

# HYPERBOLIC COMPRESSIBLE NAVIER-STOKES EQUATIONS

YUXI HU AND REINHARD RACKE

**ABSTRACT.** We consider the non-isentropic compressible Navier-Stokes equations with hyperbolic heat conduction and a law for the stress tensor which is modified correspondingly by Maxwell's law. These two relaxations, turning the whole system into a hyperbolic one, are not only treated simultaneously, but are also considered in a version having Galilean invariance. For this more complicated relaxed system, the global well-posedness is proved for small data. Moreover, for vanishing relaxation parameters the solutions are shown to converge to solutions of the classical system.

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## 1. INTRODUCTION

We consider the system of non-isentropic compressible Navier-Stokes equations in  $\mathbb{R} \times [0, \infty)$  in the following hyperbolic form:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \rho u_t + \rho u u_x + p_x = S_x, \\ \rho e_t + \rho u e_x + p u_x + q_x = S u_x, \end{cases} \quad (1.1)$$

with

$$\tau_1(q_t + u q_x) + q + \kappa \theta_x = 0 \quad (1.2)$$

and

$$\tau_2(S_t + u S_x) + S = \mu u_x. \quad (1.3)$$

Here,  $\rho$ ,  $u$ ,  $p$ ,  $S$ ,  $e$ ,  $\theta$  and  $q$  represent fluid density, velocity, pressure, stress tensor, specific internal energy per unit mass, temperature and heat flux, respectively. The equations (1.1) are the consequence of conservation of mass, momentum and energy, respectively.  $\kappa$ ,  $\mu$ ,  $\lambda$  are positive constants as well as the relaxation parameters  $\tau_1$  and  $\tau_2$ .

We investigate the Cauchy problem for the functions

$$(\rho, u, \theta, q, S) : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$$

with initial condition

$$(\rho(x, 0), u(x, 0), \theta(x, 0), q(x, 0), S(x, 0)) = (\rho_0, u_0, \theta_0, q_0, S_0). \quad (1.4)$$

For  $\tau_1 = \tau_2 = 0$  we recover the classical non-isentropic compressible Navier-Stokes equations.

Neglecting the quadratic nonlinear terms  $u q_x$  and  $u S_x$  in (1.2) resp. (1.3), the case  $\tau_2 = 0$ ,  $\tau_1 > 0$  (Cattaneo law) has been studied in  $\mathbf{R}^n$ ,  $n \geq 1$ , in our paper [15]. The case  $\tau_1 = 0$ ,  $\tau_2 > 0$  (Maxwell's law) in  $\mathbf{R}^n$ ,  $n = 2, 3$ , was treated in [16]. In the latter a splitted version

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of the relaxation law involving two relaxation for different parts of the stress tensor, which was first proposed and treated by Yong [37] for the *isentropic* case, was considered. A similar revised Maxwell model was considered by Chakraborty & Sader [1] for a compressible viscoelastic fluid (isentropic case), where one relaxation counts for the shear relaxation time, and the other counts for the compressional relaxation time. The importance of this model for describing high frequency limits is underlined together with the presentation of numerical experiments. The authors conclude that it provides a general formalism with which to characterize the fluid-structure interaction of nanoscale mechanical devices vibrating in simple liquids.

Now, here, we have as main new contributions:

- \* Taking into account the two nonlinear terms in (1.2), (1.3). This is motivated from requiring Galilean invariance of the system as suggested by Christov & Jordan [5].

- \* Discussion the two relaxations – for heat and stress – simultaneously.

- \* Deriving a global well-posedness result for small data as well as the rigorous limit as  $\tau := \tau_1 = \tau_2 \rightarrow 0$ , i.e. the convergence to the classical Navier-Stokes equations, giving convergence rates in terms of powers of the relaxation parameter  $\tau$ .

For  $\tau = 0$ , the relaxed system (1.1)–(1.3) turns into the classical Newtonian compressible Navier-Stokes system. For the latter, because of its physical importance and mathematical challenges, the well-posedness has been widely studied, see [3, 4, 10, 11, 12, 17, 18, 19, 21, 22, 23, 24, 25, 31, 35]. In particular, the local existence and uniqueness of smooth solutions was established by Serrin [31] and Nash [25] for initial data far away from vacuum. Later, Matsumura and Nishida [23] got global smooth solutions for small initial data without vacuum. For large data, Xin [35], Cho and Jin [3] showed that smooth solutions must blow up in finite time if the initial data has a vacuum state.

One should note that it is not obvious that the results which hold for the classical systems also hold for the relaxed system. Indeed, and for example, Hu and Wang [14] showed that, for the one-dimensional isentropic compressible Navier-Stokes system, classical solutions exist globally for arbitrary large initial data, while solutions blow up in finite time for some large initial data for the corresponding relaxed system. A similar qualitative change was observed before for certain thermoelastic systems, where the non-relaxed system is exponentially stable, while the relaxed one is not, see Quintanilla and Racke resp. Fernández Sare and Muñoz Rivera [26, 8] for plates, and Fernández Sare and Racke [9] for Timoshenko beams.

For *incompressible* Navier-Stokes equations the relaxation, *without* nonlinearity in (1.3), has been discussed by Racke & Saal [27, 28] and Schöwe [29, 30] proving global well-posedness for small data and rigorously investigating the singular limit as  $\tau = \tau_2 \rightarrow 0$ .

We assume the internal energy  $e$  and pressure  $p$  have the following form:

$$e = c_v \theta + \frac{\tau_1}{\kappa \theta \rho} q^2 + \frac{\tau_2}{2\mu \rho} S^2$$

and

$$p = R\rho\theta - \frac{\tau_1}{2\kappa\theta} q^2 - \frac{\tau_2}{2\mu} S^2$$

such that they satisfy the thermodynamic equation  $\rho^2 e_\rho = p - \theta p_\theta$ .

The dependence on  $q^2$  term of the internal energy is indicated in paper [6], where they rigorously prove that such constitutive equations are consistent with the second law of thermodynamics if and only if one use the relaxation equation (1.2), see also [2, 7, 33]. Since we also consider a relaxation for the stress tensor  $S$ , it is motivated, naturally, by [6] that the internal energy should also depend on  $S$  in a quadratic form. Indeed, under the above constitutive laws, we have a dissipative entropy for our system (1.1)–(1.3), see Lemma 3.1, which implies the compatibility with the second law of thermodynamics.

The main results are the following. First, we have global existence for small data.

**Theorem 1.1.** *There exists  $\varepsilon > 0$  such that if*

$$\|(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_0)\|_{H^2}^2 < \varepsilon^2, \quad (1.5)$$

*there exists a globally defined solution  $(\rho, u, \theta, q, S)(x, t) \in C^1([0, +\infty) \times \mathbb{R})$  to the initial value problem (1.1)-(1.4) satisfying*

$$\frac{3}{4} \leq \sup_{x,t} (\rho(x, t), \theta(x, t)) \leq \frac{5}{4}$$

*and*

$$\sup_{t \in [0, \infty)} \|(\rho - 1, u, \theta - 1, q, S)\|_{H^2}^2 \leq C \|(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_0)\|_{H^2}^2 \leq C\varepsilon^2, \quad (1.6)$$

*where  $C$  is a constant which is independent of  $\varepsilon$ . Moreover, the solution converges, uniformly in  $x \in \mathbb{R}$  to the constant state  $(1, 0, 1, 0, 0)$  as  $t \rightarrow \infty$ . Namely,*

$$\|(\rho - 1, u, \theta - 1, q, S)\|_{L^\infty} + \|(\rho_x, u_x, \theta_x, q_x, S_x)\|_{L^2} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Second, we have a description of the singular limit  $\tau \rightarrow 0$  where we assume for simplicity  $\tau_1 = \tau_2 =: \tau$ . We also assume the compatibility condition on the initial data,

$$S_0 = \mu(u_0)_x, \quad q_0 = -\kappa(\theta_0)_x.$$

Let  $(\rho^\tau, u^\tau, \theta^\tau, q^\tau, S^\tau)$  be solutions given by Theorem 1.1. Define

$$T_\tau = \sup\{T > 0; (\rho^\tau - 1, u^\tau, \theta^\tau - 1, q^\tau, S^\tau) \in C([0, T], H^2), \rho^\tau > 0, \theta^\tau > 0, \forall (x, t) \in \mathbb{R}^n \times [0, T]\}.$$

Then we have

**Theorem 1.2.** *Let  $(\rho, u, \theta)$  be smooth solution to the classical compressible Navier-Stokes equations with  $(\rho(x, 0), u(x, 0), \theta(x, 0)) = (\rho_0, u_0, \theta_0)$  satisfying  $\inf_{(x,t) \in \mathbb{R} \times [0, T_*]} (\rho(x, t), \theta(x, t)) > 0$  and*

$$\begin{aligned} (\rho - 1) &\in C([0, T_*], H^5) \cap C^1([0, T_*], H^4), \\ (u, \theta - 1) &\in C([0, T_*], H^5) \cap C^1([0, T_*], H^3), \end{aligned}$$

*with  $T_* > 0$  be finite. Then, there exist constants  $\tau_0$  and  $C$  such that for  $\tau \leq \tau_0$ ,*

$$\|(\rho^\tau, u^\tau, \theta^\tau)(t, \cdot) - (\rho, u, \theta)(t, \cdot)\|_{H^2} \leq C\tau, \quad (1.7)$$

*and*

$$\|(q^\tau + \kappa\theta_x, S^\tau - \mu u_x)\|_{H^2} \leq C\tau^{\frac{1}{2}}, \quad (1.8)$$

*for all  $t \in (0, \min(T_*, T_\tau))$ , and the constant  $C$  is independent of  $\tau$ .*

The  $H^5$ -regularity is needed to estimate some terms in the proof of Theorem 1.2, e.g. the terms  $F_i$  there, see below.

The paper is organized as follows. In Section 2 we prove the local well-posedness, the global existence result Theorem 1.1 is proved in Section 3. The singular limit as  $\tau \rightarrow 0$  is subject of Section 4, where Theorem 1.2 is proved.

Finally, we introduce some notation.  $W^{m,p} = W^{m,p}(\mathbb{R}^n)$ ,  $0 \leq m \leq \infty$ ,  $1 \leq p \leq \infty$ , denotes the usual Sobolev space with norm  $\|\cdot\|_{W^{m,p}}$ ,  $H^m$  and  $L^p$  stand for  $W^{m,2}(\Omega)$  resp.  $W^{0,p}(\Omega)$ .

## 2. LOCAL EXISTENCE THEOREM

In this part, we establish the local existence theorem for system (1.1)–(1.4). We rewrite the system (1.1) as follows:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \rho u_t + \rho u u_x + p_\rho \rho_x + p_\theta \theta_x + p_q q_x + (p_S - 1)S_x = 0, \\ \rho e_\theta \theta_t + (\rho u e_\theta - \frac{2q}{\theta})\theta_x + \theta p_\theta u_x + q_x = \frac{2q^2}{\kappa\theta} + \frac{S^2}{\mu}, \\ \tau_1(q_t + u q_x) + q + \kappa\theta_x = 0, \\ \tau_2(S_t + u S_x) + S = \mu u_x. \end{cases} \quad (2.1)$$

For the derivation of equation (2.1)<sub>3</sub>, we calculate

$$\begin{aligned} \rho e_t &= \rho e_\theta \theta_t + \rho e_\rho \rho_t + \rho e_q q_t + \rho e_S S_t \\ &= \rho e_\theta \theta_t + \rho e_\rho \rho_t + \rho \frac{2\tau_1}{\kappa\theta\rho} q q_t + \rho \frac{\tau_2}{\mu\rho} S S_t \\ &= \rho e_\theta \theta_t + \rho e_\rho \rho_t + \frac{2}{\kappa\theta} q (-\tau_1 u q_x - q - \kappa\theta_x) + \frac{S}{\mu} (-\tau_2 u S_x - S + \mu u_x) \\ &= \rho e_\theta \theta_t - \frac{2q}{\theta} \theta_x + \rho e_\rho \rho_t - \frac{2\tau_1}{\kappa\theta} u q q_x - \frac{2}{\kappa\theta} q^2 - \frac{\tau_2}{\mu} u S S_x - \frac{S^2}{\mu} + S u_x, \end{aligned}$$

while

$$\rho u e_x = \rho u (e_\theta \theta_x + e_\rho \rho_x + e_q q_x + e_S S_x) = \rho u e_\theta \theta_x + \rho u e_\rho \rho_x + \rho u \frac{2\tau_1}{\kappa\theta\rho} q q_x + \rho u \frac{\tau_2 S}{\mu\rho} S_x.$$

So, combining the above equalities, we derive

$$\rho e_t + \rho u e_x = \rho e_\theta \theta_t + (\rho u e_\theta - \frac{2q}{\theta})\theta_x + \rho e_\rho (\rho_t + u \rho_x) - \frac{2}{\kappa\theta} q^2 - \frac{S^2}{\mu} + S u_x.$$

On the other hand, by the thermodynamic equation, we have  $\rho u_x = (\theta p_\theta + \rho^2 e_\rho) u_x = \theta p_\theta u_x + \rho^2 e_\rho u_x$ . Combining these calculations and using the mass equation (1.1)<sub>1</sub>, we derive equation (2.1)<sub>3</sub>.

Note that the system (2.1) is non-symmetric. In order to give a local existence theorem, we will require  $p_q$  to be small enough such that the system is, for small initial data, a strictly hyperbolic system.

**Lemma 2.1.** *There exists  $\delta$  such that if  $|(\rho - 1, \theta - 1, q, S)| < \delta$ , then the system (2.1) is strictly hyperbolic.*

*Proof.* First, we choose a  $\delta_1$  small enough such that  $|(\rho - 1, \theta - 1, q, S)| < \delta_1$  implying

$$0 < \underline{\rho} < \rho < \bar{\rho}, 0 < \underline{\theta} < \theta < \bar{\theta}, \quad (2.2)$$

$$0 < \underline{p}_\rho < p_\rho < \bar{p}_\rho, 0 < \underline{p}_\theta < p_\theta < \bar{p}_\theta, 0 < \underline{e}_\theta < e_\theta < \bar{e}_\theta, |p_S| < \frac{1}{2}. \quad (2.3)$$

Now, we transform the system (2.1) into a first-order system for  $V := (\rho, u, \theta, q, S)'$ ,

$$V_t + A(V)\partial_x V + B(V)V = F(V), \quad (2.4)$$

where

$$A(V) = \begin{pmatrix} u & \rho & 0 & 0 & 0 \\ \frac{p_\rho}{\rho} & u & \frac{p_\theta}{\rho} & \frac{p_q}{\rho} & \frac{p_S - 1}{\rho} \\ 0 & \frac{\theta p_\theta}{\rho e_\theta} & u - \frac{2q}{\rho\theta e_\theta} & 1 & 0 \\ 0 & 0 & \frac{\kappa}{\tau_1} & u & 0 \\ 0 & -\frac{\mu}{\tau_2} & 0 & 0 & u \end{pmatrix}, B(V) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\tau_1} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\tau_2} \end{pmatrix} \quad (2.5)$$

and  $F(V) := \left(0, 0, \frac{2q^2}{\kappa\theta} + \frac{S^2}{\mu}, 0, 0\right)'$ . We need to show the eigenvalues of matrix  $A(V)$  are real and distinct. The characteristic polynomial for  $A(V)$  is

$$\det(A(V) - \lambda) = (u - \lambda)g(u - \lambda), \quad (2.6)$$

where

$$\begin{aligned} g(z) := & z^4 - \frac{2q}{\rho\theta e_\theta} z^3 - \left( \frac{\kappa}{\tau_1} + \frac{\theta p_\theta^2}{\rho^2 e_\theta} + \frac{\mu(1-p_S)}{\rho\tau_2} + p_\rho \right) z^2 \\ & + \left( \frac{\kappa\theta p_\theta p_q}{\tau_1 \rho^2 e_\theta} + \left( \frac{\mu(1-p_S)}{\rho\tau_2} + p_\rho \right) \cdot \frac{2q}{\rho\theta e_\theta} \right) z + \left( \frac{\mu(1-p_S)}{\rho\tau_2} + p_\rho \right) \cdot \frac{\kappa}{\tau_1}. \end{aligned} \quad (2.7)$$

Note that  $g(\pm\infty) = +\infty$  and  $g(0) = \left( \frac{\mu(1-p_S)}{\rho\tau_2} + p_\rho \right) \cdot \frac{\kappa}{\tau_1} > 0$ .

Let

$$\mu_\pm := \pm \sqrt{\frac{\mu(1-p_S)}{\rho\tau_2} + p_\rho}, \quad (2.8)$$

then

$$\mu_- < 0 < \mu_+ \quad (2.9)$$

and

$$g(\mu_\pm) = \mu_- \left( \frac{\theta p_\theta^2}{\rho^2 e_\theta} \mu_+ \mp \frac{\kappa\theta p_\theta}{\tau_1 \rho^2 e_\theta} p_q \right) \equiv \mu_- Q, \quad (2.10)$$

which implies

$$Q \geq \frac{\theta p_\theta^2 \mu_+}{2\rho^2 e_\theta} > 0 \quad (2.11)$$

if

$$|p_q| \leq \frac{\tau_1 p_\theta}{\kappa} \mu_+, \quad (2.12)$$

which is satisfied if  $|(v-1, \theta-1, q, S)| < \delta_2$  for some  $\delta_2 > 0$ . Therefore, there exists a  $\delta_2 > 0$  such that for  $|(\rho-1, \theta-1, q, S)| < \delta_2$ , we derive

$$g(\mu_\pm) < 0. \quad (2.13)$$

Hence,  $g$  has 4 different real zeros  $z_1 < z_2 < 0 < z_3 < z_4$ . Altogether we conclude that there exists a  $\delta = \min\{\delta_1, \delta_2\} > 0$  such that if  $|(\rho-1, \theta-1, q, S)|_{L^\infty} < \delta$ , the matrix  $A(V)$  has 5 different eigenvalues  $\lambda_0 := u$ ,  $\lambda_k := u - z_k$ ,  $k = 1, 2, 3, 4$ . Thus the system is strictly hyperbolic.  $\square$

The strict hyperbolicity of (2.1) now implies the local well-posedness, see e.g. [34], it also implies that (2.1) is symmetrizable. Thus, we get

**Theorem 2.2.** *Let  $s \geq 2$ . Then there is  $\delta > 0$  such that for  $(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_0) \in W^{s,2}(\mathbb{R})$  with  $\|(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_0)\|_{s,2} < \delta$ , there exists a unique local solution  $(\rho, u, \theta, q, S)$  to (2.1), (1.4) in some time interval  $[0, T]$  with*

$$(\rho - 1, u, \theta - 1, q, S) \in C^0([0, T], H^s(\mathbb{R})) \cap C^1([0, T], H^{s-1}(\mathbb{R})). \quad (2.14)$$

## 3. GLOBAL WELL-POSEDNESS: PROOF OF THEOREM 1.1

In this part, we establish the a priori estimates and prove the global existence of strong solutions by usual continuation methods. First, we introduce the following energy functional

$$\begin{aligned} E(t) := & \sup_{0 \leq s \leq t} \|(\rho - 1, u, \theta - 1, q, S)(s, \cdot)\|_{H^2}^2 + \sup_{0 \leq s \leq t} \|(\rho_t, u_t, \theta_t, q_t, S_t)\|_{H^1}^2 \\ & + \int_0^t \|(\rho_x, \rho_t, u_x, u_t, \theta_x, \theta_t, q_x, q_t, q, S_x, S_t, S)(s, \cdot)\|_{H^1}^2 ds. \end{aligned} \quad (3.1)$$

The following lemma gives the lower energy estimates.

**Lemma 3.1.** *We have the following equality*

$$\begin{aligned} & \left[ c_v \rho(\theta - \ln \theta - 1) + R(\rho \ln \rho - \rho + 1) + (1 - \frac{1}{2\theta}) \frac{\tau_1}{\kappa \theta} q^2 + \frac{1}{2} \rho u^2 + \frac{\tau_2}{2\mu} S^2 \right]_t \\ & + [\rho u c_v(\theta - \ln \theta - 1) + u(1 - \frac{1}{2\theta}) \frac{\tau_1}{\kappa \theta} q^2 + \frac{\tau_2}{2\mu} u S^2 + R \rho u \ln \rho - R \rho u - \frac{q}{\theta} + \frac{1}{2} \rho u^3 + pu + q - Su]_x \\ & + \frac{q^2}{\kappa \theta^2} + \frac{S^2}{\theta \mu} = 0. \end{aligned} \quad (3.2)$$

Moreover, if  $E(t) \leq \frac{1}{16}$ , we get the following  $L^2$ -energy estimate with a positive constant  $C > 0$ ,

$$\int_{\mathbb{R}} ((\rho - 1)^2 + u^2 + (\theta - 1)^2 + q^2 + S^2) dx + \int_0^t \int_{\mathbb{R}} (q^2 + S^2) dx dt \leq CE(0). \quad (3.3)$$

**Remark 3.1.** *When  $\tau_1 = \tau_2 = 0$ , the equality (3.2) reduce to the energy inequality for classical compressible Navier-Stokes equations, see [20]. In this regards, (3.2) can be regarded as an extension to the relaxed system (1.1)-(1.3). Moreover, if we denote, for  $U = (\rho, u, \theta, q, S)$ ,*

$$\eta(U) = c_v \rho(\theta - \ln \theta - 1) + R(\rho \ln \rho - \rho + 1) + (1 - \frac{1}{2\theta}) \frac{\tau_1}{\kappa \theta} q^2 + \frac{1}{2} \rho u^2 + \frac{\tau_2}{2\mu} S^2, \quad (3.4)$$

and

$$q(U) = \rho u c_v(\theta - \ln \theta - 1) + u(1 - \frac{1}{2\theta}) \frac{\tau_1}{\kappa \theta} q^2 + \frac{\tau_2}{2\mu} u S^2 + R \rho u \ln \rho - R \rho u - \frac{q}{\theta} + \frac{1}{2} \rho u^3 + pu + q - Su, \quad (3.5)$$

we have  $\eta(U)_t + q(U)_x \leq 0$  by (3.2). So,  $(\eta(U), q(U))$  can be regards as a convex entropy pair for system (1.1)-(1.3). Then Lemma 3.1 imply that there exists a dissipative entropy which is crucial to get the global existence of solutions, see [13, 36].

*Proof.* First, we have the energy equality:

$$(\rho e + \frac{1}{2} \rho u^2)_t + (\rho u e + \frac{1}{2} \rho u^3)_x + (pu)_x + q_x = (Su)_x \quad (3.6)$$

which gives

$$\int_{\mathbb{R}} \left( \rho e + \frac{1}{2} \rho u^2 \right) dx = \int_{\mathbb{R}} \left( \rho_0 e_0 + \frac{1}{2} \rho_0 u_0^2 \right) dx \quad (3.7)$$

or equivalently

$$\int_{\mathbb{R}} \left( c_v \rho \theta + \frac{\tau_1}{\kappa \theta} q^2 + \frac{\tau_2}{2\mu} S^2 + \frac{1}{2} \rho u^2 \right) dx = \int_{\mathbb{R}} \left( c_v \rho_0 \theta_0 + \frac{\tau_1}{\kappa \theta_0} q_0^2 + \frac{\tau_2}{2\mu} S_0^2 + \frac{1}{2} \rho_0 u_0^2 \right) dx. \quad (3.8)$$

Next, we multiply the equation (1.1)<sub>3</sub> by  $\frac{1}{\theta}$ , we get

$$\begin{aligned} \frac{\rho}{\theta} (c_v \theta + \frac{\tau_1}{\kappa \theta \rho} q^2 + \frac{\tau_2}{2\mu\rho} S^2)_t + \frac{\rho u}{\theta} (c_v \theta + \frac{\tau_1}{\kappa \theta \rho} q^2 + \frac{\tau_2}{2\mu\rho} S^2)_x \\ + R\rho u_x - \frac{\tau_1}{2\kappa\theta^2} q^2 u_x - \frac{\tau_2}{2\mu\theta} S^2 u_x + \frac{q_x}{\theta} = \frac{S u_x}{\theta}. \end{aligned} \quad (3.9)$$

For the term  $\frac{\rho}{\theta} (\frac{\tau_1}{\kappa\theta\rho} q^2)_t$ , we have

$$\begin{aligned} \frac{\rho}{\theta} (\frac{\tau_1}{\kappa\theta\rho} q^2)_t &= \rho (\frac{\tau_1}{\kappa\rho\theta^2} q^2)_t + \frac{\tau_1 \theta_t}{\kappa\theta^3} q^2 \\ &= \rho (\frac{\tau_1}{\kappa\rho\theta^2} q^2)_t - \frac{1}{2} (\frac{1}{\theta^2})_t \frac{\tau_1}{\kappa} q^2 \\ &= \rho (\frac{\tau_1}{\kappa\rho\theta^2} q^2)_t - \frac{1}{2} (\frac{\tau_1}{\kappa\theta^2} q^2)_t + \frac{\tau_1 q q_t}{\kappa\theta^2} \\ &= \rho (\frac{\tau_1}{\kappa\rho\theta^2} q^2)_t - \frac{1}{2} (\frac{\tau_1}{\kappa\theta^2} q^2)_t - u \frac{\tau_1 q q_x}{\kappa\theta^2} - \frac{q^2}{\kappa\theta^2} - q \frac{\theta_x}{\theta^2}. \end{aligned} \quad (3.10)$$

For the term  $\frac{\rho u}{\theta} (\frac{\tau_1}{\kappa\theta\rho} q^2)_x$ , we get

$$\begin{aligned} \frac{\rho u}{\theta} (\frac{\tau_1}{\kappa\theta\rho} q^2)_x &= \rho u (\frac{\tau_1}{\kappa\rho\theta^2} q^2)_x + u \frac{\theta_x}{\theta^3} \frac{\tau_1}{\kappa} q^2 \\ &= \rho u (\frac{\tau_1}{\kappa\rho\theta^2} q^2)_x - u (\frac{1}{2} \frac{\tau_1}{\kappa\theta^2} q^2)_x + u \frac{\tau_1}{\kappa\theta^2} q q_x. \end{aligned} \quad (3.11)$$

Then, we have

$$\begin{aligned} \frac{\rho}{\theta} (\frac{\tau_1}{\kappa\theta\rho} q^2)_t + \frac{\rho u}{\theta} (\frac{\tau_1}{\kappa\rho} q^2)_x - \frac{\tau_1}{2\kappa\theta^2} q^2 u_x \\ = \left[ \rho (\frac{\tau_1}{\kappa\rho\theta^2} q^2)_t + \rho u (\frac{\tau_1}{\kappa\rho\theta^2} q^2)_x \right] - \left[ \frac{1}{2} (\frac{\tau_1}{\kappa\theta^2} q^2)_t + (u \frac{1}{2} \frac{\tau_1}{\kappa\theta^2} q^2)_x \right] - \frac{q^2}{\kappa\theta^2} - q \frac{\theta_x}{\theta^2} \\ = \left[ \frac{1}{2} (\frac{\tau_1}{\kappa\theta^2} q^2)_t + (u \frac{1}{2} \frac{\tau_1}{\kappa\theta^2} q^2)_x \right] - \frac{q^2}{\kappa\theta^2} - q \frac{\theta_x}{\theta^2}, \end{aligned} \quad (3.12)$$

where we used, exploiting (1.1)<sub>1</sub>, the following fact

$$\rho(f)_t + \rho u(f)_x = (\rho f)_t + (\rho u f)_x$$

for any function  $f$ . On the other hand, we have

$$\begin{aligned} \frac{\rho}{\theta} \left( \frac{\tau_2}{2\mu\rho} S^2 \right)_t + \frac{\rho u}{\theta} \left( \frac{\tau_2}{2\mu\rho} S^2 \right)_x - \frac{\tau_2}{2\mu\theta} S^2 u_x \\ = \frac{\tau_2}{\theta\mu} S S_t - \frac{\tau_2 S^2}{2\mu\rho\theta} \rho_t + \frac{\tau_2}{\theta\mu} u S S_x - \frac{\tau_2 S^2}{2\mu\rho\theta} u \rho_x - \frac{\tau_2}{2\mu\theta} S^2 u_x \\ = \frac{S}{\theta\mu} (\tau_2 (S_t + u S_x)) - \frac{\tau_2 S^2}{2\mu\rho\theta} (\rho_t + u \rho_x) - \frac{\tau_2}{2\mu\theta} S^2 u_x \\ = \frac{S}{\theta\mu} (-S + \mu u_x) + \frac{\tau_2 S^2}{2\mu\rho\theta} \rho u_x - \frac{\tau_2}{2\mu\theta} S^2 u_x \\ = -\frac{S^2}{\theta\mu} + \frac{S}{\theta} u_x. \end{aligned} \quad (3.13)$$

Therefore, we derive the following equality

$$(\rho c_v \ln \theta + \frac{\tau_1}{2\kappa\theta^2} q^2)_t + (\rho u c_v \ln \theta + u \frac{\tau_1}{2\kappa\theta^2} q^2)_x + R\rho u_x + (\frac{q}{\theta})_x - \frac{q^2}{\kappa\theta^2} - \frac{S^2}{\theta\mu} = 0. \quad (3.14)$$

Now, we rewrite the equation (1.1)<sub>1</sub> as

$$R\rho(\ln \rho)_t + R\rho u(\ln \rho)_x + R\rho u_x = 0. \quad (3.15)$$

Combining the equations (3.14), (3.15) and (3.6), using the equation (1.1)<sub>1</sub>, we get

$$\begin{aligned} & \left[ c_v \rho(\theta - \ln \theta - 1) + R(\rho \ln \rho - \rho + 1) + \left(1 - \frac{1}{2\theta}\right) \frac{\tau_1}{\kappa\theta} q^2 + \frac{1}{2} \rho u^2 + \frac{\tau_2}{2\mu} S^2 \right]_t \\ & + [\rho u c_v(\theta - \ln \theta - 1) + u \left(1 - \frac{1}{2\theta}\right) \frac{\tau_1}{\kappa\theta} q^2 + \frac{\tau_2}{2\mu} u S^2 + R\rho u \ln \rho - R\rho u - \frac{q}{\theta} + \frac{1}{2} \rho u^3 + pu + q - Su]_x \\ & + \frac{q^2}{\kappa\theta^2} + \frac{S^2}{\theta\mu} = 0. \end{aligned}$$

Note that if  $E(t) \leq \frac{1}{16}$ , using Sobolev imbedding theorem, we have

$$\frac{3}{4} \leq \rho \leq \frac{5}{4}, \quad \frac{3}{4} \leq \theta \leq \frac{5}{4}. \quad (3.16)$$

Moreover, using Taylor expansions, we get

$$\theta - \ln \theta - 1 = \frac{1}{2\xi^2}(\theta - 1)^2, \quad (3.17)$$

$$\rho \ln \rho - \rho + 1 = \frac{1}{2\eta}(\rho - 1)^2 \quad (3.18)$$

where  $\xi \in (1, \theta), \eta \in (1, \rho)$ . Combining the above inequalities, we get the desired result.  $\square$

The next two lemmas give the first-order estimates of solutions.

**Lemma 3.2.** *Suppose that  $E(t) \leq \delta$  for  $0 \leq t \leq T$  and some sufficiently small  $\delta > 0$ , then we have*

$$\begin{aligned} & \int_{\mathbb{R}} (\rho_x^2 + \rho_t^2 + u_x^2 + u_t^2 + \theta_x^2 + \theta_t^2 + q_x^2 + q_t^2 + S_x^2 + S_t^2) dx(t) \\ & + \int_0^t \int_{\mathbb{R}} (q_x^2 + q_t^2 + S_x^2 + S_t^2) dx dt \leq C(E(0) + E(t)^{\frac{3}{2}}) \end{aligned} \quad (3.19)$$

with a positive constant  $C$ .

*Proof.* Taking derivatives of (1.1) with respect to  $x$ , we get

$$\left\{ \begin{array}{l} \rho_{tx} + u\rho_{xx} + \rho u_{xx} = -2u_x \rho_x =: f_1, \\ \rho u_{tx} + \rho u u_{xx} + p\rho\rho_{xx} + p\theta\theta_{xx} + pqq_{xx} - (1 - pS)S_{xx} \\ = -\rho_x u_t - (\rho u)_x u_x - (p\rho)_x \rho_x - (p\theta)_x \theta_x - (pq)_x q_x - (pS)_x S_x =: f_2, \\ \rho e_\theta \theta_{tx} + \left(\rho u e_\theta - \frac{2q}{\theta}\right) \theta_{xx} + \theta p_\theta u_{xx} + q_{xx} \\ = -(\rho e_\theta)_x \theta_t - \left(\rho u e_\theta - \frac{2q}{\theta}\right)_x \theta_x - (\theta p_\theta)_x u_x + \left(\frac{2q^2}{\kappa\theta}\right)_x + \left(\frac{S^2}{\mu}\right)_x =: f_3, \\ \tau_1(q_{tx} + uq_{xx}) + q_x + \kappa\theta_{xx} = -\tau_1 u_x q_x =: f_4, \\ \tau_2(S_{tx} + uS_{xx}) + S_x - \mu u_{xx} = -\tau_2 u_x S_x =: f_5. \end{array} \right. \quad (3.20)$$

Note that for  $1 \leq j \leq 5$ , there exists a constant  $C$  such that

$$|f_j| \leq C|(\rho_x, u_x, u_t, \theta_x, \theta_t, q_x, S_x, q, S)|^2. \quad (3.21)$$



Multiplying the equations in (3.20) by  $\frac{p_\rho}{\rho}\rho_x$ ,  $u_x$ ,  $\frac{1}{\theta}\theta_x$ ,  $\frac{1}{\kappa\theta}q_x$ ,  $\frac{1-pS}{\mu}S_x$ , respectively, and integrate the results over  $\mathbb{R}$ , we get

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}} \left( \frac{p_\rho}{2\rho} \rho_x^2 + \frac{\rho}{2} u_x^2 + \frac{\rho e_\theta}{2\theta} \theta_x^2 + \frac{\tau_1}{2\kappa\theta} q_x^2 + \frac{\tau_2(1-pS)}{2\mu} S_x^2 \right) dx \\
& + \int_{\mathbb{R}} \left( \frac{1}{\kappa\theta} q_x^2 + \frac{1-pS}{\mu} S_x^2 \right) dx + \int_{\mathbb{R}} p_q q_{xx} u_x dx \\
& = \int_{\mathbb{R}} \left( \frac{p_\rho}{\rho} \rho_x f_1 + u_x f_2 + \frac{\theta_x}{\theta} f_3 + \frac{1}{\kappa\theta} q_x f_4 + \frac{1-pS}{\mu} S_x f_5 \right) dx \\
& + \int_{\mathbb{R}} \left( \frac{1}{2} \rho_x^2 \left( \frac{p_\rho}{\rho} \right)_t + \frac{1}{2} u_x^2 \rho_t + \frac{1}{2} \theta_x^2 \left( \frac{\rho e_\theta}{\theta} \right)_t + \frac{1}{2} q_x^2 \left( \frac{\tau_1}{\kappa\theta} \right)_t + \frac{1}{2} S_x^2 \left( \frac{\tau_2(1-pS)}{\mu} \right)_t \right) dx \\
& + \int_{\mathbb{R}} \left( \left( \frac{p_\rho}{\rho} u \right)_x \frac{1}{2} \rho_x^2 + (\rho u)_x \frac{1}{2} u_x^2 + \left( \frac{\rho u e_\theta}{\theta} - \frac{2q}{\theta^2} \right)_x \frac{1}{2} \theta_x^2 + \left( \frac{u}{\kappa\theta} \right)_x \frac{\tau_1}{2} q_x^2 + \left( \frac{1-pS}{\mu} u \right)_x \frac{\tau_2}{2} S_x^2 \right) dx \\
& + \int_{\mathbb{R}} \left( (p_\rho)_x \rho_x u_x + (p_\theta)_x \theta_x u_x + \left( \frac{1}{\theta} \right)_x q_x \theta_x - (1-P_S)_x S_x u_x \right). \tag{3.22}
\end{aligned}$$

The integrals on the right-hand side of the above equation can be estimated by  $E(t)^{\frac{3}{2}}$  by using the Sobolev embedding inequality

$$\|(D\rho, Du, D\theta, DS, Dq)\|_{L^\infty} \leq CE(t)^{\frac{1}{2}}, \tag{3.23}$$

where  $D = (\partial_t, \partial_x)$ .

Now, we deal with one of the highest-order terms,  $\int_{\mathbb{R}} p_q q_{xx} u_x dx$ , in the following way.

$$\begin{aligned}
& \int_{\mathbb{R}} p_q q_{xx} u_x dx = - \int_{\mathbb{R}} (p_q)_x q_x u_x dx - \int_{\mathbb{R}} p_q q_x u_{xx} dx \\
& = - \int_{\mathbb{R}} (p_q)_x q_x u_x dx - \int_{\mathbb{R}} p_q q_x \left( - \frac{\rho e_\theta}{\theta p_\theta} \theta_{tx} - \frac{\rho u e_\theta - \frac{2q}{\theta}}{\theta p_\theta} \theta_{xx} - \frac{1}{\theta p_\theta} q_{xx} + \frac{1}{\theta p_\theta} f_3 \right) dx \\
& = - \int_{\mathbb{R}} (p_q)_x q_x u_x dx - \int_{\mathbb{R}} \frac{p_q}{\theta p_\theta} q_x f_3 dx - \int_{\mathbb{R}} \left( \frac{p_q}{\theta p_\theta} \right)_x \frac{1}{2} q_x^2 dx \\
& + \int_{\mathbb{R}} \frac{p_q \rho e_\theta}{\theta p_\theta} q_x \theta_{tx} dx + \int_{\mathbb{R}} \frac{p_q (\rho u e_\theta - \frac{2q}{\theta})}{\theta p_\theta} q_x \theta_{xx} dx, \tag{3.24}
\end{aligned}$$

where we have used the temperature equation (2.1)<sub>3</sub>, while, using the equation (2.1)<sub>4</sub> for  $q$ , we get for the second highest-order term

$$\begin{aligned}
& \int_{\mathbb{R}} \frac{p_q \rho e_\theta}{\theta p_\theta} q_x \theta_{tx} dx \\
& = \frac{d}{dt} \int_{\mathbb{R}} \frac{p_q \rho e_\theta}{\theta p_\theta} q_x \theta_x dx - \int_{\mathbb{R}} \left( \frac{p_q \rho e_\theta}{\theta p_\theta} \right)_t q_x \theta_x dx - \int_{\mathbb{R}} \frac{p_q \rho e_\theta}{\theta p_\theta} q_{tx} \theta_x dx \\
& = \frac{d}{dt} \int_{\mathbb{R}} \frac{p_q \rho e_\theta}{\theta p_\theta} q_x \theta_x dx - \int_{\mathbb{R}} \left( \frac{p_q \rho e_\theta}{\theta p_\theta} \right)_t q_x \theta_x dx \\
& - \int_{\mathbb{R}} \frac{p_q \rho e_\theta}{\theta p_\theta} \left( \frac{1}{\tau_1} f_4 - \frac{1}{\tau_1} q_x - \frac{\kappa}{\tau_1} \theta_{xx} - u_x q_x - u q_{xx} \right) \theta_x dx \\
& = \frac{d}{dt} \int_{\mathbb{R}} \frac{p_q \rho e_\theta}{\theta p_\theta} q_x \theta_x dx - \int_{\mathbb{R}} \frac{p_q \rho e_\theta}{\theta p_\theta} u q_x \theta_{xx} - \int_{\mathbb{R}} \left( \frac{p_q \rho e_\theta}{\theta p_\theta} \right)_t q_x \theta_x dx \\
& - \int_{\mathbb{R}} \left( \frac{p_q \rho e_\theta}{\theta p_\theta} \left( \frac{1}{\tau_1} f_4 - \frac{1}{\tau_1} q_x \right) \theta_x + \left( \frac{p_q \rho e_\theta}{\theta p_\theta} u \right)_x q_x \theta_x - \left( \frac{\kappa p_q \rho e_\theta}{\tau_1 \theta p_\theta} \right)_x \frac{1}{2} \theta_x^2 \right) dx. \tag{3.25}
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned} & \int_{\mathbb{R}} \frac{p_q \rho e_\theta}{\theta p_\theta} q_x \theta_{tx} dx + \frac{p_q (\rho u e_\theta - \frac{2q}{\theta})}{\theta p_\theta} q_x \theta_{xx} dx = \frac{d}{dt} \int_{\mathbb{R}} \frac{p_q \rho e_\theta}{\theta p_\theta} q_x \theta_x dx - \int_{\mathbb{R}} \frac{2qp_q}{\theta^2 p_\theta} q_x \theta_{xx} \\ & - \int_{\mathbb{R}} \left( \frac{p_q \rho e_\theta}{\theta p_\theta} \left( \frac{1}{\tau_1} f_4 - \frac{1}{\tau_1} q_x \right) \theta_x + \left( \frac{p_q \rho e_\theta}{\theta p_\theta} u \right)_x q_x \theta_x - \left( \frac{\kappa p_q \rho e_\theta}{\tau_1 \theta p_\theta} \right)_x \frac{1}{2} \theta_x^2 + \left( \frac{p_q \rho e_\theta}{\theta p_\theta} \right)_t q_x \theta_x \right) dx. \end{aligned} \quad (3.26)$$

For the term  $\int_{\mathbb{R}} \frac{2qp_q}{\theta^2 p_\theta} q_x \theta_{xx} dx$ , using the equation for  $q$ , we have

$$\begin{aligned} & \int_{\mathbb{R}} \frac{2qp_q}{\theta^2 p_\theta} q_x \theta_{xx} dx = \int_{\mathbb{R}} \frac{2qp_q}{\theta^2 p_\theta} q_x \left( \frac{-\tau_1}{\kappa} (q_{tx} + u q_{xx} + u_x q_x) - \frac{1}{\kappa} q_x \right) dx \\ & = -\frac{d}{dt} \int_{\mathbb{R}} \frac{\tau_1 qp_q}{\kappa \theta^2 p_\theta} q_x^2 dx + \int_{\mathbb{R}} \left( \frac{\tau_1 qp_q}{\kappa \theta^2 p_\theta} \right)_t q_x^2 dx \\ & + \int_{\mathbb{R}} \left( \frac{2qp_q \tau_1}{\theta^2 p_\theta \kappa} u \right)_x \frac{1}{2} q_x^2 - \int_{\mathbb{R}} \frac{2qp_q}{\theta^2 p_\theta} q_x^2 \frac{\tau_1}{\kappa} u_x dx - \int_{\mathbb{R}} \frac{2qp_q}{\kappa \theta^2 p_\theta} q_x^2 dx. \end{aligned} \quad (3.27)$$

Finally, we derive, integrating (3.22), that

$$\begin{aligned} & \int_{\mathbb{R}} \left( \frac{p_\rho}{2\theta} \rho_x^2 + \frac{1}{2} \rho u_x^2 + \frac{\rho e_\theta}{2\theta} \theta_x^2 + \frac{\tau_1}{2\kappa\theta} q_x^2 + \frac{\tau_1(1-p_S)}{2\mu} S_x^2 + \frac{p_q \rho e_\theta}{\theta p_\theta} q_x \theta_x + \frac{\tau_1 qp_q}{\kappa \theta^2 p_\theta} q_x^2 \right) dx \\ & + \int_0^t \int_{\mathbb{R}} \frac{1}{\kappa\theta} q_x^2 + \frac{1-p_S}{\mu} S_x^2 dx dt \leq C(E(0) + E(t)^{\frac{3}{2}}). \end{aligned} \quad (3.28)$$

Note that

$$p_q = -\frac{\tau_1}{\kappa\theta} q, \quad e_\theta = c_v - \frac{\tau_1}{\kappa\theta^2 \rho} q^2, \quad p_\theta = R\rho + \frac{\tau_1}{2\kappa\theta^2} q^2$$

It is not difficult to see that for sufficiently small  $\delta$ , if  $E(t) \leq \delta$ , then we get from (3.28) the following estimate

$$\int_{\mathbb{R}} (\rho_x^2 + u_x^2 + \theta_x^2 + q_x^2 + S_x^2) dx + \int_0^t \int_{\mathbb{R}} (q_x^2 + S_x^2) dx dt \leq C(E(0) + E(t)^{\frac{3}{2}}). \quad (3.29)$$

Using similar methods, we obtain

$$\int_{\mathbb{R}} (\rho_t^2 + u_t^2 + \theta_t^2 + q_t^2 + S_t^2) dx + \int_0^t \int_{\mathbb{R}} (q_t^2 + S_t^2) dx dt \leq C(E(0) + E(t)^{\frac{3}{2}}). \quad (3.30)$$

Thfe estimates (3.29) and (3.30) prove (3.19). □

**Lemma 3.3.** *If  $E(t) \leq \delta$  for  $0 \leq t \leq T$  and some sufficiently small  $\delta > 0$ , we have*

$$\int_0^t \int_{\mathbb{R}} (\rho_t^2 + \rho_x^2 + u_t^2 + u_x^2 + \theta_t^2 + \theta_x^2) dx dt \leq C(E(0) + E(t)^{\frac{3}{2}}). \quad (3.31)$$

*Proof.* Using Lemma 3.1 and Lemma 3.2 and the equations, we immediately get

$$\int_0^t \int_{\mathbb{R}} (u_x^2 + \theta_x^2 + \theta_t^2) dx dt \leq C(E(0) + E(t)^{\frac{3}{2}}). \quad (3.32)$$

On the other hand, multiplying the momentum equation (1.1)<sub>2</sub> by  $u_t + uu_x$  and integrating over  $(0, t) \times \mathbb{R}$ , we get

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}} \rho(u_t + uu_x)^2 dx dt \\
& \leq - \int_0^t \int_{\mathbb{R}} R\theta \rho_x (u_t + uu_x) dx dt + \frac{1}{4} \int_0^t \int_{\mathbb{R}} \rho(u_t + uu_x)^2 dx dt + C(E(0) + E(t)^{\frac{3}{2}}) \\
& \leq \int_0^t \int_{\mathbb{R}} R\theta \rho u_{xt} dx dt - \int_0^t \int_{\mathbb{R}} R\theta \rho_x uu_x dx dt + \frac{1}{2} \int_0^t \int_{\mathbb{R}} \rho(u_t + uu_x)^2 dx dt + C(E(0) + E(t)^{\frac{3}{2}}) \\
& \leq - \int_0^t \int_{\mathbb{R}} R\theta u_x (\rho_t + u\rho_x) dx dt + \frac{1}{2} \int_0^t \int_{\mathbb{R}} \rho(u_t + uu_x)^2 dx dt + C(E(0) + E(t)^{\frac{3}{2}}) \\
& \leq \frac{1}{2} \int_0^t \int_{\mathbb{R}} \rho(u_t + uu_x)^2 dx dt + C(E(0) + E(t)^{\frac{3}{2}})
\end{aligned} \tag{3.33}$$

which together with (3.32) imply

$$\int_0^t \int_{\mathbb{R}} u_t^2 dx dt \leq C(E(0) + E(t)^{\frac{3}{2}}). \tag{3.34}$$

Now, using the equations (1.1) again, we get

$$\int_0^t \int_{\mathbb{R}} (\rho_t^2 + \rho_x^2) dx dt \leq C(E(0) + E(t)^{\frac{3}{2}}). \tag{3.35}$$

□

Next we give the second-order estimates of solutions.

**Lemma 3.4.** *If  $E(t) \leq \delta$  for  $0 \leq t \leq T$  and some sufficiently  $\delta > 0$ , we have*

$$\begin{aligned}
& \int_{\mathbb{R}} (\rho_{xx}^2 + \rho_{tx}^2 + u_{xx}^2 + u_{tx}^2 + \theta_{xx}^2 + \theta_{tx}^2 + q_{xx}^2 + q_{tx}^2 + S_{xx}^2 + S_{tx}^2) dx(t) \\
& + \int_0^t \int_{\mathbb{R}} (q_{xx}^2 + q_{tx}^2 + S_{xx}^2 + S_{tx}^2) dx dt \leq C(E(0) + E(t)^{\frac{3}{2}}).
\end{aligned} \tag{3.36}$$

*Proof.* In the following proof, we shall use higher-order derivatives, like  $\rho_{txx}, u_{xxx}$ , to show the above estimate. Although these higher-order derivatives are not covered by Theorem 1.1, a standard density argument will eliminate the needs for the extra regularity of local solutions. (cf. the proof of Theorem 4.1 in [32]).

Taking derivatives of the equations(1.1) with respect to  $x$  twice, we get

$$\rho_{txx} + u\rho_{xxx} + \rho u_{xxx} = -u_x \rho_{xx} - \rho_x u_{xx} + (f_1)_x =: g_1, \tag{3.37}$$

$$\begin{aligned}
& \rho u_{txx} + \rho u u_{xxx} + p_\rho \rho_{xxx} + p_\theta \theta_{xxx} + p_q q_{xxx} - (1 - p_S) S_{xxx} \\
& = -\rho_x u_{tx} - (\rho u)_x u_{xx} - (p_\rho)_x \rho_{xx} - (p_\theta)_x \theta_{xx} - (p_q)_x q_{xx} - (p_S)_x S_{xx} + (f_2)_x =: g_2,
\end{aligned} \tag{3.38}$$

$$\begin{aligned}
& \rho e_\theta \theta_{txx} + \left( \rho u e_\theta - \frac{2q}{\theta} \right) \theta_{xxx} + \theta p_\theta u_{xxx} + q_{xxx} \\
& = -(\rho e_\theta)_x \theta_{tx} - \left( \rho u e_\theta - \frac{2q}{\theta} \right)_x \theta_{xx} - (\theta p_\theta)_x u_{xx} + (f_3)_x =: g_3,
\end{aligned} \tag{3.39}$$

$$\tau_1 (q_{txx} + u q_{xxx}) + q_{xx} + \kappa \theta_{xxx} = -\tau_1 u_x q_{xx} + (f_4)_x =: g_4, \tag{3.40}$$

$$\tau_2 (S_{txx} + u S_{xxx}) + S_{xx} - \mu u_{xxx} = -\tau_2 u_x S_{xx} + (f_5)_x =: g_5. \tag{3.41}$$

Note that for  $1 \leq j \leq 5$ , there exists a constant  $C$  such that

$$|g_j| \leq C |(\rho_{xx}, u_{xx}, u_{tx}, \theta_{xx}, \theta_{tx}, q_{xx}, S_{xx}, \rho_x, u_x, u_t, \theta_x, \theta_t, q_x, S_x, q, S)|^2. \tag{3.42}$$

Multiplying the resulting equations by  $\frac{p_\rho}{\rho}\rho_{xx}$ ,  $u_{xx}$ ,  $\frac{1}{\theta}\theta_{xx}$ ,  $\frac{1}{\kappa\theta}q_{xx}$ ,  $\frac{1-p_S}{\mu}S_{xx}$ , respectively, and integrating the results over  $\mathbb{R}$ , we get

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}} \left( \frac{p_\rho}{2\rho} \rho_{xx}^2 + \frac{\rho}{2} u_{xx}^2 + \frac{\rho e_\theta}{2\theta} \theta_{xx}^2 + \frac{\tau_1}{2\kappa\theta} q_{xx}^2 + \frac{\tau_2(1-p_S)}{2\mu} S_{xx}^2 \right) dx \\
& + \int_{\mathbb{R}} \left( \frac{1}{\kappa\theta} q_{xx}^2 + \frac{1-p_S}{\mu} S_{xx}^2 \right) dx + \int_{\mathbb{R}} p_q q_{xxx} u_{xx} dx \\
& = \int_{\mathbb{R}} \left( \frac{p_\rho}{\rho} \rho_{xx} g_1 + u_{xx} g_2 + \frac{\theta_{xx}}{\theta} g_3 + \frac{1}{\kappa\theta} q_{xx} g_4 + \frac{1-p_S}{\mu} S_{xx} g_5 \right) dx \\
& + \int_{\mathbb{R}} \left( \frac{1}{2} \rho_{xx}^2 \left( \frac{p_\rho}{\rho} \right)_t + \frac{1}{2} u_{xx}^2 \rho_t + \frac{1}{2} \theta_{xx}^2 \left( \frac{\rho e_\theta}{\theta} \right)_t + \frac{1}{2} q_{xx}^2 \left( \frac{\tau_1}{\kappa\theta} \right)_t + \frac{1}{2} S_{xx}^2 \left( \frac{\tau_2(1-p_S)}{\mu} \right)_t \right) dx \\
& \int_{\mathbb{R}} \left( \left( \frac{p_\rho}{\rho} u \right)_x \frac{1}{2} \rho_{xx}^2 + (\rho u)_x \frac{1}{2} u_{xx}^2 + \left( \frac{\rho u e_\theta}{\theta} - \frac{2q}{\theta^2} \right)_x \frac{1}{2} \theta_{xx}^2 + \left( \frac{u}{\kappa\theta} \right)_x \frac{\tau_1}{2} q_{xx}^2 + \left( \frac{1-p_S}{\mu} u \right)_x \frac{\tau_2}{2} S_{xx}^2 \right) dx \\
& + \int_{\mathbb{R}} \left( (p_\rho)_x \rho_{xx} u_{xx} + (p_\theta)_x \theta_{xx} u_{xx} + \left( \frac{1}{\theta} \right)_x q_{xx} \theta_{xx} - (1-p_S)_x S_{xx} u_{xx} \right). \tag{3.43}
\end{aligned}$$

The integrals on the right-hand side can be estimated by  $E(t)^{\frac{3}{2}}$  by using the Sobolev embedding inequality

$$\|(D\rho, Du, D\theta, DS, Dq)\|_{L^\infty}(t) \leq CE(t)^{\frac{1}{2}} \tag{3.44}$$

where  $D = (\partial_t, \partial_x)$ . Using similar idea as in the proof of Lemma 3.2, we deal with the term  $\int_{\mathbb{R}} p_q q_{xxx} u_{xx} dx$  as follows.

$$\begin{aligned}
& \int_{\mathbb{R}} p_q q_{xxx} u_{xx} dx = - \int_{\mathbb{R}} (p_q)_x q_{xx} u_{xx} dx - \int_{\mathbb{R}} p_q q_{xx} u_{xxx} dx \\
& = - \int_{\mathbb{R}} (p_q)_x q_{xx} u_{xx} dx - \int_{\mathbb{R}} p_q q_{xx} \left( -\frac{\rho e_\theta}{\theta p_\theta} \theta_{txx} - \frac{\rho u e_\theta - \frac{2q}{\theta}}{\theta p_\theta} \theta_{xxx} - \frac{1}{\theta p_\theta} q_{xxx} + \frac{1}{\theta p_\theta} g_3 \right) dx \\
& = - \int_{\mathbb{R}} (p_q)_x q_{xx} u_{xx} dx - \int_{\mathbb{R}} \frac{p_q}{\theta p_\theta} q_{xx} g_3 dx - \int_{\mathbb{R}} \left( \frac{p_q}{\theta p_\theta} \right)_x \frac{1}{2} q_{xx}^2 dx \\
& + \int_{\mathbb{R}} \frac{p_q \rho e_\theta}{\theta p_\theta} q_{xx} \theta_{txx} dx + \int_{\mathbb{R}} \frac{p_q (\rho u e_\theta - \frac{2q}{\theta})}{\theta p_\theta} q_{xx} \theta_{xxx} dx, \tag{3.45}
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\mathbb{R}} \frac{p_q \rho e_\theta}{\theta p_\theta} q_{xx} \theta_{txx} dx \\
& = \frac{d}{dt} \int_{\mathbb{R}} \frac{p_q \rho e_\theta}{\theta p_\theta} q_{xx} \theta_{xx} dx - \int_{\mathbb{R}} \left( \frac{p_q \rho e_\theta}{\theta p_\theta} \right)_t q_{xx} \theta_{xx} dx - \int_{\mathbb{R}} \frac{p_q \rho e_\theta}{\theta p_\theta} q_{txx} \theta_{xx} dx \\
& = \frac{d}{dt} \int_{\mathbb{R}} \frac{p_q \rho e_\theta}{\theta p_\theta} q_{xx} \theta_{xx} dx - \int_{\mathbb{R}} \left( \frac{p_q \rho e_\theta}{\theta p_\theta} \right)_t q_{xx} \theta_{xx} dx \\
& - \int_{\mathbb{R}} \frac{p_q \rho e_\theta}{\theta p_\theta} \left( \frac{1}{\tau_1} g_4 - \frac{1}{\tau_1} q_{xx} - \frac{\kappa}{\tau_1} \theta_{xxx} - u q_{xxx} \right) \theta_{xx} dx \\
& = \frac{d}{dt} \int_{\mathbb{R}} \frac{p_q \rho e_\theta}{\theta p_\theta} q_{xx} \theta_{xx} dx - \int_{\mathbb{R}} \frac{p_q \rho e_\theta}{\theta p_\theta} u q_{xx} \theta_{xxx} - \int_{\mathbb{R}} \left( \frac{p_q \rho e_\theta}{\theta p_\theta} \right)_t q_{xx} \theta_{xx} dx \\
& - \int_{\mathbb{R}} \left( \frac{p_q \rho e_\theta}{\theta p_\theta} \left( \frac{1}{\tau_1} g_4 - \frac{1}{\tau_1} q_{xx} \right) \theta_{xx} + \left( \frac{p_q \rho e_\theta}{\theta p_\theta} u \right)_x q_{xx} \theta_{xx} - \left( \frac{\kappa p_q \rho e_\theta}{\tau_1 \theta p_\theta} \right)_x \frac{1}{2} \theta_{xx}^2 \right) dx. \tag{3.46}
\end{aligned}$$

Therefore, we get

$$\begin{aligned} & \int_{\mathbb{R}} \frac{p_q \rho e_\theta}{\theta p_\theta} q_{xx} \theta_{txx} dx + \frac{p_q (\rho u e_\theta - \frac{2q}{\theta})}{\theta p_\theta} q_{xx} \theta_{xxx} dx = \frac{d}{dt} \int_{\mathbb{R}} \frac{p_q \rho e_\theta}{\theta p_\theta} q_{xx} \theta_{xx} dx - \int_{\mathbb{R}} \frac{2qp_q}{\theta^2 p_\theta} q_{xx} \theta_{xxx} \\ & - \int_{\mathbb{R}} \left( \frac{p_q \rho e_\theta}{\theta p_\theta} \left( \frac{1}{\tau_1} g_4 - \frac{1}{\tau_1} q_{xx} \right) \theta_{xx} + \left( \frac{p_q \rho e_\theta}{\theta p_\theta} u \right)_x q_{xx} \theta_{xx} - \left( \frac{\kappa p_q \rho e_\theta}{\tau_1 \theta p_\theta} \right)_x \frac{1}{2} \theta_{xx}^2 + \left( \frac{p_q \rho e_\theta}{\theta p_\theta} \right)_t q_{xx} \theta_{xx} \right) dx. \end{aligned} \quad (3.47)$$

For the term  $\int_{\mathbb{R}} \frac{2qp_q}{\theta^2 p_\theta} q_{xx} \theta_{xxx} dx$ , using the equation for  $q$ , we have

$$\begin{aligned} & \int_{\mathbb{R}} \frac{2qp_q}{\theta^2 p_\theta} q_{xx} \theta_{xxx} dx = \int_{\mathbb{R}} \frac{2qp_q}{\theta^2 p_\theta} q_{xx} \left( \frac{-\tau_1}{\kappa} (q_{txx} + u q_{xxx}) - \frac{1}{\kappa} q_{xx} + \frac{1}{\kappa} g_4 \right) dx \\ & = -\frac{d}{dt} \int_{\mathbb{R}} \frac{\tau_1 qp_q}{\kappa \theta^2 p_\theta} q_{xx}^2 dx + \int_{\mathbb{R}} \left( \frac{\tau_1 qp_q}{\kappa \theta^2 p_\theta} \right)_t q_{xx}^2 dx \\ & + \int_{\mathbb{R}} \left( \frac{2qp_q \tau_1}{\theta^2 p_\theta \kappa} u \right)_x \frac{1}{2} q_{xx}^2 - \int_{\mathbb{R}} \frac{2qp_q}{\kappa \theta^2 p_\theta} q_{xx}^2 dx + \int_{\mathbb{R}} \frac{2qp_q}{\theta^2 p_\theta} q_{xx} g_4. \end{aligned} \quad (3.48)$$

Finally, we derive that

$$\begin{aligned} & \int_{\mathbb{R}} \left( \frac{p_\rho}{2\rho} \rho_{xx}^2 + \frac{1}{2} \rho u_{xx}^2 + \frac{\rho e_\theta}{2\theta} \theta_{xx}^2 + \frac{\tau_1}{2\kappa\theta} q_{xx}^2 + \frac{\tau_1(1-p_S)}{2\mu} S_{xx}^2 + \frac{p_q \rho e_\theta}{\theta p_\theta} q_{xx} \theta_{xx} + \frac{\tau_1 qp_q}{\kappa \theta^2 p_\theta} q_{xx}^2 \right) dx \\ & + \int_0^t \int_{\mathbb{R}} \frac{1}{\kappa\theta} q_{xx}^2 + \frac{1-p_S}{\mu} S_{xx}^2 dx dt \leq C(E(0) + E(t)^{\frac{3}{2}}). \end{aligned} \quad (3.49)$$

Note that

$$p_q = -\frac{\tau_1}{\kappa\theta} q, e_\theta = c_v - \frac{\tau_1}{\kappa\theta^2 \rho} q^2, p_\theta = R\rho + \frac{\tau_1}{2\kappa\theta^2} q^2.$$

So, there exists a small  $\delta$  such that if  $E(t) \leq \delta$ , we can derive that

$$\int_{\mathbb{R}} (\rho_{xx}^2 + u_{xx}^2 + \theta_{xx}^2 + q_{xx}^2 + S_{xx}^2) dx + \int_0^t \int_{\mathbb{R}} (q_{xx}^2 + S_{xx}^2) dx dt \leq C(E(0) + E(t)^{\frac{3}{2}}). \quad (3.50)$$

Using similar methods, we can get

$$\int_{\mathbb{R}} (\rho_{tx}^2 + u_{tx}^2 + \theta_{tx}^2 + q_{tx}^2 + S_{tx}^2) dx + \int_0^t \int_{\mathbb{R}} (q_{tx}^2 + S_{tx}^2) dx dt \leq C(E(0) + E(t)^{\frac{3}{2}}). \quad (3.51)$$

□

**Lemma 3.5.** *If  $E(t) \leq \delta$  for  $0 \leq t \leq T$  and some sufficiently small  $\delta > 0$ , we have*

$$\int_0^t \int_{\mathbb{R}} (\rho_{tx}^2 + \rho_{xx}^2 + u_{tx}^2 + u_{xx}^2 + \theta_{tx}^2 + \theta_{xx}^2) dx dt \leq C(E(0) + E(t)^{\frac{3}{2}}). \quad (3.52)$$

*Proof.* Using Lemmas 3.1-3.4 and the system (3.20), we can immediately get

$$\int_0^t \int_{\mathbb{R}} (u_{xx}^2 + \theta_{xx}^2 + \theta_{tx}^2) dx dt \leq C(E(0) + E(t)^{\frac{3}{2}}). \quad (3.53)$$

On the other hand, multiplying the momentum equation (3.20)<sub>2</sub> by  $u_{tx} + uu_{xx}$  and integrating over  $(0, t) \times \mathbb{R}$ , we get

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}} \rho(u_{tx} + uu_{xx})^2 dx dt \\
& \leq - \int_0^t \int_{\mathbb{R}} R\theta\rho_{xx}(u_{tx} + uu_{xx}) dx dt + \frac{1}{4} \int_0^t \int_{\mathbb{R}} \rho(u_{tx} + uu_{xx})^2 dx dt + C(E(0) + E(t)^{\frac{3}{2}}) \\
& \leq \int_0^t \int_{\mathbb{R}} R\theta\rho_x u_{txx} dx dt - \int_0^t \int_{\mathbb{R}} R\theta\rho_{xx} uu_{xx} dx dt + \frac{1}{2} \int_0^t \int_{\mathbb{R}} \rho(u_{tx} + uu_{xx})^2 dx dt + C(E(0) + E(t)^{\frac{3}{2}}) \\
& \leq - \int_0^t \int_{\mathbb{R}} R\theta u_{xx}(\rho_{tx} + u\rho_{xx}) dx dt + \frac{1}{2} \int_0^t \int_{\mathbb{R}} \rho(u_{tx} + uu_{xx})^2 dx dt + C(E(0) + E(t)^{\frac{3}{2}}) \\
& \leq \frac{1}{2} \int_0^t \int_{\mathbb{R}} \rho(u_{tx} + uu_{xx})^2 dx dt + C(E(0) + E(t)^{\frac{3}{2}})
\end{aligned} \tag{3.54}$$

which together with (3.53) imply

$$\int_0^t \int_{\mathbb{R}} u_{tx}^2 dx dt \leq C(E(0) + E(t)^{\frac{3}{2}}). \tag{3.55}$$

Now, use the equations (3.20) again, we get

$$\int_0^t \int_{\mathbb{R}} (\rho_{tx}^2 + \rho_{xx}^2) dx dt \leq C(E(0) + E(t)^{\frac{3}{2}}). \tag{3.56}$$

□

With the Lemmas 3.1-3.5 we have the necessary a priori estimates to continue, as usual for small data, the local solution from Theorem 2.2 to a global one. This proves Theorem 1.1.

#### 4. RELAXATION LIMIT

In this section we prove Theorem 1.2, i.e. we show the uniform convergence of the relaxed system (1.1) to the classical compressible Navier-Stokes equations (corresponding to  $\tau = \tau_1 = \tau_2 = 0$ ).

*Proof.* Let  $\rho^d = \frac{\rho^\tau - \rho}{\tau}$ ,  $u^d = \frac{u^\tau - u}{\tau}$ ,  $\theta^d = \frac{\theta^\tau - \theta}{\tau}$ ,  $q^d = \frac{q^\tau - q}{\tau}$ ,  $S^d = \frac{S^\tau - S}{\tau}$ , where  $q = -\kappa\theta_x$  and  $S = \mu u_x$ . It suffices to show that for small  $\tau$  and  $t < \min\{T_*, T_\tau\}$ ,

$$\|(\rho^d, u^d, \theta^d)(t, \cdot)\|_{H^2} \leq C, \quad \|\sqrt{\tau}(q^d, S^d)(t, \cdot)\|_{H^2} \leq C, \tag{4.1}$$

where  $C > 0$  will denote constants which do not depend on  $\tau$ .

We define

$$E := \sup_{0 \leq t \leq T} (\|(\rho - 1, u, \theta - 1)\|_{H^5} + \|(u_t, \theta_t)\|_{H^3}),$$

and

$$E^d := \sup_{0 \leq t \leq T} \|(\rho^d, u^d, \theta^d, \sqrt{\tau}q^d, \sqrt{\tau}S^d)\|_{H^2}.$$

Note that, by assumption,  $E \leq C$  and

$$\|(\rho^\tau - 1, u^\tau, \theta^\tau - 1)\|_{H^2} \leq C + \tau E^d, \quad \|(q^\tau, S^\tau)\|_{H^2} \leq C + \sqrt{\tau} E^d.$$

The equations for the difference  $(\rho^d, u^d, \theta^d, q^d, S^d)$  have the following form:

$$\begin{aligned}
\rho_t^d + u^\tau \rho_x^d + \rho^\tau u_x^d &= -\rho^d u_x - u^d \rho_x =: f_1, \\
u_t^d + u^\tau u_x^d + \frac{p_\rho^\tau}{\rho^\tau} \rho_x^d + \frac{p_\theta^\tau}{\rho^\tau} \theta_x^d + \frac{p_q^\tau}{\rho^\tau} q_x^d + \frac{(p_S^\tau - 1)}{\rho^\tau} S_x^d \\
&= -\frac{\rho^d u_t}{\rho^\tau} - \frac{\rho^\tau u^\tau - \rho u}{\tau \rho^\tau} u_x - \frac{p_\rho^\tau - p_\rho}{\tau \rho^\tau} \rho_x - \frac{p_\theta^\tau - p_\theta}{\tau \rho^\tau} \theta_x - \frac{q_x}{\tau \rho^\tau} p_q^\tau - \frac{S_x}{\tau \rho^\tau} p_S^\tau =: f_2, \\
\theta_t^d + (u^\tau - \frac{2q^\tau}{\rho^\tau e_\theta^\tau}) \theta_x^d + \frac{\theta^\tau p_\theta^\tau}{\rho^\tau e_\theta^\tau} u_x^d + \frac{1}{\rho^\tau e_\theta^\tau} q_x^d \\
&= -\frac{\rho^\tau e_\theta^\tau - \rho e_\theta}{\tau \rho^\tau e_\theta^\tau} \theta_t - \frac{\rho^\tau u^\tau e_\theta^\tau - \rho u e_\theta}{\tau \rho^\tau e_\theta^\tau} \theta_x - \frac{\theta^\tau p_\theta^\tau - \theta p_\theta}{\tau \rho^\tau e_\theta^\tau} u_x + \frac{2q^\tau}{\kappa \theta^\tau \rho^\tau e_\theta^\tau} q^d + \frac{S^\tau + S}{\mu \rho^\tau e_\theta^\tau} S^d =: f_3, \\
\tau(q_t^d + u^\tau q_x^d) + q^d + \kappa \theta_x^d &= -\tau q_x u^d - (q_t + u q_x) =: f_4, \\
\tau(S_t^d + u^\tau S_x^d) + S^d - \mu u_x^d &= -\tau S_x u^d - (S_t + u S_x) =: f_5.
\end{aligned}$$

Taking  $\partial_x^\alpha$  ( $0 \leq \alpha \leq 2$ ) to the above system, multiplying by  $\frac{p_\rho^\tau}{\rho^\tau} \partial_x^\alpha \rho^d$ ,  $\rho^\tau \partial_x^\alpha u^d$ ,  $\frac{\rho^\tau e_\theta^\tau}{\theta^\tau} \partial_x^\alpha \theta^d$ ,  $\frac{1}{\kappa \theta^\tau} \partial_x^\alpha q^d$  and  $\frac{(1-p_S^\tau)}{\mu} \partial_x^\alpha S^d$ , respectively, and integrating the result with respect to  $x$ , we obtain

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}} \left( \frac{p_\rho^\tau}{2\rho^\tau} (\partial_x^\alpha \rho^d)^2 + \frac{\rho^\tau}{2} (\partial_x^\alpha u^d)^2 + \frac{\rho^\tau e_\theta^\tau}{2\theta^\tau} (\partial_x^\alpha \theta^d)^2 + \frac{\tau}{2\kappa \theta^\tau} (\partial_x^\alpha q^d)^2 + \frac{\tau(1-p_S^\tau)}{2\mu} (\partial_x^\alpha S^d)^2 \right) dx \\
+ \int_{\mathbb{R}} \left( \frac{1}{\kappa \theta^\tau} (\partial_x^\alpha q^d)^2 + \frac{(1-p_S^\tau)}{\mu} (\partial_x^\alpha S^d)^2 \right) dx + \int_{\mathbb{R}} \partial_x^\alpha \left( \frac{p_q^\tau}{\rho^\tau} q^d \right) \rho^\tau \partial_x^\alpha u^d dx \\
\equiv \sum_{i=1}^4 T_i + \sum_{i=1}^5 K_i + \sum_{i=1}^4 M_i + \sum_{i=1}^5 F_i, \quad (4.2)
\end{aligned}$$

where

$$\begin{aligned}
T_1 &= \int_{\mathbb{R}} \frac{1}{2} \rho_t^\tau (\partial_x^\alpha u^d)^2 dx, \quad T_2 = \int_{\mathbb{R}} \frac{1}{2} \left( \frac{\rho^\tau e_\theta^\tau}{\theta^\tau} \right)_t (\partial_x^\alpha \theta^d)^2 dx, \\
T_3 &= \int_{\mathbb{R}} \frac{\tau}{2} \left( \frac{1}{\kappa \theta^\tau} \right)_t (\partial_x^\alpha q^d)^2 dx, \quad T_4 = \int_{\mathbb{R}} \left( \frac{\tau(1-p_S^\tau)}{2\mu} \right)_t (\partial_x^\alpha S^d)^2 dx, \\
K_1 &= \int_{\mathbb{R}} \partial_x^\alpha (u^\tau \rho_x^d) \cdot \frac{p_\rho^\tau}{\rho^\tau} \partial_x^\alpha \rho^d dx, \quad K_2 = \int_{\mathbb{R}} \partial_x^\alpha (u^\tau u_x^d) \cdot (\rho^\tau \partial_x^\alpha u^d) dx, \\
K_3 &= \int_{\mathbb{R}} \partial_x^\alpha \left( (u^\tau - \frac{2q^\tau}{\rho^\tau e_\theta^\tau}) \theta_x^d \right) \cdot \frac{\rho^\tau e_\theta^\tau}{\theta^\tau} \partial_x^\alpha \theta^d dx, \\
K_4 &= \int_{\mathbb{R}} \tau \partial_x^\alpha (u^\tau q_x^d) \cdot \frac{1}{\kappa \theta^\tau} \partial_x^\alpha q^d dx, \quad K_5 = \int_{\mathbb{R}} \tau \partial_x^\alpha (u^\tau S_x^d) \cdot \frac{1-p_S^\tau}{\mu} \partial_x^\alpha S^d dx, \\
M_1 &= \int_{\mathbb{R}} \left( \partial_x^\alpha (\rho^\tau u_x^d) \frac{p_\rho^\tau}{\rho^\tau} \partial_x^\alpha \rho^d + \partial_x^\alpha \left( \frac{p_\rho^\tau}{\rho^\tau} \rho_x^d \right) \rho^\tau \partial_x^\alpha u^d \right) dx, \\
M_2 &= \int_{\mathbb{R}} \left( \partial_x^\alpha \left( \frac{p_\theta^\tau}{\rho^\tau} \theta_x^d \right) \rho^\tau \partial_x^\alpha u^d + \partial_x^\alpha \left( \frac{\theta^\tau p_\theta^\tau}{\rho^\tau e_\theta^\tau} u_x^d \right) \frac{\rho^\tau e_\theta^\tau}{\theta^\tau} \partial_x^\alpha \theta^d \right) dx, \\
M_3 &= \int_{\mathbb{R}} \left( \partial_x^\alpha \left( \frac{p_S^\tau - 1}{\rho^\tau} S_x^d \right) \rho^\tau \partial_x^\alpha u^d - \mu \partial_x^\alpha u_x^d \frac{1-p_S^\tau}{\mu} \partial_x^\alpha S^d \right) dx, \\
M_4 &= \int_{\mathbb{R}} \left( \partial_x^\alpha \left( \frac{1}{\rho^\tau e_\theta^\tau} q_x^d \right) \frac{\rho^\tau e_\theta^\tau}{\theta^\tau} \partial_x^\alpha \theta^d + \kappa \partial_x^\alpha \theta_x^d \frac{1}{\kappa \theta^\tau} \partial_x^\alpha q^d \right) dx,
\end{aligned}$$

$$F_1 = \int_{\mathbb{R}} \partial_x^\alpha f_1 \cdot \frac{p_\rho^\tau}{\rho^\tau} \partial_x^\alpha \rho^d dx, F_2 = \int_{\mathbb{R}} \partial_x^\alpha f_2 \cdot \rho^\tau \partial_x^\alpha u^\tau dx,$$

$$F_3 = \int_{\mathbb{R}} \partial_x^\alpha f_3 \cdot \frac{\rho^\tau e_\theta^\tau}{\theta^\tau} \partial_x^\alpha \theta^d dx, F_4 = \int_{\mathbb{R}} \partial_x^\alpha f_4 \cdot \frac{1}{\kappa \theta^\tau} \partial_x^\alpha q^d dx, F_5 = \int_{\mathbb{R}} \partial_x^\alpha f_5 \cdot \frac{1 - p_S^\tau}{\mu} \partial_x^\alpha S^d dx.$$

For  $1 \leq i \leq 4$ , we have

$$|T_i| \leq C \|((\rho^\tau)_t, (\theta^\tau)_t, \tau(S^\tau)_t)\|_{L^\infty} (E^d)^2 \leq C((E^d)^2 + \tau(E^d)^3 + \tau^2(E^d)^4).$$

By using Moser type inequality and Sobolev imbedding theorem, we can get

$$K_1 = \int_{\mathbb{R}} (\partial_x^\alpha (u^\tau \rho_x^d) - u^\tau \partial_x^{\alpha+1} \rho^d) \cdot \frac{p_\rho^\tau}{\rho^\tau} \partial_x^\alpha \rho^d dx + \int_{\mathbb{R}} \frac{p_\rho^\tau u^\tau}{\rho^\tau} \partial_x^{\alpha+1} \rho^d \partial_x^\alpha \rho^d$$

$$\leq C (\|\partial_x^\alpha u^\tau\|_{L^2} \|\rho_x^d\|_{L^\infty} + \|\partial_x^\alpha \rho^d\|_{L^2} \|\partial_x u^\tau\|_{L^\infty}) \cdot \|\frac{p_\rho^\tau}{\rho^\tau}\|_{L^\infty} \|\partial_x^\alpha \rho^d\|_{L^2} + \frac{1}{2} \left\| \left( \frac{p_\rho^\tau u^\tau}{\rho^\tau} \right)_x \right\|_{L^\infty} \|\partial_x^\alpha \rho^d\|_{L^2}^2$$

$$\leq (C + \tau E^d)^2 \cdot (E^d)^2 + (C + \tau E^d)(E^d)^2 \leq C((E^d)^2 + \tau(E^d)^3 + \tau^2(E^d)^4).$$

Similarly, for  $i = 2, 3, 4$ , we can get that

$$|K_i| \leq C((E^d)^2 + \tau(E^d)^3 + \tau^2(E^d)^4).$$

We estimate  $M_1$  as follows.

$$M_1 = \int_{\mathbb{R}} (\partial_x^\alpha (\rho^\tau u_x^d) - \rho^\tau \partial_x^{\alpha+1} u^d) \cdot \frac{p_\rho^\tau}{\rho^\tau} \partial_x^\alpha \rho^d + \left( \partial_x^\alpha \left( \frac{p_\rho^\tau}{\rho^\tau} \rho_x^d \right) - \frac{p_\rho^\tau}{\rho^\tau} \partial_x^{\alpha+1} \rho^d \right) \rho^\tau \partial_x^\alpha u^d dx$$

$$+ \int_{\mathbb{R}} p_\rho^\tau (\partial_x^{\alpha+1} u^d \partial_x^\alpha \rho^d + \partial_x^{\alpha+1} \rho^d \partial_x^\alpha u^d) dx$$

$$\leq (C + \tau E^d) (\|\partial_x \rho^\tau\|_{L^\infty} \|\partial_x^\alpha u^d\|_{L^2} + \|u_x^d\|_{L^\infty} \|\partial_x^\alpha \rho^\tau\|_{L^2}) \|\partial_x^\alpha \rho^d\|_{L^2}$$

$$+ (C + \tau E^d) \left( \|\partial_x \frac{p_\rho^\tau}{\rho^\tau}\|_{L^\infty} \|\partial_x^\alpha \rho^d\| + \|\rho_x^d\|_{L^\infty} \|\partial_x^\alpha \frac{p_\rho^\tau}{\rho^\tau}\|_{L^2} \right) \|\partial_x^\alpha u^d\|_{L^2} + \|\partial_x p_\rho^\tau\|_{L^\infty} \|\partial_x^\alpha u^d\|_{L^2} \|\partial_x^\alpha \rho^d\|_{L^2}$$

$$\leq (C + \tau E^d)^2 \cdot (E^d)^2 + (C + \tau E^d)(E^d)^2 \leq C((E^d)^2 + \tau(E^d)^3 + \tau^2(E^d)^4).$$

Using similar methods, we can get also for  $i = 2, 3, 4$

$$|M_i| \leq C((E^d)^2 + \tau(E^d)^3 + \tau^2(E^d)^4).$$

Now we estimate the term  $F_i$  for  $i = 1, 2, 3, 4$ , where the higher regularity assumptions on the solution are needed. First, we have

$$\|\partial_x^\alpha f_1\|_{L^2} \leq \|\partial_x^\alpha (\rho^d u_x)\|_{L^2} + \|\partial_x^\alpha (u^d \rho_x)\|_{L^2}$$

$$\leq \|(u_x, \rho_x)\|_{L^\infty} \|(\partial_x^\alpha \rho^d, \partial_x^\alpha u^d)\|_{L^2} + \|(\rho^d, u^d)\|_{L^\infty} \|\partial_x^\alpha (u_x, \rho_x)\|_{L^2} \leq C E^d,$$

$$\|\partial_x^\alpha f_2\|_{L^2} \leq \|\partial_x^\alpha \left( \frac{\rho^d u_t}{\rho^\tau} \right)\|_{L^2} + \|\partial_x^\alpha \left( \frac{u_x}{\rho^\tau} (\rho^\tau u^d + u \rho^d) \right)\|_{L^2} + \|\partial_x^\alpha \left( \frac{p_\rho^\tau - p_\rho}{\rho^\tau \tau} \rho_x \right)\|_{L^2}$$

$$+ \|\partial_x^\alpha \left( \frac{p_\theta^\tau - p_\theta}{\rho^\tau \tau} \theta_x \right)\|_{L^2} + \|\partial_x^\alpha \left( \frac{p_q^\tau}{\rho^\tau \tau} q_x \right)\|_{L^2} + \|\partial_x^\alpha \left( \frac{p_S^\tau}{\rho^\tau \tau} S_x \right)\|_{L^2}$$

$$\leq C \left( \|(u_t, u_x, uu_x, \rho_x, \theta_x, q_x, S_x)\|_{L^\infty} \|\partial_x^\alpha \left( \frac{\rho^d}{\rho^\tau}, u^d, \frac{\rho^d}{\rho^\tau}, \frac{p_\rho^\tau - p_\rho}{\rho^\tau \tau}, \frac{p_\theta^\tau - p_\theta}{\rho^\tau \tau}, \frac{p_q^\tau}{\rho^\tau \tau}, \frac{p_S^\tau}{\rho^\tau \tau} \right)\|_{L^2} \right.$$

$$\left. + \|\partial_x^\alpha (u_t, u_x, uu_x, \rho_x, \theta_x, q_x, S_x)\|_{L^2} \left\| \left( \frac{\rho^d}{\rho^\tau}, u^d, \frac{\rho^d}{\rho^\tau}, \frac{p_\rho^\tau - p_\rho}{\rho^\tau \tau}, \frac{p_\theta^\tau - p_\theta}{\rho^\tau \tau}, \frac{p_q^\tau}{\rho^\tau \tau}, \frac{p_S^\tau}{\rho^\tau \tau} \right) \right\|_{L^\infty} \right)$$

$$\leq C E^d (C + \tau E^d)$$



where we have used the fact that  $\frac{p_q^\tau}{\tau} = -\frac{q^\tau}{\kappa\theta^\tau}$  and  $\frac{p_S^\tau}{\tau} = -\frac{S^\tau}{\mu}$ . Similarly, we can get

$$\begin{aligned} \|\partial_x^\alpha f_3\|_{L^2} &\leq \|(\theta_t, \theta_x, S)\|_{L^\infty} \|\partial_x^\alpha \left( \frac{\rho^\tau e_\theta^\tau - \rho e_\theta}{\tau \rho^\tau e_\theta^\tau}, \frac{\rho^\tau u^\tau e_\theta^\tau - \rho u e_\theta}{\rho^\tau \tau}, \frac{S^d}{\mu \rho^\tau e_\theta^\tau} \right)\|_{L^2} \\ &\quad + \|\partial_x^\alpha(\theta_t, \theta_x, S)\|_{L^2} \left\| \left( \frac{\rho^\tau e_\theta^\tau - \rho e_\theta}{\tau \rho^\tau e_\theta^\tau}, \frac{\rho^\tau u^\tau e_\theta^\tau - \rho u e_\theta}{\rho^\tau \tau}, \frac{S^d}{\mu \rho^\tau e_\theta^\tau} \right) \right\|_{L^\infty} \\ &\quad + \left\| \left( \frac{2q^\tau}{\kappa\theta^\tau \rho^\tau e_\theta^\tau}, \frac{S^\tau}{\mu \rho^\tau e_\theta^\tau} \right) \right\|_{L^\infty} \|\partial_x^\alpha(q^d, S^d)\|_{L^2} + \|\partial_x^\alpha \left( \frac{2q^\tau}{\kappa\theta^\tau \rho^\tau e_\theta^\tau}, \frac{S^\tau}{\mu \rho^\tau e_\theta^\tau} \right)\|_{L^2} \|(q^d, S^d)\|_\infty \\ &\leq C(E^d + \tau(E^d)^2 + \tau^2(E^d)^3). \end{aligned}$$

Note that  $q = -\kappa\theta_x, S = \mu u_x$ , so we get

$$\|(\partial_x^\alpha(f_4, f_5))\|_{L^2} \leq C + C\tau E^d.$$

Based on the above estimates, we conclude

$$\sum_{i=1}^4 T_i + \sum_{i=1}^5 K_i + \sum_{i=1}^4 M_i + \sum_{i=1}^4 F_i \leq C(1 + (E^d)^2 + \tau(E^d)^3 + \tau^2(E^d)^4). \quad (4.3)$$

Finally, we deal with the last term on the left-hand side of (4.2), the term  $\int_{\mathbb{R}} \partial_x^\alpha \left( \frac{p_q^\tau}{\rho^\tau} q_x^d \right) \rho^\tau \partial_x^\alpha u^d dx$ .

$$\begin{aligned} &\int_{\mathbb{R}} \partial_x^\alpha \left( \frac{p_q^\tau}{\rho^\tau} q_x^d \right) \rho^\tau \partial_x^\alpha u^d dx \\ &= \int_{\mathbb{R}} \left( \partial_x^\alpha \left( \frac{p_q^\tau}{\rho^\tau} q_x^d \right) - \frac{p_q^\tau}{\rho^\tau} \partial_x^{\alpha+1} q^d \right) \rho^\tau \partial_x^\alpha u^d dx + \int_{\mathbb{R}} p_q^\tau \partial_x^{\alpha+1} q^d \partial_x^\alpha u^d dx \equiv I_1 + I_2. \end{aligned} \quad (4.4)$$

Using Moser type inequality and the Sobolev imbedding theorem, we get

$$I_1 \leq C \left( \|\partial_x \left( \frac{p_q^\tau}{\rho^\tau} \right)\|_{L^\infty} \|\partial_x^\alpha q^d\|_{L^2} + \|q_x^d\|_\infty \|\partial_x^\alpha \frac{p_q^\tau}{\rho^\tau}\|_{L^2} \right) \|\partial_x^\alpha u^d\|_{L^2} \leq C((E^d)^2 + \tau(E^d)^3). \quad (4.5)$$

For the term  $I_2$ , we use similar methods as in the proof of Lemma 3.2.

$$\begin{aligned} I_2 &= - \int_{\mathbb{R}} (p_q^\tau)_x \partial_x^\alpha q^d \partial_x^\alpha u^d dx - \int_{\mathbb{R}} p_q^\tau \partial_x^\alpha q^d \partial_x^{\alpha+1} u^d dx \\ &\leq \int_{\mathbb{R}} \frac{p_q^\tau \rho^\tau e_\theta^\tau}{\theta^\tau p_\theta^\tau} \partial_x^\alpha q^d \partial_x^\alpha \partial_t \theta^d dx + \int_{\mathbb{R}} \frac{p_q^\tau \rho^\tau e_\theta^\tau}{\theta^\tau p_\theta^\tau} \left( u^\tau - \frac{2q^\tau}{\rho^\tau e_\theta^\tau \theta^\tau} \right) \partial_x^\alpha q^d \partial_x^{\alpha+1} \theta dx \\ &\quad + C((E^d)^2 + \tau(E^d)^3 + \tau^2(E^d)^4 + \tau^3(E^d)^5) \\ &\leq \frac{d}{dt} \int_{\mathbb{R}} \frac{p_q^\tau \rho^\tau e_\theta^\tau}{\theta^\tau p_\theta^\tau} \partial_x^\alpha q^d \partial_x^\alpha \theta^d dx - \int_{\mathbb{R}} \frac{p_q^\tau \rho^\tau e_\theta^\tau}{\theta^\tau p_\theta^\tau} \partial_x^\alpha \partial_t q^d \partial_x^\alpha \theta^d dx + \int_{\mathbb{R}} \frac{p_q^\tau \rho^\tau e_\theta^\tau}{\theta^\tau p_\theta^\tau} \left( u^\tau - \frac{2q^\tau}{\rho^\tau e_\theta^\tau \theta^\tau} \right) \partial_x^\alpha q^d \partial_x^{\alpha+1} \theta dx \\ &\quad + C((E^d)^2 + \tau(E^d)^3 + \tau^2(E^d)^4 + \tau^3(E^d)^5) \\ &\leq \frac{d}{dt} \int_{\mathbb{R}} \frac{p_q^\tau \rho^\tau e_\theta^\tau}{\theta^\tau p_\theta^\tau} \partial_x^\alpha q^d \partial_x^\alpha \theta^d dx - \int_{\mathbb{R}} \frac{2p_q^\tau q^\tau}{(\theta^\tau)^2 p_\theta^\tau} \partial_x^\alpha q^d \partial_x^{\alpha+1} \theta dx + C((E^d)^2 + \tau(E^d)^3 + \tau^2(E^d)^4 + \tau^3(E^d)^5) \\ &\leq \frac{d}{dt} \int_{\mathbb{R}} \left( \frac{p_q^\tau \rho^\tau e_\theta^\tau}{\theta^\tau p_\theta^\tau} \partial_x^\alpha q^d \partial_x^\alpha \theta^d + \frac{\tau p_q^\tau q^\tau}{\kappa(\theta^\tau)^2 p_\theta^\tau} (\partial_x^\alpha q^d)^2 \right) dx + C((E^d)^2 + \tau(E^d)^3 + \tau^2(E^d)^4 + \tau^3(E^d)^5). \end{aligned} \quad (4.6)$$

Combining (4.2)–(4.6), we finally derive that

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} \left( \frac{1}{2} (\partial_x^\alpha \rho^d)^2 + \frac{\rho^\tau}{2} (\partial_x^\alpha u^d)^2 + \frac{\rho^\tau e_\theta^\tau}{2\theta^\tau} (\partial_x^\alpha \theta^d)^2 + \frac{\tau}{2\kappa\theta^\tau} (\partial_x^\alpha q^d)^2 + \frac{\tau(1-p_S^\tau)}{2\mu} (\partial_x^\alpha S^d)^2 \right. \\ & \quad \left. + \frac{p_q^\tau \rho^\tau e_\theta^\tau}{\theta^\tau p_\theta^\tau} \partial_x^\alpha q^d \partial_x^\alpha \theta^d + \frac{\tau p_q^\tau q^\tau}{\kappa(\theta^\tau)^2 p_\theta^\tau} (\partial_x^\alpha q^d)^2 \right) dx \\ & + \int_{\mathbb{R}} \left( \frac{1}{\kappa\theta^\tau} (\partial_x^\alpha q^d)^2 + \frac{(1-p_S^\tau)}{\mu} (\partial_x^\alpha S^d)^2 \right) dx \leq C(1 + (E^d)^2 + \tau(E^d)^3 + \tau^2(E^d)^4 + \tau^3(E^d)^5). \end{aligned}$$

Summing  $\alpha$  from 0 to 2 and noting that  $p_q$  is sufficiently small if we choose  $\tau$  sufficiently small, we get

$$\frac{d}{dt} (E^d)^2 \leq C(1 + (E^d)^2 + \tau(E^d)^3 + \tau^2(E^d)^4 + \tau^3(E^d)^5).$$

Thus, use the same arguments as in [15, 16], we conclude that there is a small  $\tau_0 > 0$  such that for  $0 < \tau < \tau_0$ ,

$$E^d \leq C.$$

This proves (4.1) and hence Theorem 1.2.  $\square$

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