A transmission problem for wave equations in infinite waveguides

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Abstract

We prove a decay estimate for solutions to a transmission problem for wave equations with different propagation speeds in an infinite waveguide. The problem represents the wave propagation in unbounded and layered composite materials in which different properties are given. It is a new composition of a waveguide problem and a transmission problem, motivated by a unit cell model for CFRP. The proof is based on splitting variables, partial eigenfunction expansions in the bounded cross section, and on an explicit Weyl type estimate for the eigenvalues.

Keywords: waveguide, transmission problem, decay estimate, wave equation, composite materials

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1 Introduction

In this article, we study the decay of solutions for a transmission problem of wave equations in a two-dimensional infinite waveguide, being an unbounded strip-shaped domain. Set $\mathcal{B}_1 := (0, L_0)$ and $\mathcal{B}_2 := (L_0, L)$ for $0 < L_0 < L$, and denote $\mathbb{R} \times \mathcal{B}_1$, $\mathbb{R} \times \mathcal{B}_2$ and $\mathbb{R} \times \{L_0\}$ by Ω_1, Ω_2 and Γ, respectively (see Figure 1 (a)). Let $\Delta_\alpha := \partial_x^2 + \alpha \partial_y^2$, for $\alpha > 0$ ($\partial_z := \frac{\partial}{\partial z}$, $z = t, x, y$). The unknown function $u_i = u_i(t, x, y)$ is a real-valued function in $[0, \infty) \times \Omega_i$ for $i = 1, 2$. We consider two wave equations with different propagation speeds:

$$\partial_t^2 u_1 = \Delta_{\alpha_1} u_1 \qquad \text{in } (0, \infty) \times \Omega_1, \tag{1.1}$$

$$\partial_t^2 u_2 = \Delta_{\alpha_2} u_2 \qquad \text{in } (0, \infty) \times \Omega_2, \tag{1.2}$$

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where α_1, α_2 are given positive constants, with transmission conditions on the interface Γ given by

$$u_1(t,x,L_0) = u_2(t,x,L_0), \quad \alpha_1 \partial_y u_1(t,x,L_0) = \alpha_2 \partial_y u_2(t,x,L_0) \qquad \text{on } (0,\infty)\times\mathbb{R}, \tag{1.3}$$

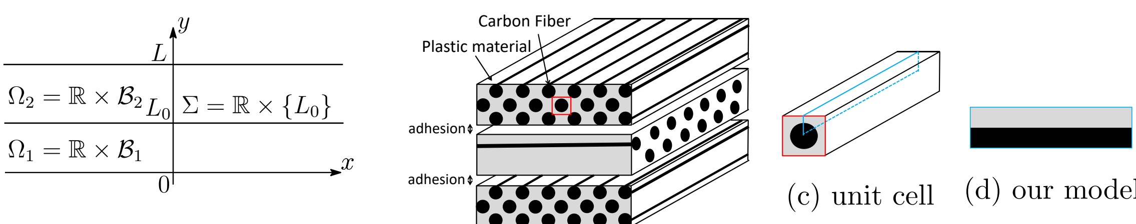
Dirichlet boundary conditions at the lower and upper end,

$$u_1(t,x,0) = u_2(t,x,L) = 0 \qquad \text{on } (0,\infty)\times\mathbb{R}, \tag{1.4}$$

and initial conditions

$$\begin{cases} u_1(0,x,y) := u_{1,0}(x,y) \\ \partial_t u_1(0,x,y) := v_{1,0}(x,y) \end{cases} \text{in } \Omega_1, \qquad \begin{cases} u_2(0,x,y) := u_{2,0}(x,y) \\ \partial_t u_2(0,x,y) := v_{2,0}(x,y) \end{cases} \text{in } \Omega_2. \tag{1.5}$$

The problem corresponds to a toy model for a composite material problem addressing *carbon fiber-reinforced plastics* (CFRP). We aim at giving a first mathematical analysis of a derived transmission problem in an infinite wave-guide. A CRFP is usually layered on multiple sheets with different angles to strengthen from each direction such as Figure 1 (b). Although there is an enormous number of mathematical models for CFRP from various aspects (see e.g. [1]), we here focus on the *unit cell model* which samples one element in periodic materials (see Figure 1 (c)), and we study the dynamical property for one sheet of the CFRP as indicated in Figure 1 (d).



(a) waveguide domain (b) CFRP (c) unit cell model (d) our model

Figure 1: waveguide domain, standard product of CFRP, unit cell model and our model

Among the many existing results on mere transmission problems and the fewer existing ones in infinite waveguides, we only mention some related to ours. It is well known that the energy norm of a solution for a (single) viscoelastic material decays exponentially in a bounded domain. However, when we consider the composite material which is a viscoelastic material sandwiched by elastic materials, it is shown in [6] that the energy norm of a solution only decays polynomially. This kind of problem is studied for other various materials and settings. We refer to e.g. [8] and the references therein for the recent development.

Infinite waveguides are domains which are bounded in some directions but are unbounded in other directions. The simplest example of a waveguide is a strip with infinite length. There are several results for the initial boundary value problems in an infinite waveguide for different partial differential equations (e.g [3] and [7]). The decay estimates for the wave equation and the elastic equation in a waveguide domain are investigated in [4] and [5], see also the monograph [10].

As far as the authors know, there are no results on the composite material (transmission) problem in an infinite waveguide, which is the motivation for the present work which should be a starting point to investigatethe stability for more complex models in bounded domains.

Our main results on the asymptotic distribution of eigenvalues in the cross section and on the asymptotic behavior as time tends to infinity will be given in Section 3 resp. 4. The latter, roughly speaking, describes the time decay of solutions as follows:

$$\|u_1(t)\|_{L^\infty(\Omega_1)} + \|u_2(t)\|_{L^\infty(\Omega_2)} \leq \frac{C}{t^{1/2}}, \qquad t \geq 0, \tag{1.6}$$

where C depends only on the initial value under appropriate assumptions on the data. It should be also noted that C depends on characteristic properties of material parameters involved via the eigenvalues λ_j of the associated time independent operator, see Sections 2 and 4 below. This dependence is also expected for future models in bounded domains.

Summarizing we present a) the first discussion of a combined transmission – infinite waveguide problem, b) explicit estimates for eigenvalues, and c) a first discussion of stability (time asymptotics) for CFRP, intended to trigger further research on the stability for bounded, more complex situations.

2 Setting and Preliminaries

Let $E(t) := \sum_{j=1}^{2} \frac{1}{2} \iint_{\Omega_j} \left(|\partial_t u_j|^2 + |\partial_x u_j|^2 + \alpha_j |\partial_y u_j|^2\right) dxdy$ be the associated energy term. We have energy conservation: $dE/dt = 0$. Writing $\mathcal{B} := (0, L)$ and $\Omega := \mathbb{R} \times \mathcal{B}$, we gather together u_1 in $(0,\infty) \times \Omega_1$ and u_2 in $(0,\infty) \times \Omega_2$ into a single function u in $(0,\infty) \times \Omega$, and define, yet formally, the operator A by

$$u = \begin{cases} u_1 & \text{in } (0,\infty) \times \Omega_1, \\ u_2 & \text{in } (0,\infty) \times \Omega_2. \end{cases} \qquad A := \begin{cases} -\Delta_{\alpha_1} & \text{in } \Omega_1, \\ -\Delta_{\alpha_2} & \text{in } \Omega_2. \end{cases}$$

Then, the problem (1.1)–(1.3) is reformulated as the single wave type equation $\partial_t^2 u + Au = 0$. More precisely, we define the operator $A : L^2(\Omega) \to L^2(\Omega)$ by

$$D(A) := \left\{ u \in H_0^1(\Omega) \mid \exists\, g = g_u \in L^2(\Omega) \mid \forall\, \varphi \in H_0^1(\Omega) : \ B(u, \varphi) = \langle g, \varphi \rangle_{L^2(\Omega)} \right\},$$

$Au := g_u$, where for $u, v \in H_0^1(\Omega)$,

$$B(u, v) := \langle \partial_x u, \partial_x v \rangle_{L^2(\Omega_1)} + \alpha_1 \langle \partial_y u, \partial_y v \rangle_{L^2(\Omega_1)} + \langle \partial_x u, \partial_x v \rangle_{L^2(\Omega_2)} + \alpha_2 \langle \partial_y u, \partial_y v \rangle_{L^2(\Omega_2)}.$$

This domain assures the transmission conditions (1.3) and the boundary conditions (1.4) in the usual weak sense.

Remark 2.1. Since $u \in H^2(\Omega_j)$, for $j = 1, 2$ by the regularity theorem [9, Theorem 6.5.2], and by the first transmission condition we conclude $u \in C^0(\overline{\Omega})$.

The operator A is self-adjoint and positive by the Lax-Milgram theorem, observing $B(u,u) \geq \min\{1, \alpha_1, \alpha_2\} \|\nabla u\|_{L^2(\Omega)}$.

Let us split the operator A into the operators A_x and A_y, where

$$A_x : \; D(A_x) := H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R}), \qquad A_x u := -\partial_x^2 u,$$

$$A_y : \; D(A_y) \subset L^2(\mathcal{B}) \to L^2(\mathcal{B}), \quad \text{(yet formally) } A_y u = \begin{cases} -\alpha_1 \partial_y^2 u & \text{in } \mathcal{B}_1, \\ -\alpha_2 \partial_y^2 u & \text{in } \mathcal{B}_2, \end{cases} \text{ and}$$

with $B_y(u,v) := \alpha_1 \langle \partial_y u, \partial_y v \rangle_{L^2(\mathcal{B}_1)} + \alpha_2 \langle \partial_y u, \partial_y v \rangle_{L^2(\mathcal{B}_2)}, \quad u, v \in H_0^1(\mathcal{B})$,

$$D(A_y) := \left\{ u \in H_0^1(\mathcal{B}) \mid \exists\, g = g_u \in L^2(\mathcal{B}) : \; \forall \varphi \in H_0^1(\mathcal{B}), \; B_y(u,\varphi) = \langle g, \varphi \rangle_{L^2(\mathcal{B})} \right\},$$

$A_y u := g_u$. The operator A_y is self-adjoint, positive with a compact inverse A_y^{-1}, the latter by Rellich's selection theorem. As a consequence, there exists a complete orthonormal system $\{\phi_m\}_m \in L^2(\mathcal{B})$ of eigenfunctions of A_y with corresponding eigenvalues $\{\lambda_m\}_m$ satisfying $A_y \phi_m = \lambda_m \phi_m$ and $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \to \infty$ as $m \to \infty$.

We also need the decay estimate for solutions to the Klein-Gordon equation from [10, Lemma 13.2].

Lemma 2.2. *Let $M \geq M_0 > 0$, $u_0 \in W^{2,1}(\mathbb{R})$, $v_0 \in W^{1,1}(\mathbb{R})$. Then the unique solution v to*

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + Mu = 0 & \text{in } [0,\infty) \times \mathbb{R}, \\ u(0,\cdot) = u_0, \quad \partial_t u(0,\cdot) = v_0 & \text{in } \mathbb{R}, \end{cases}$$

satisfies, for $t \geq \frac{1}{\sqrt{M}}$, with a constant $c > 0$ depending on M_0,

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq \frac{c}{t^{1/2}} \left(M^{1/4} \|u_0\|_{W^{2,1}(\mathbb{R})} + M^{-1/4} \|v_0\|_{W^{1,1}(\mathbb{R})} \right).$$

3 The Distribution of the Eigenvalues

The estimate for the distribution of the eigenvalues $\{\lambda_m\}_m$ of A_y is of interest in itself, but will also play an essential role in the proof of the theorem on the decay of solutions to be presented in Section 4. The following estimate can be found in [2, (1.28)] for more general *classical* elliptic boundary value problems in any space dimension. Since the proof of the estimate for *transmission* problems there consists only in the remark that all the results in five chapters before *remain true almost without alterations*, we prefer to give a separate proof here, finally obtaining more inside into relations between system parameters.

Theorem 3.1. *The eigenvalues $\{\lambda_m\}_m$ of A_y satisfy $\lambda_m \geq cm^2$ ($m \in \mathbb{N}$), where $c > 0$ is independent of m (but depends on $\mathcal{B}$).*

Proof. The eigenvalue problem $A_y\phi_=\lambda\phi$ is equivalent to the ODE transmission problem for $\tilde{\phi}_j := \phi$ in $\mathcal{B}_j$, $j = 1, 2$,

$$\begin{cases} -\alpha_1\tilde{\phi}_1'' = \lambda\tilde{\phi}_1, & y \in (0, L_0), \\ \tilde{\phi}_1(0) = 0, \end{cases} \qquad \begin{cases} -\alpha_2\tilde{\phi}_2'' = \lambda\tilde{\phi}_2, & y \in (L_0, L), \\ \tilde{\phi}_2(L) = 0, \end{cases} \tag{3.1}$$

with transmission conditions $\tilde{\phi}_1(L_0) = \tilde{\phi}_2(L_0)$ and $\alpha_1\tilde{\phi}_1'(L_0) = \alpha_2\tilde{\phi}_2'(L_0)$. With $\kappa_1 := \sqrt{\lambda/\alpha_1}$ and $\kappa_2 := \sqrt{\lambda/\alpha_2}$, and observing the boundary condition, we obtain $\tilde{\phi}_1(y) = C_1\sin(\kappa_1 y)$, $\tilde{\phi}_2(y) = C_2\sin(\kappa_2(y{-}L))$. Substituting them into the transmission conditions, we get

$$C_1\sin(\kappa_1 L_0) = C_2\sin(\kappa_2(L_0 - L)), \quad \alpha_1\kappa_1 C_1\cos(\kappa_1 L_0) = \alpha_2 C_2\cos(\kappa_2(L_0 - L)),$$

implying $\alpha_1\kappa_1\frac{\cos(\kappa_1 L_0)}{\sin(\kappa_1 L_0)} = -\alpha_2\kappa_2\frac{\cos(\kappa_2(L-L_0))}{\sin(\kappa_2(L-L_0))}$, which is equivalent to

$$f(\xi) := \sin\xi + B\sin(b\,\xi) = 0, \qquad \xi > 0, \tag{3.2}$$

where $\xi := \left(\frac{L_0}{\sqrt{\alpha_1}} + \frac{L-L_0}{\sqrt{\alpha_2}}\right)\sqrt{\lambda}$, $B := \frac{\sqrt{\alpha_1}-\sqrt{\alpha_2}}{\sqrt{\alpha_1}+\sqrt{\alpha_2}}$ and $b := (\frac{L-L_0}{\sqrt{\alpha_2}} - \frac{L_0}{\sqrt{\alpha_1}})/(\frac{L_0}{\sqrt{\alpha_1}} + \frac{L-L_0}{\sqrt{\alpha_2}})$. In the case $\alpha_1 = \alpha_2$, these are the well-known eigenvalues for the Dirichlet boundary value problem without interface, $\lambda = \lambda_k = \frac{\alpha_1\pi^2}{L^2}k^2$, $k \in \mathbb{N}$. Thus, let us consider the case $\alpha_1 \neq \alpha_2$ corresponding to the presence of a real interface. We remark that $|b|$, $|B| < 1$ and $B \neq 0$ since $\alpha_1 \neq \alpha_2$. Let $I_k = \left[-\frac{\pi}{2} + 2\pi k, \frac{\pi}{2} + 2\pi k\right]$ for $k \in \mathbb{N}$. Since $f\left(-\frac{\pi}{2} + 2\pi k\right) < 0 < f\left(\frac{\pi}{2} + 2\pi k\right)$, there exists at least one solution ξ_k for the equation (3.2) in I_k. This ξ_k satisfies $\xi_k \sim k$ namely $c^{-1}k \leq \xi_k \leq ck$ for some $c > 0$.

Denote $\sin\xi$ and $-B\sin(b\,\xi)$ by $f_1(\xi)$ and $f_2(\xi)$, respectively. Investigating the zeros of f is equivalent to investigating the intersections of f_1 and f_2. The intersection is at most two in each I_k, namely, there is at most one other intersection without ξ_k in I_k. Let us give a more detailed explanation. There are three cases: (i) f_2 is monotone decreasing in I_k, (ii) f_2 is monotone increasing in I_k, (iii) f_2 has an extremum in I_k.
Case (i): $(f_1^{-1})' = \frac{1}{\sqrt{1-\eta^2}} > 0$ and $(f_2^{-1})' < 0$ in $[f_2(\frac{\pi}{2} + 2\pi k), f_2(-\frac{\pi}{2} + 2\pi k)] \subset [-1, 1]$, because $f_1^{-1}(\eta) = \arcsin\eta + 2\pi k$ and f_2 is monotone decreasing. Therefore, it holds that $(f_1^{-1})'(\eta) > (f_2^{-1})'(\eta)$ for every η. The following elementary Lemma 3.2 implies that there is no other intersection than $f(\xi_k)$ of f_1^{-1} and f_2^{-1}:

Lemma 3.2. *Let g_1, $g_2 \in C^1([a, b])$. If $g_1'(x) > g_2'(x)$ for every $x \in [a, b]$ and there is an intersection $c \in [a, b]$ such that $g_1(c) = g_2(c)$, then there is no other intersection of g_1 and g_2 in $[a, b]$.*

Case (ii): $f_2^{-1}(\eta)$ is expressed as $\frac{1}{|b|}\arcsin\left(\frac{\eta}{|B|}\right)+c$ by some constant c. Then, we see that $(f_1^{-1})'(\eta) = \frac{1}{\sqrt{1-\eta^2}} < \frac{1}{|b|\sqrt{B^2-\eta^2}} = (f_2^{-1})'(\eta)$ for every $\eta \in [f_2(-\frac{\pi}{2}+2\pi k), f_2(\frac{\pi}{2}+2\pi k)]$, thanks to $|b|<1$ and $|B|<1$.

Case (iii): The extremum is only one in I_k because of the period of f_2 is larger than the one of f_1. Let us denote by $\widetilde{\xi}_k$ the value ξ of the extremum (that is, $\widetilde{\xi}_k$ is the point satisfying $f_2'(\widetilde{\xi}_k)=0$ in I_k. Split the interval I_k into $I_{k,l} := [-\frac{\pi}{2}+2\pi k,\ \widetilde{\xi}_k]$ and $I_{k,r} := [\widetilde{\xi}_k,\ \frac{\pi}{2}+2\pi k]$. Then f_2 is monotone in the each subintervals. Therefore, from the same argument as above, there are at most two intersections in I_k for $k\in\mathbb{N}$. Applying the same argument to the problem in $\widehat{I}_k := \left[\frac{\pi}{2}+2\pi(k-1), -\frac{\pi}{2}+2\pi k\right]$, we obtain the unique existence of a solution to $f(\xi)=0$ in $\widehat{I}_k$. Lastly, it is easily seen that there is no solution in $(0,\frac{\pi}{2}]$. This completes the proof of Theorem 3.1. □

4 The Decay Estimate

As in the classical situation for an infinite waveguide without interface situation, see [10], we will make the ansatz of an eigenfunction expansions in the bounded cross section. Modifications are necessary due to the additional difficulties coming up through the lack of regularity of the solution along the interface.

We expand the solution $u=u(t,x,y)$ as $u(t,x,y)=\sum_{j=1}^{\infty} w_j(t,x)\phi_j(y)$ with $w_j(t,x) = \langle u(t,x,\cdot),\phi_j\rangle_{L^2(\mathcal{B})}$. Substituting this into the equation for u, observing $A=A_x+A_y$, yields

$$0 = \partial_t^2 u + Au = \sum_{j=1}^{\infty}\left(\partial_t^2 w_j + A_x w_j + \lambda_j w_j\right)(t,x)\cdot\phi_j(y),$$

thus w_j satifies

$$\begin{cases}\partial_t^2 w_j - \partial_x^2 w_j + \lambda_j w_j = 0, & (t,x)\in(0,\infty)\times\mathbb{R},\\ w_j(0,x) = \langle u_0(x,\cdot),\phi_j\rangle_{L^2(\mathcal{B})},\ \partial_t w_j(0,x) = \langle v_0(x,\cdot),\phi_j\rangle_{L^2(\mathcal{B})}, & x\in\mathbb{R},\end{cases}$$

for $j=1,2,\ldots$. Observe that $\langle u,\phi_j\rangle_{L^2(\mathcal{B})} = \frac{1}{\lambda_j^{K/2}}\langle (A_y)^{\frac{K}{2}}u,\phi_j\rangle_{L^2(\mathcal{B})}$ for any K. It follows from Sobolev imbedding that

$$|u(t,x,y)|^2 \le c\|u(t,x,\cdot)\|_{H^1(\mathcal{B})}^2 \le c\|A_y u(t,x,\cdot)\|_{L^2(\mathcal{B})}^2 = c\sum_{j_1}^{\infty}\lambda_j^2|w_j(t,x)|^2,$$

where the penultimate inequality arises from the regularity result for the transmission problem, see [9, Theorems 6.5.1 and 6.5.2]. By Lemma 2.2, the quantity $\sum_{j_1}^{\infty}\lambda_j^2|w_j(t,x)|^2$ is estimated by

$$\frac{c}{t}\sum_{j=1}^{\infty}\frac{1}{\lambda_j^{K-5/2}}\left(\|\langle A_y^{K/2}u_0,\phi_j\rangle_{L^2(\mathcal{B})}\|_{W^{2,1}(\mathbb{R})}^2 + \|\langle A_y^{(K-1)/2}v_0,\phi_j\rangle_{L^2(\mathcal{B})}\|_{W^{1,1}(\mathbb{R})}^2\right),$$

where $K \in \mathbb{N}$ will be chosen appropriately below. Dividing $\mathcal{B}$ into $\mathcal{B}_1$ and $\mathcal{B}_2$, we have $|\langle A_y^{K/2} u_0(x,\cdot), \phi_j\rangle_{L^2(\mathcal{B})}| \leq \|A_y^{K/2} u_0(x,\cdot)\|_{W^{1,1}(\mathcal{B}_1)\cup W^{1,1}(\mathcal{B}_2)}$, thus, $\|\langle A_y^{K/2} u_0, \phi_j\rangle_{L^2(\mathcal{B})}\|_{W^{2,1}(\mathbb{R})} \leq c\|A_y^{K/2} u_0\|_{W^{2,1}(\Omega)}$, where the derivates in the y-direction are taken in $\mathcal{B}_1$ and $\mathcal{B}_2$ separately, not along the interface. Analogously, we have $\|\langle A_y^{(K-1)/2} v_0, \phi_j\rangle_{L^2(\mathcal{B})}\|_{W^{1,1}(\mathbb{R})} \leq c\|A_y^{(K-1)/2} v_0\|_{W^{2,1}(\Omega)}$. The series $\sum_{j=1}^{\infty} \lambda_j^{5/2-K}$ is finite if $K > 3$, because of Theorem 3.1. Hence, we have proved the following

Theorem 4.1. *Let $K > 3$. Let u_0, v_0 be such that the norms appearing below are finite. Then the solution for the initial boundary value and transmission problem* (1.1)–(1.5) *satisfies*

$$\|u(t,\cdot,\cdot)\|_{L^\infty(\Omega)} \leq \frac{C}{t^{1/2}} \left(\|A_y^{\frac{K}{2}} u_0\|_{W^{2,1}(\Omega)} + \|A_y^{\frac{K-1}{2}} v_0\|_{W^{2,1}(\Omega)} \right),$$

where C only depends on K.

The constant C naturally depends on the geometry of the domains and the material parameters α_1, α_2, as the explicit dependence on the eigenvalues visible in (4.16) shows. Of course, data $u_0,\ v_0 \in C_0^\infty(\Omega_1 \cup \Omega_2)$ are admissible.

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