Decay for thermoelastic Green-Lindsay plates in bounded and unbounded domains

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Abstract: We consider equations describing the thermoelastic behavior of plates modeled in the Green-Lindsay sense. This is done with two different type of couplings of the fourth-order plate Kirchhoff-type plate equation to a second-order heat equation of Cattaneo type, once of second, and once of first order. We investigate both systems for bounded domains and for the Cauchy problem, asking for exponential stability in bounded domains resp. polynomial decay rates for the Cauchy problem. It turns out that one system is exponentially stable, while the other is not, and that, in correspondence, one does not have and the other one has regularity loss in the Cauchy problem. This provides a new interesting example where the different couplings lead to qualitatively different behavior, as known before for classical thermoelastic plates, for Timoshenko systems, for porous elasticity or for plates with two temperatures, with Fourier resp. Cattaneo heat conduction. The optimality of the decay rates obtained is also proved.

1 Introduction

We consider different models for thermoelastic plates within the Green-Lindsay framework, as there are system (I) given by

$$\rho u_{tt} + \mu\Delta^2 u - a\Delta(\theta + \alpha\theta_t) = 0, \tag{1.1}$$

$$h\theta_{tt} + d\theta_t - k\Delta\theta + a\Delta u_t = 0, \tag{1.2}$$

and system (II) given by

$$\rho u_{tt} + \mu u_{xxxx} - bu_{xx} - a(\theta + \alpha\theta_t)_x = 0, \tag{1.3}$$

$$h\theta_{tt} + d\theta_t - k\theta_{xx} - au_{tx} = 0. \tag{1.4}$$

Here, $(u, \theta) = (u, \theta)(t, x)$ denote the displacement and the temperature deviation for either a smoothly bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$ for system (I), and $\Omega = (0, L) \subset \mathbb{R}^1$ for system (II), or for the Cauchy problem where $\Omega = \mathbb{R}^n$, $n \geq 1$ for system (I) and $n = 1$ for system (II), $t \geq 0$, $x \in \Omega$. The parameters $\rho, \mu, |a|, \alpha, h, d, k, b$ are positive constants.

From a mechanical point of view these two systems have a very different meaning. While the first one corresponds to the system of equations for a Green-Lindsay thermoelastic plate [4],

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Proposed running head: Decay for Green-Lindsay plates the second one describes the one-dimensional thermoelastic deformations for the Green-Lindsay strain gradient thermoelastic theory [7, pp. 257–261]. However, from the mathematical point of view both systems have a big similarity and only the coupling terms determine the big difference between them. In fact, the main aim of this paper is to clarify the consequences of this difference.

The following natural condition (cp. [4]) on the coefficients is fixed throughout the paper:

$$\alpha d - h > 0. \tag{1.5}$$

We ask for the consequence of the different coupling, in particular for the description of the qualitative and quantitative asymptotic behavior of solutions as time t tends to infinity. For bounded domains we are interested in investigating the possible exponential or non-exponential stability of the associated dynamical systems for which we additionally consider initial conditions

$$u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1, \quad \theta(0,\cdot) = \theta_0, \quad \theta_t(0,\cdot) = \theta_1, \qquad \text{in } \Omega, \tag{1.6}$$

and the Dirichlet type boundary conditions

$$u(t,\cdot) = \frac{\partial u}{\partial \nu}(t,\cdot) = 0, \qquad \theta(t,\cdot) = 0 \qquad \text{in } [0,\infty) \times \partial\Omega \tag{1.7}$$

for system (I), as well as one of the following two boundary conditions, either of Dirichlet type for both u and θ,

$$u(t,\cdot) = u_x(t,\cdot) = 0, \qquad \theta(t,\cdot) = 0 \qquad \text{in } [0,\infty) \times \{0, L\}, \tag{1.8}$$

or with hinged type conditions for u and a Neumann type condition for θ,

$$u(t,\cdot) = u_{xx}(t,\cdot) = 0, \qquad \theta_x(t,\cdot) = 0 \qquad \text{in } [0,\infty) \times \{0, L\}, \tag{1.9}$$

for system (II). We will exclude trivial stationary solutions for the latter one by assuming

$$\int_0^L \theta(t,x)\, dx = 0.$$

For the Cauchy problem, where $\Omega = \mathbb{R}^n$, we are interested in deriving polynomial decay rates for the solutions and in the question of (non-) regularity loss. By regularity loss we mean the necessity of assuming higher regularity for the initial data to assure a certain decay rate. The optimality of the decay rates will also be investigated.

It turns out that the exponential stability obtained for system (I) corresponds to no regularity loss for the Cauchy problem, while the loss of exponential stability that will be obtained for system (II) corresponds to a regularity loss. This correspondence is expected and known for example for the classical thermoelastic plate [17, 1, 22] and for the Timoshenko system [3, 10], for porous elasticity [13, 16, 19] as well as for plates with two temperatures [18, 23] considered for the Fourier law of heat conduction (with exponential stability in bounded domains / no regularity loss for the Cauchy problem) resp. for the Cattaneo law (non exponential stability / regularity loss).

In order to get an idea why system (II) might not be exponentially stable, we look at the associated characteristic polynomial. Computing au_{xt} from (1.4) and plugging it into (1.3), after having differentiated (1.3) by $a\partial_t\partial_x$, gives a single equation for θ,

$$\rho h\theta_{tttt} + \rho d\theta_{ttt} + \left[-(k\rho + a^2\alpha + bh)\partial_x^2 + h\mu\partial_x^4\right]\theta_{tt} + \left[-(a^2 + bd)\partial_x^2 + \mu d\partial_x^4\right]\theta_t - \mu k\partial_x^6\theta = 0. \tag{1.10}$$

The characteristic polynomial arises from this equation by formally replacing ∂_t by ω and ∂_x^2 by $\lambda_j := (j^2\pi^2)/L^2$, corresponding to the ansatz $\theta(t,x) = \mathrm{e}^{\omega t}\sin(\frac{j\pi x}{L})$ suitable for the Laplacian ∂_x^2 with Dirichlet boundary conditions,

$$\rho h\omega^4 + d\rho\omega^3 + \left[(k\rho + a^2\alpha + bh)\lambda_j + h\mu\lambda_j^2\right]\omega^2 + \left[(a^2 + bd)\lambda_j + \mu d\lambda_j^2)\right]\omega + \left[bk\lambda_j^2 + \mu k\lambda_j^3\right]. \tag{1.11}$$

Now we compare this with the system of classical thermoelastic plate with the Cattaneo law,

$$u_{tt} + \mu u_{xxxx} + \gamma\theta_{xx} = 0, \tag{1.12}$$

$$\theta_t + \delta q_x - \gamma u_{txx} = 0, \tag{1.13}$$

$$\tau q_t + q + \kappa\theta_x = 0, \tag{1.14}$$

with positive constants $\gamma, \delta, \tau, \kappa$. One obtains the single differential equation

$$\tau\theta_{tttt} + \theta_{ttt} + \left[-\kappa\delta\partial_x^2 + (\mu + \gamma^2)\tau\partial_x^4\right]\theta_{tt} + \left[(\mu + \gamma^2)\partial_x^4\right]\theta_t - \mu\kappa\delta\partial_x^6\theta = 0, \tag{1.15}$$

giving the characteristic polynomial

$$\tau\omega^4 + \omega^3 + \left[\kappa\delta\lambda_j + (\mu + \gamma^2)\tau\lambda_j^2\right]\omega^2 + \left[(\mu + \gamma^2)\lambda_j^2\right]\theta_t + \left[\mu\kappa\delta\lambda_j^3\right]. \tag{1.16}$$

Comparing the polynomials (1.11) and (1.16), we observe exactly the same order of powers of λ_j in front of the powers of ω.

In contrast, if one looks at the classical second-order thermoelastic bar with the Cattaneo law of heat conduction, also called thermoelasticity *with second sound*,

$$u_{tt} - \mu u_{xx} + \gamma\theta_x = 0, \tag{1.17}$$

$$\theta_t + \delta q_x + \gamma u_{tx} = 0, \tag{1.18}$$

$$\tau q_t + q + \kappa\theta_x = 0, \tag{1.19}$$

where we have the characteristic polynomial

$$\tau\omega^4 + \omega^3 + \left[(\tau\mu + \tau\gamma\delta + \delta\kappa)\lambda_j\right]\omega^2 + \left[(\mu + \gamma^2)\lambda_j\right]\omega + \left[\mu\kappa\delta\lambda_j^2\right], \tag{1.20}$$

see e.g. [9, 21], we observe the different powers in λ_j in front of the powers of ω. This system, indeed, is exponentially stable. Therefore, we expect no exponential stability for system (II).

We remark that within the discussion of the well-posedness of system (I) we will have another example, where the generator of the associated semigroup is not expected to have a compact inverse, due to combined regularity assumptions instead of separate regularities. A similar phenomenon is known in Kelvin-Voigt elasticity, cf. the discussion in [15].

We summarize the main new contributions as there are:

– The first discussion and comparison for the thermoelastic plate systems (I) and (II).

– Discovering the different qualitative behavior both for bounded and for unbounded domains ((no) exponential stability resp. (no) regularity loss).

– Proving decay rates and their optimality.

The paper is organized as follows. In Section 2 we study bounded domains and the systems (I) and (II), the latter with two different boundary conditions. The well-posedness for the three initial-boundary value problems is shown in Subsection 2.1. In Subsection 2.2 the exponential stability is proved for system (I), while in Subsection 2.3 system (II) is shown to be not exponentially stable. Section 3 treats the Cauchy problem for both systems. After the discussion of the well-posedness in Subsection 3.1, optimal polynomial decay rates are provided for system (I) in Subsection 3.2, and for system (II) in Subsection 3.3.

We use standard notation, in particular the Sobolev spaces $L^p = L^p(\Omega)$, $p \geq 1$, and $H^s = W^{s,2}(\Omega)$, $s \in \mathbf{N}_0$, with their associated norms $\|\cdot\|_{L^p}$ resp. $\|\cdot\|_{H^s}$. The usual Sobolev spaces H_0^1 and H_0^2, representing zero boundary conditions of first resp. second order, are also used. The inner product in a Hilbert space X is given by $\langle \cdot, \cdot \rangle_X$. By Id we denote the identity on some given space.

2 Bounded domains

We start in considering the systems (I) and (II) in bounded domains, system (I) with Dirichlet type boundary conditions, and system (II) with Dirichlet type or with mixed hinged-Neumann type boundary conditions. First we prove the well-posedness in Sobolev spaces. Here, system (I) is of particular interest, needing combined regularity considerations and possibly having a non-compact inverse of the generator of the associated semigroup. Second we show the exponential stability for system (I). System (II) is shown to be not exponentially stable (demonstrated for the boundary conditions (1.9)), according to the expectation raised in the introduction comparing its characteristic polynomial with that belonging to the classical thermoelastic plate with the Cattaneo law of heat conduction.

2.1 Well-posedness

2.1.1 System (I)

We consider system (I),

$$\rho u_{tt} + \mu \Delta^2 u - a\Delta(\theta + \alpha\theta_t) = 0, \tag{2.1}$$

$$h\theta_{tt} + d\theta_t - k\Delta\theta + a\Delta u_t = 0, \tag{2.2}$$

with initial conditions

$$u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1, \quad \theta(0,\cdot) = \theta_0, \quad \theta_t(0,\cdot) = \theta_1, \qquad \text{in } \Omega, \tag{2.3}$$

and the Dirichlet type boundary conditions

$$u(t,\cdot) = \frac{\partial u}{\partial \nu}(t,\cdot) = 0, \qquad \theta(t,\cdot) = 0 \qquad \text{in } [0,\infty) \times \partial\Omega. \tag{2.4}$$

A transformation to a first-order system by defining $W := (u, u_t, \theta, \theta_t)' \equiv (u, v, \theta, \psi)'$, where $'$ denotes the transposed matrix, yields

$$W_t = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\mu}{\rho}\Delta^2 & 0 & \frac{a}{\rho}\Delta & \frac{a}{\rho}\alpha\Delta \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{a}{h}\Delta & \frac{k}{h}\Delta & -\frac{d}{h} \end{pmatrix} W \equiv A_{I,f}W, \qquad W(0,\cdot) = W^0 := (u_0, u_1, \theta_0, \theta_1)'. \tag{2.5}$$

This system with the formal differential symbol $A_{I,f}$ will be considered as an evolution equation in the Hilbert space

$$\mathcal{H}_I := H_0^2 \times L^2 \times H_0^1 \times L^2,$$

with inner product

$$\begin{aligned} \langle W, W^* \rangle_{\mathcal{H}_I} \quad := \quad & \mu \langle \Delta u, \Delta u^* \rangle_{L^2} + \rho \langle v, v^* \rangle_{L^2} + \left(d - \frac{h}{\alpha}\right) \langle \theta, \theta^* \rangle_{L^2} + \\ & \alpha k \langle \nabla\theta, \nabla\theta^* \rangle_{L^2} + \frac{h}{\alpha} \langle \theta + \alpha\psi, \theta^* + \alpha\psi^* \rangle_{L^2}. \end{aligned}$$

The factor $\left(d - \frac{h}{\alpha}\right)$ appearing is positive because of the general assumption (1.5). Moreover, the induced norm $\|\cdot\|_{\mathcal{H}_I}$ is equivalent to the standard norm in $H_0^2 \times L^2 \times H_0^1 \times L^2$,

$$C_1 \|W\|_{\mathcal{H}_I} \le \|u\|_{H^2} + \|v\|_{L^2} + \|\theta\|_{H^1} + \|\psi\|_{L^2} \le C_2 \|W\|_{\mathcal{H}_I},$$

with positive constants C_1, C_2. While the estimate from below is obvious, the estimate from above can be obtained observing the ellipticity of the Laplace operator on $H^2 \cap H_0^1$ and the estimate

$$\left(d - \frac{h}{\alpha}\right) \|\theta\|_{L^2}^2 + \frac{h}{\alpha} \|\theta + \alpha\psi\|_{L^2}^2 \ge (d - h\varepsilon)\|\theta\|_{L^2}^2 + h\left(\alpha - \frac{1}{\varepsilon}\right) \|\psi\|_{L^2}^2,$$

for any $\varepsilon > 0$. The factors in front of the norms are both positive if $\frac{1}{\alpha} < \varepsilon < \frac{d}{h}$, for which the choice of ε is possible because of the assumption (1.5).

The proper choice of the inner product is essential for an appropriate treatment.

Then

$$W_t = A_I W, \quad W(t=0) = W^0, \tag{2.6}$$

where

$$A_I : D(A_I) \subset \mathcal{H}_I \to \mathcal{H}_I, \qquad A_I W := A_{I,f} W, \tag{2.7}$$

for $W \in D(A_I)$ with

$$D(A_I) := \{\, W = (u, v, \theta, \psi)' \in \mathcal{H}_I \,|\, v \in H_0^2,\ \psi \in H_0^1,\ \theta \in H^2,\ \Delta(\mu\Delta u - a\alpha\psi)) \in L^2 \,\}. \tag{2.8}$$

In the definition of $D(A_I)$, the problem of the missing (separate) regularity is reflected. One just has the combined regularity $\Delta(\mu\Delta u - a\alpha\psi) \in L^2$, not writing $\Delta^2 u,\ \Delta\psi \in L^2$, and this way $A_{I,f}V$ has to be interpreted. As for viscoelastic systems (e.g. [15]) this nourishes the expectation that the inverse A_I^{-1} is not a compact operator.

Lemma 2.1. *$D(A_I)$ is dense in $\mathcal{H}_I$, and for $W = (u, v, \theta, \psi)' \in D(A_I)$,*

$$\mathrm{Re}\,\langle A_I W, W\rangle_{\mathcal{H}_I} = -\left((d\alpha - h)\|\psi\|_{L^2}^2 + k\|\nabla\theta\|_{L^2}^2\right) \leq 0, \tag{2.9}$$

i.e. A_I is dissipative.

Proof: $(C_0^\infty)^4$ is contained in $D(A_I)$ and dense in $\mathcal{H}_I$. (2.9) is easy to compute.
□

Lemma 2.2. *0 is in the resolvent set $\varrho(A_I)$, $R(A_I) = \mathcal{H}_I$.*

Proof: We will show the assertion in proving that, for any $F \in \mathcal{H}_I$, the equation $A_I W = F$ has a (unique) solution satisfying $\|W\|_{\mathcal{H}_I} \leq c\|F\|_{\mathcal{H}_I}$, with c being independent of F, W. Here, and in the sequel, c will be used to denote various constants.

Let $F = (F_1, F_2, F_3, F_4)' \in \mathcal{H}_I$. To solve $A_I W = F$, we first define

$$v := F_1 \in H_0^2, \qquad \|v\|_{H^2} = \|F_1\|_{H^2} \leq \|F\|_{\mathcal{H}_I},$$

$$\psi := F_3 \in H_0^1, \qquad \|v\|_{H^1} = \|F_3\|_{H^1} \leq \|F\|_{\mathcal{H}_I},$$

then we solve

$$k\Delta\theta = hF_4 + dF_3 + a\Delta F_1 =: G_I,$$

with

$$\theta \in H^2 \cap H_0^1, \qquad \|\theta\|_{H^2} \leq c\|G_I\|_{L^2} \leq c\|F\|_{\mathcal{H}_I}.$$

Finally, we have to solve

$$\Delta\left(-\Delta u + a\alpha\psi\right) = \rho F_2 - a\Delta\theta \in L^2, \tag{2.10}$$

with $u \in H_0^2$ (and $\psi = F_3 \in H_0^1$).

We remark that we cannot solve $-\mu\Delta^2 u = \rho F_2 - a\Delta\theta - a\alpha\Delta\psi$ in H^4, since $\Delta\psi$ is not known to be in L^2. We only have the *combined* H^2-regularity of $-\Delta u + a\alpha\psi$.

To solve (2.10), we consider the following bilinear form,

$$B : H_0^2 \times H_0^2 \to \mathbb{C}, \qquad (u, \varphi) \mapsto \mu\langle\Delta u, \Delta\varphi\rangle_{L^2}$$

and the continuous linear functional

$$f : H_0^2 \to \mathbb{C}, \qquad \varphi \mapsto f(\varphi) := -\rho\langle F_2, \varphi\rangle_{L^2} + a\langle\Delta\theta, \varphi\rangle_{L^2} - a\alpha\langle\nabla\psi, \nabla\varphi\rangle_{L^2}.$$

Then B is a strongly coercive sesquilinear form, and, by Lax-Milgram, there is a unique $u \in H_0^2$ satisfying

$$\forall\,\varphi \in H_0^2 :\ B(u, \varphi) = f(\varphi), \qquad \|u\|_{H^2} \leq c\|f\|_{H^2 \to \mathbb{C}} \leq c\|F\|_{\mathcal{H}_I}.$$

This solves (2.10), and we have $W = (u, v, \theta, \psi)' \in D(A_I)$ with $A_I W = F$ and $\|W\|_{\mathcal{H}_I} \leq c\|F\|_{\mathcal{H}_I}$. □

By the Lumer-Phillips theorem we conclude

Theorem 2.3. *A_I generates a contraction semigroup, and, for any $W^0 \in D(A_I)$, the a unique solution V to (2.6) satisfying*

$$W \in C^1([0,\infty), \mathcal{H}_I) \cap C^0([0,\infty), D(A_I)).$$

Lemma 2.4. *$i\mathbb{R} \subset \varrho(A_I)$.*

Proof: We do not know if (and not expect that) A_I^{-1} is compact – due to the combined regularity mentioned above, so the spectrum of A_I may consist of more than just eigenvalues (cp. the easier arguments in the next subsections). Let

$$N := \{ R > 0 \mid [-iR, iR] \subset \varrho(A_I) \}.$$

Since $0 \in \varrho(A_I)$ we have that N is non-empty, and

$$\lambda^* := \sup N > 0.$$

If $\lambda^* = \infty$, we are done. So let us assume $0 < \lambda^* < \infty$, which will lead to a contradiction. Then, w.l.o.g., we may assume the existence of a sequence $(\lambda_n)_n \subset \mathbb{R}$ such that $i\lambda_n \in \varrho(A_I)$ with $\lambda_n \to \lambda^*$, and $(W_n)_n \subset D(A_I)$ with $\|W_n\|_{\mathcal{H}_I} = 1$ and $(i\lambda_n - A_I)W_n \to 0$, as $n \to \infty$. This implies

$$i\lambda_n u_n - v_n \quad \to \quad 0 \qquad \text{in } H^2, \tag{2.11}$$

$$i\lambda_n v_n - \frac{1}{\rho}\left(\Delta(-\mu\Delta u_n + a\alpha\psi_n) + \alpha\Delta\theta_n\right) \quad \to \quad 0 \qquad \text{in } L^2, \tag{2.12}$$

$$i\lambda_n \theta_n - \psi_n \quad \to \quad 0 \qquad \text{in } H^1, \tag{2.13}$$

$$i\lambda_n \psi_n - \frac{1}{h}\left(-a\Delta v_n + k\Delta\theta_n - d\psi_n\right) \quad \to \quad 0 \qquad \text{in } L^2. \tag{2.14}$$

Since $\mathrm{Re}\,\langle AW, W\rangle_{\mathcal{H}_I} \to 0$ we obtain by (2.9)

$$\psi_n \to 0 \quad \text{in } L^2, \quad \theta_n \to 0 \quad \text{in } H^1. \tag{2.15}$$

Combining (2.14) and (2.15) we get

$$a\Delta v_n - k\Delta\theta_n \to 0 \quad \text{in } L^2, \tag{2.16}$$

implying, using (2.11),

$$a\Delta u_n + \frac{k}{i\lambda_n}\Delta\theta_n = \underbrace{\frac{1}{i\lambda_n}\left(a\Delta v_n - k\Delta\theta_n\right)}_{=:q_n} \to 0 \quad \text{in } L^2. \tag{2.17}$$

Thus, $\langle q_n, \Delta u_n\rangle_{L^2} \to 0$, implying

$$a\|\Delta u_n\|_{L^2}^2 - \frac{k}{i\lambda_n}\langle \nabla\theta_n, \nabla\Delta u_n\rangle_{L^2} \to 0. \tag{2.18}$$

We conclude from (2.12)

$$\frac{1}{i\lambda_n\rho}\left(\Delta(-\mu\Delta u_n + a\alpha\psi_n) + a\Delta\theta_n\right) = v_n + \mathcal{O}(1) \quad \text{is bounded in } L^2. \tag{2.19}$$

Multiplying the right-hand side of (2.19) by Δu_n in L^2, we get

$$\mu\lambda_n\|\frac{\nabla\Delta u_n}{\lambda_n}\|_{L^2}^2 - a\alpha\langle\nabla\psi_n, \frac{\nabla\Delta u_n}{\lambda_n}\rangle_{L^2} - a\langle\nabla\theta_n, \nabla\Delta u_n\rangle_{L^2} \quad \text{is bounded.}$$

Writing $\nabla\psi_n = i\lambda_n\nabla\theta_n + p_n$, with $p_n \to 0$ according to (2.13), we obtain that

$$Z_n := \mu\lambda_n\|\frac{\nabla\Delta u_n}{\lambda_n}\|_{L^2}^2 - a\langle\nabla\theta_n, \frac{\nabla\Delta u_n}{\lambda_n}\rangle_{L^2} - a\alpha\lambda_n\langle i\nabla\theta_n, \frac{\nabla\Delta u_n}{\lambda_n}\rangle_{L^2} - a\alpha\langle p_n, \frac{\nabla\Delta u_n}{\lambda_n}\rangle_{L^2} \tag{2.20}$$

is bounded. Since, for n large enough,

$$Z_n \geq \frac{\mu\lambda^*}{2}\|\frac{\nabla\Delta u_n}{\lambda_n}\|_{L^2}^2 - c\|\nabla\theta_n\|_{L^2}^2 - c\|p_n\|_{L^2}^2, \tag{2.21}$$

we conclude

$$\|\frac{\nabla\Delta u_n}{\lambda_n}\|_{L^2} \quad \text{is bounded in } L^2.$$

This implies, using (2.18) and (2.15),

$$u_n \to 0 \quad \text{in } H^2. \tag{2.22}$$

Using (2.11) we finally get

$$v_n \to 0 \quad \text{in } L^2. \tag{2.23}$$

Combining (2.15), (2.22) and (2.23) we conclude $W_n \to 0$ in $\mathcal{H}_I$ contradicting $\|W_n\|_{\mathcal{H}_I} = 1$. □

The proven fact that the spectrum of A_I is *strictly* to the left of the imaginary axis fits to the exponential stability which will be proved in Subsection 2.2. But we will see in Subsection 2.1.3 in combination with Subsection 2.3 that for System (II) the spectrum is also strictly in the left-hand plane but without having exponential stability.

2.1.2 System (II) with boundary conditions (1.8)

We now consider system (II),

$$\rho u_{tt} + \mu_{xxxx} - bu_{xx} - a(\theta + \alpha\theta_t)_x = 0, \tag{2.24}$$

$$h\theta_{tt} + d\theta_t - k\theta_{xx} - au_{tx} = 0. \tag{2.25}$$

with initial conditions

$$u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1, \quad \theta(0,\cdot) = \theta_0, \quad \theta_t(0,\cdot) = \theta_1, \qquad \text{in } \Omega, \tag{2.26}$$

and the Dirichlet type boundary conditions

$$u(t,\cdot) = u_x(t,\cdot) = 0, \qquad \theta(t,\cdot) = 0 \qquad \text{in } [0,\infty)\times\{0,L\}. \tag{2.27}$$

A transformation to a first-order system by defining $W := (u, u_t, \theta, \theta_t)' \equiv (u, v, \theta, \psi)'$ yields

$$W_t = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\mu}{\rho}\partial_x^4 + \frac{b}{\rho}\partial_x^2 & 0 & \frac{a}{\rho}\partial_x & \frac{a}{\rho}\alpha\partial_x \\ 0 & 0 & 0 & 1 \\ 0 & \frac{a}{h}\partial_x & \frac{k}{h}\partial_x^2 & -\frac{d}{h} \end{pmatrix} V \equiv A_{II,f}W, \qquad W(0,\cdot) = W^0 := (u_0, u_1, \theta_0, \theta_1)'. \tag{2.28}$$

This system with the formal differential symbol $A_{II,f}$ will be considered as an evolution equation in the Hilbert space

$$\mathcal{H}_{II} := H_0^2 \times L^2 \times H_0^1 \times L^2,$$

with the same inner product as for $\mathcal{H}_I$ above,

$$\langle W, W^* \rangle_{\mathcal{H}_{II}} := \langle W, W^* \rangle_{\mathcal{H}_I}.$$

Then

$$W_t = A_{II}W, \quad W(t=0) = W^0, \tag{2.29}$$

where

$$A_{II} : D(A_{II}) \subset \mathcal{H}_{II} \to \mathcal{H}_{II}, \qquad A_{II}W := A_{II,f}W, \tag{2.30}$$

for $W \in D(A_{II})$ with

$$D(A_{II}) := \{\, W \in \mathcal{H}_{II} \,|\, v \in H_0^2,\ \psi \in H_0^1,\ \theta \in H^2,\ \partial_x^4 u \in L^2 \,\}. \tag{2.31}$$

In the definition of $D(A_{II})$, the problem of the missing (separate) regularity mentioned for A_I in system (I) is not appearing. As for system (I) we have

Lemma 2.5. *$D(A_{II})$ is dense in $\mathcal{H}_{II}$, and for $W = (u, v, \theta, \psi)' \in D(A_{II})$,*

$$\mathrm{Re}\,\langle A_{II}W, W \rangle_{\mathcal{H}_{II}} = -\left((d\alpha - h)\|\psi\|_{L^2}^2 + k\|\theta_x\|_{L^2}^2\right) \le 0, \tag{2.32}$$

i.e. A_{II} is dissipative.

Lemma 2.6. *0 is in the resolvent set $\varrho(A_{II})$, $R(A_{II}) = \mathcal{H}_{II}$.*

Proof: Again we will show the assertion in proving that, for any $F \in \mathcal{H}_{II}$, the equation $A_{II}W = F$ has a (unique) solution satisfying $\|W\|_{\mathcal{H}_{II}} \le c\|F\|_{\mathcal{H}_{II}}$, with being independent of F, W. Let $F = (F_1, F_2, F_3, F_4)' \in \mathcal{H}_{II}$. To solve $A_{II}W = F$, we first define

$$v := F_1 \in H_0^2, \qquad \|v\|_{H^2} = \|F_1\|_{H^2} \le \|F\|_{\mathcal{H}_{II}},$$

$$\psi := F_3 \in H_0^1, \qquad \|v\|_{H^1} = \|F_3\|_{H^1} \le \|F\|_{\mathcal{H}_{II}},$$

then we solve

$$k\partial_x^2\theta = hF_4 + dF_3 - a\partial F_1 =: G_{II},$$

with

$$\theta \in H^2 \cap H_0^1, \qquad \|\theta\|_{H^2} \leq c\|G_{II}\|_{L^2} \leq c\|F\|_{\mathcal{H}_{II}}.$$

Finally, we solve

$$(\mu\partial_x^4 - b\partial_x^2)u = \rho F_2 - a\theta_x - a\alpha\psi_x \in L^2, \tag{2.33}$$

with $u \in H^4 \cap H_0^2$ (and $\psi = F_3 \in H_0^1$), observing that $B := (\mu\partial_x^4 - b\partial_x^2) : H^4 \cap H_0^2 \to L^2$ is a homeomorphism, yielding

$$\|u\|_{H^4} \leq c\|F\|_{\mathcal{H}_{II}}.$$

This solves (2.33),and we have $W = (u, v, \theta, \psi)' \in D(A_{II})$ with $A_{II}W = F$ and $\|W\|_{\mathcal{H}_{II}} \leq c\|F\|_{\mathcal{H}_{II}}$. □

By the Lumer-Phillips theorem we conclude again

Theorem 2.7. *A_{II} generates a contraction semigroup, and, for any $W^0 \in D(A_{II})$, there is a unique solution V to (2.29) satisfying*

$$W \in C^1\left([0,\infty), \mathcal{H}_{II}\right) \cap C^0\left([0,\infty), D(A_{II})\right).$$

As a corollary from the estimates in the proof above we obtain

Lemma 2.8. *$A_{II}^{-1} : H_{II} \to H_{II}$ is a compact operator.*

In view of this, to prove the following property of the spectrum, it will be sufficient to exclude purely imaginary eigenvalues.

Lemma 2.9. *$i\mathbb{R} \subset \varrho(A_{II})$.*

Proof: Let $A_{II}W = i\beta W$. $\beta \in \mathbb{R} \setminus \{0\}$. Then $\mathrm{Re}\,\langle AW, W\rangle_{\mathcal{H}_{II}} = \mathrm{Re}\,\langle i\beta W, W\rangle_{\mathcal{H}_{II}} = 0$, and the dissipativity (2.32) yields $\theta = \psi = 0$. From the equations we then successively conclude $v = 0$ and $u = 0$, hence $W = 0$, i.e. there are no purely imaginary eigenvalues. □

2.1.3 System (II) with boundary conditions (1.9)

Here we consider system (II)

$$\rho u_{tt} + \mu_{xxxx} - bu_{xx} - a(\theta + \alpha\theta_t)_x = 0, \tag{2.34}$$

$$h\theta_{tt} + d\theta_t - k\theta_{xx} - au_{tx} = 0. \tag{2.35}$$

with initial conditions

$$u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1, \quad \theta(0,\cdot) = \theta_0, \quad \theta_t(0,\cdot) = \theta_1, \qquad \text{in } \Omega, \tag{2.36}$$

and the mixed hinged type (for u) and Neumann type (for θ) boundary conditions

$$u(t,\cdot) = u_{xx}(t,\cdot) = 0, \qquad \theta_x(t,\cdot) = 0 \qquad \text{in } [0,\infty) \times \{0, L\}. \tag{2.37}$$

For θ we require (to avoid the trivial solution $u = 0$, $\theta =$ constant)

$$\int_0^L \theta(t,x)\,dx = 0, \qquad t \geq 0. \tag{2.38}$$

A transformation to a first-order system by defining $W := (u, u_t, \theta, \theta_t)' \equiv (u, v, \theta, \psi)'$ yields

$$W_t = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\mu}{\rho}\partial_x^4 + \frac{b}{\rho}\partial_x^2 & 0 & \frac{a}{\rho}\partial_x & \frac{a}{\rho}\alpha\partial_x \\ 0 & 0 & 0 & 1 \\ 0 & \frac{a}{h}\partial_x & \frac{k}{h}\partial_x^2 & -\frac{d}{h} \end{pmatrix} V \equiv A_{III,f}W, \;\; W(0,\cdot) = W^0 := (u_0, u_1, \theta_0, \theta_1)'. \tag{2.39}$$

This system with the formal differential symbol $A_{III,f}$ will be considered as an evolution equation in the Hilbert space

$$\mathcal{H}_{III} := \left(H^2 \cap H_0^1\right) \times L^2 \times H_*^1 \times L_*^2,$$

where

$$L_*^2 := \{\, g \in L^2 \mid \int_0^L g(x)\,dx = 0\}, \qquad H_*^1 := H^1 \cap L_*^2,$$

and with the same inner product as for $\mathcal{H}_I$ above,

$$\langle W, W^* \rangle_{\mathcal{H}_{III}} := \langle W, W^* \rangle_{\mathcal{H}_I}.$$

Then

$$W_t = A_{III}W, \quad W(t=0) = W^0, \tag{2.40}$$

where

$$A_{III} : D(A_{III}) \subset \mathcal{H}_{III} \to \mathcal{H}_{III}, \qquad A_{III}W := A_{III,f}W, \tag{2.41}$$

for $W \in D(A_{III})$ with

$$D(A_{III}) := \{\, W \in \mathcal{H}_{III} \,|\, v \in H^2 \cap H_0^1,\, \psi \in H_*^1,\, \theta \in H^2,\, \theta_x \in H_0^1,\, u \in H^4,\, u_{xx} \in H_0^1 \,\}. \tag{2.42}$$

As for system (II) we have

Lemma 2.10. *$D(A_{III})$ is dense in $\mathcal{H}_{III}$, and for $W = (u, v, \theta, \psi)' \in D(A_{III})$,*

$$\mathrm{Re}\,\langle AW, W \rangle_{\mathcal{H}_{III}} = -\left((d\alpha - h)\|\psi\|_{L^2}^2 + k\|\theta_x\|_{L^2}^2\right) \leq 0, \tag{2.43}$$

i.e. A_{III} is dissipative.

Similarly we obtain

Lemma 2.11. 0 *is in the resolvent set $\varrho(A_{III})$.*

Proof: Let $F = (F_1, F_2, F_3, F_4)' \in \mathcal{H}_{III}$. To solve $A_{III}W = F$, we first define

$$v := F_1 \in H_0^2, \qquad \|v\|_{H^2} = \|F_1\|_{H^2} \leq \|F\|_{\mathcal{H}_{III}},$$

$$\psi := F_3 \in H_0^1, \qquad \|v\|_{H^1} = \|F_3\|_{H^1} \le \|F\|_{\mathcal{H}_{III}},$$

then we solve

$$k\partial_x^2\theta = hF_4 + dF_3 - a\partial_x F_1 =: G_{III},$$

with G_{III} belonging to L_*^2 and with

$$\theta \in H^2 \cap H_*^1, \qquad \|\theta\|_{H^2} \le c\|G_{III}\|_{L^2} \le c\|F\|_{\mathcal{H}_{III}}.$$

Finally, we solve

$$(\mu\partial_x^4 - b\partial_x^2)u = \rho F_2 - a\theta_x - a\alpha\psi_x \in L^2, \tag{2.44}$$

with $u \in H^4$, $u_{xx} \in H_0^1$ (and $\psi = F_3 \in H_0^1$), yielding

$$\|u\|_{H^4} \le c\|F\|_{\mathcal{H}_{III}}.$$

This solves (2.44),and we have $W = (u, v, \theta, \psi)' \in D(A_{III})$ with $A_{III}W = F$ and $\|W\|_{\mathcal{H}_{III}} \le c\|F\|_{\mathcal{H}_{III}}$. □

By the Lumer-Phillips theorem we conclude again

Theorem 2.12. *A_{III} generates a contraction semigroup, and, for any $W^0 \in D(A_{III})$, there is a unique solution V to* (2.40) *satisfying*

$$W \in C^1([0,\infty), \mathcal{H}_{III}) \cap C^0([0,\infty), D(A_{III})).$$

Again we have as a corollary from the estimates above

Lemma 2.13. *$A_{III}^{-1} : H_{III} \to H_{III}$ is a compact operator.*

Finally we also get

Lemma 2.14. *$i\mathbb{R} \subset \varrho(A_{III})$.*

Proof: Let $A_{III}W = i\beta W$. $\beta \in \mathbb{R} \setminus \{0\}$. Then $\mathrm{Re}\,\langle AW, W\rangle_{\mathcal{H}_{III}} = \mathrm{Re}\,\langle i\beta W, W\rangle_{\mathcal{H}_{III}} = 0$, and the dissipativity (2.43) yields $\theta_x = \psi = 0$. Since $\theta \in H_*^1$, we conclude $\theta = 0$. From the equations we then successively conclude $v = 0$ and $u = 0$, hence $W = 0$, i.e. there are no purely imaginary eigenvalues. □

As for system (II) with the Dirichlet type boundary conditions in the previous subsection as well as for system (I) we have that the spectrum is *strictly* contained in the left-hand plane, but, in contrast to system (I), we will *not* have exponential stability, see Subsection 2.3.

2.2 Exponential stability for system (I)

We consider system (I), the initial boundary value problem (2.1)–(2.4). We use the following characterization of exponential stability given in [12], going back to Gearhart [6], Huang [5] and Prüß [20].

Theorem 2.15. *Let $\{e^{t\mathcal{A}_*}\}_{t\geq 0}$ be a C_0-semigroup of contractions generated by the operator $\mathcal{A}_*$ in the Hilbert space $\mathcal{H}_*$. Then the semigroup is exponentially stable if and only if $i\mathbb{R} \subset \varrho(\mathcal{A}_*)$ and*

$$\overline{\lim}_{|\beta|\to\infty} ||(i\beta \mathrm{Id} - \mathcal{A}_*)^{-1}|| < \infty, \qquad \beta \in \mathbb{R}. \tag{2.45}$$

For $A_* = A_I$ we know already from Lemma 2.4 that $i\mathbb{R} \subset \varrho(A_I)$. We now assume that (2.45) does not hold. Then, as in the proof of Lemma 2.4, there exists a sequence $(\lambda_n)_n \subset \mathbb{R}$ such that $i\lambda_n \in \varrho(A_I)$ with $\lambda_n \to \infty$ (now), and $(W_n)_n \subset D(A_I)$ with $\|W_n\|_{\mathcal{H}_I} = 1$ and

$$(i\lambda_n - A_I)W_n \to 0,$$

as $n \to \infty$. This implies

$$i\lambda_n u_n - v_n \to 0 \quad \text{in } H^2, \tag{2.46}$$

$$i\lambda_n v_n - \frac{1}{\rho}\left(\Delta(-\mu\Delta u_n + a\alpha\psi_n) + \alpha\Delta\theta_n\right) \to 0 \quad \text{in } L^2, \tag{2.47}$$

$$i\lambda_n \theta_n - v_n \to 0 \quad \text{in } H^1, \tag{2.48}$$

$$i\lambda_n \psi_n - \frac{1}{h}\left(-a\Delta v_n + k\Delta\theta_n - d\psi_n\right) \to 0 \quad \text{in } L^2. \tag{2.49}$$

We conclude as in (2.15)

$$\psi_n \to 0 \quad \text{in } L^2, \quad \theta_n \to 0 \quad \text{in } H^1. \tag{2.50}$$

Similarly defining

$$Z_n := \mu\lambda_n \|\frac{\nabla\Delta u_n}{\lambda_n}\|^2_{L^2} - a\langle\nabla\theta_n, \frac{\nabla\Delta u_n}{\lambda_n}\rangle_{L^2} - a\alpha\lambda_n\langle i\nabla\theta_n, \frac{\nabla\Delta u_n}{\lambda_n}\rangle_{L^2} - a\alpha\langle p_n, \frac{\nabla\Delta u_n}{\lambda_n}\rangle_{L^2}, \tag{2.51}$$

we have that $(Z_n)_n$ is bounded. We now estimate as follows, for n large enough,

$$Z_n \geq \frac{\mu\lambda_n}{4}\|\frac{\nabla\Delta u_n}{\lambda_n}\|^2_{L^2} - \frac{\mu}{2}\|\nabla\theta_n\|^2_{L^2} - \frac{1}{2}\|p_n\|^2_{L^2} - \frac{\lambda_n}{2\mu}\|\nabla\theta_n\|^2_{L^2}, \tag{2.52}$$

implying that

$$\|\frac{\nabla\Delta u_n}{\lambda_n}\|_{L^2} \quad \text{is bounded in } L^2.$$

Again we conclude that

$$u_n \to 0 \quad \text{in } H^2. \tag{2.53}$$

Finally, multiplying (2.12) by u_n in L^2, we get, using (2.11), (2.50),

$$-\|v_n\|^2_{L^2} - \langle v_n, i\lambda u_n - v_n\rangle_{L^2} - \frac{1}{\rho}\|\Delta u_n\|^2_{L^2} + a\langle\theta_n, \Delta u_n\rangle_{L^2} + a\alpha\langle\psi_n, \Delta u_n\rangle_{L^2} \to 0,$$

implying

$$v_n \to 0 \quad \text{in } L^2. \tag{2.54}$$

Combining (2.50), (2.53) and (2.54) we conclude $W_n \to 0$ in $\mathcal{H}_I$ contradicting $\|W_n\|_{\mathcal{H}_I} = 1$. Thus we have proved

Theorem 2.16. *The semigroup $\{e^{tA_I}\}_{t\geq 0}$ is exponentially stable.*

We recall here the plate system for the Lord-Shulman theory [17] with non-exponential behavior, but we have seen that for the Green-Lindsay system the decay is exponential. This is similar to what we noticed in the case of thermoelasticity with two temperatures in [11]. Therefore, we can emphasize the fact that the coupling for Green-Lindsay is stronger than for Lord-Shulman, and we have two examples where these two theories (with hyperbolic heat conduction) show a different behavior.

2.3 Non-exponential stability for system (II)

As we anticipated in the introduction in comparing the characteristic polynomials of system (II)) (1.11) with the polynomial (1.16) which arises in thermoelastic plates with the Cattaneo law (1.12)–(1.14), we will now show that system (II) (1.3)–(1.4) with boundary conditions (1.9) is not exponentially stable. The proof uses an appropriate ansatz being compatible with these boundary conditions and demonstrating that there are (arbitrarily) slowly decaying solutions – *slowly* in comparison to an exponential type. The arguments will use the Hurwitz criterion similar to the situation for two-temperature plate systems with the Cattaneo law in [18].

So we discuss system (II), (1.3)–(1.4),

$$\rho u_{tt} + \mu u_{xxxx} - b u_{xx} - a(\theta + \alpha\theta_t)_x = 0, \tag{2.55}$$

$$h\theta_{tt} + d\theta_t - k\theta_{xx} - a u_{tx} = 0, \tag{2.56}$$

with the boundary conditions

$$u(t,\cdot) = u_{xx}(t,\cdot) = 0, \qquad \theta(t,\cdot) = 0 \quad \text{on } \{0, L\}. \tag{2.57}$$

Theorem 2.17. *The system (2.55)–(2.57) is not exponentially stable.*

Proof: Assuming w.l.o.g. $L = \pi$, we make the ansatz, for $j \in \mathbb{N}$,

$$u_j(t,x) = q_j(t)\sin(jx), \qquad \theta_j(t,x) = p_j(t)\cos(jx),$$

This ansatz is compatible with the differential equations (2.55)–(2.56) and with the boundary conditions (2.57). It gives a solution (u_j, θ_j) if the coefficients (q_j, p_j) satisfy the following system of ODEs, where a prime $'$ denotes here differentiation with respect to time t,

$$\left.\begin{aligned} \rho q_j'' + \mu j^4 q_j + b j^2 q_j + a j p_j + a\alpha j p_j' &= 0, \\ h p_j'' + d p_j' + k j^2 p_j - a j q_j' &= 0. \end{aligned}\right\} \tag{2.58}$$

System (2.58) is equivalent to a first-order system for the column vector $V_j := (q_j, q_j', p_j, p_j')$,

$$V_j' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\mu}{\rho}j^4 - \frac{b}{\rho}j^2 & 0 & -\frac{a}{\rho}j & -\frac{a}{\rho}\alpha j \\ 0 & 0 & 0 & 1 \\ 0 & \frac{a}{h}j & -\frac{k}{h}j^2 & -\frac{d}{h} \end{pmatrix} V_j \equiv \mathcal{A}_j V_j. \tag{2.59}$$

We are looking for solutions to (2.59) of type $V_j(t) = \mathrm{e}^{\omega_j t} V_j^0$. In other words, ω_j has to be an eigenvalue of $\mathcal{A}_j$ with eigenvector V_j^0 as initial data. It is the aim to demonstrate that, for any given small $\varepsilon > 0$, we have some j and some eigenvalue ω_j such that the real part $\mathrm{Re}\,\omega_j$ of ω_j is larger than $-\varepsilon$. This will contradict the exponential stability. We have

$$\begin{aligned}
\det(\omega\,\mathrm{Id} - \mathcal{A}_j) &= \frac{1}{\rho h)}\left\{\rho h\omega^4 + d\rho\omega^3 + \left[(k\rho + a^2\alpha + bh)j^2 + h\mu j^4\right]\omega^2\right. \\
&\quad \left. + \left[(a^2+bd)j^2 + \mu d j^4)\right]\omega + \left[bkj^4 + \mu k j^6\right]\right\} \\
&\equiv \frac{1}{\rho h} P_j(\omega) \equiv \frac{1}{\rho h}\sum_{j=0}^{4} \alpha_j w^j.
\end{aligned}$$

We remark that the polynomial P_j is, of course, the same as the characteristic polynomial obtained directly from the differential equations (2.55), (2.56) in (1.11).
To show that

$$\forall\,\varepsilon > 0 \;\; \exists\, j \;\; \exists\,\omega_j,\; P_j(\omega_j) = 0: \;\; \mathrm{Re}\,\omega_j \geq -\varepsilon,$$

we introduce, for small $\varepsilon > 0$,

$$z := \omega + \varepsilon, \qquad P_{j,\varepsilon}(z) := P_j(z - \varepsilon).$$

That is, we have to show

$$\forall\; 0 < \varepsilon \ll 1 \;\; \exists\, j \;\; \exists\, z_j,\; P_{j,\varepsilon}(z_j) = 0: \;\; \mathrm{Re}\, z_j \geq 0. \tag{2.60}$$

To prove (2.60) we start with computing

$$P_{j,\varepsilon}(z) = q_4 z^4 + q_3 z^3 + q_2 z^2 + q_1 z + q_0$$

where

$$\begin{aligned}
q_4 &:= \rho h, \\
q_3 &:= -4\rho h\varepsilon + d\rho, \\
q_2 &:= 6\rho h\varepsilon^2 - 3d\rho\varepsilon + (k\rho + a^2\alpha + bh)j^2 + h\mu j^4, \\
q_1 &:= -\rho h\varepsilon^3 + 3d\rho\varepsilon^2 - 2\left\{(k\rho + a^2\alpha + bh)j^2 + h\mu j^4\right\} + (a^2 + bd)j^2 + \mu d j^4, \\
q_0 &:= \rho h\varepsilon^4 - \rho d\varepsilon^3 - \left\{(k\rho + a^2\alpha + bh)j^2 + h\mu j^4\right\}) \\
&\qquad - \left\{(a^2 + bd)j^2 + \mu d j^4\right\}\varepsilon + bkj^4 + \mu k j^6.
\end{aligned}$$

The coefficients $q_0, \ldots, q_4$ are positive for sufficiently small ε and large j. We use the Hurwitz criterion [26]: Let

$$\mathbb{H}^j := \begin{pmatrix} q_3 & q_4 & 0 & 0 \\ q_1 & q_2 & q_3 & q_4 \\ 0 & q_0 & q_1 & q_2 \\ 0 & 0 & 0 & q_0 \end{pmatrix}$$

denote the Hurwitz matrix associated to the polynomial $P_{j,\varepsilon}$. Then (2.60) is fulfilled if we find, for given small $\varepsilon > 0$, a (sufficiently large) index j such that one of the principal minors of $\mathbb{H}^j$ is not positive. The principal minors are given by the determinants $\det D_m^j$ of the matrices D_m^j, for $m = 1, 2, 3, 4$, where D_m^j denotes the upper left square submatrix of $\mathbb{H}^j$ consisting of the elements $\mathbb{H}_{11}^j, \ldots, \mathbb{H}_{mm}^j$.

We compute

$$\det D_{j,2} = q_3 q_2 - q_1 q_4$$

with

$$\begin{aligned} q_3 q_2 &= -4\varepsilon\rho h^2 \mu j^4 + d\rho h \mu j^4 + \mathcal{O}(j^2), \\ q_1 q_4 &= -2\varepsilon\rho h^2 \mu j^4 + d\rho h \mu j^4 + \mathcal{O}(j^2), \end{aligned}$$

hence

$$\det D_{j,2} = -2\varepsilon\rho h^2 \mu j^4 + \mathcal{O}(j^2) < 0$$

as $j \to \infty$. □

It is worth noting that in the case of classical parabolic heat conduction (i.e. $h = 0$, $\alpha = 0$) the decay is exponential [2]. Furthermore, in case that we also assume the existence of microtemperatures, the semigroup is exponentially stable and analytic [8]. We also recall that one-dimensional porous-elasticity for the Green-Lindsay theory was considered in [14], and the authors proved that, generically, the decay is also slower than of exponential type.

3 The Cauchy problem

Now we look at the Cauchy problem for the systems (I) and (II). We are interested in proving pointwise estimates for the solutions in Fourier space, leading to results on polynomial decay of solutions without or with a so-called regularity loss. The latter means that one has estimates only for less derivatives of the solution than that needed for the initial data, if one wishes to reach a certain – optimal – decay rate. In view of the results for bounded domains in Section 2, we expect a regularity loss for system (II), but no loss for system (I). This correspondence is known for other systems, cp. the Introduction.

The Cauchy problem for system (I) is given by

$$\rho u_{tt} + \mu \Delta^2 u - a\Delta(\theta + \alpha\theta_t) = 0, \qquad \text{in } [0, \infty) \times \mathbb{R}^n, \tag{3.1}$$

$$h\theta_{tt} + d\theta_t - k\Delta\theta + a\Delta u_t = 0, \qquad \text{in } [0, \infty) \times \mathbb{R}^n, \tag{3.2}$$

with initial data

$$u(0, \cdot) = u_0, \quad u_t(0, \cdot) = u_1, \quad \theta(0, \cdot) = \theta_0, \quad \theta_t(0, \cdot) = \theta_1, \qquad \text{in } \mathbb{R}^n. \tag{3.3}$$

. System (II) is given by

$$\rho u_{tt} + \mu u_{xxxx} - b u_{xx} - a(\theta + \alpha\theta_t)_x = 0, \qquad \text{in } [0, \infty) \times \mathbb{R}, \tag{3.4}$$

$$h\theta_{tt} + d\theta_t - k\theta_{xx} - a u_{tx} = 0, \qquad \text{in } [0, \infty) \times \mathbb{R}, \tag{3.5}$$

together with the initial conditions (3.3).

We will reformulate the equations as systems of type

$$A^0 U_t + \sum_{j=1}^{n} A^j U_{x_j} - \sum_{j,\ell=1}^{n} B^{j\ell} U_{x_j x_\ell} + LU = 0 \tag{3.6}$$

to derive the well-posedness and the decay estimate of solutions. Here A^0 is a positive definite matrix, A^j with $j = 0, \cdots, n$ are symmetric, $B^{j\ell}$ with $j, \ell = 1, \cdots, n$ and L are positive semi-definite.

Precisely, for system (I), we introduce the new functions

$$v := \sqrt{\mu}\Delta u, \qquad w := u_t, \qquad \psi := \theta_t, \qquad q := \nabla\theta,$$

Then (3.1)–(3.2) are equivalent to

$$\begin{aligned}
v_t - \sqrt{\mu}\Delta w = 0,\\
\rho w_t + \sqrt{\mu}\Delta v - a\Delta\theta - a\alpha\Delta\psi = 0,\\
d\theta_t + h\psi_t - k\Delta\theta + a\Delta w = 0,\\
h\theta_t + h\alpha\psi_t - k\alpha \mathrm{div} q + a\alpha\Delta w + (d\alpha - h)\psi = 0,\\
k\alpha q_t - k\alpha\nabla\psi = 0,
\end{aligned}$$

with the constraint condition $q = \nabla\theta$. We can rewrite (3.1)–(3.2) in the form (3.6) with $U := (v, w, \theta, \psi, q)'$, $B^{j\ell} = 0$ for $j \neq \ell$, and

$$\begin{aligned}
&A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & d & h & 0 \\ 0 & 0 & h & h\alpha & 0 \\ 0 & 0 & 0 & 0 & k\alpha I \end{pmatrix}, \qquad \sum_{j=1}^{n} A^j \omega_j = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -k\alpha\omega \\ 0 & 0 & 0 & -k\alpha\omega^T & 0 \end{pmatrix}, \\
&B^{jj} = \begin{pmatrix} 0 & \sqrt{\mu} & 0 & 0 & 0 \\ -\sqrt{\mu} & 0 & a & a\alpha & 0 \\ 0 & -a & k & 0 & 0 \\ 0 & -a\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d\alpha - h & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{3.7}$$

On the other hand, for system (II), we introduce

$$v := \sqrt{\mu}u_{xx}, \qquad z := \sqrt{b}u_x, \qquad w := u_t, \qquad \psi := \theta_t, \qquad q := \theta_x.$$

Then (3.4)–(3.5) are equivalent to

$$
\begin{aligned}
v_t - \sqrt{\mu} w_{xx} &= 0, \\
z_t - \sqrt{b} w_x &= 0, \\
\rho w_t - \sqrt{b} z_x + \sqrt{\mu} v_{xx} - a\theta_x - a\alpha\psi_x &= 0, \\
d\theta_t + h\psi_t - k\theta_{xx} - a w_x &= 0, \\
h\theta_t + h\alpha\psi_t - k\alpha q_x - a\alpha w_x + (d\alpha - h)\psi &= 0, \\
k\alpha q_t - k\alpha\psi_x &= 0,
\end{aligned}
$$

with the constraint condition $\sqrt{b}v = \sqrt{\mu} z_x$ and $q = \theta_x$, and we can rewrite (3.4)–(3.5) in the form (3.6) with $U := (v, z, w, \theta, \psi, q)'$ and

$$
A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & d & h & 0 \\ 0 & 0 & 0 & h & h\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & k\alpha \end{pmatrix}, \qquad A^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{b} & 0 & 0 & 0 \\ 0 & -\sqrt{b} & 0 & -a & -a\alpha & 0 \\ 0 & 0 & -a & 0 & 0 & 0 \\ 0 & 0 & -a\alpha & 0 & 0 & -k\alpha \\ 0 & 0 & 0 & 0 & -k\alpha & 0 \end{pmatrix},
$$

$$
B^{11} = \begin{pmatrix} 0 & 0 & \sqrt{\mu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{\mu} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d\alpha - h & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.8}
$$

The positivity assumption (1.5) assures in both cases the positive definiteness of A^0 and the positive semi-definiteness of B and L.

Applying the Fourier transform to (3.6), we obtain

$$
A^0 \hat{U}_t + i|\xi| A(\omega)\hat{U} + |\xi|^2 B(\omega)\hat{U} + L\hat{U} = 0, \tag{3.9}
$$

where $A(\omega) := \sum_{j=1}^n A^j \omega_j$ and $B(\omega) := \sum_{j,\ell=1}^n B^{j\ell}\omega_j\omega_k$ for $\omega := \xi/|\xi| \in S^{n-1}$ and $\omega = (\omega_1, \cdots, \omega_n)$. Then the solution of (3.9) can be written as

$$
\hat{U}(t,\xi) = e^{t\hat{\Phi}(\xi)}\hat{U}_0(\xi), \tag{3.10}
$$

where $\hat{U}_0$ is defined by the initial data and

$$
\hat{\Phi}(\xi) := -\big(A^0\big)^{-1}(i|\xi|A(\omega) + |\xi|^2 B(\omega) + L). \tag{3.11}
$$

Then we define the semigroup $\{e^{t\Phi}\}_{t\geq 0}$ by the formula

$$
e^{t\Phi}\varphi := \mathcal{F}^{-1}[e^{t\hat{\Phi}(\xi)}\hat{\varphi}(\xi)]. \tag{3.12}
$$

It is easy to check that the each system with constraint condition satisfies the stability condition introduced in [24]. This means that these systems have a dissipative structure, and it guarantees the decay estimates for the solutions. Thus, our main purpose of this section is now to derive the property of the solution operator $e^{t\Phi}$. Afterwards we will derive pointwise estimates in Fourier space for the solutions leading to polynomial decay estimates.

The well-posedness of the problem can be shown in adapting the techniques used for bounded domains in the previous section, cf. [23] for similar considerations. Here we concentrate on the decay estimates.

3.1 Decay for system (I)

In this section, our purpose is to derive the following decay estimate.

Theorem 3.1. *Let $\{e^{t\Phi}\}_{t\geq 0}$ be the semigroup defined by* (3.12) *and* (3.11)*, where A is given via* (3.7)*. Then the following decay estimates hold for $1 \leq p \leq 2$ and $j \geq 0$.*

$$\|\partial_x^j e^{t\Phi}\varphi\|_{L^2} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|\varphi\|_{L^p} + Ce^{-ct}\|\partial_x^j\varphi\|_{L^2}, \tag{3.13}$$

where c and C are certain positive constants, being independent of t and φ.

The key of the proof of Theorem 3.1 is to get a pointwise estimate of the operator $e^{t\Phi}$ in Fourier space. More precisely, we get the following proposition, in which the form of $\lambda_I(\xi)$ already indicates that there will be no regularity loss. Once we derive the following pointwise estimate, it is not difficult to get (3.13), and we omit the proof. For details we refer reader e.g. to [22, 23].

Proposition 3.2. *Let $\hat{\Phi}(\xi)$ be the matrix defined in* (3.11) *where A arises via* (3.7)*. Then the corresponding matrix exponential $e^{t\hat{\Phi}(i\xi)}$ satisfies the following pointwise estimate*

$$|e^{t\hat{\Phi}(\xi)}| \leq Ce^{-c\lambda_I(\xi)t}, \tag{3.14}$$

for $t \geq 0$ and $\xi \in \mathbb{R}^n$, where $\lambda_I(\xi) := |\xi|^2/(1+|\xi|^2)$, and c and C are certain positive constants, being independent of t and ξ.

Proof. The proof is based on the energy method. We first derive the basic energy equality for the system (3.9) in the Fourier space. Taking the inner product (3.9) with $\hat{U}$, and taking real parts for the resulting equality, we arrive at the following basic energy equality

$$\frac{1}{2}\frac{\partial}{\partial t}\langle A^0\hat{U}, \hat{U}\rangle + k|\xi|^2|\hat{\theta}|^2 + (\alpha d - h)|\hat{\psi}|^2 = 0. \tag{3.15}$$

This corresponds to the dissipativity of the operators in bounded domains, cp. (2.9). We calculate

$$\begin{aligned}\langle A^0\hat{U}, \hat{U}\rangle &= \rho|\hat{w}|^2 + \sqrt{\mu}|\hat{v}|^2 + d|\hat{\theta}|^2 + \alpha h|\hat{\psi}|^2 + \alpha k|\hat{q}|^2 + 2h\mathrm{Re}(\hat{\theta}\bar{\hat{\psi}}) \\ &\geq \rho|\hat{w}|^2 + \sqrt{\mu}|\hat{v}|^2 + \alpha k|\hat{q}|^2 + c_*(|\hat{\theta}|^2 + |\hat{\psi}|^2)\end{aligned} \tag{3.16}$$

because of $\alpha d - h > 0$, where c_* is a certain positive constant which depends on α, d and h.

We next construct further dissipation terms. The system defined by (3.9) is equivalent to

$$
\begin{aligned}
\hat{v}_t + \sqrt{\mu}|\xi|^2\hat{w} &= 0,\\
\rho\hat{w}_t - \sqrt{\mu}|\xi|^2\hat{v} + a|\xi|^2\hat{\theta} + a\alpha|\xi|^2\hat{\psi} &= 0,\\
d\hat{\theta}_t + h\hat{\psi}_t + k|\xi|^2\hat{\theta} - a|\xi|^2\hat{w} &= 0,\\
h\hat{\theta}_t + h\alpha\hat{\psi}_t - k\alpha i\xi\cdot\hat{q} - a\alpha|\xi|^2\hat{w} + (d\alpha - h)\hat{\psi} &= 0,\\
k\alpha\hat{q}_t - k\alpha i\xi\hat{\psi} &= 0.
\end{aligned}
\tag{3.17}
$$

Multiplying the first equation of (3.17) resp. the second equation of (3.17) by $-\rho\bar{\hat{w}}$ resp. $-\bar{\hat{v}}$, we get

$$
-\rho\frac{\partial}{\partial t}\mathrm{Re}(\hat{v}\bar{\hat{w}}) + \sqrt{\mu}|\xi|^2(|\hat{v}|^2 - \rho|\hat{w}|^2) - a|\xi|^2\mathrm{Re}(\hat{v}\bar{\hat{\theta}}) - a\alpha|\xi|^2\mathrm{Re}(\hat{v}\bar{\hat{\psi}}) = 0. \tag{3.18}
$$

Furthermore, we multiply the second equation of (3.17) resp. the fourth equation of (3.17) by $-\alpha h\bar{\hat{\psi}}$ resp. $-\rho\bar{\hat{w}}$, and combine the resulting equations to obtain

$$
\begin{aligned}
&-\alpha\rho h(\hat{w}_t\bar{\hat{\psi}} + \bar{\hat{w}}\hat{\psi}_t) - \alpha ha|\xi|^2\hat{\theta}\bar{\hat{\psi}} + \alpha a|\xi|^2(\rho|\hat{w}|^2 - \alpha h|\hat{\psi}|^2)\\
&+ \alpha h\sqrt{\mu}|\xi|^2\hat{v}\bar{\hat{\psi}} + \alpha\rho k\bar{\hat{w}}(i\xi\cdot\hat{q}) - \alpha\rho d\bar{\hat{w}}\hat{\psi} + \rho h\bar{\hat{w}}(\hat{\psi} - \hat{\theta}_t) = 0.
\end{aligned}
$$

Moreover, using $\psi = \theta_t$ and taking the real part, we get

$$
\begin{aligned}
&-\frac{\partial}{\partial t}\big(\rho h\mathrm{Re}(\hat{w}\bar{\hat{\psi}}) + \rho d\mathrm{Re}(\hat{w}\bar{\hat{\theta}}) + \frac{1}{2}ah|\xi|^2|\hat{\theta}|^2\big) + a|\xi|^2(\rho|\hat{w}|^2 - \alpha h|\hat{\psi}|^2)\\
&+ h\sqrt{\mu}|\xi|^2\mathrm{Re}(\hat{v}\bar{\hat{\psi}}) - \rho k\xi\cdot\mathrm{Re}(i\hat{w}\bar{\hat{q}}) + \rho d\mathrm{Re}(\hat{w}_t\bar{\hat{\theta}}) = 0.
\end{aligned}
$$

For this equation, to control the term $\mathrm{Re}(\hat{w}_t\bar{\hat{\theta}})$, we multiply the second equation of (3.17) by $-d\bar{\hat{\theta}}$. Then we get

$$
-\frac{1}{2}a\alpha d|\xi|^2\frac{\partial}{\partial t}|\hat{\theta}|^2 - ad|\xi|^2|\hat{\theta}|^2 - \rho d\mathrm{Re}(\hat{w}_t\bar{\hat{\theta}}) + d\sqrt{\mu}|\xi|^2\mathrm{Re}(\hat{v}\bar{\hat{\theta}}) = 0.
$$

Thus, combining the last two equations yields

$$
\begin{aligned}
&-\frac{\partial}{\partial t}\big(\rho h\mathrm{Re}(\hat{w}\bar{\hat{\psi}}) + \rho d\mathrm{Re}(\hat{w}\bar{\hat{\theta}}) + \frac{1}{2}a(\alpha d + h)|\xi|^2|\hat{\theta}|^2\big) + a|\xi|^2(\rho|\hat{w}|^2 - \alpha h|\hat{\psi}|^2 - d|\hat{\theta}|^2)\\
&+ h\sqrt{\mu}|\xi|^2\mathrm{Re}(\hat{v}\bar{\hat{\psi}}) - \rho k\xi\cdot\mathrm{Re}(i\hat{w}\bar{\hat{q}}) + d\sqrt{\mu}|\xi|^2\mathrm{Re}(\hat{v}\bar{\hat{\theta}}) = 0.
\end{aligned}
\tag{3.19}
$$

Now, we multiply (3.18) by $a/(2\sqrt{\mu})$, and add the resulting equation to (3.19),

$$
\begin{aligned}
&-\frac{\partial}{\partial t}\Big(\frac{\rho a}{2\sqrt{\mu}}\mathrm{Re}(\hat{v}\bar{\hat{w}}) + \rho h\mathrm{Re}(\hat{w}\bar{\hat{\psi}}) + \rho d\mathrm{Re}(\hat{w}\bar{\hat{\theta}}) + \frac{1}{2}a(\alpha d + h)|\xi|^2|\hat{\theta}|^2\Big)\\
&+ \frac{a}{2}|\xi|^2(|\hat{v}|^2 + \rho|\hat{w}|^2) - a|\xi|^2(\alpha h|\hat{\psi}|^2 + d|\hat{\theta}|^2)\\
&+ \frac{1}{\sqrt{\mu}}\Big(h\mu - \frac{a^2\alpha}{2}\Big)|\xi|^2\mathrm{Re}(\hat{v}\bar{\hat{\psi}}) - \rho k\xi\cdot\mathrm{Re}(i\hat{w}\bar{\hat{q}}) + \frac{1}{\sqrt{\mu}}\Big(d\mu - \frac{a^2}{2}\Big)|\xi|^2\mathrm{Re}(\hat{v}\bar{\hat{\theta}}) = 0.
\end{aligned}
\tag{3.20}
$$

Finally, we multiply (3.15) and (3.20) by $(1 + |\xi|^2)$ and κ_0, respectively, and combine these equations, where κ_0 is a positive constant to be determined. This yields

$$
\frac{\partial}{\partial t}E + D = 0, \tag{3.21}
$$

where

$$E := \frac{1}{2}(1+|\xi|^2)\langle A^0\hat{U},\hat{U}\rangle - \kappa_0\Big\{\frac{\rho a}{2\sqrt{\mu}}\mathrm{Re}(\hat{v}\bar{\hat{w}}) + \rho h\mathrm{Re}(\hat{w}\bar{\hat{\psi}}) + \rho d\mathrm{Re}(\hat{w}\bar{\hat{\theta}}) + \frac{1}{2}a(\alpha d+h)|\xi|^2|\hat{\theta}|^2\Big\},$$

$$D := \frac{\kappa_0 a}{2}|\xi|^2(|\hat{v}|^2+\rho|\hat{w}|^2) + \big(k(1+|\xi|^2)-\kappa_0 ad\big)|\xi|^2|\hat{\theta}|^2 + ((\alpha d-h)(1+|\xi|^2)-\kappa_0\alpha ah|\xi|^2)|\hat{\psi}|^2$$
$$+ \frac{\kappa_0}{\sqrt{\mu}}\Big(h\mu - \frac{a^2\alpha}{2}\Big)|\xi|^2\mathrm{Re}(\hat{v}\bar{\hat{\psi}}) - \kappa_0\rho k\xi\cdot\mathrm{Re}(i\hat{w}\bar{\hat{q}}) + \frac{\kappa_0}{\sqrt{\mu}}\Big(d\mu - \frac{a^2}{2}\Big)|\xi|^2\mathrm{Re}(\hat{v}\bar{\hat{\theta}}).$$

Taking κ_0 suitably small and employing (3.16), we obtain

$$c_0(1+|\xi|^2)|\hat{U}|^2 \le E \le C_0(1+|\xi|^2)|\hat{U}|^2,$$

with positive constants c_0, C_0. Similarly, taking κ_0 suitably small and using $q=\nabla\theta$, we get

$$D \ge c_1(|\xi|^2|\hat{w}|^2 + |\xi|^2|\hat{v}|^2 + |\xi|^2|\hat{\theta}|^2 + |\xi|^2|\hat{q}|^2 + (1+|\xi|^2)|\hat{\psi}|^2) \ge c_1|\xi|^2|\hat{U}|^2,$$

with a positive constant c_1. Therefore, substituting these estimates into (3.21), we can derive the following energy estimate in Fourier space

$$c_0|\hat{U}(t,\xi)|^2 + c_1\int_0^t\Big(\frac{|\xi|^2}{1+|\xi|^2}(|\hat{w}(\tau,\xi)|^2+|\hat{v}(\tau,\xi)|^2) + |\xi|^2|\hat{\theta}(\tau,\xi)|^2 + |\hat{\psi}(\tau,\xi)|^2\Big)d\tau \le C_0|\hat{U}(0,\xi)|^2,$$

and this gives

$$c_0(\|(\sqrt{\mu}\Delta u, u_t, \theta_t)(t)\|_{H^1}^2 + \|\theta(t)\|_{H^2}^2) + c_1\int_0^t(\|\partial_x(\sqrt{\mu}\Delta u, u_t)(\tau)\|_{L^2}^2 + \|\partial_x\theta(\tau)\|_{H^1}^2 + \|\theta_t(\tau)\|_{H^1}^2)d\tau$$
$$\le C_0(\|(\sqrt{\mu}\Delta u_0, u_1, \theta_1)\|_{H^1}^2 + \|\theta_0\|_{H^2}^2).$$

Moreover, we also get $E(t) \le E(0)e^{-c_1C_0^{-1}\lambda_I(|\xi|)t}$, and hence

$$|\hat{U}(t,\xi)|^2 \le c_0^{-1}C_0e^{-c_1C_0^{-1}\lambda(|\xi|)t}|\hat{U}(0,\xi)|^2.$$

In particular, we remark that $\lambda_I(\xi)$ has the standard dissipative structure, leading to estimates on the decay rates without loss of regularity. This corresponds to the exponential stability in the bounded domain case. □

3.2 Optimality for system (I)

Here we investigate the optimality of the pointwise estimates in Theorem 3.1. For this purpose, we consider the characteristic equation $\det(\lambda I - \hat{\Phi}(i\xi)) = 0$ for the system (3.6) with (3.7), which is equivalent to

$$\rho h\lambda^4 + \rho d\lambda^3 + \{\rho k + (\mu h + \alpha a^2)|\xi|^2\}|\xi|^2\lambda^2 + (\mu d + a^2)|\xi|^4\lambda + \mu\kappa|\xi|^6 = 0. \tag{3.22}$$

We consider the asymptotic expansion of $\lambda = \lambda(|\xi|)$ for $|\xi|\to 0$ and for $|\xi|\to\infty$. These expansions essentially determine the asymptotic behavior of solutions. Using the Newton polygon method, see e.g. [25], we have the following asymptotic expansion for $|\xi|\to 0$, :

$$\lambda_j(|\xi|) = \sum_{\ell=0}^{\infty}\alpha_{\ell,j}|\xi|^{2\ell}, \quad j=1,2,3,4. \tag{3.23}$$

Substituting (3.23) into (3.22), we compare the terms of the same order in $|\xi|$. Then we obtain

$$\lambda_j(|\xi|) = z_j|\xi|^2 + O(|\xi|^4), \qquad \lambda_4(|\xi|) = -\frac{d}{h} + \frac{k}{d}|\xi|^2 + O(|\xi|^4) \tag{3.24}$$

for $j = 1, 2, 3$. Here, z_j is a solution for $f(z) = 0$ with

$$f(z) := \rho d z^3 + \rho k z^2 + (\mu d + a^2)z + \mu k. \tag{3.25}$$

Remark that these solutions satisfy $z_1 + z_2 + z_3 = -k/d$. Since $f(0) = \mu k > 0$ and $f(-k/d) = -a^2/d < 0$, we get $\mathrm{Re}(z_j) < 0$ for $j = 1, 2, 3$.

Analogously, we consider the asymptotic expansion for $|\xi| \to \infty$. For this purpose, we introduce ν by $\lambda = |\xi|^2\nu$, and we get from (3.22)

$$\rho h\nu^4 + \rho d|\xi|^{-2}\nu^3 + (\mu h + \alpha a^2 + \rho k|\xi|^{-2})\nu^2 + (\mu d + a^2)|\xi|^{-2}\nu + \mu k|\xi|^{-2} = 0. \tag{3.26}$$

Using again the Newton polygon method, we make the ansatz

$$\nu_j(|\xi|) = \sum_{\ell=0}^{\infty} \beta_{\ell,j}|\xi|^{-\ell}, \quad j = 1, 2, 3, 4,$$

and substitute this into (3.26). Then we obtain

$$\begin{aligned} \lambda_j(|\xi|) &= \pm\sqrt{\frac{\mu h + \alpha a^2}{\rho h}}\, i|\xi|^2 - \frac{a^2(\alpha d - h)}{2h(\mu h + \alpha a^2)} \pm \frac{k}{2}\sqrt{\frac{\rho}{h(\mu h + \alpha a^2)}}\, i + O(|\xi|^{-2}), \\ \lambda_{j+2}(|\xi|) &= \pm\sqrt{\frac{\mu k}{\mu h + \alpha a^2}}\, i|\xi| - \frac{\mu d + a^2}{2(\mu h + \alpha a^2)} + O(|\xi|^{-1}), \end{aligned} \tag{3.27}$$

for $j = 1, 2$.

Consequently, the asymptotic expansions (3.24) and (3.27) tell us that the pointwise estimate (3.14) is optimal.

3.3 Decay for system (II)

Now we will derive the decay estimate for the solution of the initial value problem (3.4), (3.5), (3.3). Our decay estimate is stated as follows

Theorem 3.3. *Let $e^{t\Phi}$ be the semigroup defined by* (3.12) *with* (3.8)*. Then the following decay estimates hold for* $1 \le p \le 2$ *and* $j, \ell \ge 0$,

$$\|\partial_x^j e^{t\Phi}\varphi\|_{L^2} \le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{j}{2}}\|\varphi\|_{L^p} + C(1+t)^{-\frac{\ell}{2}}\|\partial_x^{j+\ell}\varphi\|_{L^2}, \tag{3.28}$$

where c and C are certain positive constants, being independent of t and φ.

To get the decay estimate, we derive the corresponding pointwise estimate, in which the form of $\lambda_{II}(|\xi|)$ already indicates that there will be a regularity loss.

Proposition 3.4. *Let $\hat{\Phi}(\xi)$ be the matrix defined in* (3.11) *with* (3.8)*. Then the corresponding matrix exponential $e^{t\hat{\Phi}(\xi)}$ satisfies the following pointwise estimate*

$$|e^{t\hat{\Phi}(\xi)}| \le Ce^{-c\lambda_{II}(|\xi|)t}, \tag{3.29}$$

for $t \ge 0$ and $\xi \in \mathbb{R}^n$, where $\lambda_{II}(|\xi|) := |\xi|^2/(1+|\xi|^2)^2$, and c and C are certain positive constants, being independent of t and ξ.

Proof. The argument is same as the previous proof. We derive the basic energy equality for the system (3.9) with (3.8) in the Fourier space using the same argument as in the proof of Proposition 3.2. Namely we get

$$\frac{1}{2}\frac{\partial}{\partial t}\langle A^0\hat{U}, \hat{U}\rangle + k|\xi|^2|\hat{\theta}|^2 + (\alpha d - h)|\hat{\psi}|^2 = 0. \tag{3.30}$$

This corresponds to the dissipativity of the operators in bounded domains, cp. (2.32). Here, we compute

$$\begin{aligned} \langle A^0\hat{U}, \hat{U}\rangle &= \rho|\hat{w}|^2 + \sqrt{b}|\hat{z}|^2 + \sqrt{\mu}|\hat{v}|^2 + d|\hat{\theta}|^2 + \alpha h|\hat{\psi}|^2 + \alpha k|\hat{q}|^2 + 2h\mathrm{Re}(\hat{\theta}\bar{\hat{\psi}}) \\ &\ge \rho|\hat{w}|^2 + \sqrt{b}|\hat{z}|^2 + \sqrt{\mu}|\hat{v}|^2 + \alpha k|\hat{q}|^2 + c_*(|\hat{\theta}|^2 + |\hat{\psi}|^2), \end{aligned} \tag{3.31}$$

because of $\alpha d - h > 0$, where c_* is a certain positive constant which depends on α, d and h.

We construct further dissipation terms. The system defined by (3.9) with (3.8) is equivalent to

$$\begin{aligned} \hat{v}_t + \sqrt{\mu}\xi^2\hat{w} &= 0, \\ \hat{z}_t - \sqrt{b}i\xi\hat{w} &= 0, \\ \rho\hat{w}_t - \sqrt{b}i\xi\hat{z} - \sqrt{\mu}\xi^2\hat{v} - ai\xi\hat{\theta} - a\alpha i\xi\hat{\psi} &= 0, \\ d\hat{\theta}_t + h\hat{\psi}_t + k\xi^2\hat{\theta} - ai\xi\hat{w} &= 0, \\ h\hat{\theta}_t + h\alpha\hat{\psi}_t - k\alpha i\xi\hat{q} - a\alpha i\xi\hat{w} + (d\alpha - h)\hat{\psi} &= 0, \\ k\alpha\hat{q}_t - k\alpha i\xi\hat{\psi} &= 0, \end{aligned} \tag{3.32}$$

and $\sqrt{\mu}i\xi\hat{z} = \sqrt{b}\hat{v}$. Multiplying the first equation of (3.32) resp. the third equation of (3.32) by $-\rho\bar{\hat{w}}$ resp. $-\bar{\hat{v}}$, we get

$$-\rho\frac{\partial}{\partial t}\mathrm{Re}(\hat{v}\bar{\hat{w}}) + \sqrt{\mu}|\xi|^2(|\hat{z}|^2 + |\hat{v}|^2 - \rho|\hat{w}|^2) + a\xi\mathrm{Re}(i\hat{v}\bar{\hat{\theta}}) + a\alpha\xi\mathrm{Re}(i\hat{v}\bar{\hat{\psi}}) = 0. \tag{3.33}$$

Furthermore, we multiply the third equation of (3.32) resp. the fifth equation of (3.32) by $-\alpha hi\xi\bar{\hat{\psi}}$ resp. $\rho i\xi\bar{\hat{w}}$, Then, combining the resulting equations, we obtain

$$\begin{aligned} &-\frac{\partial}{\partial t}\big(\rho h\xi\mathrm{Re}(i\hat{w}\bar{\hat{\psi}}) + \rho d\xi\mathrm{Re}(i\hat{w}\bar{\hat{\theta}}) + \frac{1}{2}ah\xi^2|\hat{\theta}|^2\big) + a\xi^2(\rho|\hat{w}|^2 - \alpha h|\hat{\psi}|^2) \\ &+ h\sqrt{\mu}\xi^3\mathrm{Re}(i\hat{v}\bar{\hat{\psi}}) + \rho k\xi^2\mathrm{Re}(\hat{w}\bar{\hat{q}}) + \rho d\xi\mathrm{Re}(i\hat{w}_t\bar{\hat{\theta}}) - h\sqrt{b}\xi^2\mathrm{Re}(\hat{z}\bar{\hat{\psi}}) = 0. \end{aligned}$$

For this equation, to control the term $\xi\mathrm{Re}(i\hat{w}_t\bar{\hat{\theta}})$, we multiply the third equation of (3.32) by $-di\xi\bar{\hat{\theta}}$. Then we get

$$-\frac{1}{2}a\alpha d\xi^2\frac{\partial}{\partial t}|\hat{\theta}|^2 - ad\xi^2|\hat{\theta}|^2 - \rho d\xi\mathrm{Re}(i\hat{w}_t\bar{\hat{\theta}}) - d\sqrt{b}\xi^2\mathrm{Re}(\hat{z}\bar{\hat{\theta}}) + d\sqrt{\mu}\xi^3\mathrm{Re}(i\hat{v}\bar{\hat{\theta}}) = 0.$$

Similarly, to control the term $\xi^2\mathrm{Re}(\hat{w}\bar{\hat{q}})$, we multiply the second equation of (3.32) resp. the sixth equation of (3.32) by $-\alpha k i\xi\bar{\hat{q}}$ resp. $i\xi\bar{\hat{z}}$, Then, combining the resulting equations, we obtain

$$-\xi\frac{\partial}{\partial t}\mathrm{Re}(i\hat{z}\bar{\hat{q}})-\sqrt{b}\xi^2\mathrm{Re}(\hat{w}\bar{\hat{q}})+\xi^2\mathrm{Re}(\hat{z}\bar{\hat{\psi}})=0.$$

Thus, combining the last three equations yields

$$\begin{aligned}&-\frac{\partial}{\partial t}\Big(\rho h\xi\mathrm{Re}(i\hat{w}\bar{\hat{\psi}})+\rho d\xi\mathrm{Re}(i\hat{w}\bar{\hat{\theta}})+\frac{\rho k}{\sqrt{b}}\xi\mathrm{Re}(i\hat{z}\bar{\hat{q}})+\frac{1}{2}a(\alpha d+h)\xi^2|\hat{\theta}|^2\Big)\\&+a\xi^2(\rho|\hat{w}|^2-\alpha h|\hat{\psi}|^2-d|\hat{\theta}|^2)-\frac{d}{\sqrt{b}}(b+\mu\xi^2)\xi^2\mathrm{Re}(\hat{z}\bar{\hat{\theta}})+\frac{1}{\sqrt{b}}(\rho k-bh-\mu h\xi^2)\xi^2\mathrm{Re}(\hat{z}\bar{\hat{\psi}})=0.\end{aligned}\tag{3.34}$$

Here, we also used $\sqrt{\mu}i\xi\hat{z}=\sqrt{b}\hat{v}$. Now, we multiply (3.33) by $a/(2\sqrt{\mu})$, and add the resulting equation to (3.34),

$$\begin{aligned}&-\frac{\partial}{\partial t}\Big(\frac{\rho a}{2\sqrt{\mu}}\mathrm{Re}(\hat{v}\bar{\hat{w}})+\rho h\xi\mathrm{Re}(i\hat{w}\bar{\hat{\psi}})+\rho d\xi\mathrm{Re}(i\hat{w}\bar{\hat{\theta}})+\frac{\rho k}{\sqrt{b}}\xi\mathrm{Re}(i\hat{z}\bar{\hat{q}})+\frac{1}{2}a(\alpha d+h)\xi^2|\hat{\theta}|^2\Big)\\&+\frac{a}{2}\xi^2(|\hat{v}|^2+|\hat{z}|^2+\rho|\hat{w}|^2)-a\xi^2(\alpha h|\hat{\psi}|^2+d|\hat{\theta}|^2)\\&-\frac{d}{\sqrt{b}}(\frac{a^2}{2d}+b+\mu\xi^2)\xi^2\mathrm{Re}(\hat{z}\bar{\hat{\theta}})+\frac{1}{\sqrt{b}}(-\frac{\alpha a^2}{2}+\rho k-bh-\mu h\xi^2)\xi^2\mathrm{Re}(\hat{z}\bar{\hat{\psi}})=0.\end{aligned}\tag{3.35}$$

Finally, we multiply (3.30) and (3.35) by $(1+|\xi|^2)^2$ and κ_0, respectively, and combine these equations, where κ_0 is a positive constant to be determined. This yields

$$\frac{\partial}{\partial t}E+D=0,\tag{3.36}$$

where

$$\begin{aligned}E:=&\frac{1}{2}(1+|\xi|^2)^2\langle A^0\hat{U},\hat{U}\rangle\\&-\kappa_0\Big(\frac{\rho a}{2\sqrt{\mu}}\mathrm{Re}(\hat{v}\bar{\hat{w}})+\rho h\xi\mathrm{Re}(i\hat{w}\bar{\hat{\psi}})+\rho d\xi\mathrm{Re}(i\hat{w}\bar{\hat{\theta}})+\frac{\rho k}{\sqrt{b}}\xi\mathrm{Re}(i\hat{z}\bar{\hat{q}})+\frac{1}{2}a(\alpha d+h)\xi^2|\hat{\theta}|^2\Big),\end{aligned}$$

$$\begin{aligned}D:=&\frac{\kappa_0 a}{2}\xi^2(|\hat{v}|^2+|\hat{z}|^2+\rho|\hat{w}|^2)+\big(k(1+\xi^2)^2-\kappa_0 ad\big)\xi^2|\hat{\theta}|^2+((\alpha d-h)(1+\xi^2)^2-\kappa_0\alpha ah\xi^2)|\hat{\psi}|^2\\&-\frac{\kappa_0 d}{\sqrt{b}}(\frac{a^2}{2d}+b+\mu\xi^2)\xi^2\mathrm{Re}(\hat{z}\bar{\hat{\theta}})+\frac{\kappa_0}{\sqrt{b}}(-\frac{\alpha a^2}{2}+\rho k-bh-\mu h\xi^2)\xi^2\mathrm{Re}(\hat{z}\bar{\hat{\psi}}).\end{aligned}$$

Taking κ_0 suitably small and employing (3.31), we obtain

$$c_0(1+|\xi|^2)^2|\hat{U}|^2\le E\le C_0(1+|\xi|^2)^2|\hat{U}|^2,\tag{3.37}$$

with positive constants c_0, C_0. On the other hand, to derive the estimate for D, we prepare the following estimates,

$$\begin{aligned}|\frac{\kappa_0 d}{\sqrt{b}}(\frac{a^2}{2d}+b+\mu\xi^2)\xi^2\mathrm{Re}(\hat{z}\bar{\hat{\theta}})|&\le\frac{\kappa_0 d}{\sqrt{b}}|\frac{a^2}{2d}+b||\xi|^2|\hat{z}||\hat{\theta}|+\kappa_0 d\sqrt{\mu}|\xi|^3|\hat{v}||\hat{\theta}|\\&\le\varepsilon_1\xi^2(|\hat{v}|^2+|\hat{z}|^2)+\frac{\kappa_0^2d^2}{4\varepsilon_1}\{\frac{1}{b}(\frac{a^2}{2d}+b)^2+\mu\xi^2\}\xi^2|\hat{\theta}|^2,\end{aligned}$$

$$|\frac{\kappa_0}{\sqrt{b}}(-\frac{\alpha a^2}{2}+\rho k-bh-\mu h\xi^2)\xi^2\mathrm{Re}(\hat{z}\bar{\hat{\psi}})|\leq\frac{\kappa_0}{\sqrt{b}}|\rho k-bh-\frac{\alpha a^2}{2}||\xi|^2|\hat{z}||\hat{\psi}|+\kappa_0 h\sqrt{\mu}|\xi|^3|\hat{v}||\hat{\psi}|$$
$$\leq\varepsilon_2\xi^2(|\hat{v}|^2+|\hat{z}|^2)+\frac{\kappa_0^2}{4\varepsilon_2}\{\frac{1}{b}(\rho k-bh-\frac{\alpha a^2}{2})^2+\mu h^2\xi^2\}\xi^2|\hat{\psi}|^2.$$

By virtue of these estimate with $\varepsilon_1=\varepsilon_2=\kappa_0 a/8$, D is estimated by

$$\begin{aligned}D\geq{}&\frac{\kappa_0 a}{8}\xi^2(|\hat{v}|^2+|\hat{z}|^2)+\frac{\kappa_0 a\rho}{2}\xi^2|\hat{w}|^2+\Big\{k(1+\xi^2)^2-\kappa_0\Big(ad+\frac{2d^2}{a}\{\frac{1}{b}(\frac{a^2}{2d}+b)^2+\mu\xi^2\}\Big)\Big\}\xi^2|\hat{\theta}|^2\\&+\Big\{(\alpha d-h)(1+\xi^2)^2-\kappa_0\Big(\alpha ah+\frac{2}{a}\{\frac{1}{b}(\rho k-bh-\frac{\alpha a^2}{2})^2+\mu h^2\xi^2\}\Big)\xi^2\Big\}|\hat{\psi}|^2.\\&\geq c_1\xi^2(|\hat{w}|^2+|\hat{v}|^2+|\hat{z}|^2)+c_1(1+\xi^2)\xi^2(|\hat{\theta}|^2+|\hat{q}|^2)+c_1(1+\xi^2)^2|\hat{\psi}|^2\geq c_1\xi^2|\hat{U}|^2,\end{aligned}\tag{3.38}$$

if we take take κ_0 suitably small, where c_1 is a positive constant. Therefore, substituting (3.37) and (3.38) into (3.36), we can obtain the energy estimate in Fourier space

$$c_0|\hat{U}(t,\xi)|^2+c_1\int_0^t\Big(\frac{\xi^2}{(1+\xi^2)^2}(|\hat{w}(\tau,\xi)|^2+|\hat{v}(\tau,\xi)|^2+|\hat{z}(\tau,\xi)|^2)+\xi^2|\hat{\theta}(\tau,\xi)|^2+|\hat{\psi}(\tau,\xi)|^2\Big)d\tau\leq C_0|\hat{U}(0,\xi)|^2,$$

and this gives

$$\begin{aligned}&c_0(\|(\sqrt{\mu}u_{xx},\sqrt{b}u_x,u_t,\theta_t)(t)\|_{H^2}^2+\|\theta(t)\|_{H^3}^2)\\&\qquad+c_1\int_0^t(\|\partial_x(\sqrt{\mu}u_{xx},\sqrt{b}u_x,u_t)(\tau)\|_{L^2}^2+\|\partial_x\theta(\tau)\|_{H^2}^2+\|\theta_t(\tau)\|_{H^2}^2)d\tau\\&\leq C_0(\|(\sqrt{\mu}u_{0xx},\sqrt{b}u_{0x},u_1,\theta_1)\|_{H^2}^2+\|\theta_0\|_{H^3}^2).\end{aligned}$$

Moreover, we also get $E(t)\leq E(0)e^{-c_1C_0^{-1}\lambda_{II}(|\xi|)t}$, and hence

$$|\hat{U}(t,\xi)|^2\leq c_0^{-1}C_0e^{-c_1C_0^{-1}\lambda_{II}(|\xi|)t}|\hat{U}(0,\xi)|^2.$$

In particular, we remark that $\lambda_{II}(|\xi|)$ has regularity loss structure, leading to estimates on the decay rates with loss of regularity. This corresponds to the lack of exponential stability in the bounded domain case. □

3.4 Optimality for system (II)

Here we discuss the optimality of the decay estimates in Theorem 3.3. Analogously as for system (I) above, we consider the corresponding characteristic equation

$$\rho h\lambda^4+\rho d\lambda^3+(\rho k+bh+\alpha a^2+\mu h\xi^2)\xi^2\lambda^2+(bd+a^2+\mu d\xi^2)\xi^2\lambda+k(b+\mu\xi^2)\xi^4=0.\tag{3.39}$$

We study the asymptotic expansion of $\lambda=\lambda(|\xi|)$ for $|\xi|\to 0$ and for $|\xi|\to\infty$. These expansions essentially determine the asymptotic behavior of solutions.

First, we consider the asymptotic expansion for $|\xi|\to 0$:

$$\lambda_j(|\xi|)=\sum_{\ell=0}^{\infty}\alpha_{\ell,j}\xi^\ell,\quad j=1,2,3,4.\tag{3.40}$$

Substituting (3.40) into (3.39), we compare the terms of the same order in $|\xi|$. Then we obtain

$$\begin{aligned} \lambda_j(|\xi|) &= \pm\sqrt{\frac{bd+a^2}{\rho d}}i\xi - \frac{a^2}{2d}\Big\{\frac{k}{bd+a^2} + \frac{\alpha d - h}{\rho d}\Big\}\xi^2 + O(|\xi|^3), \\ \lambda_3(|\xi|) &= -\frac{kb}{bd+a^2}\xi^2 + O(\xi^4), \qquad \lambda_4(|\xi|) = -\frac{d}{h} + \Big\{\frac{k}{d} + \frac{a^2(\alpha d - h)}{\rho d^2}\Big\}\xi^2 + O(\xi^4), \end{aligned} \tag{3.41}$$

for $j = 1, 2$.

Second, we consider the asymptotic expansion for $|\xi| \to \infty$. We introduce again ν by $\lambda = \xi^2\nu$, and we obtain

$$\rho h\nu^4 + \rho d\xi^{-2}\nu^3 + \{\mu h + (\rho k + bh + \alpha a^2)\xi^{-2}\}\nu^2 + \{\mu d + (bd + a^2)\xi^{-2}\}\xi^{-2}\nu + k(\mu + b\xi^{-2})\xi^{-2} = 0. \tag{3.42}$$

With the ansatz

$$\nu_j(|\xi|) = \sum_{\ell=0}^{\infty} \beta_{\ell,j}\xi^{-\ell}, \quad j = 1, 2, 3, 4,$$

in (3.26), we get

$$\begin{aligned} \lambda_j(|\xi|) &= \pm\sqrt{\frac{\mu}{\rho}}i\xi^2 \pm \frac{bh + \alpha a^2}{2\mu h}\sqrt{\frac{\mu}{\rho}}i \\ &\quad + \frac{1}{2\mu h^2}\Big\{ -a^2(\alpha d - h) \pm \frac{1}{\mu}\Big(k\rho\alpha a^2 - \frac{(bh+\alpha a^2)^2}{4}\Big)\sqrt{\frac{\mu}{\rho}}i\Big\}\xi^{-2} + O(|\xi|^{-3}), \\ \lambda_{j+2}(|\xi|) &= \pm\sqrt{\frac{k}{h}}i\xi - \frac{d}{2h} \mp \frac{1}{2h}\Big(\frac{\alpha a^2}{\mu} + \frac{d^2}{4k}\Big)\sqrt{\frac{k}{h}}i\xi^{-1} + O(\xi^{-2}) \end{aligned} \tag{3.43}$$

for $j = 1, 2$.

Consequently, the asymptotic expansions (3.41) and (3.43) tell us that the pointwise estimate (3.29) is optimal.

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References

[1] H.D. Fernández Sare and J.E. Muñoz Rivera, Optimal rates of decay in 2-d thermoelasticity with second sound, *J. Math. Phys.* **53** (2012), 073509-1–13.

[2] H.D. Fernández Sare, J.E. Muñoz Rivera and R. Quintanilla, Decay of solutions in nonsimple thermoelastic bars, *Int. J. Eng. Sci.* **48** (2010), 1233–1241.

[3] H.D. Fernández Sare and R. Racke, On the stability of damped Timoshenko systems — Cattaneo versus Fourier law. *Arch. Rational Mech. Anal.* **194** (2009), 221–251.

[4] A.E. Green and K.A. Lindsay, Thermoelasticity, *J. Elasticity* **2** (1972), 1–7.

[5] F. Huang, Characteristic conditions for exponential staility of linear dynamical systems in Hilbert spaces. *Ann. Diff. Equations* **1** (1985), 43–55.

[6] L. Gearhart, Spectral theory for contraction semigroups on Hilbert spaces. *Trans. Amer. Math. Soc.* **236** (1978), 385–394.

[7] D. Ieşan, *Thermoelastic models of continua,* Springer (2004).

[8] D. Ieşan and R. Quintanilla, Qualitative properties in strain gradient thermoelasticity with microtemperatures, *Math. Mech. Solids* **23** (2018), 240–258.

[9] T. Irmscher and R. Racke, Sharp decay rates in parabolic and hyperbolic thermoelasticity *IMA J. Appl. Math.* **71** (2006), 459–478.

[10] A. Kasimov, B. Said-Houari, Damping by heat conduction in the Timoshenko system: Fourier and Cattaneo are the same, *J. Differential Equations* **255** (2013), 611–632.

[11] M. C. Leseduarte, R. Quintanilla and R. Racke, On (non-)exponential decay in generalized thermoelasticity with two temperatures, *Appl. Math. Letters* **70** (2017), 18–25.

[12] Z. Liu and S. Zheng, *Semigroups associated with dissipative systems.* Chapman & Hall/CRC Res. Notes Math. **398** (1999).

[13] A. Magaña and R. Quintanilla, On the time dacay of solutions in one-dimensional theories of porous materials. *Int. J. Solids and Structures* **43** (2006), 3414–3427.

[14] A. Magaña and R. Quintanilla, On the exponential decay of solutions in one-dimensional generalized porous-thermo-elasticity, *Asymptotic Analysis* **49** (2006), 173–187.

[15] J.E. Muñoz Rivera and R. Racke, Transmission problems in (thermo-)viscoelasticity with Kelvin-Voigt damping: non-exponential, strong and polynomial stability, *SIAM J. Math. Anal.* **49** (2017), 3741–3765.

[16] R. Quintanilla, Slow decay for one-dimensional porous dissipation elasticity. *Appl. Math. Letters* **16** (2003), 487–491.

[17] R. Quintanilla and R. Racke, Addendum to: Qualitative aspects of solutions in resonators. *Arch. Mech.* **63** (2011), 429–435.

[18] R. Quintanilla and R. Racke, Stability for thermoelastic plates with two temperatures, *Discr. Cont. Syn. Sys.*, **37**(12) (2017), 6333–6352.

[19] R. Quintanilla and Y. Ueda, Decay structures for the equations of porous elasticity in one-dimensional whole space, *J. Dynamics Differential Equations*, https://doi.org/10.1007/s10884-019-09767-w (2019).

[20] J. Prüß, On the spectrum of C_0-semigroups. *Trans. Amer. Math. Soc.* **284** (1984), 847–857.

[21] R. Racke, Thermoelasticity, Chapter **4** in: *Handbook of Differential Equations* **5**. Evolutionary Equations. Eds.: C.M. Dafermos, M. Pokorný. Elsevier (2009), 315–420.

[22] R. Racke and Y. Ueda, Dissipative structures for thermoelastic plate equations in $\mathbb{R}^n$. *Advances Differential Equations* **21** (2016), 601–630.

[23] R. Racke and Y. Ueda, The Cauchy problem for thermoelastic plates with two temperatures, *J. Analysis Applications* (accepted) (2019).

[24] Y. Ueda, New stability criterion for the dissipative linear system and analysis of Bresse system, *Symmetry* **10**(11) (2018), 542.

[25] R.J. Walker, *Algebraic curves*, Springer-Verlag, Berlin (1978).

[26] Yang, X.: Generalized Form of Hurwitz-Routh criterion and Hopf bifurcation of higher order, *Appl. Math. Letters*, **15** (2002), 615–621.

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