Decay properties for the Cauchy problem of the linear JMGT-viscoelastic plate with heat conduction

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Abstract

We consider the Cauchy problem related to the JMGT-viscoelastic plate coupled with a heat equation with two kinds of thermal laws, which are thermoelasticity of type III and the Gurtin-Pipkin thermal law, respectively. We prove optimal results on decay rates for both the thermoelasticity type III system and the Gurtin-Pipkin thermal law system. More precisely, for the type III system, we show that the decay property is not of regularity-loss type in both the subcritical and critical cases. The result matches with the system in a bounded domain, where the system is known to be exponentially stable in the subcritical case. For the Gurtin-Pipkin thermal law system, there is a regularity-loss phenomenon in the critical case. We also study the asymptotic expansion of the eigenvalues to prove the optimality of the obtained decay rates for both models.

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1 Introduction

The linear Jordan-Moore-Gibson-Thompson equation (JMGT) has the following form

$$\tau u_{ttt} + \delta u_{tt} + \beta A u_t + \gamma A u = 0, \tag{1.1}$$

where $\tau, \delta, \beta, \gamma$ are strictly positive constants and A is a strictly positive operator defined in a Hilbert space H. The equation arises as a model for wave propagation in viscous thermally relaxing fluids and can be regarded as the linearized version of the Jordan-Moore-Gibson-Thompson equation, which comes from the combination of the usual balance equations, the equation of state as well as Maxwell-Cattaneo law (cp.[11, 12]). In this application, $A = -\Delta$ with appropriate boundary conditions in a bounded domain.

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There are other interpretations of the same equation. One is that the equation is known as the standard linear solid model or the standard linear model of viscoelasticity(cp. [9]). The other one is that the equation is obtained through the introduction of a relaxation parameter in the Green-Naghdi type III theory (cp. [5, 20]).

Increasing attention was paid to the problem (1.1) in a *bounded* domain. In [11], the authors considered the problem (1.1) and obtained a critical parameter

$$\chi = \delta - \frac{\gamma\tau}{\beta},$$

which is critical for the stability of this problem. More precisely, they established the exponential decay result in the subcritical case ($\chi > 0$) and showed that the energy remains constant in the critical case ($\chi = 0$). Alves et al. [1] studied the standard linear solid model of viscoelasticity coupled with Fourier law and established an exponential stability result by using multiplier techniques in the subcritical case. Apalara et al. [3] investigated the standard linear solid vibrating systems of thermoelasticity of type III in a bounded domain. They established the well-posedness and the exponential stability result in the subcritical case.

In [4], the authors considered the MGT-viscoelastic *plate* coupled with the Fourier law and heat conduction of type III, respectively. They proved the well-posedness and that the corresponding semigroups are analytic, for both models in the subcritical case.

For the results about the MGT equation with memory in a bounded domain, see [2, 7, 13, 14, 15] and the references therein.

In recent years, the *Cauchy problem* related to the problem (1.1) in all of $\mathbb{R}^n$ has also become an active area of research. Pellicer and Said-Houari [17] studied the Cauchy problem for the problem (1.1). The authors obtained well-posedness and the optimal decay rate by using the energy method in Fourier space in the subcritical case. Recently, they investigated the standard linear solid model coupled with heat conduction modeled by the Fourier law in [18] and by the Cattaneo law in [19], respectively. The authors proved the well-posedness of both models in the subcritical case and in the critical case. They established the optimal decay rate and found that the decay results of the Cattaneo system exhibit a regularity-loss phenomenon. We refer the readers to [10, 21, 22, 23, 25] for further results for the Cauchy problem.

Motivated by the results above, we study the following Cauchy problems for the JMGT-thermoviscoelastic plate, first with thermoelasticity of type III,

$$\begin{cases} \tau\rho u_{ttt} + \rho u_{tt} = -k^*\Delta^2 u - k\Delta^2 u_t - m\Delta\theta, & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \\ c\theta_t = l^*\Delta\alpha + l\Delta\theta + m\tau\Delta u_{tt} + m\Delta u_t & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \end{cases} \tag{1.2}$$

where α represents the time primitive of the empirical temperature θ and has the following integral form

$$\alpha(x,t) = \alpha(x,0) + \int_0^t \theta(x,s)\mathrm{d}s,$$

with initial data

$$(u, u_t, u_{tt}, \alpha, \theta)(x,0) = (u_0, u_1, u_2, \alpha_0, \theta_0)(x), \quad x \in \mathbb{R}^n, \tag{1.3}$$

and, second, the following Cauchy problem for the JMGT-thermoviscoelastic plate with the Gurtin-Pipkin thermal law,

$$\begin{cases} \tau\rho u_{ttt} + \rho u_{tt} = -k^*\Delta^2 u - k\Delta^2 u_t - m\Delta\theta, & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \\ \theta_t - \dfrac{1}{l}\displaystyle\int_0^\infty g(s)\Delta\theta(t-s)\mathrm{d}s - m\tau\Delta u_{tt} - m\Delta u_t = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \end{cases} \tag{1.4}$$

with initial data

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ u_{tt}(x,0) = u_2(x),\ \theta(x,t)|_{t\le 0} = \theta_0(x,t), \quad x \in \mathbb{R}^n, \tag{1.5}$$

where θ_0 is a prescribed past history of θ for $t \le 0$ and $g(s)$ is the heat conductivity relaxation kernel. We want to point out that the corresponding critical parameter is given by

$$K := k - \tau k^*.$$

We have

$$\chi > 0 \Leftrightarrow K > 0.$$

We are interested in the optimal decay rates for both models and whether the decay properties of both models are of regularity-loss type. To achieve our goals, we use the energy method in Fourier space with sophisticated functionals to get the decay results for both models in two cases: the subcritical case $K > 0$ and the critical case $K = 0$. We obtain that the decay property of the type III system is not of regularity-loss type for both the subcritical and the critical cases. When $K = 0$, the decay property of the Gurtin-Pipkin thermal law system shows a regularity-loss phenomenon. Furthermore, we analyse eigenvalues to show that the decay results are optimal. It is the first time we discuss the optimality of the decay rates for the Gurtin-Pipkin thermal law system. Summarizing we contribute

- the first discussion of the Cauchy problem of a coupled system of JMGT type with the heat conduction model of type III and of Gurtin-Pipkin, resprectively,
- the proof of optimal decay rates for these systems,
- the discussion of the relationship to bounded domains in view of a loss of regularity (or not), and
- the discussion of both the subcritical and the critical case.

The paper is organized as follows. In Section 2, we first prove the optimal decay result for the type III system in the subcritical case $K > 0$, and then in the critical case $K = 0$. Afterwards we prove the optimality of the decay rates obtained. In Section 3, we study the system with the Gurtin-Pipkin thermal law. We prove the decay estimates for both cases, which are the subcritical case $K > 0$ in Subsection 3.1.1 and the critical case $K = 0$ in Subsection 3.1.2. Finally, we discuss the optimality of the decay rates by giving the asymptotic expansions of the eigenvalues.

Throughout this paper, we denote the Fourier transform $\hat{f} = \hat{f}(\xi)$ of a function $f = f(x)$ by

$$\mathcal{F}[f](\xi) \equiv \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\mathrm{d}x.$$

By C we denote a generic positive constant, the value of which may vary from one place to another.

2 The JMGT-thermoviscoelastic plate with thermoelasticity of type III

We can rewrite the system (1.2), (1.3) as

$$\begin{cases} \tau\rho u_{ttt}+\rho u_{tt}=-k^*\Delta^2 u-k\Delta^2 u_t-m\Delta\alpha_t, & (x,t)\in\mathbb{R}^n\times\mathbb{R}^+, \\ c\alpha_{tt}=l^*\Delta\alpha+l\Delta\alpha_t+m\tau\Delta u_{tt}+m\Delta u_t, & (x,t)\in\mathbb{R}^n\times\mathbb{R}^+, \end{cases} \tag{2.1}$$

with initial data

$$(u,u_t,u_{tt},\alpha,\alpha_t)(x,0)=(u_0,u_1,u_2,\alpha_0,\theta_0)(x), \quad x\in\mathbb{R}^n. \tag{2.2}$$

Taking the Fourier transform of system (2.1), (2.2), we obtain

$$\begin{cases} \tau\rho\hat{u}_{ttt}+\rho\hat{u}_{tt}+k^*\xi^4\hat{u}+k\xi^4\hat{u}_t-m\xi^2\hat{\alpha}_t=0, \\ c\hat{\alpha}_{tt}+l^*\xi^2\hat{\alpha}+l\xi^2\hat{\alpha}_t+m\tau\xi^2\hat{u}_{tt}+m\xi^2\hat{u}_t=0, \end{cases} \tag{2.3}$$

with initial data

$$(\hat{u},\hat{u}_t,\hat{u}_{tt},\hat{\alpha},\hat{\alpha}_t)(\xi,0)=(\hat{u}_0,\hat{u}_1,\hat{u}_2,\hat{\alpha}_0,\hat{\theta}_0)(\xi), \tag{2.4}$$

where $\xi\in\mathbb{R}^n$. The well-posedness is easy to obtain (cp.[18, 19]), so we omit the details here. In this section, we first consider the decay result of the norm related to (2.1), (2.2). We discuss two cases, the subcritical case $K>0$ in Subsection 2.1.1 and the critical case $K=0$ in Subsection 2.1.2. After that, we investigate the associated characteristic equation and prove the optimality of the decay estimates.

2.1 Decay estimates

We introduce the following new variables

$$\hat{\varphi}=\hat{u}_t, \quad \hat{w}=\hat{u}_{tt}, \quad \hat{\psi}=\hat{\alpha}_t, \quad \hat{z}=\sqrt{\frac{l^*}{c}}i\xi\hat{\alpha}.$$

Thus, (2.3) takes the form

$$\begin{cases} \hat{u}_t-\hat{\varphi}=0, \\ \hat{\varphi}_t-\hat{w}=0, \\ \hat{w}_t+\frac{1}{\tau}\hat{w}+\frac{k^*}{\tau\rho}\xi^4\hat{u}+\frac{k}{\tau\rho}\xi^4\hat{\varphi}-\frac{m}{\tau\rho}\xi^2\hat{\psi}=0, \\ \hat{\psi}_t-\sqrt{\frac{l^*}{c}}i\xi\hat{z}+\frac{l}{c}\xi^2\hat{\psi}+\frac{m\tau}{c}\xi^2\hat{w}+\frac{m}{c}\xi^2\hat{\varphi}=0, \\ \hat{z}_t-\sqrt{\frac{l^*}{c}}i\xi\hat{\psi}=0. \end{cases} \tag{2.5}$$

2.1.1 The case $K > 0$

In this subsection, we assume that $K > 0$ and we have the following pointwise estimate and decay result:

Theorem 2.1. *Assume that $K > 0$. Let $\hat{U} := (\hat{u}_t + \tau\hat{u}_{tt}, \Delta\hat{u}_t, \Delta(\hat{u} + \tau\hat{u}_t), \hat{\alpha}_t, \nabla\hat{\alpha})^T$, where $(\hat{u}(\xi,t), \hat{\alpha}(\xi,t))$ is the Fourier image of the solution $(u(x,t), \alpha(x,t))$. Then $\hat{U}$ satisfies the following pointwise estimate*

$$|\hat{U}(\xi,t)|^2 \leq Ce^{-c\rho_1(\xi)t}|\hat{U}_0(\xi)|^2, \tag{2.6}$$

for any $t \geq 0$, where $\rho_1(\xi) := \frac{\xi^4}{(1+\xi^2)^2}$, and where $C > 0$ is independent of t, ξ and the initial data.

Furthermore, let $U = (u_t + \tau u_{tt}, \Delta u_t, \Delta(u + \tau u_t), \alpha_t, \nabla\alpha)^T$, where $(u(x,t), \alpha(x,t))$ is the solution of problem (2.1), (2.2), and $U_0 = U(x,0) \in H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, where s is nonnegative, then U satisfies the following decay estimate

$$\|\nabla^k U(t)\|^2_{L^2(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}}\|U_0\|^2_{L^1(\mathbb{R}^n)} + Ce^{-Ct}\|\nabla^k U_0\|^2_{L^2(\mathbb{R}^n)}, \tag{2.7}$$

for all $0 \leq k \leq s$.

Remark 2.2. *The decay estimate (2.7) indicates there is no regularity-loss phenomenon, which is the decay result does not require a higher regularity of the initial data. Taking the result in bounded domain into account, the decay estimate here is consistent with the exponential stability for the MGT-viscoelastic plate with the heat conduction law of type III (cf.[4]).*

For our purpose, we state and prove some lemmas needed to establish our main result.

Lemma 2.3. *Assume that $K > 0$. Let $(\hat{u}, \hat{\varphi}, \hat{w}, \hat{\psi}, \hat{z})$ be the solution of (2.5), and define the energy functional of system (2.5) as*

$$\hat{E}(\xi,t) := |\hat{\varphi} + \tau\hat{w}|^2 + \frac{\tau}{\rho}K\xi^4|\hat{\varphi}|^2 + \frac{k^*}{\rho}\xi^4|\hat{u} + \tau\hat{\varphi}|^2 + \frac{c}{\rho}|\hat{\psi}|^2 + \frac{c}{\rho}|\hat{z}|^2,$$

then there exist two positive constants C_1 and C_2 such that

$$C_1|\hat{U}(\xi,t)|^2 \leq \hat{E}(\xi,t) \leq C_2|\hat{U}(\xi,t)|^2,$$

and $\hat{E}(\xi,t)$ satisfies

$$\frac{d}{dt}\hat{E}(\xi,t) = -\frac{1}{\rho}K\xi^4|\hat{\varphi}|^2 - \frac{l}{\rho}\xi^2|\hat{\psi}|^2.$$

Remark 2.4. *Although the functionals $F_1(t)$ and $F_2(t)$ have the same form compared with [18, 19], the estimates are different. The functionals $F_3(t)$ and $\bar{F}_2(t)$ are different from those in previous papers.*

Lemma 2.5. *The functional*

$$F_1(t) := \mathrm{Re}((\hat{\varphi} + \tau\hat{w})(\hat{u}^* + \tau\hat{\varphi}^*))$$

satisfies

$$\frac{d}{dt}F_1(t) + \left(\frac{k^*}{\rho} - 2\varepsilon_1\right)\xi^4|\hat{u}+\tau\hat{\varphi}|^2 \le |\hat{\varphi}+\tau\hat{w}|^2 + C(\varepsilon_1)\xi^4|\hat{\varphi}|^2 + C(\varepsilon_1)|\hat{\psi}|^2, \tag{2.8}$$

for any $\varepsilon_1 > 0$.

Proof. Adding $(2.5)_2$ and $(2.5)_3 \times \tau$ together, we have

$$(\hat{\varphi}+\tau\hat{w})_t = -\frac{k^*}{\rho}\xi^4\hat{u} - \frac{k}{\rho}\xi^4\hat{\varphi} + \frac{m}{\rho}\xi^2\hat{\psi}. \tag{2.9}$$

Multiplying $(2.5)_2$ by τ, and summing up the resulting equality and $(2.5)_1$, we obtain

$$(\hat{u}+\tau\hat{\varphi})_t = \tau\hat{w} + \hat{\varphi}. \tag{2.10}$$

Multiplying (2.9) and (2.10) by $(\hat{u}^*+\tau\hat{\varphi}^*)$ and $(\hat{\varphi}^*+\tau\hat{w}^*)$, respectively, combining the resulting equations and taking real parts, we have

$$\frac{d}{dt}F_1(t) + \frac{k^*}{\rho}\xi^4|\hat{u}+\tau\hat{\varphi}|^2 - |\hat{\varphi}+\tau\hat{w}|^2 = -\frac{1}{\rho}K\xi^4\mathrm{Re}(\hat{\varphi}(\hat{u}^*+\tau\hat{\varphi}^*)) + \frac{m}{\rho}\xi^2\mathrm{Re}(\hat{\psi}(\hat{u}^*+\tau\hat{\varphi}^*)). \tag{2.11}$$

Thanks to Young's inequality our conclusion holds. □

Lemma 2.6. *Define the functional*

$$F_2(t) := \mathrm{Re}(-\tau(\hat{\varphi}+\tau\hat{w})\hat{\varphi}^*),$$

then

$$\frac{d}{dt}F_2(t) + (1-\varepsilon_3)|\hat{\varphi}+\tau\hat{w}|^2 \le C(\varepsilon_2,\varepsilon_3)(1+\xi^2+\xi^4)|\hat{\varphi}|^2 + \varepsilon_2\xi^4|\hat{u}+\tau\hat{\varphi}|^2 + \varepsilon_3\xi^2|\hat{\psi}|^2, \tag{2.12}$$

for any $\varepsilon_2, \varepsilon_3 > 0$ *and some positive constant* $C = C(\varepsilon_2, \varepsilon_3)$.

Proof. Multiplying $(2.5)_2$ and (2.9) by $-\tau(\hat{\varphi}^* + \tau\hat{w}^*)$ and $-\tau\hat{\varphi}^*$, respectively, adding the resulting equations up and taking the real part, we arrive at

$$\frac{d}{dt}F_2(t) + |\hat{\varphi}+\tau\hat{w}|^2 - \frac{\tau K}{\rho}\xi^4|\hat{\varphi}|^2 = \frac{\tau k^*}{\rho}\xi^4\mathrm{Re}((\hat{u}+\tau\hat{\varphi})\hat{\varphi}^*) - \mathrm{Re}((\hat{\varphi}+\tau\hat{w})\hat{\varphi}^*) - \frac{\tau m}{\rho}\xi^2\mathrm{Re}(\hat{\psi}\hat{\varphi}^*). \tag{2.13}$$

Thus we deduce (2.12). □

Lemma 2.7. *The following inequality holds true:*

$$\frac{d}{dt}F_3(t) + \left(\sqrt{\frac{l^*}{c}} - \varepsilon_4\right)\xi^2|\hat{z}|^2 \le C(\varepsilon_4,\varepsilon_5)(\xi^2+\xi^4)|\hat{\psi}|^2 + \varepsilon_5\xi^6|\hat{u}+\tau\hat{\varphi}|^2, \tag{2.14}$$

where

$$F_3(t) := \mathrm{Re}\left(i\xi\hat{\psi}\hat{z}^* + i\frac{m}{c}\xi^3(\hat{u}+\tau\hat{\varphi})\hat{z}^*\right).$$

Proof. We multiply $(2.5)_4$ and $(2.5)_5$ by $i\xi\hat{z}^*$ and $-i\xi\hat{\psi}^*$, respectively, add the results and take the real part. This yields

$$\frac{d}{dt}\mathrm{Re}(i\xi\hat{\psi}\hat{z}^*) + \sqrt{\frac{l^*}{c}}\xi^2|\hat{z}|^2 - \sqrt{\frac{l^*}{c}}\xi^2|\hat{\psi}|^2 = -\frac{l}{c}\mathrm{Re}(i\xi^3\hat{\psi}\hat{z}^*) - \frac{m}{c}\mathrm{Re}(i\xi^3(\hat{\varphi}+\tau\hat{w})\hat{z}^*). \quad (2.15)$$

Multiplying (2.10) and $(2.5)_5$ by $i\frac{m}{c}\xi^3\hat{z}^*$ and $-i\frac{m}{c}\xi^3(\hat{u}^*+\tau\hat{\varphi}^*)$, respectively, combining the resulting equations and taking real parts, we have

$$\frac{d}{dt}\mathrm{Re}\left(i\frac{m}{c}\xi^3(\hat{u}+\tau\hat{\varphi})\hat{z}^*\right) = \frac{m}{c}\mathrm{Re}(i\xi^3(\hat{\varphi}+\tau\hat{w})\hat{z}^*) + \sqrt{\frac{l^*m^2}{c^3}}\xi^4\mathrm{Re}(\hat{\psi}(\hat{u}^*+\tau\hat{\varphi}^*)). \quad (2.16)$$

Adding (2.15) and (2.16) up, we arrive at

$$\frac{d}{dt}F_3(t) + \sqrt{\frac{l^*}{c}}\xi^2|\hat{z}|^2 = \sqrt{\frac{l^*}{c}}\xi^2|\hat{\psi}|^2 - \frac{l}{c}\mathrm{Re}(i\xi^3\hat{\psi}\hat{z}^*) + \sqrt{\frac{l^*m^2}{c^3}}\xi^4\mathrm{Re}(\hat{\psi}(\hat{u}^*+\tau\hat{\varphi}^*)). \quad (2.17)$$

Hence we arrive at (2.14). □

Now, we give the proof of our main result.

Proof of Theorem 2.1. We define the Lyapunov functional

$$L_1(t) := N\hat{E}(t) + N_1\frac{\xi^4}{(1+\xi^2)^2}F_1(t) + N_2\frac{\xi^4}{(1+\xi^2)^2}F_2(t) + N_3\frac{\xi^2}{(1+\xi^2)^2}F_3(t),$$

where N, N_1, N_2 and N_3 are positive constants that will be fixed later. Taking advantage of the above lemmas, we have

$$\begin{aligned}
&\frac{d}{dt}L_1(t) + \left[N_1\left(\frac{k^*}{\rho} - 2\varepsilon_1\right) - N_2\varepsilon_2 - N_3\varepsilon_5\right]\frac{\xi^4}{(1+\xi^2)^2}\xi^4|\hat{u}+\tau\hat{\varphi}|^2\\
&+\left[N_2(1-\varepsilon_3) - N_1\right]\frac{\xi^4}{(1+\xi^2)^2}|\hat{\varphi}+\tau\hat{w}|^2 + \left[N_3\left(\sqrt{\frac{l^*}{c}} - \varepsilon_4\right)\right]\frac{\xi^4}{(1+\xi^2)^2}|\hat{z}|^2\\
&+\left[\frac{N}{\rho}K - N_1C(\varepsilon_1) - N_2C(\varepsilon_2,\varepsilon_3)\right]\xi^4|\hat{\varphi}|^2\\
&+\left[\frac{Nl}{\rho} - N_1C(\varepsilon_1) - N_2\varepsilon_3 - N_3C(\varepsilon_4,\varepsilon_5)\right]\xi^2|\hat{\psi}|^2\\
\le& 0,
\end{aligned} \quad (2.18)$$

where we have used the fact that $\frac{\xi^2}{1+\xi^2} \le 1$. At this moment, we want to choose the constants in (2.18). First, we choose

$$\varepsilon_1 < \frac{k^*}{2\rho}, \quad \varepsilon_3 < 1, \quad \varepsilon_4 < \sqrt{\frac{l^*}{c}}.$$

Next, we fix $N_1 = N_3 = 1$ and $N_2 > \frac{1}{1-\varepsilon_3}$. Then, we pick ε_2 and ε_5 satisfying

$$\varepsilon_2 < \frac{k^*}{2\rho N_2} - \frac{\varepsilon_1}{N_2} \quad \text{and} \quad \varepsilon_5 < \frac{k^*}{2\rho} - \varepsilon_1.$$

Finally, we choose N large enough such that

$$N > \max\left\{\frac{\rho[N_1C(\varepsilon_1) + N_2C(\varepsilon_2,\varepsilon_3)]}{K}, \frac{\rho[N_1C(\varepsilon_1) + N_2\varepsilon_3 + N_3C(\varepsilon_4,\varepsilon_5)]}{l}\right\}.$$

So we arrive at, with a positive constant α_1,

$$\frac{d}{dt}L_1(t) + \alpha_1 M_1(t) \le 0, \tag{2.19}$$

where

$$\begin{aligned} M_1(t) &= \frac{\xi^4}{(1+\xi^2)^2}\xi^4|\hat{u}+\tau\hat{\varphi}|^2 + \frac{\xi^4}{(1+\xi^2)^2}|\hat{\varphi}+\tau\hat{w}|^2 + \frac{\xi^4}{(1+\xi^2)^2}|\hat{z}|^2 + \xi^4|\hat{\varphi}|^2 + \xi^2|\hat{\psi}|^2 \\ &\ge C\frac{\xi^4}{(1+\xi^2)^2}\hat{E}(t). \end{aligned}$$

On the other hand, we find that

$$|L_1(t) - N\hat{E}(t)| \le C\hat{E}(t).$$

Therefore, (2.19) becomes

$$\frac{d}{dt}\hat{E}(t) + C\frac{\xi^4}{(1+\xi^2)^2}\hat{E}(t) \le 0. \tag{2.20}$$

At last, estimate (2.20) gives the desired result (2.6) and we can obtain the decay estimate (2.7). □

2.1.2 The case $K = 0$

For the problem (2.1)-(2.2) in the case $K = 0$, we have the following result:

Theorem 2.8. *Assume that $K = 0$. Let $\hat{V} = (\hat{u}_t + \tau\hat{u}_{tt}, \Delta(\hat{u}+\tau\hat{u}_t), \hat{\alpha}_t, \nabla\hat{\alpha})^T$, where $(\hat{u}(\xi,t), \hat{\alpha}(\xi,t))$ is the Fourier image of the solution $(u(x,t), \alpha(x,t))$. Then $\hat{V}$ has the following pointwise estimate*

$$|\hat{V}(\xi,t)|^2 \le Ce^{-c\rho_2(\xi)t}|\hat{V}_0(\xi)|^2, \tag{2.21}$$

for any $t \ge 0$, where $\rho_2(\xi) := \frac{\xi^6}{(1+\xi^2)^3}$. Furthermore, let $V = (u_t + \tau u_{tt}, \Delta(u+\tau u_t), \alpha_t, \nabla\alpha)^T$, where $(u(x,t), \alpha(x,t))$ is the solution of problem (2.1)-(2.2), and $V_0 = V(x,0) \in H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, where s is nonnegative, then V satisfies the following decay estimate

$$\|\nabla^k V(t)\|^2_{L^2(\mathbb{R}^n)} \le C(1+t)^{-\frac{n}{6}-\frac{k}{3}}\|V_0\|^2_{L^1(\mathbb{R}^n)} + Ce^{-Ct}\|\nabla^k V_0\|^2_{L^2(\mathbb{R}^n)}, \tag{2.22}$$

for all $0 \le k \le s$.

According to Lemma 2.3, 2.5 and $K = 0$, we have the following result:

Lemma 2.9. *Assume that $K = 0$. Let $(\hat{u}, \hat{\varphi}, \hat{w}, \hat{\psi}, \hat{z})$ be the solution of (2.5), and the energy functional of system (2.5) becomes*

$$\hat{\mathcal{E}}(\xi,t) := |\hat{\varphi}+\tau\hat{w}|^2 + \frac{k^*}{\rho}\xi^4|\hat{u}+\tau\hat{\varphi}|^2 + \frac{c}{\rho}|\hat{\psi}|^2 + \frac{c}{\rho}|\hat{z}|^2,$$

then $\hat{\mathcal{E}}(\xi,t)$ and $F_1(t)$ satisfy

$$\begin{aligned} &\frac{d}{dt}\hat{\mathcal{E}}(\xi,t) = -\frac{l}{\rho}\xi^2|\hat{\psi}|^2, \\ &\frac{d}{dt}F_1(t) + \left(\frac{k^*}{\rho} - \varepsilon_1'\right)\xi^4|\hat{u}+\tau\hat{\varphi}|^2 \le |\hat{\varphi}+\tau\hat{w}|^2 + C(\varepsilon_1')|\hat{\psi}|^2. \end{aligned}$$

Lemma 2.10. *The functional*

$$\bar{F}_2(t) := \mathrm{Re}\left((\hat{\varphi} + \tau\hat{w})\hat{\psi}^* + \sqrt{\frac{l^*}{c}} i\xi(\hat{u} + \tau\hat{\varphi})\hat{z}^*\right)$$

satisfies

$$\frac{d}{dt}\bar{F}_2(t) + \left(\frac{m}{c} - \varepsilon_2'\right)\xi^2|\hat{\varphi} + \tau\hat{w}|^2 \leq C(\varepsilon_2')\left(\xi^2 + \frac{1+\xi^2}{\xi^4}\right)|\hat{\psi}|^2 + 2\varepsilon_2'\xi^2\xi^4|\hat{u}^* + \tau\hat{\varphi}^*|^2, \quad (2.23)$$

for any $\varepsilon_2' > 0$.

Proof. Multiplying (2.9) and $(2.5)_4$ by $\hat{\psi}^*$ and $(\hat{\varphi}^* + \tau\hat{w}^*)$, respectively, adding the results and taking the real part, we have

$$\begin{aligned}&\frac{d}{dt}\mathrm{Re}((\hat{\varphi} + \tau\hat{w})\hat{\psi}^*) + \frac{m}{c}\xi^2|\hat{\varphi} + \tau\hat{w}|^2\\ =&\frac{m}{\rho}\xi^2|\hat{\psi}|^2 - \frac{k^*}{\rho}\xi^4\mathrm{Re}((\hat{u} + \tau\hat{\varphi})\hat{\psi}^*) + \sqrt{\frac{l^*}{c}}\mathrm{Re}(i\xi\hat{z}(\hat{\varphi}^* + \tau\hat{w}^*)) - \frac{l}{c}\xi^2\mathrm{Re}(\hat{\psi}(\hat{\varphi}^* + \tau\hat{w}^*)).\end{aligned} \quad (2.24)$$

Multiplying (2.10) and $(2.5)_5$ by $\sqrt{\frac{l^*}{c}} i\xi\hat{z}^*$ and $-\sqrt{\frac{l^*}{c}} i\xi(\hat{u}^* + \tau\hat{\varphi}^*)$, respectively, combining the resulting equations and taking real parts, we arrive at

$$\frac{d}{dt}\mathrm{Re}\left(\sqrt{\frac{l^*}{c}} i\xi(\hat{u} + \tau\hat{\varphi})\hat{z}^*\right) = \mathrm{Re}\left(\sqrt{\frac{l^*}{c}} i\xi(\hat{\varphi} + \tau\hat{w})\hat{z}^*\right) + \frac{l^*}{c}\xi^2\mathrm{Re}(\hat{\psi}(\hat{u}^* + \tau\hat{\varphi}^*)). \quad (2.25)$$

Summing up (2.24) and (2.25), we have

$$\begin{aligned}&\frac{d}{dt}\bar{F}_2(t) + \frac{m}{c}\xi^2|\hat{\varphi} + \tau\hat{w}|^2\\ =&\frac{m}{\rho}\xi^2|\hat{\psi}|^2 - \frac{k^*}{\rho}\xi^4\mathrm{Re}((\hat{u} + \tau\hat{\varphi})\hat{\psi}^*) + \frac{l^*}{c}\xi^2\mathrm{Re}(\hat{\psi}(\hat{u}^* + \tau\hat{\varphi}^*)) - \frac{l}{c}\xi^2\mathrm{Re}(\hat{\psi}(\hat{\varphi}^* + \tau\hat{w}^*)).\end{aligned}$$

Since $\frac{\xi^2}{1+\xi^2} \leq 1$ we obtain (2.23). The proof is complete. □

Proof of Theorem 2.8. We define the Lyapunov functional as

$$L_2(t) := \bar{N}(1+\xi^2)^3\hat{\mathcal{E}}(t) + \bar{N}_1\xi^6F_1(t) + \bar{N}_2\xi^4\bar{F}_2(t) + \bar{N}_3\xi^4F_3(t),$$

which is obviously equivalent to the energy functional $\hat{\mathcal{E}}(t)$. Now, a combination of the above lemmas, we obtain

$$\begin{aligned}&\frac{d}{dt}L_2(t) + \left[\bar{N}_1\left(\frac{k^*}{\rho} - \varepsilon_1'\right) - 2\bar{N}_2\varepsilon_2' - \bar{N}_3\varepsilon_5\right]\xi^6\xi^4|\hat{u} + \tau\hat{\varphi}|^2\\ &+\left[\bar{N}_2\left(\frac{m}{c} - \varepsilon_2'\right) - \bar{N}_1\right]\xi^6|\hat{\varphi} + \tau\hat{w}|^2 + \bar{N}_3\left(\sqrt{\frac{l^*}{c}} - \varepsilon_4\right)\xi^6|\hat{z}|^2\\ &+\left[\frac{\bar{N}l}{\rho} - \bar{N}_1C(\varepsilon_1') - \bar{N}_2C(\varepsilon_2') - \bar{N}_3C(\varepsilon_4, \varepsilon_5)\right]\xi^2(1+\xi^2)^3|\hat{\psi}|^2\\ \leq&0.\end{aligned} \quad (2.26)$$

By choosing our constants carefully like before, we can derive

$$\frac{d}{dt}L_2(t) + \alpha_2 M_2(t) \leq 0, \tag{2.27}$$

where α_2 is a positive constant and

$$\begin{aligned} M_2(t) &= \xi^6\xi^4|\hat{u} + \tau\hat{\varphi}|^2 + \xi^6|\hat{\varphi} + \tau\hat{w}|^2 + \xi^6|\hat{z}|^2 + \xi^2(1+\xi^2)^3|\hat{\psi}|^2 \\ &\geq C\xi^6\hat{\mathcal{E}}(t). \end{aligned}$$

Exploiting the equivalence $L_2(t) \sim (1+\xi^2)^3\hat{\mathcal{E}}(t)$, we arrive at

$$\frac{d}{dt}\hat{\mathcal{E}}(t) + C\frac{\xi^6}{(1+\xi^2)^3}\hat{\mathcal{E}}(t) \leq 0. \tag{2.28}$$

Therefore, making use of Gronwall's inequality, we obtain the desired pointwise estimate (2.21) and we conclude the desired decay estimate (2.22). □

2.2 Eigenvalue Expansions

In what follows, we study the asymptotic expansion of the eigenvalues to confirm that our pointwise estimates (2.6) and (2.21) are optimal.

Putting $\hat{Z} = (\hat{u}, \hat{\varphi}, \hat{w}, \hat{\psi}, \hat{z})^T$ and $\hat{Z}_0 = (\hat{u}_0, \hat{\varphi}_0, \hat{w}_0, \hat{\psi}_0, \hat{z}_0)^T$, we can rewrite system (2.3)-(2.4) as

$$\begin{cases} \hat{Z}_t + L\hat{Z} + i\xi A\hat{Z} + \xi^2 B\hat{Z} + \xi^4 D\hat{Z} = 0, \\ \hat{Z}(\xi, 0) = \hat{Z}_0(\xi), \end{cases} \tag{2.29}$$

where

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{\frac{l^*}{c}} \\ 0 & 0 & 0 & -\sqrt{\frac{l^*}{c}} & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{\tau} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{m}{\tau\rho} & 0 \\ 0 & \frac{m}{c} & \frac{m\tau}{c} & \frac{l}{c} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{k^*}{\tau\rho} & \frac{k}{\tau\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The solution to (2.29) is $\hat{Z}(\xi, t) = e^{t\hat{\Phi}(i\xi)}\hat{Z}_0(\xi)$, where $\hat{\Phi}(i\xi) = -(L + i\xi A + \xi^2 B + \xi^4 D)$.

(1) Setting $\zeta = i\xi$, we get

$$\hat{\Phi}(\zeta) = -(L + \zeta A - \zeta^2 B + \zeta^4 D).$$

Let $\lambda_j(\zeta)$ denote the eigenvalues of the matrix $\hat{\Phi}(\zeta)$. Then the eigenvalues $\lambda_j(\zeta)$, $j = 1, 2, 3, 4, 5$, are the solutions of the characteristic equation

$$\tau\rho c \det(\lambda I - \hat{\Phi}(\zeta))$$

$$
\begin{aligned}
=&\tau\rho c\lambda^5+(\rho c-\tau\rho l\zeta^2)\lambda^4+[(\tau m^2+ck)\zeta^4-(\tau\rho l^*+\rho l)\zeta^2]\lambda^3\\
&+[-kl\zeta^6+(ck^*+m^2)\zeta^4-\rho l^*\zeta^2]\lambda^2-(k^*l+l^*k)\zeta^6\lambda-k^*l^*\zeta^6\\
=&0.
\end{aligned}
\tag{2.30}
$$

When $|\zeta|\to 0$, $\lambda_j(\zeta)$ has the following asymptotic expansion:

$$
\lambda_j(\zeta)=\lambda_j^{(0)}+\lambda_j^{(1)}\zeta+\lambda_j^{(2)}\zeta^2+\lambda_j^{(3)}\zeta^3+\cdots. \tag{2.31}
$$

Substituting (2.31) into (2.30) and calculating the coefficients $\lambda_j^{(h)}(h=0,1,2,\cdots)$, we have

$$
\begin{aligned}
&\lambda_j^{(0)}=-\frac{1}{\tau},\quad j=1,\\
&\lambda_j^{(0)}=\lambda_j^{(1)}=0,\quad \lambda_j^{(2)}=\pm\sqrt{\frac{k^*}{\rho}},\quad \lambda_j^{(3)}=0,\quad \lambda_j^{(4)}=-\frac{K}{2\rho}\pm\frac{m^2}{2\rho l^*}\sqrt{\frac{k^*}{\rho}}i,\\
&\qquad\qquad j=2,3,\quad \text{when}\quad \mathrm{K}>0;\\
&\lambda_j^{(0)}=\lambda_j^{(1)}=0,\quad \lambda_j^{(2)}=\pm\sqrt{\frac{k^*}{\rho}},\quad \lambda_j^{(3)}=0,\quad \lambda_j^{(4)}=\pm\frac{m^2}{2\rho l^*}\sqrt{\frac{k^*}{\rho}}i,\\
&\lambda_j^{(5)}=0,\quad \lambda_j^{(6)}=\frac{lk^*m^2}{2\rho^2l^{*2}}\pm\frac{m^4-ck^*m^2}{2\rho^2l^{*2}}\sqrt{\frac{k^*}{\rho}}i,\quad j=2,3,\quad \text{when}\quad \mathrm{K}=0;\\
&\lambda_j^{(0)}=0,\quad \lambda_j^{(1)}=\pm\sqrt{\frac{l^*}{c}},\quad \lambda_j^{(2)}=\frac{l}{2c}\quad j=4,5.
\end{aligned}
$$

Consequently, when $K>0$, we have

$$
\mathrm{Re}\lambda_j(i\xi)=\begin{cases}
-\dfrac{1}{\tau}+O(|\xi|^2), & j=1,\\
-\dfrac{K}{2\rho}|\xi|^4+O(|\xi|^5), & j=2,3,\\
-\dfrac{l}{2c}|\xi|^2+O(|\xi|^3), & j=4,5,
\end{cases}
\tag{2.32}
$$

for $|\xi|\to 0$. And when $K=0$, we have

$$
\mathrm{Re}\lambda_j(i\xi)=\begin{cases}
-\dfrac{1}{\tau}+O(|\xi|^2), & j=1,\\
-\dfrac{lk^*m^2}{2\rho^2l^{*2}}|\xi|^6+O(|\xi|^7), & j=2,3,\\
-\dfrac{l}{2c}|\xi|^2+O(|\xi|^3), & j=4,5,
\end{cases}
\tag{2.33}
$$

for $|\xi|\to 0$.

(2) When $|\zeta|\to\infty$, we define the matrix $\hat{\Psi}(\zeta^{-1})=B-\zeta^{-1}A-\zeta^{-2}L-\zeta^2D$, which satisfies the relation $\hat{\Phi}(\zeta)=\zeta^2\hat{\Psi}(\zeta^{-1})$.

Let $\mu_j(\zeta^{-1})$, for $j=1,2,3,4,5$, be the eigenvalues of the matrix $\hat{\Psi}(\zeta^{-1})$, which are the solutions to the characteristic equation

$$
\begin{aligned}
&\tau\rho c\det(\mu I-\hat{\Phi}(\zeta^{-1}))\\
=&\tau\rho c\mu^5+(\rho c\zeta^{-2}-\tau\rho l)\mu^4+[(\tau m^2+ck)-(\tau\rho l^*+\rho l)\zeta^{-2}]\mu^3
\end{aligned}
$$

$$+[-kl+(ck^*+m^2)\zeta^{-2}-\rho l^*\zeta^{-4}]\mu^2-(k^*l+l^*k)\zeta^{-2}\mu-k^*l^*\zeta^{-4}$$
$$=0.$$

When $|\zeta|\to\infty$, $\mu_j(\zeta^{-1})$ has the following asymptotic expansion:

$$\mu_j(\zeta^{-1})=\mu_j^{(2)}+\mu_j^{(1)}\zeta^{-1}+\mu_j^{(0)}\zeta^{-2}+\mu_j^{(-1)}\zeta^{-3}+\cdots.$$

From the relation $\lambda_j(\zeta)=\zeta^2\mu_j(\zeta^{-1})$, we have the asymptotic expansion of $\lambda_j(\zeta)$ for $|\zeta|\to\infty$:

$$\lambda_j(\zeta)=\mu_j^{(2)}\zeta^2+\mu_j^{(1)}\zeta+\mu_j^{(0)}+\mu_j^{(-1)}\zeta^{-1}+\cdots.$$

By direct computations, we have

$$\begin{aligned}
&\mu_j^{(2)}=\chi_j,\quad j=1,2,3,\\
&\mu_j^{(2)}=0,\quad \mu_j^{(1)}=0,\quad \mu_j^{(0)}=-\frac{k^*}{k},\quad j=4,\\
&\mu_j^{(2)}=0,\quad \mu_j^{(1)}=0,\quad \mu_j^{(0)}=-\frac{l^*}{l},\quad j=5,
\end{aligned}$$

where χ_j are the roots of equation $\tau\rho cX^3-\tau\rho lX^2+(\tau m^2+ck)X-kl=0$. To see that $\mathrm{Re}(\chi_j)>0$, we set $f(X):=\tau\rho cX^3-\tau\rho lX^2+(\tau m^2+ck)X-kl$. Since

$$f(0)=-kl<0\quad\text{and}\quad f\left(\frac{l}{c}\right)=\frac{\tau m^2 l}{c}>0,$$

we have that f has at least least one real root $X=\chi_1$ and $\chi_1\in\left(0,\frac{l}{c}\right)$. We rewrite f as

$$f(X)=(X-\chi_1)(\tau\rho cX^2+d_1X+d_0),$$

where $d_1=-\tau\rho l+\chi_1\tau\rho c<0$ and $d_0=\frac{kl}{\chi_1}$. For the other two roots χ_2 and χ_3, we have

$$\chi_2+\chi_3=-\frac{d_1}{\tau\rho c}>0,\quad \chi_2\chi_3=\frac{d_0}{\tau\rho c}>0.$$

We conclude that if χ_2 and χ_3 are real, they are both positive; if χ_2 and χ_3 are complex conjugate and

$$\mathrm{Re}(\chi_2)=\mathrm{Re}(\chi_3)=\frac{1}{2}\left(\frac{l}{c}-\chi_1\right)>0.$$

Consequently, for $|\xi|\to\infty$, we have

$$\mathrm{Re}\lambda_j(i\xi)=\begin{cases}-\mathrm{Re}(\chi_j)+O(1),\quad j=1,2,3,\\ -\dfrac{k^*}{k}+O(|\xi|^{-1}),\quad j=4,\\ -\dfrac{l^*}{l}+O(|\xi|^{-1}),\quad j=5.\end{cases}\tag{2.34}$$

Remark 2.11. *It follows from Theorem 2.1 that*

$$\rho_1(\xi)=\frac{\xi^4}{(1+\xi^2)^2},\quad\text{when}\quad \mathrm{K}>0.$$

Then $\rho_1(\xi) \sim |\xi|^4$ for $|\xi| \to 0$ and $\rho_1(\xi) \sim 1$ for $|\xi| \to \infty$. We find that it is consistent with the real parts of the "slowest" eigenvalues, which behave like $|\xi|^4$ for $|\xi| \to 0$ and 1 for $|\xi| \to \infty$ from (2.32) *and* (2.34)*, respectively. Therefore, the pointwise estimate in Theorem 2.1 is the optimal pointwise estimates of solutions in Fourier space.*
Theorem 2.8 yields

$$\rho_2(\xi) = \frac{\xi^6}{(1+\xi^2)^3}, \quad \text{when} \quad \mathrm{K} = 0.$$

Then $\rho_2(\xi) \sim |\xi|^6$ for $|\xi| \to 0$ and $\rho_2(\xi) \sim 1$ for $|\xi| \to \infty$. Since it matches with the real parts of the slowest eigenvalues in (2.33) *and* (2.34)*, the pointwise estimate in Theorem 2.8 is also optimal.*

3 The JMGT-viscoelastic plate with Gurtin-Pipkin thermal law

Following the same process (cp.[18, 19]) and the treatment of Gurtin-Pipkin thermal law (cp.[26]), it is easy to prove the well-posedness of system (1.4)-(1.5). We start directly with decay estimates.

3.1 Decay estimate

Here we prove the decay estimates in two cases, which are the subcritical case $K > 0$ in Subsection 3.1.1 and the critical case $K = 0$ in Subsection 3.1.2. Before we state our main result about system (1.4)-(1.5), we need some notations and hypotheses to deal with the memory term. First we introduce the following new variable (cp. [6, 8])

$$\eta(x,t,s) = \int_0^s \theta(x,t-\sigma)\mathrm{d}\sigma = \int_{t-s}^t \theta(x,\sigma)\mathrm{d}\sigma, \quad (x,t,s) \in \mathbb{R}^n \times [0,+\infty) \times \mathbb{R}^+, \tag{3.1}$$

which satisfies

$$\eta_t = -\eta_s + \theta, \quad (x,t,s) \in \mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{R}^+,$$

and the conditions

$$\lim_{s\to 0} \eta(x,t,s) = 0, \quad (x,t) \in \mathbb{R}^n \times [0,+\infty),$$

and

$$\eta(x,0,s) = \eta_0(s) = \int_0^s \theta_0(x,\sigma)\mathrm{d}\sigma, \quad (x,s) \in \mathbb{R}^n \times \mathbb{R}^+.$$

Assuming $g(\infty) = 0$, it follows from integration by parts that

$$\int_0^\infty g(s)\Delta\theta(t-s)\mathrm{d}s = -\int_0^\infty g'(s)\Delta\eta(s)\mathrm{d}s.$$

Setting $\mu(s) = -g'(s)$, we obtain

$$\int_0^\infty g(s)\Delta\theta(t-s)\mathrm{d}s = \int_0^\infty \mu(s)\Delta\eta(s)\mathrm{d}s.$$

Let the linear operator T defined as $T\eta = -\eta_s$. Then, system (1.4), (1.5) is equivalent to the following:

$$\begin{cases} \tau\rho u_{ttt} + \rho u_{tt} + k^*\Delta^2 u + k\Delta^2 u_t + m\Delta\theta = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \\ \theta_t - \dfrac{1}{l}\displaystyle\int_0^\infty \mu(s)\Delta\eta(s)\mathrm{d}s - m\tau\Delta u_{tt} - m\Delta u_t = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \\ \eta_t = T\eta + \theta, & (x,t,s) \in \mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{R}^+, \end{cases} \tag{3.2}$$

with the conditions

$$\begin{cases} (u, u_t, u_{tt}, \theta)(x,0) = (u_0, u_1, u_2, \theta_0)(x), & x \in \mathbb{R}^n, \\ \displaystyle\lim_{s\to 0} \eta(x,t,s) = 0, & (x,t) \in \mathbb{R}^n \times [0,+\infty), \\ \eta_0(x,s) = \displaystyle\int_0^s \theta_0(x,\sigma)\mathrm{d}\sigma, & (x,s) \in \mathbb{R}^n \times \mathbb{R}^+. \end{cases} \tag{3.3}$$

For the memory kernel g, we use the following assumptions as in [8]:
(G1) g' is an absolutely continuous function on $\mathbb{R}^+$ so that

$$g'(s) \le 0, \quad g''(s) \ge 0, \quad g'(0) = \lim_{s\to 0} g'(s) \in (-\infty, 0).$$

(G2) There exists $\nu > 0$ such that the differential inequality

$$g''(s) + \nu g'(s) \ge 0$$

holds for almost every $s > 0$. A typical example is given by $g(s) = \mathrm{e}^{-\nu s}$.

We take the Fourier transform of system (3.2)-(3.3) to obtain

$$\begin{cases} \tau\rho\hat{u}_{ttt} + \rho u_{tt} + k^*\xi^4\hat{u} + k\xi^4\hat{u}_t - m\xi^2\hat{\theta} = 0, \\ \hat{\theta}_t + \dfrac{\xi^2}{l}\displaystyle\int_0^\infty \mu(s)\hat{\eta}(s)\mathrm{d}s + m\tau\xi^2\hat{u}_{tt} + m\xi^2\hat{u}_t = 0, \\ \hat{\eta}_t + \hat{\eta}_s = \hat{\theta}, \end{cases} \tag{3.4}$$

with initial data

$$\begin{cases} (\hat{u}, \hat{u}_t, \hat{u}_{tt}, \hat{\theta})(\xi,0) = (\hat{u}_0, \hat{u}_1, \hat{u}_2, \hat{\theta}_0)(\xi), \\ \hat{\eta}_0(\xi,s) = \displaystyle\int_0^s \hat{\theta}_0(\xi,\sigma)\mathrm{d}\sigma, \end{cases} \tag{3.5}$$

where $\xi \in \mathbb{R}^n$. By introducing the following new variables

$$\hat{v} = \hat{u}_t, \quad \hat{w} = \hat{u}_{tt},$$

(3.4) can be rewritten as

$$\begin{cases} \hat{u}_t - \hat{v} = 0, \\ \hat{v}_t - \hat{w} = 0, \\ \tau\hat{w}_t + \hat{w} + \dfrac{k^*}{\rho}\xi^4\hat{u} + \dfrac{k}{\rho}\xi^4\hat{v} - \dfrac{m}{\rho}\xi^2\hat{\theta} = 0, \\ \hat{\theta}_t + \dfrac{\xi^2}{l}\displaystyle\int_0^\infty \mu(s)\hat{\eta}(s)\mathrm{d}s + m\tau\xi^2\hat{w} + m\xi^2\hat{v} = 0, \\ \hat{\eta}_t + \hat{\eta}_s = \hat{\theta}. \end{cases} \tag{3.6}$$

3.1.1 The case $K > 0$

In this subsection, we first state the following pointwise estimate and decay result in the case $K > 0$.

Theorem 3.1. *Assume that $K > 0$. Let $\hat{U} = (\hat{u}_t + \tau\hat{u}_{tt}, \Delta\hat{u}_t, \Delta(\hat{u} + \tau\hat{u}_t), \hat{\theta}, \nabla\hat{\eta})^T$, where $(\hat{u}(\xi,t), \hat{\theta}(\xi,t))$ is the Fourier image of the solution $(u(x,t), \theta(x,t))$. Then $\hat{U}$ has the following pointwise estimate*

$$|\hat{U}(\xi,t)|^2 \le Ce^{-c\rho_1(\xi)t}|\hat{U}_0(\xi)|^2, \tag{3.7}$$

for any $t \ge 0$, where $\rho_1(\xi) := \frac{\xi^2}{(1+\xi^2)^2}$. Furthermore, let $U = (u_t + \tau u_{tt}, \Delta u_t, \Delta(u + \tau u_t), \theta, \nabla\eta)^T$, where $(u(x,t), \theta(x,t))$ is the solution of problem (3.2)-(3.3), and $U_0 = U(x,0) \in H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, where s is nonnegative, then U satisfies the following decay estimate

$$\|\nabla^k U(t)\|^2_{L^2(\mathbb{R}^n)} \le C(1+t)^{-\frac{n}{2}-k}\|U_0\|^2_{L^1(\mathbb{R}^n)} + C(1+t)^{-l}\|\nabla^{k+l}U_0\|^2_{L^2(\mathbb{R}^n)}, \tag{3.8}$$

for all $0 \le k \le s$.

Remark 3.2. *The above decay estimate is of regularity-loss type. From the asymptotic expansion of the eigenvalues later in this section, we observe that the asymptotic behavior in (3.29) is not the same as in the exponent $\rho_1(\xi)$ in (3.7), which means that the exponent is not optimal. Although the above pointwise estimate is not optimal, we notice that the decay rate in (3.8) is optimal. The regularity-loss estimate of the solutions may be improved.*

Lemma 3.3. *Assume that $K > 0$. Let $(\hat{u}, \hat{v}, \hat{w}, \hat{\theta}, \hat{\eta})$ be the solution of (3.6), and define the energy functional of system (3.6) as*

$$\hat{E}(\xi,t) := |\hat{v} + \tau\hat{w}|^2 + \frac{\tau K}{\rho}\xi^4|\hat{v}|^2 + \frac{k^*}{\rho}\xi^4|\hat{u} + \tau\hat{v}|^2 + \frac{1}{\rho}|\hat{\theta}|^2 + \frac{\xi^2}{l}\int_0^\infty \mu(s)|\hat{\eta}(\xi,t,s)|^2 \mathrm{d}s,$$

then there exist two positive constants C_1 and C_2 such that

$$C_1|\hat{U}(\xi,t)|^2 \le \hat{E}(\xi,t) \le C_2|\hat{U}(\xi,t)|^2,$$

then $\hat{E}(\xi,t)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{E}(\xi,t) = -\frac{K}{\rho}\xi^4|\hat{v}|^2 + \frac{\xi^2}{2\rho l}\int_0^\infty \mu'(s)|\hat{\eta}(\xi,t,s)|^2 \mathrm{d}s.$$

Lemma 3.4. *Define the functional*

$$F_1(\xi,t) := \mathrm{Re}((\hat{v} + \tau\hat{w})(\hat{u}^* + \tau\hat{v}^*)),$$

then for any $\varepsilon_1 > 0$, there exists $C(\varepsilon_1) > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}F_1(\xi,t) + \left(\frac{k^*}{\rho} - 2\varepsilon_1\right)\xi^4|\hat{u} + \tau\hat{v}|^2 \le |\hat{v} + \tau\hat{w}|^2 + C(\varepsilon_1)\xi^4|\hat{v}|^2 + C(\varepsilon_1)|\hat{\theta}|^2. \tag{3.9}$$

Lemma 3.5. *The functional*

$$F_2(\xi,t) := \mathrm{Re}(-\tau(\hat{v} + \tau\hat{w})\hat{v}^*)$$

satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}F_2(\xi,t) + (1-\varepsilon_3)|\hat{v} + \tau\hat{w}|^2 \le C(\varepsilon_2,\varepsilon_3)\xi^2(1+\xi^2)^2|\hat{v}|^2 + \varepsilon_2\xi^4|\hat{u} + \tau\hat{v}|^2 + \varepsilon_3\frac{\xi^2}{1+\xi^2}|\hat{\theta}|^2, \tag{3.10}$$

for any $\varepsilon_2, \varepsilon_3 > 0$ and $C(\varepsilon_2,\varepsilon_3) > 0$.

Lemma 3.6. *For any $\varepsilon_4, \varepsilon_5 > 0$, there exist $C(\varepsilon_4), C(\varepsilon_5) > 0$ such that the following inequality holds true*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}F_3(\xi,t) + (g(0)-\varepsilon_4)|\hat{\theta}|^2 \\
\leq&\Big(\frac{1}{l}+C(\varepsilon_5)\Big)g(0)\xi^2(1+\xi^2)\int_0^\infty \mu(s)|\hat{\eta}(s)|^2\mathrm{d}s + \varepsilon_5\frac{\xi^2}{1+\xi^2}|\hat{v}+\tau\hat{w}|^2 \\
&+C(\varepsilon_4)g'(0)\int_0^\infty(-\mu'(s))|\hat{\eta}(s)|^2\mathrm{d}s, \qquad (3.11)
\end{aligned}$$

where

$$F_3(\xi,t) := \mathrm{Re}\Big(-\int_0^\infty \mu(s)\hat{\eta}^*(s)\mathrm{d}s\hat{\theta}\Big).$$

Proof of Theorem 3.1. We define the Lyapunov functional

$$\mathcal{L}_1(t) := N_1\xi^2F_1(t) + N_2\xi^2F_2(t) + N_3\xi^2F_3(t),$$

where N_1, N_2 and N_3 are positive constants that will be fixed later. It follows from the above estimates that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_1(\xi,t) + \Big[N_1\Big(\frac{k^*}{\rho}-2\varepsilon_1\Big)-N_2\varepsilon_2\Big]\xi^2\xi^4|\hat{u}+\tau\hat{v}|^2 \\
&+\Big[N_2(1-\varepsilon_3)-N_1-N_3\varepsilon_5\Big]\xi^2|\hat{v}+\tau\hat{w}|^2 \\
&+\Big[N_3(g(0)-\varepsilon_4)-N_1C(\varepsilon_1)-N_2\varepsilon_3\Big]\xi^2|\hat{\theta}|^2 \\
&+\Big[-N_1C(\varepsilon_1)-N_2C(\varepsilon_2,\varepsilon_3)\Big](1+\xi^2)^2\xi^4|\hat{v}|^2 \\
&-N_3\Big(\frac{1}{l}+C(\varepsilon_5)\Big)g(0)\xi^2(1+\xi^2)\int_0^\infty \xi^2\mu(s)|\hat{\eta}(s)|^2\mathrm{d}s \\
&-N_3C(\varepsilon_4)\xi^2g'(0)\int_0^\infty(-\mu'(s))|\hat{\eta}(s)|^2\mathrm{d}s \\
\leq&0, \qquad (3.12)
\end{aligned}$$

where we used the fact that $\frac{\xi^2}{1+\xi^2}\leq 1$. Moreover, according to (G2), we deduce

$$\int_0^\infty \xi^2\mu(s)|\hat{\eta}(\xi,t,s)|^2\mathrm{d}s \leq \frac{1}{\nu}\int_0^\infty(-\mu'(s))\xi^2|\hat{\eta}(\xi,t,s)|^2\mathrm{d}s. \qquad (3.13)$$

Hence, (3.12) becomes

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_1(\xi,t) + \Big[N_1\Big(\frac{k^*}{\rho}-2\varepsilon_1\Big)-N_2\varepsilon_2\Big]\xi^2\xi^4|\hat{u}+\tau\hat{v}|^2 \\
&+\Big[N_2(1-\varepsilon_3)-N_1-N_3\varepsilon_5\Big]\xi^2|\hat{v}+\tau\hat{w}|^2+\Big[N_3(g(0)-\varepsilon_4)-N_1C(\varepsilon_1)-N_2\varepsilon_3\Big]\xi^2|\hat{\theta}|^2 \\
&-C(N_1,N_2,\varepsilon_1,\varepsilon_2,\varepsilon_3)(1+\xi^2)^2\xi^4|\hat{v}|^2-C(N_3,\varepsilon_4,\varepsilon_5,\nu)(1+\xi^2)^2\int_0^\infty\xi^2(-\mu'(s))|\hat{\eta}(\xi,t,s)|^2\mathrm{d}s \\
\leq&0. \qquad (3.14)
\end{aligned}$$

By choosing the constants carefully as before, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_1(\xi,t)+\alpha_1M_1(\xi,t)$$

$$\leq C(N_1,N_2,\varepsilon_1,\varepsilon_2,\varepsilon_3)(1+\xi^2)^2\xi^4|\hat{v}|^2+C(N_3,\varepsilon_4,\varepsilon_5,\nu)(1+\xi^2)^2\int_0^\infty \xi^2(-\mu'(s))|\hat{\eta}(\xi,t,s)|^2\mathrm{d}s, \tag{3.15}$$

where

$$M_1(\xi,t)=\xi^2(\xi^4|\hat{u}+\tau\hat{v}|^2+|\hat{v}+\tau\hat{w}|^2+|\hat{\theta}|^2).$$

We define the Lyapunov functional $L_1(\xi,t)$ as

$$L_1(\xi,t):=N(1+\xi^2)^2\hat{E}(\xi,t)+\mathcal{L}_1(\xi,t),$$

where N is a positive constant that will be fixed later. By virtue of Lemma 3.3 and (3.15), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}L_1(\xi,t)+\alpha_1M_1(\xi,t)+\left[\frac{NK}{\rho}-C(N_1,N_2,\varepsilon_1,\varepsilon_2,\varepsilon_3)\right](1+\xi^2)^2\xi^4|\hat{v}|^2\\&+\left[\frac{N}{2\rho l}-C(N_3,\varepsilon_3,\varepsilon_5,\nu)\right](1+\xi^2)^2\xi^2\int_0^\infty(-\mu'(s))|\hat{\eta}(\xi,t,s)|^2\mathrm{d}s\\&\leq 0.\end{aligned} \tag{3.16}$$

Finally, (3.16) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}L_1(\xi,t)+C\xi^2\hat{E}(\xi,t)\leq 0,\quad \forall t\geq 0 \tag{3.17}$$

by choosing N large enough.

On the other hand, for N large enough, we can find two positive constants C_1 and C_2 such that

$$C_1(1+\xi^2)^2\hat{E}(\xi,t)\leq L_1(\xi,t)\leq C_2(1+\xi^2)^2\hat{E}(\xi,t),\quad \forall\ t\geq 0.$$

Therefore, (3.17) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{E}(\xi,t)+C\frac{\xi^2}{(1+\xi^2)^2}\hat{E}(\xi,t)\leq 0,\quad \forall t\geq 0.$$

Consequently, making use of the equivalence of $\hat{E}(\xi,t)$ and $|\hat{U}|^2$ and Gronwall's inequality, we obtain the desired result (3.7). Hence we arrive at the desired decay estimate (3.8). □

3.1.2 The case $K=0$

In this subsection, we assume that $K=0$ and we have the following pointwise estimate and decay result:

Theorem 3.7. *Assume that $K=0$. Let $\hat{V}=(\hat{u}_t+\tau\hat{u}_{tt},\Delta(\hat{u}+\tau\hat{u}_t),\hat{\theta},\nabla\hat{\eta})^T$, where $(\hat{u}(\xi,t),\hat{\theta}(\xi,t))$ is the Fourier image of the solution $(u(x,t),\theta(x,t))$. Then $\hat{V}$ has the following pointwise estimate*

$$|\hat{V}(\xi,t)|^2\leq Ce^{-c\rho_2(\xi)t}|\hat{V}_0(\xi)|^2, \tag{3.18}$$

for any $t\geq 0$, where $\rho_2(\xi):=\frac{\xi^2}{(1+\xi^2)^2}$. Furthermore, let $V=(u_t+\tau u_{tt},\Delta(u+\tau u_t),\theta,\nabla\eta)^T$, where $(u(x,t),\theta(x,t))$ is the solution of problem (3.2)-(3.3), and $V_0=V(x,0)\in H^s(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$, where s is nonnegative, then V satisfies the following decay estimate

$$\|\nabla^kV(t)\|^2_{L^2(\mathbb{R}^n)}\leq C(1+t)^{-\frac{n}{2}-k}\|V_0\|^2_{L^1(\mathbb{R}^n)}+C(1+t)^{-l}\|\nabla^{k+l}V_0\|^2_{L^2(\mathbb{R}^n)}, \tag{3.19}$$

for all $0\leq k\leq s$.

Remark 3.8. *We note that the above result exhibits a regularity-loss phenomenon. And according to the asymptotic expansion of the eigenvalues in the next section, the exponent in pointwise estimate* (3.18) *is optimal. Hence, the decay estimate* (3.19) *is also optimal.*

We get the following consequence from Lemma 3.3 and 3.4.

Lemma 3.9. *Assume that $K=0$. Let $(\hat{u},\hat{v},\hat{w},\hat{\theta},\hat{\eta})$ be the solution of* (3.6)*, and the energy functional of system* (3.6) *becomes*

$$\hat{\mathcal{E}}(\xi,t):=|\hat{v}+\tau\hat{w}|^2+\frac{k^*}{\rho}\xi^4|\hat{u}+\tau\hat{v}|^2+\frac{1}{\rho}|\hat{\theta}|^2+\frac{\xi^2}{l}\int_0^\infty\mu(s)|\hat{\eta}(\xi,t,s)|^2\mathrm{d}s,$$

then $\hat{\mathcal{E}}(\xi,t)$ and $F_1(t)$ satisfy

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathcal{E}}(\xi,t)=\frac{\xi^2}{2\rho l}\int_0^\infty\mu'(s)|\hat{\eta}(\xi,t,s)|^2\mathrm{d}s,\\&\frac{\mathrm{d}}{\mathrm{d}t}F_1(\xi,t)+\left(\frac{k^*}{\rho}-\varepsilon_1'\right)\xi^4|\hat{u}+\tau\hat{v}|^2\le|\hat{v}+\tau\hat{w}|^2+C(\varepsilon_1')|\hat{\theta}|^2.\end{aligned}$$

Lemma 3.10. *The following inequality holds true:*

$$\frac{d}{dt}\bar{F}_2(t)+(m-\varepsilon_2')\xi^2|\hat{v}+\tau\hat{w}|^2\le\varepsilon_3'\xi^6|\hat{u}+\tau\hat{v}|^2+C(\varepsilon_3')\xi^2|\hat{\theta}|^2+C(\varepsilon_2')\xi^2g(0)\int_0^\infty\mu(s)|\hat{\eta}(s)|^2\mathrm{d}s, \tag{3.20}$$

for any $\varepsilon_2',\varepsilon_3'>0$, where

$$\bar{F}_2(t):=\mathrm{Re}((\hat{v}+\tau\hat{w})\hat{\theta}^*).$$

Proof. Combining $(3.6)_2+(3.6)_3$ with $(3.6)_4$, we arrive at

$$\begin{cases}(\hat{v}+\tau\hat{w})_t+\dfrac{k^*}{\rho}\xi^4(\hat{u}+\tau\hat{v})-\dfrac{m}{\rho}\xi^2\hat{\theta}=0,\\ \hat{\theta}_t+\dfrac{\xi^2}{l}\displaystyle\int_0^\infty\mu(s)\hat{\eta}(s)\mathrm{d}s+m\xi^2(\hat{v}+\tau\hat{w})=0.\end{cases} \tag{3.21}$$

Multiplying $(3.21)_1$ and $(3.21)_2$ by $\hat{\theta}^*$ and $(\hat{v}^*+\tau\hat{w}^*)$, respectively, adding the results and taking the real part, we get

$$\begin{aligned}&\frac{d}{dt}\bar{F}_2(t)+m\xi^2|\hat{v}+\tau\hat{w}|^2\\=&\frac{m}{\rho}\xi^2|\hat{\theta}|^2-\frac{k^*}{\rho}\xi^4\mathrm{Re}((\hat{u}+\tau\hat{v})\hat{\theta}^*)-\mathrm{Re}\left(\frac{\xi^2}{l}\int_0^\infty\mu(s)\hat{\eta}(s)\mathrm{d}s(\hat{v}^*+\tau\hat{w}^*)\right).\end{aligned}$$

Young's inequality yields, for any $\varepsilon_2',\varepsilon_3'>0$,

$$-\xi^4\mathrm{Re}((\hat{u}+\tau\hat{v})\hat{\theta}^*)\le\varepsilon_3'\xi^6|\hat{u}+\tau\hat{v}|^2+C(\varepsilon_3')\xi^2|\hat{\theta}|^2,$$

$$-\mathrm{Re}\left(\frac{\xi^2}{l}\int_0^\infty\mu(s)\hat{\eta}(s)\mathrm{d}s(\hat{v}^*+\tau\hat{w}^*)\right)\le\varepsilon_2'\xi^2|\hat{v}+\tau\hat{w}|^2+C(\varepsilon_2')\xi^2g(0)\int_0^\infty\mu(s)|\hat{\eta}(s)|^2\mathrm{d}s.$$

A combination of all the above estimates gives the desired result. □

At this position, we are ready to prove Theorem 3.7.

Proof of Theorem 3.7. We define the Lyapunov functional

$$\mathcal{L}_2(t) := \bar{N}_1 \frac{\xi^2}{1+\xi^2} F_1(t) + \bar{N}_2 \frac{1}{1+\xi^2} \bar{F}_2(t) + \bar{N}_3 F_3(t),$$

where $\bar{N}_1, \bar{N}_2$ and $\bar{N}_3$ are positive constants that will be fixed later. By virtue of the above estimates, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_2(\xi,t) + \left[\bar{N}_1\left(\frac{k^*}{\rho} - \varepsilon_1'\right) - \bar{N}_2\varepsilon_3'\right]\frac{\xi^2}{1+\xi^2}\xi^4|\hat{u}+\tau\hat{v}|^2 \\
&+ \left[\bar{N}_2(m-\varepsilon_2') - \bar{N}_1 - \bar{N}_3\varepsilon_5\right]\frac{\xi^2}{1+\xi^2}|\hat{v}+\tau\hat{w}|^2 \\
&+ \left[\bar{N}_3(g(0)-\varepsilon_4) - \bar{N}_1 C(\varepsilon_1') - \bar{N}_2 C(\varepsilon_3')\right]|\hat{\theta}|^2 \\
&- \bar{N}_3\left(\frac{1}{l} + C(\varepsilon_5)\right)g(0)(1+\xi^2)\int_0^\infty \xi^2\mu(s)|\hat{\eta}(s)|^2\mathrm{d}s \\
&- \bar{N}_3 C(\varepsilon_4)g'(0)\int_0^\infty (-\mu'(s))|\hat{\eta}(s)|^2\mathrm{d}s \\
&- \bar{N}_2 C(\varepsilon_2')\frac{1}{1+\xi^2}g(0)\int_0^\infty \xi^2\mu(s)|\hat{\eta}(s)|^2\mathrm{d}s \\
\le& 0,
\end{aligned}$$

where we used the fact that $\frac{\xi^2}{1+\xi^2} \le 1$. It follows from (3.13) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_2(\xi,t) + \left[\bar{N}_1\left(\frac{k^*}{\rho} - \varepsilon_1'\right) - \bar{N}_2\varepsilon_3'\right]\frac{\xi^2}{1+\xi^2}\xi^4|\hat{u}+\tau\hat{v}|^2 \\
&+ \left[\bar{N}_2(m-\varepsilon_2') - \bar{N}_1 - \bar{N}_3\varepsilon_5\right]\frac{\xi^2}{1+\xi^2}|\hat{v}+\tau\hat{w}|^2 \\
&+ \left[\bar{N}_3(g(0)-\varepsilon_4) - \bar{N}_1 C(\varepsilon_1') - \bar{N}_2 C(\varepsilon_3')\right]|\hat{\theta}|^2 \\
&- C(\bar{N}_2, \bar{N}_3, \varepsilon_2', \varepsilon_4, \varepsilon_5, \nu)(1+\xi^2)\int_0^\infty \xi^2(-\mu'(s))|\hat{\eta}(\xi,t,s)|^2\mathrm{d}s \\
\le& 0.
\end{aligned}$$

By choosing our constants carefully as what we did before, we can derive

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_2(\xi,t) + \alpha_2 M_2(\xi,t) \\
\le& C(\bar{N}_2, \bar{N}_3, \varepsilon_2', \varepsilon_4, \varepsilon_5, \nu)(1+\xi^2)\int_0^\infty \xi^2(-\mu'(s))|\hat{\eta}(\xi,t,s)|^2\mathrm{d}s,
\end{aligned} \tag{3.22}$$

where

$$M_2(\xi,t) = \frac{\xi^2}{1+\xi^2}(\xi^4|\hat{u}+\tau\hat{v}|^2 + |\hat{v}+\tau\hat{w}|^2 + |\hat{\theta}|^2).$$

Then we define the Lyapunov functional $L_2(\xi,t)$

$$L_2(\xi,t) := \bar{N}(1+\xi^2)\hat{\mathcal{E}}(\xi,t) + \mathcal{L}_2(\xi,t),$$

where $\bar{N}$ is a positive constant that will be fixed later. Applying Lemma 3.9 and (3.22), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}L_2(\xi,t)+\alpha_2 M_2(\xi,t)\\&+\left[\frac{\bar{N}}{2\rho l}-C(\bar{N}_2,\bar{N}_3,\varepsilon_2',\varepsilon_4,\varepsilon_5,\nu)\right](1+\xi^2)\xi^2\int_0^\infty(-\mu'(s))|\hat{\eta}(\xi,t,s)|^2\mathrm{d}s\\&\le 0.\end{aligned}\tag{3.23}$$

Finally, by choosing N large enough, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}L_2(\xi,t)+C\frac{\xi^2}{1+\xi^2}\hat{\mathcal{E}}(\xi,t)\le 0,\quad \forall t\ge 0.$$

Moreover, by the equivalence of $L_2(\xi,t)$ and $(1+\xi^2)\hat{\mathcal{E}}(\xi,t)$, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathcal{E}}(\xi,t)+C\frac{\xi^2}{(1+\xi^2)^2}\hat{\mathcal{E}}(\xi,t)\le 0,\quad \forall t\ge 0.$$

Therefore, we obtain the desired pointwise estimate (3.18) and the desired decay estimate (3.19). □

3.2 Asymptotic behavior of the eigenvalues

We assume here

$$g(s)=\delta_1 e^{-\delta_2 s},\tag{3.24}$$

where for $\delta_1,\delta_2>0$, as typical memory type kernel. From $(1.4)_1$ and $(1.4)_2$, it follows that

$$-m\Delta\theta=\tau\rho u_{ttt}+\rho u_{tt}+k^*\Delta^2 u+k\Delta^2 u_t\tag{3.25}$$

and

$$-m\Delta\theta_t+\frac{m}{l}\int_0^\infty g(s)\Delta^2\theta(t-s)\mathrm{d}s+m^2\tau\Delta^2 u_{tt}+m^2\Delta^2 u_t=0.\tag{3.26}$$

Substituting (3.25) into (3.26), we arrive at

$$\begin{aligned}&\tau\rho u_{tttt}+\rho u_{ttt}+k^*\Delta^2 u_t+k\Delta^2 u_{tt}+m^2\tau\Delta^2 u_{tt}+m^2\Delta^2 u_t\\&-\frac{1}{l}\int_0^\infty\delta_1 e^{-\delta_2 s}(\tau\rho\Delta u_{ttt}+\rho\Delta u_{tt}+k^*\Delta^3 u+k\Delta^3 u_t)(t-s)\mathrm{d}s=0,\end{aligned}$$

or

$$\begin{aligned}&\tau\rho u_{tttt}+\rho u_{ttt}+k^*\Delta^2 u_t+k\Delta^2 u_{tt}+m^2\tau\Delta^2 u_{tt}+m^2\Delta^2 u_t\\&-\frac{1}{l}\int_{-\infty}^t\delta_1 e^{-\delta_2(t-r)}(\tau\rho\Delta u_{ttt}+\rho\Delta u_{tt}+k^*\Delta^3 u+k\Delta^3 u_t)(r)\mathrm{d}r=0.\end{aligned}$$

Taking the derivative of the above equation with respect to t, we have

$$\begin{aligned}&\tau\rho u_{ttttt}+(k+m^2\tau)\Delta^2 u_{ttt}+(k^*+m^2+k\delta_2+m^2\tau\delta_2)\Delta^2 u_{tt}+(k^*\delta_2+m^2\delta_2)\Delta^2 u_t\\&-\frac{\tau\rho\delta_1}{l}\Delta u_{ttt}-\frac{\rho\delta_1}{l}\Delta u_{tt}-\frac{\delta_1 k}{l}\Delta^3 u_t-\frac{\delta_1 k^*}{l}\Delta^3 u+\rho\delta_2 u_{ttt}+(\rho+\tau\rho\delta_2)u_{tttt}=0.\end{aligned}\tag{3.27}$$

Taking the Fourier transform of (3.27), we arrive at

$$
\begin{aligned}
&\tau\rho\frac{\mathrm{d}^5}{\mathrm{d}t^5}\hat{u}+(\rho+\tau\rho\delta_2)\frac{\mathrm{d}^4}{\mathrm{d}t^4}\hat{u}+\left[(m^2\tau+k)\xi^4+\frac{\tau\rho\delta_1}{l}\xi^2+\rho\delta_2\right]\frac{\mathrm{d}^3}{\mathrm{d}t^3}\hat{u}\\
&+\left[(k^*+m^2+k\delta_2+m^2\tau\delta_2)\xi^4+\frac{\delta_1\rho}{l}\xi^2\right]\frac{\mathrm{d}^2}{\mathrm{d}t^2}\hat{u}\\
&+\left[\frac{\delta_1 k}{l}\xi^6+(k^*\delta_2+m^2\delta_2)\xi^4\right]\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}+\frac{\delta_1 k^*}{l}\xi^6\hat{u}=0.
\end{aligned}
$$

Let $\zeta=i\xi$, then the characteristic equation is

$$
\begin{aligned}
&\tau\rho l\lambda^5+(\rho l+\tau\rho\delta_2 l)\lambda^4+\left[(m^2\tau l+kl)\zeta^4-\tau\rho\delta_1\zeta^2+\rho\delta_2 l\right]\lambda^3\\
&+\left[(k^*l+m^2l+k\delta_2 l+m^2\tau\delta_2 l)\zeta^4-\delta_1\rho\zeta^2\right]\lambda^2\\
&+\left[-\delta_1 k\zeta^6+(k^*\delta_2 l+m^2\delta_2 l)\zeta^4\right]\lambda-\delta_1 k^*\zeta^6=0.
\end{aligned}
$$

(1)When $|\zeta|\to 0$, we denote the eigenvalues of the above equation as $\lambda_j(\zeta),\ j=1,\cdots,5$, which has the following asymptotic expansion:

$$\lambda_j(\zeta)=\lambda_j^{(0)}+\lambda_j^{(1)}\zeta+\lambda_j^{(2)}\zeta^2+\lambda_j^{(3)}\zeta^3+\cdots.$$

Here each coefficient is given by direct computations as

$$
\begin{aligned}
\lambda_j^{(0)}&=\lambda_j^{(1)}=0,\quad \lambda_j^{(2)}=\phi_j,\quad j=1,2,3,\\
\lambda_j^{(0)}&=-\frac{1}{\tau},\quad j=4,\\
\lambda_j^{(0)}&=-\delta_2,\quad j=5,
\end{aligned}
$$

where $\mathrm{Re}(\phi_j)>0$.

Consequently, we have

$$
\mathrm{Re}\lambda_j(i\xi)=\begin{cases}-\mathrm{Re}(\phi_j)|\xi|^2+O(|\xi|^3), & j=1,2,3,\\ -\dfrac{1}{\tau}+O(|\xi|^2), & j=4,\\ -\delta_2+O(|\xi|^2), & j=5.\end{cases}\tag{3.28}
$$

(2) When $|\zeta|\to\infty$, taking $\nu=\zeta^{-1}=(i\xi)^{-1}$, the characteristic equation is

$$
\begin{aligned}
&\tau\rho l\mu^5+(\rho l+\tau\rho\delta_2 l)\zeta^{-2}\mu^4+\left[(m^2\tau l+kl)-\tau\rho\delta_1\zeta^{-2}+\rho\delta_2 l\zeta^{-4}\right]\mu^3\\
&+\left[(k^*l+m^2l+k\delta_2 l+m^2\tau\delta_2 l)\zeta^{-2}-\delta_1\rho\zeta^{-4}\right]\mu^2\\
&+\left[-\delta_1 k\zeta^{-2}+(k^*\delta_2 l+m^2\delta_2 l)\zeta^{-4}\right]\mu-\delta_1 k^*\zeta^{-4}=0,
\end{aligned}
$$

where $\mu(\nu)=\nu^2\lambda=\zeta^{-2}\lambda$ is a solution. $\lambda_j(\zeta)$ has the following asymptotic expansion:

$$\lambda_j(\zeta)=\mu_j^{(2)}\zeta^2+\mu_j^{(1)}\zeta+\mu_j^{(0)}+\mu_j^{(-1)}\zeta^{-1}+\cdots.$$

Here each coefficient is given by direct computations as

$$\mu_j^{(2)}=\pm\sqrt{\frac{\tau lm^2+kl}{\tau\rho l}}i,\quad \mu_j^{(1)}=0,\quad \mu_j^{(0)}=-\frac{Kl}{2\tau(\tau lm^2+kl)^2}\mp 2\tau\delta_1 m^2\frac{(\tau\rho l)^{\frac{1}{2}}}{(\tau lm^2+kl)^{\frac{3}{2}}}i,$$

$$j=1,2,\quad \text{when}\quad \mathrm{K}>0;$$

$$\mu_j^{(2)}=\pm\sqrt{\frac{\tau lm^2+kl}{\tau\rho l}}i,\quad \mu_j^{(1)}=0,\quad \mu_j^{(0)}=\mp 2\tau\delta_1 m^2\frac{(\tau\rho l)^{\frac{1}{2}}}{(\tau lm^2+kl)^{\frac{3}{2}}}i,\quad \mu_j^{(-1)}=0,$$

$$\mathrm{Re}\left(\mu_j^{(-2)}\right)=\frac{4(\tau^3\rho\delta_1\delta_2 lm^4+\tau^2\rho\delta_1\delta_2 l^2m^2k)+3(\tau^2\rho\delta_1 l^2m^4+\tau\rho\delta_1 kl^2m^2)}{2(\tau lm^2+kl)^3},$$

$$j=1,2,\quad \text{when}\quad \mathrm{K}=0;$$

$$\mu_j^{(2)}=0,\quad \mu_j^{(1)}=\pm\sqrt{\frac{\delta_1 k}{\tau lm^2+kl}},\quad \mu_j^{(0)}=-\frac{\delta_1\delta_2 kl(\tau m^2+k)+K\delta_1 lm^2}{2\delta_1 k(\tau lm^2+kl)},\quad j=3,4,$$

$$\mu_j^{(2)}=\mu_j^{(1)}=0,\quad \mu_j^{(0)}=-\frac{k^*}{k},\quad j=5.$$

Consequently, when $K>0$, we have

$$\mathrm{Re}\lambda_j(i\xi)=\begin{cases}-\dfrac{Kl}{2\tau(\tau lm^2+kl)^2}+O(|\xi|^{-1}), & j=1,2,\\ -\dfrac{\delta_1\delta_2 kl(\tau m^2+k)+K\delta_1 lm^2}{2\delta_1 k(\tau lm^2+kl)}+O(|\xi|^{-1}), & j=3,4,\\ -\dfrac{k^*}{k}+O(|\xi|^{-1}), & j=5,\end{cases}\tag{3.29}$$

for $|\xi|\to\infty$. And when $K=0$, we have

$$\mathrm{Re}\lambda_j(i\xi)=\begin{cases}-\mathrm{Re}\left(\mu_j^{(-2)}\right)|\xi|^{-2}+O(|\xi|^{-3}), & j=1,2,\\ -\dfrac{\delta_1\delta_2 kl(\tau m^2+k)}{2\delta_1 k(\tau lm^2+kl)}+O(|\xi|^{-1}), & j=3,4,\\ -\dfrac{k^*}{k}+O(|\xi|^{-1}), & j=5,\end{cases}\tag{3.30}$$

for $|\xi|\to\infty$.

Remark 3.11. *From (3.28) and (3.29), we find that the real parts of the slowest eigenvalues are $|\xi|^2$ for $|\xi|\to 0$ and 1 for $|\xi|\to\infty$ respectively. Observe that the exponent is*

$$\rho_1(\xi)=\frac{\xi^2}{(1+\xi^2)^2},\quad \text{when}\quad \mathrm{K}>0$$

in Theorem 3.1. Then $\rho_1(\xi)\sim|\xi|^2$ for $|\xi|\to 0$ and $\rho_1(\xi)\sim|\xi|^{-2}$ for $|\xi|\to\infty$, which means the exponent does not match with the real parts of the slowest eigenvalues. Hence, the pointwise estimate in Theorem 3.1 is not yet optimal with respect to ρ_1.

It follows from Theorem 3.7 that

$$\rho_2(\xi)=\frac{\xi^2}{(1+\xi^2)^2},\quad \text{when}\quad \mathrm{K}=0.$$

Then $\rho_2(\xi)\sim\xi^2$ for $|\xi|\to 0$ and $\rho_2(\xi)\sim|\xi|^{-2}$ for $|\xi|\to\infty$. Since it is in line with the real parts of the slowest eigenvalues in (3.28) and (3.30), the pointwise estimate in Theorem 3.7 is optimal.

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