}\left(\|\nabla\left(v+\tau w\right)\|_{L^2}+\|\nabla v\|_{L^2}\right)+\|v+\tau w\|_{L^\infty}\|\nabla v\|_{L^2}\right)\|\nabla(v+\tau w)\|_{L^2}\\ &\quad+C\left(\|\nabla v\|_{L^\infty}\left(\|\Delta(u+\tau v)\|_{L^2}+\|\Delta v\|_{L^2}\right)+\|\nabla(u+\tau v)\|_{L^\infty}\|\Delta v\|_{L^2}\right)\|\nabla(v+\tau w)\|_{L^2}\end{aligned}$$

This implies that

$$\begin{aligned}\int_0^t |\mathbf{R}_2(\sigma)|d\sigma &\leq \sup_{0\leq\sigma\leq t}\Big(\|v(\sigma)\|_{L^\infty}+\|\nabla v(\sigma)\|_{L^\infty}\\ &\quad+\|(v+\tau w)(\sigma)\|_{L^\infty}+\|\nabla(u+\tau v)(\sigma)\|_{L^\infty}\Big)\mathcal{D}_0^2(t).\end{aligned} \tag{2.36}$$

For $\tilde{\mathbf{R}}_2$, we have the estimate

$$|\tilde{\mathbf{R}}_2| \leq C\|v\|_{L^\infty}\|w\|_{L^2}\|\Delta\left(u+\tau v\right)\|_{L^2}+C\|\nabla u\|_{L^\infty}\|\nabla v\|_{L^2}\|\Delta\left(u+\tau v\right)\|_{L^2}.$$

This implies

$$\int_0^t |\tilde{\mathbf{R}}_2(\sigma)|d\sigma \leq C\sup_{0\leq\sigma\leq t}\left(\|v(\sigma)\|_{L^\infty}+\|\nabla u(\sigma)\|_{L^\infty}\right)\mathcal{D}_0^2(t). \tag{2.37}$$

For $\mathbf{R}_3$, we have, as in $\tilde{\mathbf{R}}_2$,

$$\int_0^t |\mathbf{R}_3(\sigma)|d\sigma \leq C \sup_{0\leq\sigma\leq t} \big(\|v(\sigma)\|_{L^\infty} + \|\nabla u(\sigma)\|_{L^\infty} \big)\mathcal{D}_0^2(t). \tag{2.38}$$

Plugging all the estimates (2.34)–(2.38) into (2.22), we obtain

$$\mathcal{E}_0^2(t) + \mathcal{D}_0^2(t) \leq \mathcal{E}_0^2(0) + C\mathcal{E}_0(t)\mathcal{D}_0^2(t) + C\Lambda_0(t)\mathcal{D}_0^2(t), \tag{2.39}$$

where

$$\begin{aligned} \Lambda_0(t) &:= \sup_{0\leq s\leq t} \Big(\|v(s)\|_{L^\infty} + \|(v+\tau w)(s)\|_{L^\infty} \\ &\qquad + \|\nabla(u+\tau v)(s)\|_{L^\infty} + \|\nabla u(s)\|_{L^\infty} + \|\nabla v(s)\|_{L^\infty} \Big). \end{aligned}$$

2.2. **Higher-order energy estimates.** Applying the operator ∇^k, $k \geq 1$ to (1.12), we get for $U := \nabla^k u$, $V := \nabla^k v$ and $W := \nabla^k w$

$$\begin{cases} \partial_t U = V, \\ \partial_t V = W, \\ \tau\partial_t W = \Delta U + \beta\Delta V - W + \dfrac{B}{A}[\nabla^k, v]w + \dfrac{B}{A}vW + 2[\nabla^k, \nabla u]\nabla v + 2\nabla u\nabla V, \end{cases} \tag{2.40}$$

where $[A, B] = AB - BA$.

We define the first energy of order k as in the case $k = 0$ by

$$\begin{aligned} E_1^{(k)}(t) &:= \frac{1}{2}\int_{\mathbb{R}^3} \big(|\nabla^k v + \tau\nabla^k w|^2 + \tau(\beta-\tau)|\nabla^{k+1} v|^2 + |\nabla^{k+1} u + \tau\nabla^{k+1} v|^2\big)\, dx \\ &= \frac{1}{2}\int_{\mathbb{R}^3} \big(|V + \tau W|^2 + \tau(\beta-\tau)|\nabla V|^2 + |\nabla(U + \tau V)|^2\big)\, dx. \end{aligned}$$

Hence, we have the following estimate.

Lemma 2.9. *For all $t \geq 0$, it holds*

$$\frac{d}{dt}E_1^{(k)}(t) + (\beta - \tau)\, \|\nabla V\|_{L^2}^2 = \int_{\mathbb{R}^3} R_1^{(k)}(t)\,(V + \tau W)\, dx, \tag{2.41}$$

where

$$R_1^{(k)}(t) = \frac{B}{A}[\nabla^k, v]w + \frac{B}{A}vW + 2[\nabla^k, \nabla u]\nabla v + 2\nabla u\nabla V. \tag{2.42}$$

We omit the proof of Lemma 2.9 since it can be done using the same steps as in Lemma 2.3.

As in the case $k = 0$, we define the second energy of order k as follows:

$$E_2^{(k)}(t) := \frac{1}{2}\int_{\mathbb{R}^3} \big(|\nabla(V + \tau W)|^2 + \tau(\beta-\tau)|\Delta V|^2 + |\Delta(U + \tau V)|^2\big)\, dx.$$

Then, we have the following lemma.

Lemma 2.10. *The energy functional $E_2(t)$ satisfies, for all $t \geq 0$, the identity*

$$\frac{d}{dt}E_2^{(k)}(t) + (\beta - \tau)\,\|\Delta V\|_{L^2}^2 = -\int_{\mathbb{R}^3} R_1^{(k)}\Delta(V + \tau W)dx. \tag{2.43}$$

The proof of the above lemma is similar to that of Lemma 2.4. We omit the details. Now, adding (2.41) to (2.43), we get for

$$E^{(k)}(t) := E_1^{(k)}(t) + E_2^{(k)}(t)$$

$$\begin{aligned} &\frac{d}{dt}E^{(k)}(t) + (\beta - \tau)(\|\nabla V\|_{L^2}^2 + \|\Delta V\|_{L^2}^2) \\ = &\int_{\mathbb{R}^3} R_1^{(k)}(t)\,(V + \tau W)\,dx + \int_{\mathbb{R}^3} \nabla R_1^{(k)}\nabla(V + \tau W)dx. \end{aligned} \tag{2.44}$$

Now, multiplying the third equation in (2.40) by W and integrating over $\mathbb{R}^3$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} \tau\,|W|^2\,dx + \int_{\mathbb{R}^3} |W|^2 dx \\ = &\int_{\mathbb{R}^3} (\Delta U + \beta\Delta V)W dx + \int_{\mathbb{R}^3} R_1^{(k)} W dx \\ \leq \ &C(\|\Delta U\|_{L^2} + \|\Delta V\|_{L^2})\|W\|_{L^2} + \int_{\mathbb{R}^3} |R_1^{(k)}||W|dx \\ \leq \ &C(\|\Delta(U + \tau V)_{L^2} + \|\Delta V\|_{L^2})\|W\|_{L^2} + \int_{\mathbb{R}^3} |R_1^{(k)}||W|dx. \end{aligned} \tag{2.45}$$

We define now the functional $F_1^{(k)}(t)$ as

$$F_1^{(k)}(t) := \int_{\mathbb{R}^3} \nabla(U + \tau V)\nabla(V + \tau W)dx.$$

Then, we have the following estimate, the proof of which can be done following the same strategy as in Lemma 2.5.

Lemma 2.11. *For any $\epsilon_0' > 0$, we have*

$$\begin{aligned} &\frac{d}{dt}F_1^{(k)}(t) + (1 - \epsilon_0')\int_{\mathbb{R}^3} |\Delta\,(U + \tau V)\,|^2 dx \\ \leq \ &\int_{\mathbb{R}^3} |\nabla(V + \tau W)|^2 dx + C(\epsilon_0')\int_{\mathbb{R}^3} |\Delta V|^2 dx \\ &+ \int_{\mathbb{R}^3} |R_1^{(k)}||\Delta\,(U + \tau V)\,|dx \end{aligned} \tag{2.46}$$

As in the case $k = 0$, we define the functional $F_2^{(k)}(t)$ as

$$F_2^{(k)}(t) := -\tau\int_{\mathbb{R}^3} \nabla V\nabla(V + \tau W)dx.$$

Hence, we have the following estimate.

Lemma 2.12. *For any $\epsilon_1', \epsilon_2' > 0$, we have*

$$\begin{aligned} &\frac{d}{dt}F_2^{(k)}(t) + (1-\epsilon_1')\int_{\mathbb{R}^3}|\nabla(V+\tau W)|^2dx \\ \leq\ & C(\epsilon_1',\epsilon_2')\int_{\mathbb{R}^3}(|\nabla V|^2+|\Delta V|^2)dx \\ &+\epsilon_2'\int_{\mathbb{R}^3}|\Delta(U+\tau V)|^2dx+\tau\int_{\mathbb{R}^3}|R_1^{(k)}||\Delta V|dx. \end{aligned} \tag{2.47}$$

We can prove Lemma 2.12 following the same steps as in the proof of Lemma 2.6, we omit the details.

As in the case $k=0$, if we define the functional

$$\mathcal{H}^{(k)}(t) := F_1^{(k)}(t) + \gamma_1' F_2^{(k)}(t),$$

and we proceed exactly as in the case $k=0$ and fixing γ_1' as we did for γ_1 to get the following estimate, which is similar to (2.17),

$$\begin{aligned} &\frac{d}{dt}\mathcal{H}^{(k)}(t) + \|\Delta\left(U+\tau V\right)\|_{L^2}^2 + \|\nabla(V+\tau W)\|_{L^2}^2 \\ \leq\ & C\left(\left(\|\nabla V\|_{L^2}^2+\|\Delta V\|_{L^2}^2\right)+\int_{\mathbb{R}^3}|\nabla R_1^{(k)}||\nabla\left(U+\tau V\right)|dx+\int_{\mathbb{R}^3}|\nabla R_1^{(k)}||\nabla V|dx\right). \end{aligned} \tag{2.48}$$

Now, we define the Lyapunov functional $L^{(k)}(t)$ as

$$L^{(k)}(t) := \gamma_0'(E^{(k)}(t)+\tau\varepsilon_0'\|W\|_{L^2}^2)+\mathcal{H}^{(k)}(t),$$

and selecting γ_0' large enough and ε_0' small enough, we obtain as in the case $k=0$ (see inequality (2.20))

$$\begin{aligned} &\frac{d}{dt}L^{(k)}(t) + \|\Delta\left(U+\tau V\right)\|_{L^2}^2 + \|\nabla(V+\tau W)\|_{L^2}^2 + \|\nabla V\|_{L^2}^2 + \|\Delta V\|_{L^2}^2 + \|W\|_{L^2}^2 \\ &\leq C\int_{\mathbb{R}^3}|R_1^{(k)}(t)||\left(V+\tau W\right)|dx + C\int_{\mathbb{R}^3}|\nabla R_1^{(k)}||\nabla(V+\tau W)|dx \\ &+C\int_{\mathbb{R}^3}|R_1^{(k)}||\Delta\left(U+\tau V\right)|dx + C\int_{\mathbb{R}^3}|\nabla R_1^{(k)}||\nabla V|dx + \int_{\mathbb{R}^3}|R_1^{(k)}||W|dx \\ &\equiv I_1^{(k)}+I_2^{(k)}+I_3^{(k)}+I_4^{(k)}+I_5^{(k)}. \end{aligned} \tag{2.49}$$

Now, integrating (2.49) with respect to t and using the fact that

$$C_1(E^{(k)}+\|W\|_{L^2}^2)(t) \leq L^{(k)}(t) \leq C_2(E^{(k)}(t)+\|W\|_{L^2}^2)$$

for some constants C_1 and C_2, cf. Lemma 2.7, we obtain

$$\mathcal{E}_k^2(t)+\mathcal{D}_k^2(t) \leq \mathcal{E}_k^2(0)+\sum_{i=1}^{5}\int_0^t I_i^{(k)}(\sigma)d\sigma. \tag{2.50}$$

Our goal now is to estimate the terms $\int_0^t I_i^{(k)}(\sigma)d\sigma$, $i = 1,\dots,5$ on the right-hand side of (2.50). First, we estimate $I_2^{(k)}$ and $I_4^{(k)}$. We have

$$I_2^{(k)} + I_4^{(k)} \leq \|\nabla R_1^{(k)}\|_{L^2}\Big(\|\nabla\left(V+\tau W\right)\|_{L^2} + \|\nabla V\|_{L^2}\Big). \tag{2.51}$$

In order to estimate the term $\|\nabla R_1^{(k)}\|_{L^2}$, we have the following lemma.

Lemma 2.13. *For $k \geq 1$, it holds*

$$\|\nabla R_1^{(k)}\|_{L^2} \leq C\Lambda_1\left(\|\nabla V\|_{L^2} + \|\nabla W\|_{L^2} + \|\Delta V\|_{L^2} + \|\Delta\left(U+\tau V\right)\|_{L^2}\right) \tag{2.52}$$

where

$$\Lambda_1 = \|v\|_{W^{1,\infty}} + \|w\|_{L^\infty} + \|\nabla u\|_{L^\infty}.$$

Proof. We have

$$\nabla R_1^{(k)} = \nabla^{k+1}\left(\frac{B}{A}vw + 2\nabla u\nabla v\right).$$

Thus, applying (A.1), we get

$$\begin{aligned}\|\nabla R_1^{(k)}\|_{L^2} \;&\leq\; C\left(\|w\|_{L^\infty}\|\nabla^{k+1}v\|_{L^2} + \|v\|_{L^\infty}\|\nabla^{k+1}w\|_{L^2}\right)\\ &\quad +C\left(\|\nabla u\|_{L^\infty}\|\nabla^{k+2}v\|_{L^2} + \|\nabla v\|_{L^\infty}\|\nabla^{k+2}u\|_{L^2}\right).\end{aligned} \tag{2.53}$$

Now, we estimate

$$\begin{aligned}&\|\nabla u\|_{L^\infty}\|\nabla^{k+2}v\|_{L^2} + \|\nabla v\|_{L^\infty}\|\nabla^{k+2}u\|_{L^2}\\ \leq\;& C\left(\|\nabla u\|_{L^\infty}\|\Delta\nabla^k v\|_{L^2} + \|\nabla v\|_{L^\infty}\|\Delta\nabla^k u\|_{L^2}\right)\\ \leq\;& C\|\nabla u\|_{L^\infty}\|\Delta\nabla^k v\|_{L^2} + C\|\nabla v\|_{L^\infty}\left(\|\Delta\nabla^k\left(u+\tau v\right)\|_{L^2} + \|\Delta\nabla^k v\|_{L^2}\right).\end{aligned}$$

Inserting the above estimates into (2.53), we get (2.52). This completes the proof of Lemma 2.13 □

Consequently, (2.51) together with (2.52) imply that

$$\int_0^t (I_2^{(k)}(\sigma) + I_4^{(k)}(\sigma))d\sigma \leq C\sup_{0\leq\sigma\leq t}\Lambda_1(\sigma)\mathcal{D}_k^2(t). \tag{2.54}$$

The next step is to provide nice estimates for the terms $I_1^{(k)}$ and $I_3^{(k)}$. The main difficulty in controlling these terms is that the dissipation term $\mathcal{D}_k^2(t)$ does not contain terms like $\|\left(V+\tau W\right)\|_{L^2}$ and $\|\Delta\left(U+\tau V\right)\|_{L^2}$. First, we have the following lemma.

Lemma 2.14. *Assume $n = 3$. Then, we have the estimate*

$$\int_0^t I_1^{(k)}(\sigma)d\sigma \leq C\Big(\sup_{0\leq\sigma\leq t}\|\nabla u(\sigma)\|_{L^\infty} + \mathcal{E}_0(t) + \mathcal{E}_k(t)\Big)(\mathcal{D}_0^2(t) + \mathcal{D}_k^2(t)). \tag{2.55}$$

Proof. First, we have

$$
\begin{aligned}
I_1^{(k)} &= \int_{\mathbb{R}^3} |R_1^{(k)}(t)| \left| (V+\tau W) \right| dx \\
&\leq C\int_{\mathbb{R}^3} |[\nabla^k, v]w| \left| (V+\tau W) \right| dx + C\int_{\mathbb{R}^3} |vW| \left| (V+\tau W) \right| dx \\
&\quad + C\int_{\mathbb{R}^3} |[\nabla^k, \nabla u]\nabla v| \left| (V+\tau W) \right| dx + C\int_{\mathbb{R}^3} |\nabla u \nabla V| \left| (V+\tau W) \right| dx \\
&\equiv \mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3+\mathcal{T}_4. \qquad (2.56)
\end{aligned}
$$

Now, we estimate $\mathcal{T}_4$ as in (2.29). Indeed, we write

$$
\begin{aligned}
\mathcal{T}_4 &\leq \left| \int_{\mathbb{R}^3} V\nabla u \nabla V dx \right| + \left| \int_{\mathbb{R}^3} \tau \nabla u \nabla V W dx \right| \\
&\equiv \mathcal{J}_1+\mathcal{J}_2. \qquad (2.57)
\end{aligned}
$$

We estimate $\mathcal{J}_1$ and $\mathcal{J}_2$ as we did for J_1, J_2 in the case $k=0$, it holds (see Lemma 2.8)

$$
\int_0^t \mathcal{J}_1(\sigma)d\sigma \leq C\mathcal{E}_0(t)\mathcal{D}_k^2(t),. \qquad (2.58)
$$

and

$$
\int_0^t \mathcal{J}_2(\sigma)d\sigma \leq C \sup_{0\leq\sigma\leq t} \|\nabla u(\sigma)\|_{L^\infty}\mathcal{D}_k^2(t). \qquad (2.59)
$$

Consequently, we deduce from (2.58) and (2.59) that

$$
\int_0^t \mathcal{T}_4(\sigma)d\sigma \leq C\big(\sup_{0\leq\sigma\leq t} \|\nabla u(\sigma)\|_{L^\infty} + \mathcal{E}_0(t)\big)\mathcal{D}_k^2(t). \qquad (2.60)
$$

The next step is to estimate $\mathcal{T}_2$. We have

$$
\begin{aligned}
\mathcal{T}_2 &= \int_{\mathbb{R}^3} |vW| \left| (V+\tau W) \right| dx \\
&\leq \int_{\mathbb{R}^3} |vWV| dx + \tau \int_{\mathbb{R}^3} |vW^2| dx \\
&\equiv \mathcal{J}_3+\mathcal{J}_4. \qquad (2.61)
\end{aligned}
$$

For $\mathcal{J}_4$, we have (see (2.25)) As for $k=0$ we obtain

$$
\int_0^t \mathcal{J}_4(\sigma)d\sigma \leq C\mathcal{E}_0(t)\mathcal{D}_k^2(t). \qquad (2.62)
$$

and

$$
\begin{aligned}
\int_0^t \mathcal{J}_3(\sigma)d\sigma &\leq C \sup_{0\leq\sigma\leq t} \left(\|v(\sigma)\|_{L^2} + \|\nabla v(\sigma)\|_{L^2} \right) \int_0^t \|\nabla V(\sigma)\|_{L^2} \|W(\sigma)\|_{L^2} d\sigma \\
&\leq C\mathcal{E}_0(t)\mathcal{D}_k^2(t). \qquad (2.63)
\end{aligned}
$$

Consequently, from (2.62) and (2.63), we conclude

$$
\int_0^t \mathcal{T}_2(\sigma)d\sigma \leq C\mathcal{E}_0(t)\mathcal{D}_k^2(t). \qquad (2.64)
$$

Now, to estimate $\mathcal{T}_1$, we apply the inequality:

$$\mathcal{T}_1=\int_{\mathbb{R}^3}|[\nabla^k,v]w||\left(V+\tau W\right)|dx\leq\left\|[\nabla^k,v]w\right\|_{L^{6/5}}\|(V+\tau W)\|_{L^6}\,. \tag{2.65}$$

Applying, the commutator estimate (A.2), we have

$$\left\|[\nabla^k,v]w\right\|_{L^{6/5}}\leq C(\|\nabla v\|_{L^2}\|\nabla^{k-1}w\|_{L^3}+\|w\|_{L^2}\|\nabla^k v\|_{L^3}) \tag{2.66}$$

To estimate the term $\|\nabla^{k-1}w\|_{L^3}$, we apply (A.3) to find

$$\|\nabla^{k-1}w\|_{L^3}\leq C\|\nabla^k w\|_{L^2}^{\frac{2k-1}{2k}}\|w\|_{L^2}^{\frac{1}{2k}}. \tag{2.67}$$

Similarly, we have

$$\|\nabla^k v\|_{L^3}\leq C\|\nabla^{k+1}v\|_{L^2}^{\frac{2k+1}{2(k+1)}}\|v\|_{L^2}^{\frac{1}{2(k+1)}}. \tag{2.68}$$

Plugging (2.67) and (2.68) into (2.66), we obtain

$$\begin{aligned}\left\|[\nabla^k,v]w\right\|_{L^{6/5}}&\leq C\|\nabla v\|_{L^2}\|\nabla^k w\|_{L^2}^{\frac{2k-1}{2k}}\|w\|_{L^2}^{\frac{1}{2k}}+C\|w\|_{L^2}\|\nabla^{k+1}v\|_{L^2}^{\frac{2k+1}{2(k+1)}}\|v\|_{L^2}^{\frac{1}{2(k+1)}}\\&=C\|\nabla v\|_{L^2}^{\frac{2k-1}{2k}}\|w\|_{L^2}^{\frac{1}{2k}}\|\nabla v\|_{L^2}^{\frac{1}{2k}}\|\nabla^k w\|_{L^2}^{\frac{2k-1}{2k}}\\&\quad+C\|w\|_{L^2}^{\frac{2k+1}{2(k+1)}}\|v\|_{L^2}^{\frac{1}{2(k+1)}}\|\nabla^{k+1}v\|_{L^2}^{\frac{2k+1}{2(k+1)}}\|w\|_{L^2}^{\frac{1}{2(k+1)}}.\end{aligned} \tag{2.69}$$

Plugging (2.69) into (2.65), making use of Sobolev embedding theorem and applying Young's inequality, we get

$$\begin{aligned}\int_0^t\mathcal{T}_1(\sigma)d\sigma&\leq C\sup_{0\leq\sigma\leq t}(\|\nabla v(\sigma)\|_{L^2}+\|w(\sigma)\|_{L^2})\\&\quad\times\int_0^t\|\nabla v(\sigma)\|_{L^2}^{\frac{1}{2k}}\|\nabla^k w(\sigma)\|_{L^2}^{\frac{2k-1}{2k}}\|\nabla(V+\tau W)\|_{L^2}d\sigma\\&\quad+C\sup_{0\leq\sigma\leq t}(\|v(\sigma)\|_{L^2}+\|w(\sigma)\|_{L^2})\\&\quad\times\int_0^t\|\nabla^{k+1}v(\sigma)\|_{L^2}^{\frac{2k+1}{2(k+1)}}\|w(\sigma)\|_{L^2}^{\frac{1}{2(k+1)}}\|\nabla(V+\tau W)\|_{L^2}d\sigma.\end{aligned} \tag{2.70}$$

Applying Hölder's inequality together with Young's inequality, we obtain

$$\int_0^t\mathcal{T}_1(\sigma)d\sigma\leq C\mathcal{E}_0(t)(\mathcal{D}_0^2(t)+\mathcal{D}_k^2(t)). \tag{2.71}$$

Finally, we treat the term $\mathcal{T}_3$. We have as in (2.65)

$$\begin{aligned}\mathcal{T}_3=\int_{\mathbb{R}^3}|[\nabla^k,\nabla u]\nabla v||\left(V+\tau W\right)|dx&\leq\left\|[\nabla^k,\nabla u]\nabla v\right\|_{L^{6/5}}\|(V+\tau W)\|_{L^6}\\&\leq C\left\|[\nabla^k,\nabla u]\nabla v\right\|_{L^{6/5}}\|\nabla\left(V+\tau W\right)\|_{L^2}\,.\end{aligned} \tag{2.72}$$

Now, as we have done for the estimate of the commutator in $\mathcal{T}_1$, we have by applying (A.2),

$$\left\|[\nabla^k,\nabla u]\nabla v\right\|_{L^{6/5}}\leq C(\|\nabla^2 u\|_{L^2}\|\nabla^k v\|_{L^3}+\|\nabla v\|_{L^2}\|\nabla^{k+1}u\|_{L^3}). \tag{2.73}$$

Applying (A.3), we obtain (see (2.32))

$$\|\nabla^{k+1}u\|_{L^3} \leq C\|\nabla^{k+1}u\|_{L^2}^{1/2}\|\nabla^{k+2}u\|_{L^2}^{1/2}. \tag{2.74}$$

Hence, (2.74) together with (2.68) lead to

$$\begin{aligned}
\big\|[\nabla^k, \nabla u]\nabla v\big\|_{L^{6/5}} &\leq C\|\nabla^2 u\|_{L^2}\|\nabla^{k+1}v\|_{L^2}^{\frac{2k+1}{2(k+1)}}\|v\|_{L^2}^{\frac{1}{2(k+1)}}\\
&\quad +C\|\nabla v\|_{L^2}\|\nabla^{k+1}u\|_{L^2}^{1/2}\|\nabla^{k+2}u\|_{L^2}^{1/2}\\
&= C\|\nabla^2 u\|_{L^2}^{\frac{2k+1}{2(k+1)}}\|v\|_{L^2}^{\frac{1}{2(k+1)}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2(k+1)}}\|\nabla^{k+1}v\|_{L^2}^{\frac{2k+1}{2(k+1)}}\\
&\quad +C\|\nabla v\|_{L^2}^{1/2}\|\nabla^{k+1}u\|_{L^2}^{1/2}\|\nabla v\|_{L^2}^{1/2}\|\nabla^{k+2}u\|_{L^2}^{1/2}.
\end{aligned}$$

Young's inequality yields

$$\begin{aligned}
\big\|[\nabla^k, \nabla u]\nabla v\big\|_{L^{6/5}} &\leq C\Big(\|\nabla^2 u\|_{L^2}+\|v\|_{L^2}\Big)\Big(\|\nabla^2 u\|_{L^2}+\|\nabla^{k+1}v\|_{L^2}\Big)\\
&\quad +C\Big(\|\nabla v\|_{L^2}+\|\nabla^{k+1}u\|_{L^2}\Big)\Big(\|\nabla v\|_{L^2}+\|\nabla^{k+2}u\|_{L^2}\Big).
\end{aligned} \tag{2.75}$$

Consequently, we get from (2.72) and (2.75)

$$\int_0^t \mathcal{T}_3(\sigma)d\sigma \leq C(\mathcal{E}_0(t)+\mathcal{E}_k(t))(\mathcal{D}_0^2(t)+\mathcal{D}_k^2(t)). \tag{2.76}$$

This completes the proof of Lemma 2.14, by (2.60), (2.64), (2.71) and (2.75). □

In the following lemma, we estimate $I_5^{(k)}$.

Lemma 2.15. *We have the estimate*

$$\int_0^t I_5^{(k)}(\sigma)d\sigma \leq C\big(\sup_{0\leq\sigma\leq t}\|\nabla u(\sigma)\|_{L^\infty}+\mathcal{E}_0(t)+\mathcal{E}_k(t)\big)(\mathcal{D}_0^2(t)+\mathcal{D}_k^2(t)). \tag{2.77}$$

The proof of Lemma 2.15 can be done exactly as the one of Lemma 2.14. We omit the details.

Let us now focus on the estimate of $I_3^{(k)}$.

Lemma 2.16. *We have for $k \geq 1$*

$$\int_0^t I_3^{(k)}(\sigma)d\sigma \leq C\Lambda_2(t)(\mathcal{D}_{k-1}^2(t)+\mathcal{D}_k^2(t)) \tag{2.78}$$

with

$$\Lambda_2(t) := (\|\nabla v\|_{L^\infty}+\|w\|_{L^\infty}+\|\nabla^2 u\|_{L^\infty})(t).$$

To prove Lemma 2.16, we first give the proof of the following lemma.

Lemma 2.17. *For $k \geq 1$, it holds*

$$\begin{aligned}
\|R_1^{(k)}\|_{L^2} &\leq C\Lambda_2(t)\Big(\|\nabla^{k-1}w\|_{L^2}+\|\nabla V\|_{L^2}\\
&\quad +\|\nabla^{k-1}\nabla v\|_{L^2}+\|\nabla^{k-1}\Delta\left(u+\tau v\right)\|_{L^2}\Big).
\end{aligned} \tag{2.79}$$

Proof. We have from (2.42),

$$\begin{aligned} \|R_1^{(k)}\|_{L^2} &\leq C\left(\|[\nabla^k, v]w\|_{L^2} + C\|v\|_{L^\infty}\|W\|_{L^2}\right) \\ &\quad + C\left(\|[\nabla^k, \nabla u]\nabla v\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla V\|_{L^2}\right). \end{aligned} \tag{2.80}$$

Now applying the commutator estimate in Lemma A.1, we get

$$\|[\nabla^k, v]w\|_{L^2} \leq C(\|\nabla v\|_{L^\infty}\|\nabla^{k-1}w\|_{L^2} + \|w\|_{L^\infty}\|\nabla^k v\|_{L^2}).$$

Similarly,

$$\begin{aligned} \|[\nabla^k, \nabla u]\nabla v\|_{L^2} &\leq C(\|\nabla v\|_{L^\infty}\|\nabla^{k+1}u\|_{L^2} + \|\nabla^2 u\|_{L^\infty}\|\nabla^k v\|_{L^2}) \\ &\leq C(\|\nabla v\|_{L^\infty}\left(\|\nabla^{k+1}(u+\tau v)\|_{L^2} + \|\nabla^{k+1}v\|_{L^2}\right) + \|\nabla^2 u\|_{L^\infty}\|\nabla^k v\|_{L^2}). \end{aligned}$$

Plugging the last two estimates into (2.80), then (2.79) holds. □

Proof of Lemma 2.16. We have

$$\begin{aligned} I_3^{(k)} &\leq C\int_{\mathbb{R}^3} |R_1^{(k)}||\Delta(U+\tau V)|dx \\ &\leq C\|R_1^{(k)}\|_{L^2}\|\Delta(U+\tau V)\|_{L^2}. \end{aligned}$$

Applying (2.79) together with Young's inequality yield (2.78). This completes the proof of Lemma 2.16. □

Proof of Proposition 2.2. Let

$$\Lambda_3(t) = (\|v\|_{W^{1,\infty}} + \|w\|_{L^\infty} + \|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_{L^\infty})(t).$$

Using the estimates (2.54), (2.55), (2.77), (2.78), and using the fact that $\Lambda_i(t) \leq C\Lambda_3(t)$, $i = 1, 2$, then (2.50) becomes, for any $1 \leq k \leq s$,

$$\begin{aligned} \mathcal{E}_k^2(t) + \mathcal{D}_k^2(t) &\leq \mathcal{E}_k^2(0) + C(\Lambda_3(t) + \mathcal{E}_0(t) + \mathcal{E}_k(t))(\mathcal{D}_0^2(t) + \mathcal{D}_{k-1}^2(t) + \mathcal{D}_k^2(t)) \\ &\leq \mathcal{E}_k^2(0) + C(\Lambda_3(t) + \mathcal{E}_s(t))(\mathcal{D}_0^2(t) + \mathcal{D}_{k-1}^2(t) + \mathcal{D}_k^2(t)). \end{aligned} \tag{2.81}$$

Summing up (2.81) over $k = 1, \ldots, s$, adding the result to (2.39) ($k = 0$) and using the fact that $\Lambda_0(t) \leq C\Lambda_3(t)$, we find

$$\mathcal{E}_s^2(t) + \mathcal{D}_s^2(t) \leq \mathcal{E}_k^2(0) + C(\Lambda_3(t) + \mathcal{E}_s(t))\mathcal{D}_s^2(t). \tag{2.82}$$

Now, we need to estimate $\Lambda_3(t)$ by $\mathcal{E}_s(t)$ by using Sobolev embeddings. Due to the embedding $H^s(\mathbb{R}^3) \hookrightarrow W^{1,\infty}(\mathbb{R}^3)$ for $s > 5/2$ we have

$$\Lambda_3(t) \leq C\mathcal{E}_s(t). \tag{2.83}$$

Plugging (2.83) into (2.82), we conclude that (2.5) holds true. This completes the proof of Proposition 2.2. □

3. A local existence theorem – Proof of Theorem 1.2

In this section, we use the contraction mapping theorem to show Theorem 1.2.

Proof. First, we write problem (1.1) as a first-order evolution equation of the form

$$\begin{cases} \dfrac{d}{dt}\mathbf{U}(t) = \mathcal{A}\mathbf{U}(t) + \mathcal{F}(\mathbf{U}, \nabla\mathbf{U}), \quad t > 0 \\ \mathbf{U}(0) = \mathbf{U}_0, \end{cases} \tag{3.1}$$

where $\mathbf{U}(t) = (u, v, w)^T = (u, u_t, u_{tt})^T$, $\mathbf{U}_0 = (u_0, u_1, u_2)^T$ and $\mathcal{A}$ is the following linear operator which generates a semigroup (see [28]):

$$\mathcal{A}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} v \\ w \\ \dfrac{1}{\tau}\Delta(u + \beta v) - \dfrac{1}{\tau}w \end{pmatrix}$$

and $\mathcal{F}$ is the nonlinear term

$$\mathcal{F}(\mathbf{U}, \nabla\mathbf{U}) = \begin{pmatrix} 0 \\ 0 \\ \dfrac{1}{\tau}\dfrac{B}{A}vw + \dfrac{2}{\tau}\nabla u\nabla v \end{pmatrix}.$$

If $\mathbf{U}$ is a smooth solution of (1.1), then

$$\mathbf{U}(t) = \Phi(\mathbf{U})(t) = e^{t\mathcal{A}}\mathbf{U}_0 + \int_0^t e^{(t-r)\mathcal{A}}\mathcal{F}(\mathbf{U}, \nabla\mathbf{U})(r)dr. \tag{3.2}$$

We define for $s > s_0 = \frac{5}{2}$,

$$\begin{aligned} X \quad := \quad & \Big\{\mathbf{U} = (u, v, w) \mid \nabla u, \Delta u \in C^0([0,T], H^s), u \in W,\ v \in C^0([0,T], H^{s+2}), \\ & w \in C^0(0,T], H^{s+1}),\ \|\mathbf{U}\|_X^2 \equiv \mathcal{E}_s^2(T) < \infty\Big\}, \end{aligned}$$

where W is the completion of C_0^∞ under the seminorm $\|\nabla\cdot\|_{L^2}$, and where $T > 0$ will be chosen small enough later on. It is clear that X is a Banach space.

We also define

$$Y := \left\{\mathbf{U} \mid \|\mathbf{U}\|_Y^2 \equiv \mathcal{D}_s^2(T) < \infty\right\},$$

where $\mathcal{E}_s(T)$ and $\mathcal{D}_s(T)$ are defined in (2.3). It is clear that

$$\|\mathbf{U}\|_Y \leq CT\|\mathbf{U}\|_X,$$

for some $C > 0$. Hence, $X \hookrightarrow Y$, therefore, $X \cap Y$ is a Banach space. Define the complete metric space

$$Z := \{\mathbf{U} \in X \cap Y; \mathbf{U}(x, 0) = \mathbf{U}_0(x)\}.$$

Hence, from the above computation, especially the estimate (2.5) (see also see (3.5) and (3.6) below) we deduce that $\Phi(\mathbf{U})$ is well defined and it maps Z into $X \cap Y$. We define the ball D_R as:

$$D_R = \big\{\mathbf{U} \in Z : \|\mathbf{U}\|_{X\cap Y} \leq R\big\}.$$

It is clear that D_R is a closed subset of the space Z and non-empty for all $R > R_0$ with $R_0 = \mathcal{E}_s(0)$.

Our goal is to show that:

(1) Φ maps the ball D_R into itself,
(2) Φ is a contraction in D_R.

As we will, see properties (1) and (2) are valid for R large enough, depending on the initial data, and for T sufficiently small, and its choice is given later. Once the properties (1)-(2) are verified, the application of the contraction mapping theorem gives the existence of a unique solution of (3.1).

For a given U we write

$$\Phi(\mathbf{U}) = \mathbf{U}^0 + \mathcal{G}(\mathbf{U}),$$

where $\mathbf{U}^0$ and $\mathcal{G}(\mathbf{U})$ are given by

$$\mathbf{U}^0 = e^{t\mathcal{A}}\mathbf{U}_0 \qquad \text{and} \qquad \mathcal{G}(\mathbf{U}) = \int_0^t e^{(t-r)\mathcal{A}}\mathcal{F}(\mathbf{U},\nabla\mathbf{U})(r)dr.$$

$\mathbf{U}^0$ satisfies the linear equation

$$\mathbf{U}^0_t - \mathcal{A}\mathbf{U}^0 = 0, \qquad \mathbf{U}^0(x) = \mathbf{U}_0(x)\,, \tag{3.3}$$

and $\mathcal{G}(\mathbf{U})$ satisfies the nonlinear equation with zero initial data, that is

$$\partial_t\mathcal{G}(\mathbf{U}) - \mathcal{A}\mathcal{G}(\mathbf{U}) = \mathcal{F}(\mathbf{U},\nabla\mathbf{U}), \qquad \mathcal{G}(\mathbf{U})(x,0) = 0. \tag{3.4}$$

As in the proof of Theorem 1.1, we have for all $t \in [0,T)$,

$$\|\mathbf{U}^0(t)\|_X^2 + \|\mathbf{U}^0(t)\|_Y^2 \leq C\mathcal{E}_s^2(0). \tag{3.5}$$

Now, to bound $\mathcal{G}$ in D_R, we have by imitating once again the proof of Theorem 1.1, (especially the estimate (2.5)) for all $0 \leq t \leq T$,

$$\|\mathcal{G}(\mathbf{U}(t))\|_X^2 + \|\mathcal{G}(\mathbf{U}(t))\|_Y^2 \leq Ct\mathcal{E}_s^{3/2}(t). \tag{3.6}$$

Hence, (3.6) yields

$$\begin{aligned}\|\mathcal{G}(\mathbf{U}(t))\|_X^2 + \|\mathcal{G}(\mathbf{U}(t))\|_Y^2 &\leq CT\|\mathbf{U}(t)\|_X^{3/2} \\ &\leq TCR^{3/2}.\end{aligned} \tag{3.7}$$

Collecting (3.5) and (3.7), we obtain

$$\|\Phi(\mathbf{U})\|_{X\cap Y}^2 \leq C\mathcal{E}_s^2(0) + CTR^{3/2}.$$

Choosing R sufficiently large and T sufficiently small such that

$$C\mathcal{E}_s^2(0) + TCR^{3/2} \leq R^2,$$

which can be achieved by choosing $R^2 > C\mathcal{E}_s^2(0)$ and

$$T \leq \frac{R^2 - C\mathcal{E}_s^2(0)}{CR^{3/2}},$$

we obtain

$$\|\Phi(\mathbf{U})\|_{X\cap Y} \leq R.$$

Hence, we have prove that $\Phi\left(D_R\right) \subset D_R$.

Now, we need to prove that Φ is contractive. We have, as above for $\mathbf{U}$ and $\mathbf{V}$ in D_R, $\mathcal{G}(\mathbf{U})$ and $\mathcal{G}(\mathbf{V})$ solve the equation, hence, we obtain (3.4)

$$\begin{aligned} &\partial_t\Big(\mathcal{G}(\mathbf{U}) - \mathcal{G}(\mathbf{V})\Big) - \mathcal{A}\big(\mathcal{G}(\mathbf{U}) - \mathcal{G}(\mathbf{V})\big) = \mathcal{F}(\mathbf{U}, \nabla\mathbf{U}) - \mathcal{F}(\mathbf{V}, \nabla\mathbf{V}), \\ &\mathcal{G}(\mathbf{U})\,(t=0) = \mathcal{G}(\mathbf{V})\,(t=0) = 0. \end{aligned} \tag{3.8}$$

We put $\mathbf{W}(t) = \mathcal{G}(\mathbf{U}(t)) - \mathcal{G}(\mathbf{V}(t))$. Then, we obtain

$$\begin{cases} \partial_t\mathbf{W} - \mathcal{A}(\mathbf{W}) = \mathcal{F}(\mathbf{U}, \nabla\mathbf{U}) - \mathcal{F}(\mathbf{V}, \nabla\mathbf{V}), \\ \mathbf{W}(t=0) = 0. \end{cases} \tag{3.9}$$

Let $\mathbf{U} = (u, v, w)^T$ and $\mathbf{V} = (\bar{u}, \bar{v}, \bar{w})^T$, then we have

$$\begin{aligned} &\mathcal{F}(\mathbf{U}, \nabla\mathbf{U}) - \mathcal{F}(\mathbf{V}, \nabla\mathbf{V}) \\ &= \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\tau}\frac{B}{A}(vw - \bar{v}\,\bar{w}) + \frac{2}{\tau}(\nabla u\nabla v - \nabla\bar{u}\nabla\bar{v}) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\tau}\frac{B}{A}\big((v-\bar{v})\bar{w} + v(w-\bar{w})\big) + \frac{2}{\tau}\big((\nabla u - \nabla\bar{u})\nabla\bar{v} + \nabla u(\nabla v - \nabla\bar{v})\big) \end{pmatrix}. \end{aligned}$$

Lemma 3.1. *For all $0 \leq t \leq T$, we get*

$$\begin{aligned} &\mathcal{E}_s^2(\mathbf{W})(t) + \mathcal{D}_s^2(\mathbf{W})(t) \\ \leq\ &CT\Big(\mathcal{E}_s(\mathbf{U})(t) + \mathcal{E}_s(\mathbf{V})(t)\Big)\mathcal{E}_s(\mathbf{W})(t)\mathcal{E}_s(\mathbf{U}-\mathbf{V})(t). \end{aligned} \tag{3.10}$$

It is clear that (3.10) yields

$$\|\mathbf{W}\|_{X\cap Y} \leq CT\|\mathbf{U}-\mathbf{V}\|_X\left(\|\mathbf{U}\|_X + \|\mathbf{V}\|_X\right),$$

which in turns implies

$$\|\mathcal{G}(\mathbf{U}(t)) - \mathcal{G}(\mathbf{V}(t))\|_{X\cap Y} \leq 2CRT\|\mathbf{U}-\mathbf{V}\|_{X\cap Y}.$$

Now, we fix T small enough such that $2CRT = \kappa < 1$. Hence, we deduce that

$$\|\Phi(\mathbf{U}) - \Phi(\mathbf{V})\|_{X\cap Y} \leq \kappa\|\mathbf{U}-\mathbf{V}\|_{X\cap Y}.$$

Thus, we conclude that Φ is a contraction in D_R. The application of the contraction mapping principle shows that there exists a unique solution $\mathbf{U} \in Z$ of (1.1). This finishes the proof of Theorem 1.2. $\square$

Proof of Lemma 3.1. The estimate (3.10) can be obtained following the steps that used to obtain (2.5). We will just give the proof for the first order energy estimates.

Let $\mathbf{W} = (\mathbf{u}, \mathbf{v}, \mathbf{w})^T$. Then (3.9) can be rewritten as

(3.11)
$$\begin{cases} \mathbf{u}_t = \mathbf{v}, \\ \mathbf{v}_t = \mathbf{w}, \\ \tau \mathbf{w}_t = \Delta \mathbf{u} + \beta \Delta \mathbf{v} - \mathbf{w} + \dfrac{B}{A}\big((v-\bar{v})\bar{w} + v(w-\bar{w})\big) + 2\big((\nabla u - \nabla \bar{u})\nabla \bar{v} + \nabla u(\nabla v - \nabla \bar{v})\big), \end{cases}$$

We define

$$\begin{aligned} \mathcal{E}_k^2(\mathbf{W})(t) &= \sup_{0\le\sigma\le t}\Big(\big\|\nabla^k(\mathbf{v}+\tau\mathbf{w})(\sigma)\big\|_{H^1}^2 + \big\|\Delta\nabla^k\mathbf{v}(\sigma)\big\|_{L^2}^2 + \big\|\nabla^{k+1}\mathbf{v}(\sigma)\big\|_{L^2}^2 \\ &\qquad + \big\|\Delta\nabla^k(\mathbf{u}+\tau\mathbf{v})(\sigma)\big\|_{L^2}^2 + \big\|\nabla^{k+1}(\mathbf{u}+\tau\mathbf{v})(\sigma)\big\|_{L^2}^2 + \|\nabla^k\mathbf{w}(\sigma)\|_{L^2}^2\Big), \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} \mathcal{D}_k^2(\mathbf{W})(t) &= \int_0^t \Big(\big\|\nabla^{k+1}\mathbf{v}(\sigma)\big\|_{L^2}^2 + \big\|\Delta\nabla^k\mathbf{v}(\sigma)\big\|_{L^2}^2 + \|\nabla^k\mathbf{w}(\sigma)\|_{L^2}^2 \\ &\qquad + \big\|\Delta\nabla^k\left(\mathbf{u}+\tau\mathbf{v}\right)(\sigma)\big\|_{L^2}^2 + \big\|\nabla^{k+1}(\mathbf{v}+\tau\mathbf{w})(\sigma)\big\|_{L^2}^2\Big)d\sigma. \end{aligned} \tag{3.13}$$

For the first order energy estimate, we get as in (2.22):

$$\begin{aligned} \mathcal{E}_0^2(\mathbf{W})(t) + \mathcal{D}_0^2(\mathbf{W})(t) &\le C\int_0^t \Big(|\mathbf{R}_1(\sigma)| + |\tilde{\mathbf{R}}_1(\sigma)| + |\mathbf{R}_2(\sigma)| \\ &\qquad + |\tilde{\mathbf{R}}_2(\sigma)| + |\mathbf{R}_3(\sigma)|\Big)d\sigma \end{aligned} \tag{3.14}$$

with for instance (we will estimate just $\int_0^t |\mathbf{R}_1(\sigma)|d\sigma$, the other terms can be estimated similarly)

$$\begin{aligned} |\mathbf{R}_1| &= \Big|\int_{\mathbb{R}^3} \frac{B}{A}\big((v-\bar{v})\bar{w} + v(w-\bar{w})\big) + 2\big((\nabla u - \nabla\bar{u})\nabla\bar{v} + \nabla u(\nabla v - \nabla\bar{v})\big)\left(\mathbf{v}+\tau\mathbf{w}\right)dx\Big| \\ &\le C\Big|\int_{\mathbb{R}^3}\big((v-\bar{v})\bar{w} + v(w-\bar{w})\big)(\mathbf{v}+\tau\mathbf{w})dx\Big| \\ &\quad + C\Big|\int_{\mathbb{R}^3}\big((\nabla u - \nabla\bar{u})\nabla\bar{v} + \nabla u(\nabla v - \nabla\bar{v})\big)\left(\mathbf{v}+\tau\mathbf{w}\right)dx\Big| \\ &\equiv \underbrace{\mathbf{I}_{1,1} + \mathbf{I}_{1,2}}_{:=\mathbf{I}_1} + \underbrace{\mathbf{I}_{2,1} + \mathbf{I}_{2,2}}_{:=\mathbf{I}_2}. \end{aligned}$$

For instance, we have

$$\begin{aligned}\mathbf{I}_{1,1} &= \int_{\mathbb{R}^3}(v-\bar{v})\bar{w}(\mathbf{v}+\tau\mathbf{w})dx\\ &\leq C\|\bar{w}\|_{L^2}\|v-\bar{v}\|_{L^4}\|\mathbf{v}\|_{L^4}+\|v-\bar{v}\|_{L^2}\|\bar{w}\|_{L^4}\|\mathbf{w}\|_{L^4}.\end{aligned}$$

This gives by using (2.23)

$$\|\bar{w}\|_{L^2}\|v-\bar{v}\|_{L^4}\|\mathbf{v}\|_{L^4}\leq C\|\bar{w}\|_{L^2}\|v-\bar{v}\|_{L^2}^{1/4}\|\nabla(v-\bar{v})\|_{L^2}^{3/4}\|\mathbf{v}\|_{L^2}^{1/4}\|\nabla\mathbf{v}\|_{L^2}^{3/4}.$$

Consequently, we have

$$\begin{aligned}\int_0^t\|\bar{w}\|_{L^2}\|v-\bar{v}\|_{L^4}\|\mathbf{v}\|_{L^4}d\sigma &\leq Ct\mathcal{E}_0(\mathbf{V})(t)\mathcal{E}_0(\mathbf{V}-\mathbf{U})(t)\mathcal{E}_0(\mathbf{W})(t)\\ &\leq CT\mathcal{E}_0(\mathbf{V})(t)\mathcal{E}_0(\mathbf{V}-\mathbf{U})(t)\mathcal{E}_0(\mathbf{W})(t).\end{aligned}$$

Similarly, we have

$$\|v-\bar{v}\|_{L^2}\|\bar{w}\|_{L^4}\|\mathbf{w}\|_{L^4}\leq\|v-\bar{v}\|_{L^2}\|\bar{w}\|_{L^2}^{1/4}\|\bar{w}\|_{L^2}^{3/4}\|\mathbf{w}\|_{L^2}^{1/4}\|\nabla\mathbf{w}\|_{L^2}^{3/4}.$$

Hence, we have

$$\int_0^t\|v-\bar{v}\|_{L^2}\|\bar{w}\|_{L^4}\|\mathbf{w}\|_{L^4}d\sigma\leq CT\mathcal{E}_0(\mathbf{V})(t)\mathcal{E}_0(\mathbf{V}-\mathbf{U})(t)\mathcal{E}_0(\mathbf{W})(t).$$

Therefore, we have

$$\int_0^t\mathbf{I}_{1,1}d\sigma\leq CT\mathcal{E}_0(\mathbf{V})(t)\mathcal{E}_0(\mathbf{V}-\mathbf{U})(t)\mathcal{E}_0(\mathbf{W})(t).$$

Similarly, we have

$$\int_0^t\mathbf{I}_{1,2}d\sigma\leq CT\mathcal{E}_0(\mathbf{U})(t)\mathcal{E}_0(\mathbf{V}-\mathbf{U})(t)\mathcal{E}_0(\mathbf{W})(t).$$

Therefore,

$$\int_0^t\mathbf{I}_1d\sigma\leq CT\left(\mathcal{E}_0(\mathbf{U})+\mathcal{E}_0(\mathbf{V})\right)(t)\mathcal{E}_0(\mathbf{V}-\mathbf{U})(t)\mathcal{E}_0(\mathbf{W})(t).$$

The estimate for $\int_0^t\mathbf{I}_2$ can be done similarly. Consequently, we deduce that

$$\mathcal{E}_0^2(\mathbf{W})(t)+\mathcal{D}_0^2(\mathbf{W})(t)\leq CT\Big(\mathcal{E}_0(\mathbf{U})(t)+\mathcal{E}_0(\mathbf{V})(t)\Big)\mathcal{E}_0(\mathbf{W})(t)\mathcal{E}_0(\mathbf{U}-\mathbf{V})(t).$$

Higher order energy estimates can be done as before. We omit the details. □

4. Decay rates – Proof of Theorem 1.3

In this section, we prove decay rates for solutions to (1.1) given in Theorem 1.3. Recall that

$$\mathbf{V} = (v + \tau w, \nabla(u + \tau v), \nabla v)$$

where u is the global solution according to Theorem 1.1, with $v = u_t$ and $w = u_{tt}$. Let

$$\|(u, v, w)\|^2_{\mathcal{H}} = \tau(\beta - \tau)\|\nabla v\|^2_{L^2(\mathbb{R}^3)} + \|\nabla(u + \tau v)\|^2_{L^2(\mathbb{R}^3)} + \|v + \tau w\|^2_{L^2(\mathbb{R}^3)}. \tag{4.1}$$

define a norm in $\mathcal{H} = H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. It is clear that the norm above is equivalent to the norm $\|\mathbf{V}(t)\|^2_{L^2}$.

Inspired by the decay estimates of the linear problem (see Propositions A.4 and A.5), we define

$$\mathcal{M}(t) := \sup_{0\le\sigma\le t} \sum_{j=0}^{s} (1 + \tau)^{3/4+j/2}(\|\nabla^j \mathbf{V}(\sigma)\|_{L^2} + \|\nabla^j v(\sigma)\|_{L^2}).$$

We also define the quantities

$$\begin{aligned} M_0(t) &:= \sup_{0\le\sigma\le t} (1+\sigma)^{\frac{3}{2}} \left(\|\mathbf{V}(\sigma)\|_{L^\infty} + \|v(\sigma)\|_{L^\infty}\right), \\ M_1(t) &:= \sup_{0\le\sigma\le t} (1+\sigma)^{2} \|\nabla \mathbf{V}(\sigma)\|_{L^\infty}, \\ M_2(t) &:= \sup_{0\le\sigma\le t} (1+\sigma)^{\frac{5}{2}} \left\|\nabla^2 \mathbf{V}(\sigma)\right\|_{L^\infty}. \end{aligned}$$

Our goal is to show that $\mathcal{M}(t)$ is bounded uniformly in t if $\|\mathbf{V}_0\|_{H^s\cap L^1} = \|\mathbf{V}_0\|_{H^s} + \|\mathbf{V}_0\|_{L^1}$ is small enough. From (3.2), we have for $\mathbf{U} = (u, v, w)$, and for $0 \le j \le s$,

$$\begin{aligned} \|\nabla^j \mathbf{U}(t)\|_{\mathcal{H}} &\le \|\nabla^j e^{t\mathcal{A}} \mathbf{U}_0\|_{\mathcal{H}} + \int_0^t \left\|\nabla^j e^{(t-r)\mathcal{A}} \mathcal{F}(\mathbf{U}, \nabla \mathbf{U})(r)\right\|_{\mathcal{H}} dr \\ &= \|\nabla^j e^{t\mathcal{A}} \mathbf{U}_0\|_{\mathcal{H}} + \int_0^{t/2} \left\|\nabla^j e^{(t-r)\mathcal{A}} \mathcal{F}(\mathbf{U}, \nabla \mathbf{U})(r)\right\|_{\mathcal{H}} dr \\ &\quad + \int_{t/2}^t \left\|\nabla^j e^{(t-r)\mathcal{A}} \mathcal{F}(\mathbf{U}, \nabla \mathbf{U})(r)\right\|_{\mathcal{H}} dr \\ &\equiv \|\nabla^j e^{t\mathcal{A}} \mathbf{U}_0\|_{\mathcal{H}} + J_1 + J_2. \end{aligned}$$

This gives, by using the estimate (A.5),

$$\|\nabla^j e^{t\mathcal{A}} \mathbf{U}_0\|_{\mathcal{H}} \le C(1+t)^{-3/4-j/2} \left(\|\mathbf{V}_0\|_{L^1(\mathbb{R}^3)} + \|\nabla^j \mathbf{V}_0\|_{L^2(\mathbb{R}^3)}\right).$$

Now, for J_1, we have (also using the estimate (A.5))

$$\begin{aligned} J_1 &\leq C\int_0^{t/2}(1+t-r)^{-3/4-j/2}\|\tilde{\mathcal{F}}(\mathbf{U},\nabla\mathbf{U})(r)\|_{L^1(\mathbb{R}^3)}dr \\ &\quad +C\int_0^{t/2}e^{-c(t-r)}\|\nabla^j\tilde{\mathcal{F}}(\mathbf{U},\nabla\mathbf{U})(r)\|_{L^2(\mathbb{R}^3)}dr \\ &\equiv J_{11}+J_{12}, \end{aligned}$$

where $\tilde{\mathcal{F}}(\mathbf{U},\nabla\mathbf{U})(t)=\Big(\dfrac{B}{A}vw+2\nabla u\nabla v,0,0\Big)$.

To estimate J_1 it is convenient to divide the integral into two parts J_{11} and J_{12} corresponding to $[0,t/2]$ and $[t/2,t]$ and then estimate each term separately, cf. Lemma 7.4 in [30]. First, we have by using Hölder's inequality,

$$\|\tilde{\mathcal{F}}(\mathbf{U},\nabla\mathbf{U})(t)\|_{L^1(\mathbb{R}^3)}\leq\|\mathbf{V}(t)\|_{L^2}^2+\|v(t)\|_{L^2}^2.$$

Hence, J_{11} can be estimated as follow:

$$\begin{aligned} J_{11} &= \int_0^{t/2}(1+t-r)^{-3/4-j/2}\|\tilde{\mathcal{F}}(\mathbf{U},\nabla\mathbf{U})(r)\|_{L^1(\mathbb{R}^3)}dr \\ &\leq C\mathcal{M}^2(t)\int_0^{t/2}(1+t-r)^{-3/4-j/2}(1+r)^{-3/2}dr \\ &\leq C\mathcal{M}^2(t)\int_0^{t/2}(1+t)^{-3/4-j/2}(1+r)^{-3/2}dr \\ &\leq C\mathcal{M}^2(t)(1+t)^{-3/4-j/2}\int_0^{t/2}(1+r)^{-3/2}dr \\ &\leq C\mathcal{M}^2(t)(1+t)^{-3/4-j/2}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|\nabla^j\tilde{\mathcal{F}}(\mathbf{U},\nabla\mathbf{U})(r)\|_{L^2} &\leq C\left(\|\nabla^j(vw)\|_{L^2}+\|\nabla^j(\nabla u\nabla v)\|_{L^2}\right) \\ &\leq C\left(\|\nabla^j(v(v+\tau w))\|_{L^2}+\|\nabla^j(v^2)\|_{L^2}+\|\nabla^j(\nabla u\nabla v)\|_{L^2}\right). \end{aligned}$$

This gives, by applying (A.1),

$$\begin{aligned} &\|\nabla^j\tilde{\mathcal{F}}(\mathbf{U},\nabla\mathbf{U})(t)\|_{L^2} \\ \leq\ & C\|v\|_{L^\infty}\left(\|\nabla^j(v+\tau w)\|_{L^2}+\|\nabla^j v\|_{L^2}\right) \\ &+C\|v+\tau w\|_{L^\infty}\|\nabla^j v\|_{L^2}+C\|\nabla v\|_{L^\infty}\|\nabla^j\nabla v\|_{L^2} \\ &+C\|\nabla v\|_{L^\infty}\|\nabla^j\nabla(u+\tau v)\|_{L^2}+C\|\nabla(u+\tau v)\|_{L^\infty}\|\nabla^j\nabla v\|_{L^2} \\ \leq\ & C(1+t)^{-3/2}(1+t)^{-3/4-j/2}M_0(t)\mathcal{M}(t) \\ &+C(1+t)^{-2}(1+t)^{-3/4-j/2}M_1(t)\mathcal{M}(t). \end{aligned}$$

Consequently, using these estimates, we deduce that

$$J_{12}\leq C(1+t)^{-9/2-j/2}(M_0(t)+M_1(t))\mathcal{M}(t).$$

Next, J_2 is estimated by applying (A.5) with $j=1$ and using $\nabla^{j-1}\tilde{\mathcal{F}}(\mathbf{U},\nabla\mathbf{U})(t)$ instead of $\mathbf{V}_0$, to obtain, for $j\geq 1$,

$$\begin{aligned} J_2 &= \int_{t/2}^{t}\left\|\nabla e^{(t-r)\mathcal{A}}\nabla^{j-1}\tilde{\mathcal{F}}(\mathbf{U},\nabla\mathbf{U})(r)\right\|_{L^2}dr\\ &\leq C\int_{t/2}^{t}(1+t-r)^{-\frac{3}{4}-\frac{1}{2}}\left\|\nabla^{j-1}\tilde{\mathcal{F}}(\mathbf{U},\nabla\mathbf{U})(r)\right\|_{L^1}dr\\ &\quad +C\int_{t/2}^{t}e^{-c(t-r)}\left\|\nabla^{j}\tilde{\mathcal{F}}(\mathbf{U},\nabla\mathbf{U})(r)\right\|_{L^2}dr\\ &\equiv J_{21}+J_{22}. \end{aligned}$$

On the other hand, we have by applying (A.1),

$$\begin{aligned} \|\nabla^{j-1}\tilde{\mathcal{F}}(\mathbf{U},\nabla\mathbf{U})(t)\|_{L^1} &\leq C(\|\mathbf{V}(t)\|_{L^2}+\|v(t)\|_{L^2})(\|\nabla^{j-1}\mathbf{V}(t)\|_{L^2}+\|\nabla^{j-1}v(t)\|_{L^2})\\ &\leq C\mathcal{M}^2(t)(1+t)^{-\frac{3}{2}-\frac{j-1}{2}}. \end{aligned}$$

Thus,

$$\begin{aligned} J_{21} &\leq C\mathcal{M}^2(t)\int_{t/2}^{t}(1+t-r)^{-\frac{3}{4}-\frac{1}{2}}(1+r)^{-\frac{3}{2}-\frac{j-1}{2}}dr\\ &\leq C\mathcal{M}^2(t)(1+t/2)^{-\frac{3}{2}-\frac{j-1}{2}}\int_{t/2}^{t}(1+t-r)^{-\frac{3}{4}-\frac{1}{2}}dr\\ &\leq (1+t/2)^{-\frac{3}{2}-\frac{j-1}{2}}\int_{0}^{t/2}(1+r)^{-\frac{3}{4}-\frac{1}{2}}dr\\ &\leq C(1+t)^{-\frac{3}{2}-\frac{j-1}{2}}(1+t)^{-1/4}+1\\ &\leq C(1+t)^{-\frac{3}{4}-\frac{j}{2}}. \end{aligned} \tag{4.2}$$

For J_{22}, we have as in the estimate of J_{12},

$$J_{22}\leq C(1+t)^{-9/2-j/2}(M_0(t)+M_1(t))\mathcal{M}(t).$$

Therefore, collecting the above estimates, we have

$$\begin{aligned} \|\nabla^j\mathbf{U}(t)\|_{\mathcal{H}} &\leq C(1+t)^{-3/4-j/2}\left(\|\mathbf{V}_0\|_{L^1}+\|\nabla^j\mathbf{V}_0\|_{L^2}\right)\\ &\quad +C\mathcal{M}^2(t)(1+t)^{-3/4-j/2}+C(1+t)^{-3/4-j/2}(M_0(t)+M_1(t))\mathcal{M}(t) \end{aligned} \tag{4.3}$$

This yields

$$\begin{aligned} \mathcal{M}(t) &\leq C\left(\|\mathbf{V}_0\|_{L^1}+\|\nabla^j\mathbf{V}_0\|_{L^2}\right)\\ &\quad +C\mathcal{M}^2(t)+C(M_0(t)+M_1(t))\mathcal{M}(t). \end{aligned} \tag{4.4}$$

Applying Lemma A.3 with $\alpha=\frac{3}{2m}$, $q=r=2$, $j=0$ and $p=\infty$, we get for $m>\frac{3}{2}$

$$\|\mathbf{V}\|_{L^\infty}\leq C\,\|\nabla^m\mathbf{V}\|_{L^2}^{\frac{3}{2m}}\,\|\mathbf{V}\|_{L^2}^{1-\frac{3}{2m}},$$

and similar estimates can be used for $\|v\|_{L^\infty}$. This yields

$$M_0(t) \leq C\mathcal{M}(t),$$

provided that $s \geq m > \frac{3}{2}$.

Next, to estimate $M_1(t)$, we apply Lemma A.3 with $\alpha = \frac{5}{2m}$, $q = r = 2$, $j = 1$ and $p = \infty$, we get for $m > \frac{5}{2}$,

$$\|\nabla \mathbf{V}\|_{L^\infty} \leq C \|\nabla^m \mathbf{V}\|_{L^2}^{\frac{5}{2m}} \|\mathbf{V}\|_{L^2}^{1-\frac{5}{2m}}.$$

This leads to

$$M_1(t) \leq C\mathcal{M}(t),$$

provided that $s \geq m > \frac{5}{2}$. Hence, since $M_0(t) + M_1(t) \leq C\mathcal{M}(t)$, then (4.4) implies that

$$\mathcal{M}(t) \leq C \left(\|\mathbf{V}_0\|_{L^1(\mathbb{R}^3)} + \|\nabla^j \mathbf{V}_0\|_{L^2(\mathbb{R}^3)} \right) + C\mathcal{M}^2(t).$$

Consequently, applying Lemma A.2 gives the desired result, provided that $\|\mathbf{V}_0\|_{L^1(\mathbb{R}^3)} + \|\nabla^j \mathbf{V}_0\|_{L^2(\mathbb{R}^3)}$ is small enough for all $0 \leq j \leq s$. This finishes the proof of Theorem 1.3.

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Appendix A. Useful inequalities and linear decay

In the next three lemmas, we recall without proof some important inequalities which are very useful in the proof of our results. The following lemma has been proved for instance in [12, Lemma 4.1].

Lemma A.1. *Let $1 \leq p, q, r \leq \infty$ and $1/p = 1/q + 1/r$. Then, we have*

$$\|\nabla^k(uv)\|_{L^p} \leq C(\|u\|_{L^q}\|\nabla^k v\|_{L^r} + \|v\|_{L^q}\|\nabla^k u\|_{L^r}), \quad k \geq 0, \tag{A.1}$$

and the commutator estimate

$$\begin{aligned} \|[\nabla^k, f]g\|_{L^p} &= \|\nabla^k(fg) - f\nabla^k g\|_{L^p} \\ &\leq C(\|\nabla f\|_{L^q}\|\nabla^{k-1} g\|_{L^r} + \|g\|_{L^q}\|\nabla^k f\|_{L^r}), \quad k \geq 1, \end{aligned} \tag{A.2}$$

for some constant $C > 0$.

The next lemma has been proved in [32, Lemma 3.7].

Lemma A.2. *Let $M = M(t)$ be a non-negative continuous function satisfying the inequality*

$$M(t) \leq c_1 + c_2 M(t)^\kappa,$$

in some interval containing 0*, where* c_1 *and* c_2 *are positive constants and* $\kappa > 1$*. If* $M(0) \le c_1$ *and*

$$c_1 c_2^{1/(\kappa-1)} < (1 - 1/\kappa)\kappa^{-1/(\kappa-1)},$$

then in the same interval

$$M(t) < \frac{c_1}{1 - 1/\kappa}.$$

We will use the Gagliardo–Nirenberg interpolation inequality as follows.

Lemma A.3. *([27]) Let* $1 \le p, q, r \le \infty$*, and let* m *be a positive integer. Then for any integer* j *with* $0 \le j < m$*, we have*

$$\left\|\nabla^j u\right\|_{L^p} \le C \left\|\nabla^m u\right\|_{L^r}^{\alpha} \left\|u\right\|_{L^q}^{1-\alpha} \tag{A.3}$$

where

$$\frac{1}{p} = \frac{j}{n} + \alpha\left(\frac{1}{r} - \frac{m}{n}\right) + \frac{1-\alpha}{q}$$

for α *satisfying* $j/m \le \alpha \le 1$ *and* C *is a positive constant depending only on* n*,* m*,* j*,* q*,* r *and* α*. There are the following exceptional cases:*

1. *If* $j = 0$*,* $rm < n$ *and* $q = \infty$*, then we made the additional assumption that either* $u(x) \to 0$ *as* $|x| \to \infty$ *or* $u \in L^{q'}$ *for some* $0 < q' < \infty$*.*
2. *If* $1 < r < \infty$ *and* $m - j - n/r$ *is a nonnegative integer, then (A.3) holds only for* $j/m \le \alpha < 1$*.*

We also recall the decay estimates of the linearized problem associated to (1.1a)

Proposition A.4. *([28]) Let* u *be the solution of the linear problem*

$$\tau u_{ttt} + u_{tt} - c^2\Delta u - \beta\Delta u_t = 0. \tag{A.4}$$

Assume that $0 < \tau < \beta$*. Let* $\mathbf{V} = (u_t + \tau u_{tt}, \nabla(u + \tau u_t), \nabla u_t)$ *and assume in addition that* $\mathbf{V}_0 \in L^1(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$*. Then, for all* $0 \le j \le s$*, we have*

$$\|\nabla^j \mathbf{V}(t)\|_{L^2} \le C(1+t)^{-n/4-j/2}\|\mathbf{V}_0\|_{L^1} + Ce^{-ct}\|\nabla^j \mathbf{V}_0\|_{L^2}. \tag{A.5}$$

Also, differentiating (A.4) with respect t and following the same steps as in the proof of [28, Theorem 5.5], we have the following result.

Proposition A.5. *Let* $0 < \tau < \beta$ *and let* $v_0, v_1, v_2 \in L^1(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$*. Also, let* $(v_1, v_2) \in L^{1,1}(\mathbb{R}^n)$ *with* $\int_{\mathbb{R}^n} v_i(x)dx = 0$*,* $i = 1, 2$*. Then, for* $0 \le j \le s$*, the following decay estimate holds:*

$$\begin{aligned} \left\|\nabla^j v(t)\right\|_{L^2} &\le C(\|v_0\|_{L^1} + \|v_1\|_{L^{1,1}} + \|v_2\|_{L^{1,1}})(1+t)^{-n/4-j/2} \\ &+ C(\|\nabla^j v_0\|_{L^2} + \|\nabla^j v_1\|_{L^2} + \|\nabla^j v_2\|_{L^2})e^{-ct}. \end{aligned} \tag{A.6}$$

Here $v_1 = w_0$ *and* $v_2 = v_{tt}(t = 0)$*.*

Appendix B. Derivation of the model

The motion of a viscous, heat-conducting fluid can be described by four equations: the conservation of mass (the continuity equation), the conservation of momentum (Newton's second law), conservation of energy (first law of thermodynamics or entropy balance) and an equation of state. Thus, the first three equations: conservation of mass, conservation of momentum and entropy balance in the model of thermo-viscous flow in compressible fluid, for the mass density ϱ, the acoustic particle velocity v and the absolute temperature θ, can be written as

$$\left\{\begin{array}{l} \varrho_t + \nabla\cdot(\varrho v) = 0, \\ \varrho(v_t + (v\cdot\nabla)v) = \nabla\cdot\mathbb{T}, \\ \varrho\theta(\eta_t + (v\cdot\nabla)\eta) = -\nabla\cdot q + \mathbb{T}:\mathbb{D}. \end{array}\right. \tag{B.1}$$

Here, η is the entropy and q is the heat flux vector. Moreover, $\mathbb{D}$ is the deformation tensor given by

$$\mathbb{D} = \frac{1}{2}(\nabla v + (\nabla v)^T),$$

anf $\mathbb{T}$ is the Cauchy–Poisson stress tensor given by

$$\begin{aligned} \mathbb{T} &= (-p + \lambda(\nabla\cdot v))\mathbb{I} + 2\mu\mathbb{D} \\ &= -p\mathbb{I} + 2\mu\Big(\mathbb{D} - \frac{1}{3}(\nabla\cdot v)\mathbb{I}\Big) + \sigma(\nabla\cdot v)\mathbb{I}, \end{aligned}$$

where p is the acoustic pressure $\mathbb{I}$ is the identity matrix, μ is the shear viscosity (the first coefficient of viscosity), $\lambda = \sigma - \frac{2}{3}\mu$, where σ is the second coefficient of viscosity (the bulk viscosity) and the components of $\mathbb{T}:\mathbb{D}$ are $T_{ij}D_{ij}$ where T_{ij} are the components of the matrix $\mathbb{T}$ and D_{ij} are the components of the matrix $\mathbb{D}$. Hence, we can recast (B.1) as

$$\left\{\begin{array}{l} \varrho_t + \nabla\cdot(\varrho v) = 0, \\ \varrho(v_t + (v\cdot\nabla)v) = -\nabla p + (\lambda+\mu)\nabla(\nabla\cdot v) + \mu\Delta u, \\ \varrho\theta(\eta_t + (v\cdot\nabla)\eta) = -\nabla\cdot q + 2\mu\mathbb{D}:\mathbb{D} + \lambda(\nabla\cdot v)^2. \end{array}\right. \tag{B.2}$$

The equation of state (which describes the relationship between the pressure, the density and the entropy) is

$$p = p(\varrho,\eta). \tag{B.3}$$

First, we assume that the deviations of ϱ, p, η and θ from their equilibrium values ϱ_0, p_0, η_0 and θ_0 are small.

By taking the Taylor series expansion of (B.3) around values at rest ϱ_0 and η_0 and ignoring the higher-order terms, we get

$$p(\varrho,\eta) = p(\varrho_0,\eta_0) + \left(\frac{\partial p}{\partial\varrho}(\varrho_0,\eta_0)\right)(\varrho-\varrho_0) + \frac{1}{2}\left(\frac{\partial^2 p}{\partial\varrho^2}(\varrho_0,\eta_0)\right)(\varrho-\varrho_0)^2 + \left(\frac{\partial p}{\partial\eta}(\varrho_0,\eta_0)\right)(\eta-\eta_0).$$

We have

$$p_0 = p(\varrho_0, \eta_0), \quad A := \varrho_0 \frac{\partial p}{\partial \varrho}(\varrho_0, \eta_0) = \varrho_0 c^2, \quad B := \varrho_0^2 \frac{\partial^2 p}{\partial \varrho^2}(\varrho_0, \eta_0), \quad \varrho_0 \frac{\gamma - 1}{\chi} = \frac{\partial p}{\partial \eta}(\varrho_0, \eta_0),$$

and the pressure p is given by

$$p(\varrho, \eta) = p_0 + \varrho_0 c^2 \left[\frac{\varrho - \varrho_0}{\varrho_0} + \frac{B}{2A} \left(\frac{\varrho - \varrho_0}{\varrho_0} \right)^2 + \frac{\gamma - 1}{\chi c^2}(\eta - \eta_0) \right], \tag{B.4}$$

where $\nabla p_0 = 0$. In the above equations, c is the speed of sound, B/A the parameter of nonlinearity, χ the coefficient of volume expansion and $\gamma = c_p/c_v$ is the ratio of specific heat, where c_p and c_v are the specific heat capacities at constant pressure and constant volume. Assuming that the flow is rotation free, that is $\nabla \times v = 0$, by introducing the acoustic velocity potential $v = -\nabla u$, it has been shown in [13, 24] that equation (1.1a) can be derived from the above set of equations by assuming the Cattaneo law of heat conduction

$$\tau q_t + q = -K \nabla \theta, \tag{B.5}$$

where K is the thermal conductivity.

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