INITIAL BOUNDARY VALUE PROBLEMS FOR 3-D NAVIER-STOKES EQUATIONS WITH HYPERBOLIC HEAT CONDUCTION

YUXI HU AND REINHARD RACKE

Abstract. We study an initial boundary value problem for the 3-dimensional compressible Navier-Stokes equations with hyperbolic heat conduction, where the classical Fourier law is replaced by the Cattaneo-Christov constitutive relation. We focus on spherically symmetric solutions. We establish the existence of uniform global small solutions to the resulting system. Furthermore, based on uniform a priori estimates, we rigorously justify both the relaxation limit and the vanishing viscosity limit.
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1. Introduction

The basic equations governing three-dimensional compressible fluid dynamics are given by

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \operatorname{div} S, \\ \partial_t(\rho \mathcal{E}) + \operatorname{div}(\rho u \mathcal{E} + pu + q - Su) = 0, \end{cases} \tag{1.1}$$

for $(x,t) \in \Omega(\subset \mathbb{R}^3) \times (0,\infty)$. The quantities ρ, u, p, θ, and $\mathcal{E}$ represent the fluid density, velocity, pressure, temperature, and specific energy, respectively. The quantities q and S denote the heat flux and stress tensor, respectively, and must be specified through constitutive relations to close the system (1.1).

We assume the flow is Newtonian, which implies that the stress tensor S is given by

$$S = \mu(\theta)\left(\nabla u + (\nabla u)^T - \frac{2}{3}\operatorname{div} u I_3\right) + \lambda(\theta)\operatorname{div} u I_3, \tag{1.2}$$

where μ and λ are the shear and bulk viscosities, respectively, and both are functions of the temperature θ.

The formulation of the heat flux q is more intricate. Instead of employing the classical Fourier law for heat conduction, we adopt the Cattaneo-Christov (CC) constitutive relation:

$$\tau(\theta)\rho\left(\partial_t q + u\cdot\nabla q - q\cdot\nabla u + (\operatorname{div} u) q\right) + q + \kappa(\theta)\nabla\theta = 0, \tag{1.3}$$

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where $\tau(\theta)$ is the relaxation parameter accounting for the time lag in the response of heat flux to the temperature gradient, and $\kappa(\theta)$ is the heat conductivity coefficient. Both parameters depend on the temperature.

The Cattaneo-Christov relation (1.3) was originally proposed in the linearized case by Cattaneo [1] to resolve the paradox of infinite heat propagation speed implied by Fourier's law, and was later extended to an objective (frame-indifferent) nonlinear formulation by Christov [3]. The CC model has been shown to be particularly relevant in extreme physical scenarios, such as instantaneous heating by laser pulses [22], heat conduction in nanometer-scale materials [20], and the Rayleigh-Bénard convection near MEMS devices [8]. In fact, the dimensionless Cattaneo number, defined by $C = \frac{\tau\kappa}{L^2}$ where L is a reference length scale, is used to quantify the significance of the CC model relative to Fourier's law [21]. When κ is large or L is small, the CC model can become particularly important.

Note that the second law of thermodynamics may not hold for the new CC model if the classical thermodynamic equation remains unchanged; see [4, 13, 14, 26]. Inspired by Coleman *et al.* [4], we assume that the specific total energy is given by

$$\mathcal{E} = \frac{1}{2}u^2 + e, \tag{1.4}$$

with the specific internal energy e and the pressure p given by

$$e = C_v\theta + a(\theta)q^2, \quad p = R\rho\theta, \tag{1.5}$$

where $a(\theta) = \frac{Z(\theta)}{\theta} - \frac{1}{2}Z'(\theta)$ and $Z(\theta) = \frac{\tau(\theta)}{\kappa(\theta)}$. C_v and R denote the heat capacity at constant volume and the gas constant, respectively. The quantities p and e satisfy the usual thermodynamic relation

$$\rho^2 e_\rho = p - \theta p_\theta.$$

For $\tau(\theta) = 0$, the system (1.1)-(1.5) reduces to the classical compressible Navier-Stokes-Fourier system, for which the global well-posedness, blow-up of smooth solutions, and large-time behavior have been extensively studied. In particular, the local existence and uniqueness of smooth solutions were established by Serrin [30], Nash [27] and Itaya [17] for initial data away from vacuum. Later, Matsumura and Nishida [25] obtained global smooth solutions for small initial data without vacuum. For large initial data, Xin [32], and Cho and Jin [2], showed that smooth solutions must blow up in finite time if the initial data contains vacuum. See [9, 10, 23, 18, 19, 6] for results on the global existence of weak solutions.

On the other hand, for $\tau(\theta) > 0$, there are relatively few results. The authors first studied in [12] the well-posedness of strong solutions in Sobolev spaces, as well as the relaxation limit, under a linearized form of (1.3). The decay properties of the resulting solutions were established in [24, 31]. Crin-Barat, Kawashima, and Xu [5] obtained global-in-time well-posedness for small initial data in Besov spaces, along with a strong global relaxation limit. In this sense, the Cauchy problem for the system (1.1)-(1.5) has been extensively investigated, cf. the survey on our results in [28]. However, to the best of our knowledge, initial boundary value problems have not yet been addressed. We note that the initial boundary value problem for compressible Navier-Stokes equations with Maxwell's law – i.e. relaxation for the stress tensor – has been considered recently, see [11, 15, 16]. As a first step in this direction, we study the initial boundary value problem for system (1.1)-(1.5) under the assumption of spherical symmetry. We consider spherically symmetric solutions of the following form:

$$\rho(t,x)=\rho(t,r),\quad u(t,x)=u(t,r)\frac{x}{r},\quad \theta(t,x)=\theta(t,r),\quad q(t,x)=q(t,r)\frac{x}{r},\tag{1.6}$$

where $x\in\Omega;=\overline{B(0,1)}^c$ or $\Omega:=B(0,2)\setminus\overline{B(0,1)}$, with $B(0,r)$ denoting the open ball of radius r centered at the origin. Under this symmetry, the system (1.1)-(1.5) reduces to:

$$\begin{cases}\rho_t+(\rho u)_r+\frac{2}{r}\rho u=0,\\ \rho u_t+\rho u u_r+p_r=\left(\left(\frac{4}{3}\mu(\theta)+\lambda(\theta)\right)\left(u_r+\frac{2}{r}u\right)\right)_r,\\ \rho e_\theta\theta_t+(\rho u e_\theta-\frac{2a(\theta)}{Z(\theta)}q)\theta_r+p\left(u_r+\frac{2}{r}u\right)+q_r+\frac{2}{r}q=\\ \quad\left(\frac{2}{\tau(\theta)}+\frac{4}{r}u\right)a(\theta)q^2+\mu(\theta)\left(2u_r^2+\frac{4}{r^2}u^2-\frac{2}{3}\left(u_r+\frac{2}{r}u\right)^2\right)+\lambda(\theta)\left(u_r+\frac{2}{r}u\right)^2,\\ \tau(\theta)\rho\left(q_t+uq_r+\frac{2}{r}uq\right)+q+\kappa(\theta)\theta_r=0.\end{cases}\tag{1.7}$$

We supplement system (1.7) with the initial data

$$(\rho,u,\theta,q)|_{t=0}=(\rho_0,u_0,\theta_0,q_0),\tag{1.8}$$

possibly depending on τ, and the boundary conditions

$$u|_{\partial\Omega}=q|_{\partial\Omega}=0,\tag{1.9}$$

with the domain Ω reduced to either $(1,\infty)$ or to $(1,2)$.

To establish estimates independent of $\tau(\theta)$, $\mu(\theta)$, and $\lambda(\theta)$ and to facilitate taking the limit, we assume the following relations:

$$\tau(\theta)=\tau g(\theta),\quad \mu(\theta)=\mu h(\theta),\quad \lambda(\theta)=\lambda l(\theta),$$

where g, h, and l are smooth (and positive) functions of θ, and τ, μ, and λ are positive constants. The following assumptions are used throughout the paper:

Assumption 1.1.
(1) The initial and boundary data satisfy the usual compatibility conditions:

$$\partial_t^k u(0,x)\big|_{\partial\Omega}=\partial_t^k q(0,x)\big|_{\partial\Omega}=0,\quad k=0,1,\tag{1.10}$$

where $\partial_t u(0,x)$ and $\partial_t q(0,x)$ are defined recursively using equations $(1.7)_2$ and $(1.7)_4$, respectively.

(2) The possible dependence of the initial data on τ is compatible in the following sense:

$$\|r\left(q_0+\kappa(\theta_0)(\theta_0)_r\right)\|_{H^1}=O(\sqrt{\tau}),\quad \textit{as } \tau\to 0.\tag{1.11}$$

To formulate our theorem, we introduce the following energy functional:

$$\begin{aligned}E(t):=\sup_{0\le s\le t}&\|r(\rho-1,u,\theta-1,q)\|_{L^2}^2+\|r(\rho_r,u_r,\theta_r,q_r)\|_{L^2}^2+\|(u,q)\|_{L^2}^2+\|(u_r,q_r)\|_{L^2}^2\\ &+\|r(\rho_{rr},u_{rr},\theta_{rr},q_{rr})\|_{L^2}^2+\|r(\rho_t,u_t,\theta_t,\sqrt{\tau}q_t)\|_{L^2}^2+\|r(\rho_{tr},u_{tr},\theta_{tr},\sqrt{\tau}q_{tr})\|_{L^2}^2\\ &+\tau^2\|r(\rho_{tt},u_{tt},\theta_{tt},\sqrt{\tau}q_{tt})\|_{L^2}^2+\left(\frac{4}{3}\mu+\lambda\right)^2\|r(u_{rrr},\tau u_{trr})\|_{L^2}^2\end{aligned}$$

The corresponding dissipation functional is defined as:

$$\begin{aligned}\mathcal{D}(t) :=& \|rq\|_{L^2}^2 + \|rD(\rho,u,\theta,q)\|_{L^2}^2 + \|(u,q)\|_{L^2}^2 + \|r(\rho_{rr},u_{rr},\theta_{rr},q_{rr})\|_{L^2}^2 \\ &+ \|(u_r,q_r,u_t,q_t)\|_{L^2}^2 + \|r(\rho_{tr},u_{tr},\theta_{tr},q_{tr})\|_{L^2}^2 + \|r(\rho_{tt},u_{tt},\theta_{tt})\|_{L^2}^2 \\ &+ \tau^2\|rq_{tt}\|_{L^2}^2 + \left(\frac{4}{3}\mu+\lambda\right)\|r(u_{trr},u_{ttr})\|_{L^2}^2\end{aligned}$$

where $D := (\partial_t, \partial_r)$.

Our first result concerns the global-in-time small well-posedness of the initial boundary value problem (1.7)-(1.9), with estimates that are uniform with respect to the parameters τ, λ, and μ.

Theorem 1.2. *Let Assumption 1.1 hold and $r(\rho_0-1,\theta_0-1,q_0)\in H^3$, $ru_0\in H^4$. Then there exists a small constant $\epsilon_0>0$ such that if*

$$\|r(\rho_0-1,\theta_0-1,q_0)\|_{H^3}^2 + \|ru_0\|_{H^4}^2 \le \epsilon_0,$$

there exists a unique global solution (ρ,u,θ,q) to the initial boundary value problem (1.7)-(1.9) *such that*

$$(\rho-1,\theta-1,q)\in\bigcap_{k=0}^{2}C^k([0,\infty),H^{2-k}),\; u\in\bigcap_{k=0}^{1}C^k([0,\infty),H^{3-k})\bigcap C^2([0,\infty),L^2)$$

and satisfy the uniform estimate

$$E(t)+\int_0^\infty \mathcal{D}(t)\mathrm{d}t \le CE(0),$$

where C is a constant independent of the parameters τ, λ, and μ.

Building on the uniform estimates established in Theorem 1.2, we now state our second result regarding the vanishing viscosity limit.

Theorem 1.3. *Fix $\tau>0$. Let $\varepsilon=(\mu,\lambda)$ and $(\rho_\tau^\varepsilon,u_\tau^\varepsilon,\theta_\tau^\varepsilon,q_\tau^\varepsilon)$ be the global solutions obtained in Theorem 1.2, then there exists $(\rho_\tau^0,u_\tau^0,\theta_\tau^0,q_\tau^0)\in L^\infty((0,\infty);H^2)\cap C^0((0,\infty);H^{2-\delta})$ for any $\delta>0$, such that, as $\varepsilon\to 0$, up to subsequences, for any $T>0$,*

$$(\rho_\tau^\varepsilon,u_\tau^\varepsilon,\theta_\tau^\varepsilon,q_\tau^\varepsilon)\to(\rho_\tau^0,u_\tau^0,\theta_\tau^0,q_\tau^0)\quad\text{strongly in}\quad C([0,T],H_{loc}^{2-\delta}) \tag{1.12}$$

where $(\rho_\tau^0,u_\tau^0,\theta_\tau^0,q_\tau^0)$ is a strong solution to the three-dimensional hyperbolized compressible Euler-Cattaneo-Christov system (4.2) *(below), satisfying the initial and boundary conditions* (1.8) *and* (1.9).

The next result shows the global relaxation limit.

Theorem 1.4. *Fix $\mu>0,\lambda>0$. Let $\varepsilon=(\mu,\lambda)$ and $(\rho_\tau^\varepsilon,u_\tau^\varepsilon,\theta_\tau^\varepsilon,q_\tau^\varepsilon)$ be the global solutions obtained in Theorem 1.2, then there exists $(\rho_0^\epsilon-1,\theta_0^\epsilon-1,q_0^\epsilon)\in L^\infty((0,\infty);H^2)\cap C^0((0,\infty);H^{2-\delta})$ and $u_0^\epsilon\in L^\infty((0,\infty);H^3)\cap C^0((0,\infty);H^{3-\delta})$for any $\delta>0$, such that, as $\tau\to 0$, up to subsequences, for any $T>0$,*

$$(\rho_\tau^\varepsilon,\theta_\tau^\varepsilon,q_\tau^\varepsilon)\to(\rho_0^\epsilon,\theta_0^\epsilon,q_0^\epsilon)\quad\text{strongly in}\quad C([0,T],H_{loc}^{2-\delta}) \tag{1.13}$$

and

$$u_\tau^\varepsilon\to u_0^\epsilon\quad\text{strongly in}\quad C([0,T],H_{loc}^{3-\delta})$$

where $q_0^\epsilon = -\kappa(\theta_0^\epsilon)\partial_r\theta_0^\epsilon, a.e.,$ *and* $(\rho_0^\epsilon, u_0^\epsilon, \theta_0^\epsilon)$ *is a strong solution to the three-dimensional compressible Navier-Stokes-Fourier system in spherical symmetry* (4.5) *(below), satisfying the initial and boundary conditions* (1.8) *and* (1.9) *with* $q_0 = -\kappa(\theta_0)(\theta_0)_r$.

The paper is organized as follows. In Section 2 we state the local existence theorem for the system (1.7)-(1.9). The central Section 3 provides a priori estimates, being necessary for extending a local solution to a global one and for carrying out the singular limits. In Section 4 the proofs the main theorems are given.

Finally, we introduce some notation. $W^{m,p} = W^{m,p}(\Omega), 0 \le m \le \infty, 1 \le p \le \infty$, denotes the usual Sobolev space with norm $\|\cdot\|_{W^{m,p}}$, H^m and L^p stand for $W^{m,2}$ resp. $W^{0,p}$.

2. Local well-posedness

In this section, we establish the local well-posedness for system (1.7)-(1.9). We note that, to our knowledge, general results for the initial boundary value problem of hyperbolic-parabolic coupled systems are lacking. To obtain the desired well-posedness, we follow the approach in [33] (pp. 119-124). First, Let $M_0 := \sum_{k=0}^{2} \|D^k(\rho-1, u, \theta-1, q)|_{t=0}\|_{L^2}^2 < \infty$. Let M_1 be a positive constant such that when $\sup_{0\le t\le h} \|(\rho-1, u, \theta-1, q)\|_{H^2} \le M_1$,

$$\min\{\min_{[0,h]\times\Omega} \rho(t,r), \min_{[0,h]\times\Omega} \theta(t,r), \min_{[0,h]\times\Omega} e_\theta(t,r)\} =: \gamma > 0.$$

Introduce the space

$$\begin{aligned} X_h(M_2, M_3) = \Bigg\{ &(\rho, u, \theta, q) | (\rho-1, \theta-1, q) \in \bigcap_{k=0}^{2} C^k([0,h], H^{2-k}), u \in C([0,h], H^3), \\ &u_t \in C([0,h], H^2), u_{tt} \in C([0,h], L^2) \cap L^2([0,h], H^1)) \\ &\sup_{0\le t\le h} \sum_{k=0}^{2} \|D^k(\rho-1, u, \theta-1, q)\|_{L^2}^2 \le M_2, \\ &\sup_{0\le t\le h} (\|u_{xxx}\|_{L^2}^2 + \|u_{txx}\|_{L^2}^2) + \int_0^h \|u_{xtt}\|_{L^2}^2 \mathrm{d}t \le M_3 \\ &(\rho, u, \theta, q)|_{t=0} = (\rho_0, u_0, \theta_0, q_0), \quad (u, q)|_{\partial\Omega} = 0, \} \end{aligned}$$

where $M_2 < M_1$, M_3 are positive constant specified in the course of the proof.

Now, we are ready to state the local well-posedness theorem.

Theorem 2.1. *Assume* $(\rho_0 - 1, \theta_0 - 1, q_0) \in H^3$, $u_0 \in H^4$ *and the compatibility condition* (1.10) *hold. Then, there exists a positive constant* ϵ_0 *and* t_0 *depending on* ϵ_0 *such that when* $M_0 \le \epsilon_0$, *problem* (1.7)-(1.9) *admits a unique local solution* $(\rho, u, \theta, q) \in X_{t_0}(C_1 M_0, C_2 M_0)$ *where* C_1, C_2 *only depends on* γ.

Proof. We present a concise proof of this theorem, though detailed proofs for similar systems can be found in [33] and references therein. Specifically, we divide the proof into four steps.

Step 1: For any $(\tilde\rho,\tilde u,\tilde\theta,\tilde q)\in X_h(M_2,M_3)$, consider the auxiliary linear problem

$$\begin{cases}\tilde\rho\tilde e_\theta\theta_t+(\tilde\rho\tilde u\tilde e_\theta-\frac{2a(\tilde\theta)}{Z(\tilde\theta)}\tilde q)\theta_r+q_r=f_1,\\ \tau(\tilde\theta)\tilde\rho(q_t+\tilde u q_r)+\kappa(\tilde\theta)\theta_r=f_2,\\ (\theta,q)|_{t=0}=(\theta_0,q_0),\quad q|_{\partial\Omega}=0,\end{cases}\tag{2.1}$$

$$\begin{cases}\tilde\rho u_t-(\frac{4}{3}\mu(\tilde\theta)+\lambda(\tilde\theta))u_{rr}=f_3,\\ u|_{t=0}=u_0, u|_{\partial\Omega}=0,\end{cases}\tag{2.2}$$

and

$$\begin{cases}\rho_t+\tilde u\rho_r=f_4,\\ \rho|_{t=0}=\rho_0,\end{cases}\tag{2.3}$$

where

$$\begin{cases}f_1=\left(\frac{2}{\tau(\tilde\theta)}+\frac{4}{r}\tilde u\right)a(\tilde\theta)\tilde q^2+\mu(\tilde\theta)\left(2\tilde u_r^2+\frac{4}{r^2}\tilde u^2-\frac{2}{3}\left(\tilde u_r+\frac{2}{r}\tilde u\right)^2\right)\\ \qquad\qquad +\lambda(\tilde\theta)\left(\tilde u_r+\frac{2}{r}\tilde u\right)^2-(\tilde u_r+\frac{2}{r}\tilde u)\tilde p-\frac{2}{r}\tilde q,\\ f_2=-\tau(\tilde\theta)\tilde\rho\cdot\frac{2}{r}\tilde u\tilde q-\tilde q,\\ f_3=\left(\frac{4}{3}\mu(\tilde\theta)+\lambda(\tilde\theta)\right)\left(\frac{2}{r}\tilde u_r-\frac{2}{r^2}\tilde u\right)+\lambda'(\tilde\theta)\tilde\theta_r\left(\tilde u_r+\frac{2}{r}\tilde u\right)+\frac{4}{3}\mu'(\tilde\theta)\tilde\theta_r\left(\tilde u_r-\frac{\tilde u}{r}\right)\\ \qquad\qquad -R\tilde\rho\tilde\theta_r-R\tilde\theta\tilde\rho_r-\tilde\rho\tilde u\tilde u_r,\\ f_4=-\frac{2}{r}\tilde\rho\tilde u-\tilde\rho\tilde u_r.\end{cases}$$

We deal with the system (2.1), (2.2), (2.3), respectively. First, problem (2.1) is a linear symmetric hyperbolic system of first order (by dividing $(2.1)_2$ by $\kappa(\tilde\theta)$). We first check that the boundary condition $q|_{\partial\Omega}=0$ is non-characteristic and satisfy the maximal nonnegative condition (admissible in the sense of Friedrich, see [33, 7]). Indeed, we rewrite system (2.1) with $U=(\theta,q)$ as

$$A^0U_t+A^1U_r=F,\tag{2.4}$$

where

$$A^0=\begin{pmatrix}\tilde\rho\tilde e_\theta & 0\\ 0 & \frac{\tau(\tilde\theta)\tilde\rho}{\kappa(\tilde\theta)}\end{pmatrix}, A^1=\begin{pmatrix}\left(\tilde\rho\tilde u\tilde e_\theta-\frac{2a(\tilde\theta)}{Z(\tilde\theta)}\tilde q\right) & 1\\ 1 & \frac{\tau(\tilde\theta)\tilde\rho\tilde u}{\kappa(\tilde\theta)}\end{pmatrix}, F=\mathrm{diag}\{f_1,f_2\}.$$

The boundary condition reduces to

$$PU|_{\partial\Omega}=0,\text{ with } P=\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

Note that by virtue of $(\tilde u,\tilde q)|_{\partial\Omega}=0$, we have $\det\left((A^0)^{-1}A^1\right)|_{\partial\Omega}\neq 0$. Consequently, the boundary condition $q|_{\partial\Omega}=0$ is non-characteristic. Now, we show the boundary condition is maximally nonnegative, i.e., the matrix $A^1\cdot\nu|_{\partial\Omega}$ is positive semidefinite on the null space K of P but not on any space containing K. Here $\nu=-1$. Let $\xi=(\xi_1,0)^T\in K=\mathrm{span}\{(1,0)^T\}$, then,

$$\xi^TA^1\cdot\nu|_{\partial\Omega}\xi=0.$$

On the other hand, $\mathbb{R}^2$ is the only space containing K as a proper subspace. Take $\psi = (1,1)^T$, we get

$$\psi^T A^1 \cdot \nu|_{\partial\Omega}\psi = -2 < 0.$$

Thus, the maximally nonnegative property is satisfied. Furthermore, since the coefficients belong to $\bigcap_{k=0}^{2} C^k([0,h],H^{2-k})$ and $D^2 f_1 \in L^2([0,h],L^2)$ and the compatibility condition satisfied up to first-order, by Theorem 1.3.10 in [33] (pp. 18-19) (originally coming from [29]) , problem (2.1) admits a unique solution $(\theta-1,q) \in \bigcap_{k=0}^{2} C^k([0,h],H^{2-k})$ satisfying

$$\begin{aligned}&\sum_{k=0}^{2}\|D^k(\theta-1,q)\|_{L^2}^2\\&\le C\left(\sum_{k=0}^{2}\|D^k(\theta-1,q)|_{t=0}\|_{L^2}^2+\sum_{k=0}^{1}\|D_t^k f_1|_{t=0}\|_{H^{1-k}}^2+t\int_0^t\sum_{l=0}^{2}\|D^l f_1\|_{L^2}^2\mathrm{d}t\right)e^{CM_2 t},\end{aligned}$$

where C is a positive constant depending on γ.

Concerning the initial-boundary value problem for linear parabolic system (2.2), we deduce from energy methods that the unique solution u satisfies

$$\sum_{k=0}^{2}\|D^k u\|_{L^2}^2+\int_0^t\|(u_x,u_{xx},u_{tx},u_{ttx},u_{txx})\|_{L^2}^2\mathrm{d}t \le Ce^{CM_2 t}\left\{\sum_{k=0}^{2}\|D^k u|_{t=0}\|_{L^2}+M_2 t\right\},$$

$$\|u_{xxx}\|_{L^2}^2+\|u_{txx}\|_{L^2}^2 \le C(M_2)(1+\sum_{k=1}^{2}\|D^k u\|_{L^2}^2),$$

where $C(M_2)$ is a positive constant depending on M_2.

Finally, for the transport equation $(2.3)_1$, we define the characteristic line

$$\frac{dX(t,r)}{dt}=\tilde{u}(t,X(t,r)), \quad X(0,r)=r.$$

Then, we can derive an explicit solution for (2.3),

$$\rho(t,r)=\rho_0(r_0)+\int_0^t f_4(s,X(s,r_0))\mathrm{d}s, \tag{2.5}$$

where r_0 denotes the unique point on the r-axis ($t=0$) such that the characteristic line passing through $(0,r_0)$ and (t,r). Moreover, from usual energy estimates, we can easily get

$$\begin{aligned}&\sum_{k=0}^{2}\|D^k(\rho-1)\|_{L^2}^2\\&\le C\left(\sum_{k=0}^{2}\|D^k(\rho-1)|_{t=0}\|_{L^2}^2+\sum_{k=0}^{1}\|D_t^k f_4|_{t=0}\|_{H^{1-k}}^2+t\int_0^t\sum_{l=0}^{2}\|D^l f_4\|_{L^2}^2\mathrm{d}t\right)e^{CM_2 t}.\end{aligned}$$

Step 2: We first choose

$$M_2 = 2C\left(\sum_{k=0}^{2}\|D^k(\rho-1,\theta-1,q)|_{t=0}\|_{L^2}^2 + \sum_{k=0}^{1}\|D_t^k(f_1,f_4)|_{t=0}\|_{H^{1-k}}^2 + \sum_{k=0}^{2}\|D^k u|_{t=0}\|_{L^2}\right)$$
$$=: C_1 M_0.$$

Then, choose

$$M_3 = \max\{M_2, C(M_2)(1+M_2)\} =: C_2 M_0.$$

Finally, choose h depending on M_0 such that for $t \le h$,

$$Cte^{CM_2t}\int_0^t \sum_{l=0}^{2}\|D^l(f_1,f_4)\|_{L^2}^2 \mathrm{d}t \le \frac{M_2}{2}, \quad e^{CM_2t} \le \frac{5}{4}, \quad Ce^{CM_2t}t \le \frac{1}{4}.$$

Therefore, the linear operator defined by (2.1)-(2.3) maps $X_h(M_2, M_3)$ into itself.

Step 3: Following the above two steps, we can get an iterative sequence $(\rho_n, u_n, \theta_n, q_n) \in X_h(M_2, M_3)$. By usual compactness argument, the sequence$(\rho_n - 1, \theta_n - 1, q_n)$ converge in $C([0,h], H^1) \cap C^1([0,h], L^2)$ and u_n converge in $C([0,h], H^2) \cap C^1([0,h], L^2) \cap L^2([0,h], H^3)$ and the limit function (ρ, u, θ, q) satisfy the system (1.7)-(1.9). Moreover, the function (ρ, u, θ, q) belongs to the class: for $0 \le k \le 2$,

$$D^k(\rho-1, u, \theta-1, q) \in L^\infty((0,h), L^2),\ u_{xxx}, u_{txx} \in L^\infty((0,h), L^2), u_{xtt} \in L^2((0,h), L^2)$$

and is the unique pair of solutions for problem (1.7)-(1.9) in this class.

Step 4: Substituting this (ρ, u, θ, q) into the coefficients in (1.7), the reduced linearized problems admits a unique pair of solutions in $X_h(M_2, M_3)$. By uniqueness, $(\rho, u, \theta, q) \in X_h(M_2, M_3)$.

Therefore, by choosing M_0 sufficiently small so that $M_2 := C_1 M_0 \le M_1$, the proof of the theorem is complete.

□

3. A priori estimates

In order to extend the local solution obtained in Theorem 2.1 to a global solution, we need the following a priori estimates.

Proposition 3.1. *Let Assumption 1.1 hold and (ρ, u, θ, q) be the local solution to system $(1.7)-(1.9)$ given in Theorem 2.1. Then, there exists a $\delta > 0$ such that if $E(t) < \delta$,*

$$E(t) + \int_0^t \mathcal{D}(s)\mathrm{d}s \le CE(0), \tag{3.1}$$

where C is a constant independent of τ, λ, μ.

Without loss of generality, we assume $g(1) = \kappa(1) = h(1) = l(1) = 1$, $\tau \leq 1$. Moreover, there exists δ_1 such that if $E(t) \leq \delta_1$, one has

$$\frac{3}{4} \leq \rho(t,x), \theta(t,x) \leq \frac{5}{4}, \tag{3.2}$$

$$2C_v \geq e_\theta \geq \frac{C_v}{2},\ 2\tau \geq Z(\theta) = \frac{\tau(\theta)}{\kappa(\theta)} \geq \frac{\tau}{2}, |e_{\theta\theta}| + |e_{\theta q}| + |e_{\theta\theta\theta}| + |e_{\theta\theta q}| + |e_{\theta qq}| \leq C, \tag{3.3}$$

$$a(\theta) + \left(\frac{Z(\theta)}{2\theta}\right)' = \frac{1}{\theta}(1 - \frac{1}{2\theta})Z(\theta) + \frac{1}{2}(\frac{1}{\theta} - 1)Z'(\theta) \geq \frac{1}{4}\tau, \tag{3.4}$$

where C denotes a universal constant which is independent of τ, μ and λ. Note that (3.4) can be satisfied by choosing θ sufficiently close to 1.

The proof of Proposition 3.1 is divided into the following Lemmas.

Lemma 3.2. *There exists some constant C such that*

$$\int_\Omega r^2\left((\rho-1)^2 + u^2 + (\theta-1)^2 + \tau q^2\right)\mathrm{d}r + \int_0^t\int_\Omega r^2 q^2 \mathrm{d}r\mathrm{d}s \leq CE_0. \tag{3.5}$$

Proof. From $(1.7)_{3,4}$, we get

$$\rho e_t + \rho u e_r + p(u_r + \frac{2}{r}u) + q_r + \frac{2}{r}q = \mu(\theta)\left(2u_r^2 + \frac{4}{r^2}u^2 - \frac{2}{3}\left(u_r + \frac{2}{r}u\right)^2\right) + \lambda(\theta)\left(u_r + \frac{2}{r}u\right)^2. \tag{3.6}$$

Dividing the above equation by θ, and using formula (1.5), one has

$$\frac{\rho}{\theta}(C_v\theta + a(\theta)q^2)_t + \frac{\rho u}{\theta}(C_v\theta + a(\theta)q^2)_r + R\rho(u_r + \frac{2}{r}u) + \frac{1}{\theta}(q_r + \frac{2}{r}q) = \frac{\mu(\theta)}{\theta}\left(2u_r^2 + \frac{4}{r^2}u^2 - \frac{2}{3}\left(u_r + \frac{2}{r}u\right)^2\right) + \frac{\lambda(\theta)}{\theta}\left(u_r + \frac{2}{r}u\right)^2.$$

For the above equations, first, we have

$$\frac{1}{\theta}C_v\theta_t = C_v(\ln\theta)_t,$$

and

$$
\begin{aligned}
&\frac{\rho}{\theta}(a(\theta)q^2)_t+\frac{\rho u}{\theta}(a(\theta)q^2)_r\\
&=\rho\left(\frac{a(\theta)}{\theta}q^2\right)_t+\rho\frac{\theta_t}{\theta^2}a(\theta)q^2+\rho u\left(\frac{a(\theta)}{\theta}q^2\right)_r+\rho u\frac{\theta_r}{\theta^2}a(\theta)q^2\\
&=\rho\left(\frac{a(\theta)}{\theta}q^2\right)_t-\rho\left(\frac{Z(\theta)}{2\theta^2}\right)_t q^2+\rho u\left(\frac{a(\theta)}{\theta}q^2\right)_r-\rho u\left(\frac{Z(\theta)}{2\theta^2}\right)_r q^2\\
&=\rho\left(\frac{a(\theta)}{\theta}q^2\right)_t-\rho\left(\frac{Z(\theta)}{2\theta^2}q^2\right)_t+\rho\frac{Z(\theta)}{\theta^2}\left(\frac{1}{2}q^2\right)_t\\
&\qquad+\rho u\left(\frac{a(\theta)}{\theta}q^2\right)_r-\rho u\left(\frac{Z(\theta)}{2\theta^2}q^2\right)_r+\rho u\frac{Z(\theta)}{\theta^2}\left(\frac{1}{2}q^2\right)_r\\
&=\rho\left(\left(\frac{a(\theta)}{\theta}-\frac{Z(\theta)}{2\theta^2}\right)q^2\right)_t+\rho u\left(\left(\frac{a(\theta)}{\theta}-\frac{Z(\theta)}{2\theta^2}\right)q^2\right)_r+\frac{Z(\theta)}{\theta^2}q(\rho q_t+\rho u q_r),
\end{aligned}
$$

where we have used the identity

$$\left(\frac{Z(\theta)}{2\theta^2}\right)'=-\frac{a(\theta)}{\theta^2}.$$

Using the equation $(1.7)_4$, one has

$$\frac{Z(\theta)}{\theta^2}q(\rho q_t+\rho u q_r)=\frac{1}{\kappa(\theta)\theta^2}\left(-\tau(\theta)\rho u\frac{2}{r}-1\right)q^2-\frac{\theta_r}{\theta^2}q.$$

Thus, we derive that

$$
\begin{aligned}
&\rho(C_v\ln\theta+A(\theta)q^2)_t+\rho u(C_v\ln\theta+A(\theta)q^2)_r+R\rho(u_r+\frac{2}{r}u)+\left(\frac{q}{\theta}\right)_r+\frac{2}{r}\frac{q}{\theta}\\
&=\frac{1}{\kappa(\theta)\theta^2}\left(1+\tau(\theta)\rho u\frac{2}{r}\right)q^2+\frac{\mu(\theta)}{\theta}\left(2u_r^2+\frac{4}{r^2}u^2-\frac{2}{3}\left(u_r+\frac{2}{r}u\right)^2\right)+\frac{\lambda(\theta)}{\theta}\left(u_r+\frac{2}{r}u\right)^2,
\end{aligned}
$$

where

$$A(\theta):=\frac{a(\theta)}{\theta}-\frac{Z(\theta)}{2\theta^2}=-\left(\frac{Z(\theta)}{2\theta}\right)'.$$

Multiplying the above equation by r^2, it yields

$$
\begin{aligned}
&\left(r^2\rho\left(C_v\ln\theta+A(\theta)q^2\right)\right)_t+\left(r^2\rho u\left(C_v\ln\theta+A(\theta)q^2\right)\right)_r+R\rho(r^2u)_r+\left(r^2\frac{q}{\theta}\right)_r\\
&=\frac{1}{\kappa(\theta)\theta^2}\left(r^2+\tau(\theta)\rho u2r\right)q^2+\frac{\mu(\theta)}{\theta}r^2\left(2u_r^2+\frac{4}{r^2}u^2-\frac{2}{3}\left(u_r+\frac{2}{r}u\right)^2\right)+\frac{\lambda(\theta)}{\theta}r^2\left(u_r+\frac{2}{r}u\right)^2.
\end{aligned}
$$

Then, using the mass equation,

$$(r^2\rho)_t+(r^2\rho u)_r=0,$$

and the energy equation,

$$
\begin{aligned}
&\left(r^2\rho(C_v\theta+a(\theta)q^2+\frac{1}{2}u^2)\right)_t+\left(r^2\rho u\left(C_v\theta+a(\theta)q^2+\frac{1}{2}u^2\right)+r^2(pu+q)-\right.\\
&\qquad\left.r^2u\left(\frac{4}{3}\mu(\theta)(u_r-\frac{u}{r})+\lambda(\theta)(u_r+\frac{2}{r}u)\right)\right)_r=0,
\end{aligned}
$$

we conclude that

$$
\begin{aligned}
&\left(r^2\rho C_v(\theta-\ln\theta-1)+r^2R(\rho\ln\rho-\rho+1)+r^2\rho(a(\theta)-A(\theta))q^2+\frac{1}{2}r^2\rho u^2\right)_t\\
&+\left(r^2\rho u\left(C_v(\theta-\ln\theta-1)+R\ln\rho-1+\frac{1}{2}u^2+(a(\theta)-A(\theta))q^2\right)+r^2(pu+q)-r^2\frac{q}{\theta}-\right.\\
&\left.r^2u\left(\frac{4}{3}\mu(\theta)(u_r-\frac{u}{r})+\lambda(\theta)(u_r+\frac{2}{r}u)\right)\right)_r+\frac{1}{\kappa(\theta)\theta^2}\left(r^2+\tau(\theta)\rho u2r\right)q^2\\
&+\frac{\mu(\theta)}{\theta}r^2\left(2u_r^2+\frac{4}{r^2}u^2-\frac{2}{3}\left(u_r+\frac{2}{r}u\right)^2\right)+\frac{\lambda(\theta)}{\theta}r^2\left(u_r+\frac{2}{r}u\right)^2=0. \quad (3.7)
\end{aligned}
$$

A Taylor expansion together with (3.2) imply

$$C_0(\rho-1)^2\le\rho\ln\rho-\rho+1\le C_1(\rho-1)^2, \quad (3.8)$$

$$C_0(\theta-1)^2\le\theta-\ln\theta-1\le C_1(\theta-1)^2, \quad (3.9)$$

where C_0, C_1 are two positive constants. Furthermore, since $|u|_{L^\infty}\le CE^{\frac{1}{2}}$, we can choose δ (in Proposition 3.1) small enough such that

$$r^2+\tau(\theta)\rho u2r\ge\frac{1}{2}r^2.$$

Therefore, integrating the equation (3.7) over $\Omega\times(0,t)$ and noting (3.4), we get the desired result (3.5) in Lemma 3.2. □

Lemma 3.3. *There exists some constant C such that*

$$
\begin{aligned}
&\int_\Omega\left(r^2\left((u_t)^2+(\rho_t)^2+(\theta_t)^2+\tau(q_t)^2\right)\right)\mathrm{d}r+\int_0^t\int_\Omega r^2((q_t)^2+(\frac{4}{3}\mu+\lambda)u_{tr}^2)\mathrm{d}r\mathrm{d}t\\
&\qquad\le C(E_0+E^{\frac{1}{2}}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s). \quad (3.10)
\end{aligned}
$$

Proof. Taking the t-derivative of (1.7), we have

$$
\begin{cases}
\rho_{tt}+\rho u_{rt}+u\rho_{rt}+\frac{2}{r}\rho u_t=F_1,\\
\rho u_{tt}+\rho uu_{rt}+R\theta\rho_{rt}+R\rho\theta_{rt}-(\frac{4}{3}\mu(\theta)+\lambda(\theta))\left(u_{rr}+\frac{2}{r}u_r-\frac{2}{r^2}u\right)_t=F_2,\\
\rho e_\theta\theta_{tt}+B(\rho,u,\theta,q)\theta_{rt}+p\left(u_r+\frac{2}{r}u\right)_t+\left(q_r+\frac{2}{r}q\right)_t=F_3,\\
\tau(\theta)\rho\left(q_{tt}+uq_{rt}+u\left(\frac{2}{r}q\right)_t\right)+q_t+\kappa(\theta)\theta_{rt}=F_4,
\end{cases} \quad (3.11)
$$

where we denote $B := \rho u e_\theta - \frac{2a(\theta)}{Z(\theta)} q$ and

$$
\begin{aligned}
F_1 :&= -\rho_t u_r - u_t\rho_r - \frac{2}{r}\rho_t u,\\
F_2 :&= -\rho_t u_t - (\rho u)_t u_r - R\theta_t\rho_r - R\rho_t\theta_r\\
&+\left(\frac{4}{3}\mu(\theta)+\lambda(\theta)\right)_t\left(u_{rr}+\frac{2}{r}u_r-\frac{2}{r^2}u\right)+\left(\lambda'(\theta)\theta_r\left(u_r+\frac{2}{r}u\right)+\frac{4}{3}\mu'(\theta)\theta_r\left(u_r-\frac{u}{r}\right)\right)_t,\\
F_3 :&= -(\rho e_\theta)_t\theta_t - B_t\theta_r - \rho_t\left(u_r+\frac{2}{r}u\right)+\left(\left(\frac{2}{\tau(\theta)}+\frac{4}{r}u\right)a(\theta)q^2\right)_t\\
&\qquad+\left(\mu(\theta)\left(2u_r^2+\frac{4}{r^2}u^2-\frac{2}{3}\left(u_r+\frac{2}{r}u\right)^2\right)+\lambda(\theta)\left(u_r+\frac{2}{r}u\right)^2\right)_t,\\
F_4 :&= -(\tau(\theta)\rho)_t q_t - (\tau(\theta)\rho u)_t q_r - (\tau(\theta)\rho u)_t\frac{2}{r}q - \kappa(\theta)_t\theta_r.
\end{aligned}
$$

Multiplying $(3.11)_2$ by r^2u_t, integrating with respect to r, we get

$$
\begin{aligned}
&\int_\Omega(\rho u_{tt}+\rho u u_{rt})r^2u_t\mathrm{d}r+\int_\Omega R\theta\rho_{rt}r^2u_t\mathrm{d}r+\int_\Omega R\rho\theta_{rt}r^2u_t\mathrm{d}r\\
&\qquad-\int_\Omega(\frac{4}{3}\mu(\theta)+\lambda(\theta))\left(u_{rr}+\frac{2}{r}u_r-\frac{2}{r^2}u\right)_t r^2u_t=\int_\Omega F_2r^2u_t\mathrm{d}r.
\end{aligned}
\tag{3.12}
$$

We estimate each term as follows.

First, the mass equation $(1.7)_1$ gives

$$
\int_\Omega(\rho u_{tt}+\rho u u_{rt})r^2u_t\mathrm{d}r=\int_\Omega\rho r^2\left(\frac{1}{2}u_t^2\right)_t+r^2\rho u\left(\frac{1}{2}u_t^2\right)_r\mathrm{d}r=\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{1}{2}\rho r^2u_t^2\mathrm{d}r.
\tag{3.13}
$$

Second, using again the boundary condition $u_t|_{\partial\Omega}=0$, one has

$$
\begin{aligned}
\int_\Omega R\theta r^2\rho_{rt}u_t\mathrm{d}r&=-\int_\Omega\rho_t(R\theta r^2u_t)_r\mathrm{d}r=-\int_\Omega R\rho_t\theta_r r^2u_t\mathrm{d}r-\int_\Omega R\rho_t\theta(r^2u_t)_r\mathrm{d}r\\
&\geq -CE^{\frac{1}{2}}(t)\mathcal{D}(t)-\int_\Omega R\rho_t\theta\left(-\frac{1}{\rho}r^2\rho_{tt}-\frac{1}{\rho}r^2u\rho_{tr}+\frac{1}{\rho}r^2F_1\right)\mathrm{d}r\\
&\geq -CE^{\frac{1}{2}}(t)\mathcal{D}(t)+\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{R\theta}{2\rho}r^2\rho_t^2\mathrm{d}r.
\end{aligned}
\tag{3.14}
$$

Similarly, one has

$$
\begin{aligned}
&-\int_\Omega(\frac{4}{3}\mu(\theta)+\lambda(\theta))\left(u_{rr}+\frac{2}{r}u_r-\frac{2}{r^2}u\right)_t r^2u_t\\
&=-\int_\Omega(\frac{4}{3}\mu(\theta)+\lambda(\theta))\left((r^2u_r)_{rt}-2u_t\right)u_t\mathrm{d}r\\
&\geq\int_\Omega(\frac{4}{3}\mu(\theta)+\lambda(\theta))(r^2u_{rt}^2+2u_t^2)\mathrm{d}r-CE^{\frac{1}{2}}(t)\mathcal{D}(t).
\end{aligned}
\tag{3.15}
$$

So, we have from (3.12)-(3.15)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{1}{2}\rho r^2u_t^2+\frac{\theta}{2\rho}r^2\rho_t^2\right)\mathrm{d}r+\int_\Omega R\rho\theta_{rt}r^2u_t\mathrm{d}r+\int_\Omega(\frac{4}{3}\mu(\theta)+\lambda(\theta))(r^2u_{rt}^2+2u_t^2)\mathrm{d}r \leq CE^{\frac{1}{2}}(t)\mathcal{D}(t). \tag{3.16}$$

Multiplying $(3.11)_3$ by $\frac{1}{\theta}r^2\theta_t$, we have

$$\int_\Omega\frac{\rho e_\theta}{\theta}r^2\theta_{tt}\theta_t\mathrm{d}r+\int_\Omega\frac{1}{\theta}r^2B\theta_{rt}\theta_t\mathrm{d}r+\int_\Omega R\rho r^2(u_r+\frac{2}{r}u)_t\theta_t\mathrm{d}r+\int_\Omega\frac{1}{\theta}r^2(q_r+\frac{2}{r}q)_t\theta_t\mathrm{d}r =\int_\Omega\frac{1}{\theta}r^2F_3\theta_t\mathrm{d}r. \tag{3.17}$$

We estimate each term as follows. First,

$$\begin{aligned}\int_\Omega r^2\frac{\rho e_\theta}{\theta}\theta_{tt}\theta_t\mathrm{d}r&=\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{\rho e_\theta}{2\theta}r^2\theta_t^2\mathrm{d}r-\int_\Omega\left(\frac{\rho e_\theta}{2\theta}\right)_tr^2\theta_t^2\mathrm{d}r\\&\geq\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{\rho e_\theta}{2\theta}r^2\theta_t^2\mathrm{d}r-CE^{\frac{1}{2}}(t)\mathcal{D}(t).\end{aligned} \tag{3.18}$$

Second, using the fact $B|_{\partial\Omega}=0$, one has

$$\int_\Omega\frac{1}{\theta}r^2B\theta_{rt}\theta_t\mathrm{d}r=\int_\Omega\frac{1}{\theta}r^2B\left(\frac{1}{2}\theta_t^2\right)_r\mathrm{d}r=-\int_\Omega\left(\frac{1}{\theta}r^2B\right)_r\frac{1}{2}\theta_t^2\mathrm{d}r\geq-CE^{\frac{1}{2}}(t)\mathcal{D}(t). \tag{3.19}$$

Third, using the fact $u_t|_{\partial\Omega}=0$, we get

$$\begin{aligned}\int_\Omega R\rho r^2(u_r+\frac{2}{r}u)_t\theta_t\mathrm{d}r&=\int_\Omega R\rho(r^2u)_{rt}\theta_t\mathrm{d}r=-\int_\Omega R(r^2u)_t(\rho_r\theta_t+\rho\theta_{tr})\mathrm{d}r\\&\geq-\int_\Omega R\rho r^2u_t\theta_{tr}\mathrm{d}r-CE^{\frac{1}{2}}(t)\mathcal{D}(t).\end{aligned} \tag{3.20}$$

The last term on the left-hand side of equation (3.17) reduces to

$$\int_\Omega\frac{1}{\theta}r^2\left(q_r+\frac{2}{r}q\right)_t\theta_t\mathrm{d}r=\int_\Omega\frac{1}{\theta}(r^2q)_{rt}\theta_t\mathrm{d}r. \tag{3.21}$$

Thus, we derive from (3.17)-(3.21)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{\rho e_\theta}{2\theta}r^2\theta_t^2\mathrm{d}r-\int_\Omega R\rho r^2u_t\theta_{tr}\mathrm{d}r+\int_\Omega\frac{1}{\theta}(r^2q)_{rt}\theta_t\mathrm{d}r\leq CE^{\frac{1}{2}}(t)\mathcal{D}(t). \tag{3.22}$$

Multiplying $(3.11)_4$ by $\frac{1}{\kappa(\theta)\theta}r^2q_t$, one gets

$$\int_\Omega\frac{\tau(\theta)}{\kappa(\theta)\theta}\rho r^2q_{tt}q_t\mathrm{d}r+\int_\Omega\frac{\tau(\theta)}{\kappa(\theta)\theta}\rho ur^2q_{rt}q_t\mathrm{d}r+\int_\Omega\frac{\tau(\theta)}{\kappa(\theta)\theta}\rho ur^2\left(\frac{2}{r}q\right)_tq_t\mathrm{d}r +\int_\Omega\frac{1}{\kappa(\theta)\theta}r^2q_t^2\mathrm{d}r+\int_\Omega\frac{1}{\theta}r^2q_t\theta_{rt}\mathrm{d}r=\int_\Omega\frac{1}{\kappa(\theta)\theta}r^2F_4q_t\mathrm{d}r. \tag{3.23}$$

We estimate each term as follows.

First,

$$\begin{aligned}\int_\Omega \frac{\tau(\theta)}{\kappa(\theta)\theta}\rho r^2 q_{tt}q_t \mathrm{d}r &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\tau(\theta)}{\kappa(\theta)\theta}\rho r^2 \frac{1}{2}q_t^2 \mathrm{d}r - \int_\Omega \left(\frac{\tau(\theta)}{\kappa(\theta)\theta}\rho r^2\right)_t \frac{1}{2}q_t^2 \mathrm{d}r \\ &\geq \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\tau(\theta)}{\kappa(\theta)\theta}\rho r^2 \frac{1}{2}q_t^2 \mathrm{d}r - CE^{\frac{1}{2}}(t)\mathcal{D}(t).\end{aligned}$$

Second, using the boundary condition $u|_{\partial\Omega} = 0$, one has

$$\begin{aligned}\int_\Omega \frac{\tau(\theta)}{\kappa(\theta)\theta}\rho u r^2 q_{tr}q_t \mathrm{d}r &= \int_\Omega \frac{\tau(\theta)}{\kappa(\theta)\theta}\rho u r^2 \left(\frac{1}{2}q_t^2\right)_r \mathrm{d}r \\ &= -\int_\Omega \left(\frac{\tau(\theta)}{\kappa(\theta)\theta}\rho u r^2\right)_r \frac{1}{2}q_t^2 \mathrm{d}r \geq -CE^{\frac{1}{2}}(t)\mathcal{D}(t).\end{aligned}$$

Third,

$$\int_\Omega \frac{\tau(\theta)}{\kappa(\theta)\theta}\rho u r^2 \left(\frac{2}{r}q\right)_t q_t + \frac{1}{\kappa(\theta)\theta}r^2 q_t^2 \mathrm{d}r = \int_\Omega \left(\tau(\theta)\rho u\frac{2}{r}+1\right)\frac{1}{\kappa(\theta)\theta}r^2 q_t^2 \mathrm{d}r \geq \int_\Omega \frac{1}{2\kappa(\theta)\theta}r^2 q_t^2 \mathrm{d}r.$$

For the last term on the left hand side of equation (3.23), one has

$$\int_\Omega \frac{1}{\theta}r^2 q_t \theta_{rt}\mathrm{d}r = \int_\Omega \left(\frac{1}{\theta}\right)_r r^2 q_t\theta_t \mathrm{d}r - \int_\Omega \frac{1}{\theta}(r^2 q_t)_r\theta_t \mathrm{d}r \geq -\int_\Omega \frac{1}{\theta}(r^2 q)_{rt}\theta_t \mathrm{d}r - CE^{\frac{1}{2}}(t)\mathcal{D}(t).$$

Thus, we derive that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\rho\tau(\theta)}{2\kappa(\theta)\theta}r^2 q_t^2 \mathrm{d}r - \int_\Omega \frac{1}{\theta}(r^2 q)_{rt}\theta_t \mathrm{d}r + \int_\Omega \frac{1}{2\kappa(\theta)\theta}r^2 q_t^2 \mathrm{d}r \leq CE^{\frac{1}{2}}(t)\mathcal{D}(t). \tag{3.24}$$

Therefore, combining (3.16), (3.22) and (3.24), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{\rho}{2}r^2 u_t^2 + \frac{R\theta}{2\rho}r^2\rho_t^2 + \frac{\rho e_\theta}{2\theta}r^2\theta_t^2 + \frac{\rho\tau(\theta)}{2\kappa(\theta)\theta}r^2 q_t^2\right)\mathrm{d}r + \int_\Omega \frac{1}{2\kappa(\theta)\theta}r^2 q_t^2 \mathrm{d}r \\ + \int_\Omega (\frac{4}{3}\mu(\theta)+\lambda(\theta))(r^2 u_{rt}^2 + 2u_t^2)\mathrm{d}r \leq CE^{\frac{1}{2}}(t)\mathcal{D}(t).\end{aligned} \tag{3.25}$$

Integrating over $(0,t)$, and noticing the fact that

$$\int_\Omega \left(\frac{\rho}{2}r^2 u_t^2 + \frac{R\theta}{2\rho}r^2\rho_t^2 + \frac{\rho e_\theta}{2\theta}r^2\theta_t^2 + \frac{\rho\tau(\theta)}{2\kappa(\theta)\theta}r^2 q_t^2\right)(t=0,r)\mathrm{d}r \leq CE_0,$$

we get the desired result (3.10) in Lemma 3.3. □

Lemma 3.4. *There exists some constant C such that*

$$\begin{aligned}\int_\Omega \left((r^2u_r^2 + 2u^2) + r^2\rho_r^2 + r^2\theta_r^2 + \tau(r^2q_r^2 + 2q^2)\right)\mathrm{d}r + \int_0^t\int_\Omega (r^2q_r^2 + 2q^2)\mathrm{d}r\mathrm{d}t \\ \leq C(E_0 + E^{\frac{1}{2}}(t)\int_0^t \mathcal{D}(s)\mathrm{d}s).\end{aligned} \tag{3.26}$$

Proof. Taking the r-derivative of (1.7), we get

$$\begin{cases} \rho_{tr} + u\rho_{rr} + \rho u_{rr} + \frac{2}{r}\rho u_r - \frac{2}{r^2}\rho u = G_1, \\ \rho u_{tr} + \rho u u_{rr} + p_{rr} - \left((\frac{4}{3}\mu(\theta) + \lambda(\theta)) \left(u_{rr} + \frac{2}{r}u_r - \frac{2}{r^2}u \right) \right)_r = G_2, \\ \rho e_\theta \theta_{tr} + B(\rho, u, \theta, q)\theta_{rr} + p \left(u_r + \frac{2}{r}u \right)_r + \left(q_r + \frac{2}{r}q \right)_r = G_3, \\ \tau(\theta)\rho(q_{tr} + u q_{rr}) + q_r + \kappa(\theta)\theta_{rr} = G_4, \end{cases} \tag{3.27}$$

where

$$\begin{aligned} G_1 :&= -2\rho_r u_r - \frac{2}{r}\rho_r u, \\ G_2 :&= -\rho_r u_t - (\rho u)_r u_r + \left(\lambda'(\theta)\theta_r \left(u_r + \frac{2}{r}u \right) + \frac{4}{3}\mu'(\theta)\theta_r \left(u_r - \frac{u}{r} \right) \right)_r, \\ G_3 :&= -(\rho e_\theta)_r \theta_t - B_r(\rho, u, \theta, q)\theta_r - p_r(u_r + \frac{2}{r}u) + \left(\left(\frac{2}{\tau(\theta)} + \frac{4}{r}u \right) a(\theta) q^2 \right)_r \\ &\quad + \left(\mu(\theta) \left(2u_r^2 + \frac{4}{r^2}u^2 - \frac{2}{3}\left(u_r + \frac{2}{r}u \right)^2 \right) + \lambda(\theta) \left(u_r + \frac{2}{r}u \right)^2 \right)_r \\ G_4 :&= -(\tau(\theta)\rho)_r q_t - (\tau(\theta)\rho u)_r q_r - \left(\tau(\theta)\rho u \frac{2}{r} q \right)_r - \kappa'(\theta)\theta_r^2. \end{aligned}$$

Multiplying $(3.27)_2$ by $r^2 u_r$, we get

$$\int_\Omega \rho u_{tr} r^2 u_r \mathrm{d}r + \int_\Omega \rho u u_{rr} r^2 u_r \mathrm{d}r + \int_\Omega p_{rr} r^2 u_r \mathrm{d}r = \int_\Omega G_2 r^2 u_r \mathrm{d}r. \tag{3.28}$$

We estimate the different terms as follows.
First, one has

$$\int_\Omega \left(\rho u_{tr} r^2 u_r + \rho u u_{rr} r^2 u_r \right) \mathrm{d}r = \int_\Omega \rho r^2 \left(\frac{1}{2}u_r^2 \right)_t + \rho u r^2 \left(\frac{1}{2}u_r^2 \right)_r \mathrm{d}r = \frac{\mathrm{d}}{\mathrm{d}r} \int_\Omega \frac{1}{2}\rho r^2 u_r^2 \mathrm{d}r.$$

Second, since $(u, u_t, \theta_r)|_{\partial\Omega} = 0$, we derive from the momentum equation (1.7) that

$$\left(p_r - (\frac{4}{3}\mu(\theta) + \lambda(\theta)) \left(u_{rr} + \frac{2}{r}u_r - \frac{2}{r^2}u \right) \right) \Big|_{\partial\Omega} = 0,$$

so one gets

$$\begin{aligned} &\int_\Omega \left(p_{rr} - \left((\frac{4}{3}\mu(\theta) + \lambda(\theta)) \left(u_{rr} + \frac{2}{r}u_r - \frac{2}{r^2}u \right) \right)_r \right) r^2 u_r \mathrm{d}r \\ &= -\int_\Omega \left(R(\theta\rho_r + \rho\theta_r) - (\frac{4}{3}\mu(\theta) + \lambda(\theta)) \left(u_{rr} + \frac{2}{r}u_r - \frac{2}{r^2}u \right) \right) (r^2 u_r)_r \mathrm{d}r. \end{aligned}$$

Using the following equation, arising from the mass equation,

$$r^2 \rho_{tr} + 2r^2 \rho_r u_r + r^2 u \rho_{rr} + \rho(r^2 u_r)_r + 2r\rho_r u - 2\rho u = 0, \tag{3.29}$$

we have

$$
\begin{aligned}
&-\int_\Omega R\theta\rho_r(r^2u_r)_r\mathrm{d}r=-\int_\Omega R\theta\rho_r(2ru_r+r^2u_{rr})\mathrm{d}r\\
&=\int_\Omega R\theta\rho_r\frac{1}{\rho}\left(r^2\rho_{tr}+2r^2\rho_ru_r+r^2u\rho_{rr}+2r\rho_ru-2\rho u\right)\mathrm{d}r\\
&=\int_\Omega \frac{R\theta}{\rho}r^2\left(\frac{1}{2}\rho_r^2\right)_t\mathrm{d}r+\int_\Omega\frac{R\theta}{\rho}2r^2\rho_r^2u_r\mathrm{d}r+\int_\Omega\frac{R\theta}{\rho}\rho_rr^2u\rho_{rr}\mathrm{d}r+\int_\Omega\frac{R\theta}{\rho}\rho_r(2r\rho_ru-2\rho u)\mathrm{d}r\\
&\geq\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{R\theta}{2\rho}r^2\rho_r^2\mathrm{d}r-\int_\Omega 2R\theta\rho_ru\mathrm{d}r-CE^{\frac12}(t)\mathcal{D}(t).
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&\int_\Omega\left(\frac43\mu(\theta)+\lambda(\theta)\right)\left(u_{rr}+\frac{2}{r}u_r-\frac{2}{r^2}u\right)(r^2u_r)_r\mathrm{d}r\\
&=\int_\Omega\left(\frac43\mu(\theta)+\lambda(\theta)\right)\left(u_{rr}+\frac{2}{r}u_r-\frac{2}{r^2}u\right)(r^2u_{rr}+2ru_r)\mathrm{d}r\\
&=\int_\Omega\left(\frac43\mu(\theta)+\lambda(\theta)\right)\left(r^2u_{rr}^2+4u_r^2+4ru_{rr}u_r-2uu_{rr}-\frac{4}{r}uu_r\right)\mathrm{d}r\\
&\geq\int_\Omega\left(\frac43\mu(\theta)+\lambda(\theta)\right)\left(\frac15r^2u_{rr}^2+u_r^2\right)\mathrm{d}r-\int_\Omega\left(\frac43\mu(\theta)+\lambda(\theta)\right)\frac{2}{r^2}u^2-CE^{\frac12}(t)\mathcal{D}(t).
\end{aligned}
$$

Thus, we derive

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac12\rho r^2u_r^2+\frac{\theta}{2\rho}r^2\rho_r^2\right)\mathrm{d}r-\int_\Omega R\rho\theta_r(r^2u_r)_r\mathrm{d}r-\int_\Omega 2R\theta\rho_ru\mathrm{d}r\\
&+\int_\Omega\left(\frac43\mu(\theta)+\lambda(\theta)\right)\left(\frac15r^2u_{rr}^2+u_r^2\right)\mathrm{d}r-\int_\Omega\left(\frac43\mu(\theta)+\lambda(\theta)\right)\frac{2}{r^2}u^2\leq CE^{\frac12}(t)\mathcal{D}(t).
\end{aligned}\tag{3.30}
$$

Multiplying $(3.27)_3$ by $\frac{1}{\theta}r^2\theta_r$, one has

$$
\begin{aligned}
&\int_\Omega\rho e_\theta\theta_{tr}\frac{1}{\theta}r^2\theta_r\mathrm{d}r+\int_\Omega B(\rho,u,\theta,q)\theta_{rr}\frac{1}{\theta}r^2\theta_r\mathrm{d}r+\int_\Omega R\rho r^2\left(u_r+\frac{2}{r}u\right)_r\theta_r\mathrm{d}r\\
&\qquad+\int_\Omega\left(q_r+\frac{2}{r}q\right)_r\frac{1}{\theta}r^2\theta_r\mathrm{d}r=\int_\Omega G_3\frac{1}{\theta}r^2\theta_r\mathrm{d}r.
\end{aligned}\tag{3.31}
$$

We estimate each term in the above equation as follows.
First, one has

$$
\int_\Omega\rho e_\theta\theta_{tr}\frac{1}{\theta}r^2\theta_r\mathrm{d}r=\int_\Omega\frac{\rho e_\theta}{\theta}r^2\left(\frac12\theta_r^2\right)_t\mathrm{d}r\geq\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{\rho e_\theta}{2\theta}r^2\theta_r^2\mathrm{d}r-CE^{\frac12}(t)\mathcal{D}(t).
$$

Second, using the fact $B(\rho,u,\theta,q)|_{\partial\Omega}=0$, we have

$$
\int_\Omega B(\rho,u,\theta,q)\theta_{rr}\frac{1}{\theta}r^2\theta_r\mathrm{d}r=-\int_\Omega\left(\frac{B(\rho,u,\theta,q)}{\theta}r^2\right)_r\frac12\theta_r^2\mathrm{d}r\geq-CE^{\frac12}(t)\mathcal{D}(t).
$$

Third,

$$\int_\Omega R\rho r^2\left(u_r+\frac{2}{r}u\right)_r\theta_r\mathrm{d}r=\int_\Omega R\rho r^2(u_{rr}+\frac{2}{r}u_r-\frac{2}{r^2}u)\theta_r\mathrm{d}r$$
$$=\int_\Omega R\rho((r^2u_r)_r-2u)\theta_r\mathrm{d}r=\int_\Omega R\rho(r^2u_r)_r\theta_r\mathrm{d}r-\int_\Omega 2R\rho u\theta_r\mathrm{d}r.$$

Similarly, one has

$$\int_\Omega\left(q_r+\frac{2}{r}q\right)_r\frac{1}{\theta}r^2\theta_r\mathrm{d}r=\int_\Omega\frac{1}{\theta}(r^2q_r)_r\theta_r\mathrm{d}r-\int_\Omega\frac{2}{\theta}q\theta_r\mathrm{d}r.$$

So, we derive that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{\rho e_\theta}{2\theta}r^2\theta_r^2\mathrm{d}r+\int_\Omega\left(R\rho(r^2u_r)_r\theta_r+\frac{1}{\theta}(r^2q_r)_r\theta_r\right)\mathrm{d}r-\int_\Omega(2R\rho u\theta_r+\frac{2}{\theta}q\theta_r)\mathrm{d}r$$
$$\le CE^{\frac12}(t)\mathcal{D}(t).\quad(3.32)$$

Multiplying $(3.27)_4$ by $\frac{1}{\kappa(\theta)\theta}r^2q_r$, one get

$$\int_\Omega\frac{\tau(\theta)}{\kappa(\theta)\theta}\rho r^2(q_{tr}+uq_{rr})q_r\mathrm{d}r+\int_\Omega\frac{1}{\kappa(\theta)\theta}r^2q_r^2\mathrm{d}r+\int_\Omega\frac{1}{\theta}\theta_{rr}(r^2q_r)\mathrm{d}r=\int_\Omega\frac{1}{\kappa(\theta)\theta}r^2q_rG_4\mathrm{d}r.\quad(3.33)$$

We estimate each term in the above equation as follows.

First, using the mass equation, one has

$$\int_\Omega\frac{\tau(\theta)}{\kappa(\theta)\theta}\rho r^2(q_{tr}+uq_{rr})q_r\mathrm{d}r=\int_\Omega\frac{\tau(\theta)}{\kappa(\theta)\theta}\rho r^2((\frac12q_r^2)_t+u(\frac12q_r^2)_r)\mathrm{d}r$$
$$\ge\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{\tau(\theta)}{\kappa(\theta)\theta}\rho r^2\frac12q_r^2\mathrm{d}r-CE^{\frac12}(t)\mathcal{D}(t).$$

Second, use the fact $\theta_r|_{\partial\Omega}=0$, we have

$$\int_\Omega\frac{1}{\theta}\theta_{rr}r^2q_r\mathrm{d}r\ge-\int_\Omega\frac{1}{\theta}\theta_r(r^2q_r)_r\mathrm{d}r-CE^{\frac12}(t)\mathcal{D}(t).$$

Thus, we derive that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{\tau(\theta)}{2\kappa(\theta)\theta}\rho r^2q_r^2\mathrm{d}r-\int_\Omega\frac{1}{\theta}\theta_r(r^2q_r)_r\mathrm{d}r+\int_\Omega\frac{1}{\kappa(\theta)\theta}r^2q_r^2\mathrm{d}r\le CE^{\frac12}(t)\mathcal{D}(t).\quad(3.34)$$

Combining (3.30), (3.32) and (3.34), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac12\rho r^2u_r^2+\frac{\theta}{2\rho}r^2\rho_r^2+\frac{\rho e_\theta}{2\theta}r^2\theta_r^2+\frac{\tau(\theta)}{2\kappa(\theta)\theta}\rho r^2q_r^2\right)\mathrm{d}r+\int_\Omega\frac{1}{\kappa(\theta)\theta}r^2q_r^2\mathrm{d}r-\int_\Omega\left(2up_r+\frac{2}{\theta}q\theta_r\right)\mathrm{d}r$$
$$+\int_\Omega\left(\frac43\mu(\theta)+\lambda(\theta)\right)\left(\frac15r^2u_{rr}^2+u_r^2\right)\mathrm{d}r-\int_\Omega\left(\frac43\mu(\theta)+\lambda(\theta)\right)\frac{2}{r^2}u^2\le CE^{\frac12}(t)\mathcal{D}(t).\quad(3.35)$$

On the other hand, multiplying $(1.7)_2$ by $2u$, one gets

$$\int_\Omega 2\rho u_tu\mathrm{d}r+\int_\Omega 2\rho uu_ru\mathrm{d}r+\int_\Omega 2p_ru\mathrm{d}r$$
$$=\int_\Omega\left(\left(\frac43\mu(\theta)+\lambda(\theta)\right)\left(u_{rr}+\frac{2}{r}u_r-\frac{2}{r^2}u\right)+\lambda'(\theta)\theta_r\left(u_r+\frac{2}{r}u\right)+\frac43\mu'(\theta)\theta_r\left(u_r-\frac{u}{r}\right)\right)2u\mathrm{d}r,$$

For the term on the right hand side of the above equation, we have

$$\begin{aligned}&\int_\Omega \left(\left(\frac{4}{3}\mu(\theta)+\lambda(\theta)\right)\left(u_{rr}+\frac{2}{r}u_r-\frac{2}{r^2}u\right)+\lambda'(\theta)\theta_r\left(u_r+\frac{2}{r}u\right)+\frac{4}{3}\mu'(\theta)\theta_r\left(u_r-\frac{u}{r}\right)\right)2u\mathrm{d}r\\ &\le -\int_\Omega\left(\frac{4}{3}\mu(\theta)+\lambda(\theta)\right)\left(2u_r^2+\frac{2}{r^2}u^2\right)\mathrm{d}r+CE^{\frac12}(t)\mathcal{D}(t).\end{aligned}$$

Thus, we derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho u^2\mathrm{d}r+\int_\Omega 2p_r u\mathrm{d}r+\int_\Omega\left(\frac{4}{3}\mu(\theta)+\lambda(\theta)\right)\left(2u_r^2+\frac{2}{r^2}u^2\right)\mathrm{d}r\le CE^{\frac12}(t)\mathcal{D}(t). \tag{3.36}$$

Then, multiplying $(1.7)_4$ by $\frac{2}{\kappa(\theta)\theta}q$, one has

$$\int_\Omega \tau(\theta)\rho(q_t+uq_r+u\cdot\frac{2}{r}q)\frac{2}{\kappa(\theta)\theta}q\mathrm{d}r+\int_\Omega\frac{2}{\kappa(\theta)\theta}q^2\mathrm{d}r+\int_\Omega\frac{2}{\theta}q\theta_r\mathrm{d}r=0,$$

from which we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{\rho\tau(\theta)}{\kappa(\theta)\theta}q^2\mathrm{d}r+\int_\Omega\frac{2}{\kappa(\theta)\theta}q^2\mathrm{d}r+\int_\Omega\frac{2}{\theta}q\theta_r\mathrm{d}r\le CE^{\frac12}(t)\mathcal{D}(t). \tag{3.37}$$

Therefore, combining (3.35), (3.36) and (3.37), we conclude that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac12\rho(r^2u_r^2+2u^2)+\frac{\theta}{2\rho}r^2\rho_r^2+\frac{\rho e_\theta}{2\theta}r^2\theta_r^2+\frac{\tau(\theta)}{2\kappa(\theta)\theta}(r^2q_r^2+2q^2)\right)\mathrm{d}r\\ &+\int_\Omega\left(\frac{4}{3}\mu(\theta)+\lambda(\theta)\right)\left(\frac15 r^2u_{rr}^2+3u_r^2\right)\mathrm{d}r+\int_\Omega\frac{1}{\kappa(\theta)\theta}(r^2q_r^2+2q^2)\mathrm{d}r\le CE^{\frac12}(t)\mathcal{D}(t).\end{aligned}$$

Integrating the above result with respect to t, the proof of Lemma 3.4 is finished. □

Using the equation $(1.7)_3$ and $(1.7)_4$ and Lemmas 3.2-3.4, we obtain

Lemma 3.5. *There exists some constant C such that*

$$\int_\Omega r^2(q^2+q_r^2)\mathrm{d}r\le C(E_0+E^{\frac12}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s+E^2(t)). \tag{3.38}$$

Now we continue with

Lemma 3.6. *There exists some constant C such that*

$$\int_0^t\int_\Omega(r^2|D(\rho,u,\theta,q)|^2+u^2+q^2)\mathrm{d}r\mathrm{d}t\le C(E_0+E^{\frac12}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s). \tag{3.39}$$

Proof. Using equation $(1.7)_4$ and Lemmas 3.2-3.4, we have

$$\begin{aligned}\int_0^t\int_\Omega\kappa^2(\theta)r^2\theta_r^2\mathrm{d}r\mathrm{d}t&\le C\int_0^t\int_\Omega\left(r^2q^2+\tau^2(\theta)\rho^2r^2q_t^2+\tau^2(\theta)\rho^2r^2q_r^2+\tau^2(\theta)\rho^2u^24q^2\right)\mathrm{d}r\mathrm{d}t\\ &\le C(E_0+E^{\frac12}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s).\end{aligned} \tag{3.40}$$

Multiplying $(1.7)_3$ by $\frac{1}{\rho e_\theta}(r^2u)_r$ and integrating the result, we get

$$
\begin{aligned}
&\int_0^t\int_\Omega \frac{p}{\rho e_\theta}(u_r+\frac{2}{r}u)(r^2u)_r\mathrm{d}r\\
=&-\int_0^t\int_\Omega \theta_t(r^2u)_r\mathrm{d}r\mathrm{d}t-\int_0^t\int_\Omega \frac{B(\rho,u,\theta,q)}{\rho e_\theta}\theta_r(r^2u)_r\mathrm{d}r\mathrm{d}t-\int_0^t\int_\Omega \frac{1}{\rho e_\theta}(q_r+\frac{2}{r}q)(r^2u)_r\mathrm{d}r\mathrm{d}t\\
&+\int_0^t\int_\Omega \frac{1}{\rho e_\theta}\left(\frac{2}{\tau(\theta)}+\frac{4}{r}u\right)a(\theta)q^2(r^2u)_r\mathrm{d}r\mathrm{d}t\\
&+\int_0^t\int_\Omega\left(\mu(\theta)\left(2u_r^2+\frac{4}{r^2}u^2-\frac{2}{3}\left(u_r+\frac{2}{r}u\right)^2\right)+\lambda(\theta)\left(u_r+\frac{2}{r}u\right)^2\right)(r^2u)_r\mathrm{d}r\mathrm{d}t.
\end{aligned}
\tag{3.41}
$$

We estimate each term in the above equation as follows. First, one has

$$
\begin{aligned}
&\int_0^t\int_\Omega \frac{R\theta}{e_\theta}(u_r+\frac{2}{r}u)(r^2u_r+2ru)\mathrm{d}r\mathrm{d}t\\
&=\int_0^t\int_\Omega \frac{R\theta}{e_\theta}(r^2u_r^2+4u^2+4ruu_r)\mathrm{d}r\mathrm{d}t\\
&=\int_0^t\int_\Omega \frac{R\theta}{e_\theta}(r^2u_r^2+2u^2)\mathrm{d}r\mathrm{d}t-\int_0^t\int_\Omega\left(\frac{R\theta}{e_\theta}\right)_r r2u^2\mathrm{d}r\mathrm{d}t\\
&\geq\int_0^t\int_\Omega \frac{R}{C_v}(r^2u_r^2+2u^2)\mathrm{d}r\mathrm{d}t-E^{\frac12}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s,
\end{aligned}
$$

where we use the fact that

$$\|r(\rho_r,u_r,\theta_r,q_r)\|_{L^\infty}\leq CE^{\frac12}(t).$$

For the right hand side of (3.41), using the boundary condition $u_t|_{\partial\Omega}=0$, we have

$$
\begin{aligned}
&-\int_0^t\int_\Omega\theta_t(r^2u)_r\mathrm{d}r\mathrm{d}t=-\int_0^t\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\theta(r^2u)_r\mathrm{d}r\mathrm{d}t+\int_0^t\int_\Omega\theta(r^2u)_{rt}\mathrm{d}r\mathrm{d}t\\
&\leq-\int_0^t\int_\Omega\theta_r r^2u_t\mathrm{d}r\mathrm{d}t+C(E_0+E^{\frac12}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s)\\
&\leq\eta\int_0^t\int_\Omega r^2u_t^2\mathrm{d}r\mathrm{d}t+C(\eta)(E_0+E^{\frac12}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s).
\end{aligned}
$$

where η is a small constant to be determined later.

For the second term of the right hand side of (3.41), one has

$$
\begin{aligned}
&-\int_0^t\int_\Omega\frac{B}{\rho e_\theta}\theta_r(r^2u)_r\mathrm{d}r\mathrm{d}t=-\int_0^t\int_\Omega\frac{B}{\rho e_\theta}\theta_r(r^2u_r+2ru)\mathrm{d}r\mathrm{d}t\\
&\leq\frac12\int_0^t\int_\Omega\frac{R}{C_v}(r^2u_r^2+2u^2)\mathrm{d}r\mathrm{d}t+C(E_0+E^{\frac12}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&-\int_0^t\int_\Omega \frac{1}{\rho e_\theta}(q_r+\frac{2}{r}q)(r^2u)_r\mathrm{d}r\mathrm{d}t\\
&\le \frac{1}{4}\int_0^t\int_\Omega \frac{R}{C_v}(r^2u_r^2+2u^2)\mathrm{d}r\mathrm{d}t + C\int_0^t\int_\Omega (r^2q_r^2+q^2)\mathrm{d}r\mathrm{d}t\\
&\le \frac{1}{4}\int_0^t\int_\Omega \frac{R}{C_v}(r^2u_r^2+2u^2)\mathrm{d}r\mathrm{d}t + C(E_0+E^{\frac{1}{2}}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t\int_\Omega \frac{1}{\rho e_\theta}\left(\frac{2}{\tau(\theta)}+\frac{4}{r}u\right)a(\theta)q^2(r^2u)_r\mathrm{d}r\mathrm{d}t\\
&\le \frac{1}{16}\int_0^t\int_\Omega \frac{R}{C_v}(r^2u_r^2+2u^2)\mathrm{d}r\mathrm{d}t + C\int_0^t\int_\Omega (r^2q^4)\mathrm{d}r\mathrm{d}t\\
&\le \frac{1}{16}\int_0^t\int_\Omega \frac{R}{C_v}(r^2u_r^2+2u^2)\mathrm{d}r\mathrm{d}t + C(E_0+E^{\frac{1}{2}}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s).
\end{aligned}
$$

In the same way, one has

$$
\begin{aligned}
&\int_0^t\int_\Omega \left(\mu(\theta)\left(2u_r^2+\frac{4}{r^2}u^2-\frac{2}{3}\left(u_r+\frac{2}{r}u\right)^2\right)+\lambda(\theta)\left(u_r+\frac{2}{r}u\right)^2\right)(r^2u)_r\mathrm{d}r\mathrm{d}t\\
&\le \frac{1}{16}\int_0^t\int_\Omega \frac{R}{C_v}(r^2u_r^2+2u^2)\mathrm{d}r\mathrm{d}t + C(E_0+E^{\frac{1}{2}}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s).
\end{aligned}
$$

Thus, we derive that

$$
\frac{1}{8}\int_0^t\int_\Omega \frac{R}{C_v}(r^2u_r^2+2u^2)\mathrm{d}r\mathrm{d}t \le \eta\int_0^t\int_\Omega r^2u_t^2\mathrm{d}r\mathrm{d}t + C(E_0+E^{\frac{1}{2}}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s). \tag{3.42}
$$

Now, multiplying the mass equation $(1.7)_1$ by $r^2\rho_t$, one has

$$
\begin{aligned}
\int_0^t\int_\Omega r^2\rho_t^2\mathrm{d}r\mathrm{d}t &= -\int_0^t\int_\Omega u\rho_r r^2\rho_t\mathrm{d}r\mathrm{d}t - \int_0^t\int_\Omega \rho u_r r^2\rho_t\mathrm{d}r\mathrm{d}t - \int_0^t\int_\Omega 2r\rho u\rho_t\mathrm{d}r\mathrm{d}t\\
&\le \frac{1}{2}\int_0^t\int_\Omega r^2\rho_t^2\mathrm{d}r\mathrm{d}t + C\int_0^t\int_\Omega r^2(u_r^2+2u^2)\mathrm{d}r\mathrm{d}t + E^{\frac{1}{2}}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s,
\end{aligned}
$$

which, combined with (3.42), implies

$$
\frac{1}{2}\int_0^t\int_\Omega r^2\rho_t^2\mathrm{d}r\mathrm{d}t \le C\eta\int_0^t\int_\Omega r^2u_t^2\mathrm{d}r\mathrm{d}t + C(E_0+E^{\frac{1}{2}}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s). \tag{3.43}
$$

Multiplying the equation $(1.7)_2$ by r^2u_t, we get

$$
\begin{aligned}
&\int_0^t\int_\Omega \rho r^2u_t^2\mathrm{d}r\mathrm{d}t = -\int_0^t\int_\Omega \rho uu_r r^2u_t\mathrm{d}r\mathrm{d}t - \int_0^t\int_\Omega R\theta\rho_r r^2u_t\mathrm{d}r\mathrm{d}t - \int_0^t\int_\Omega R\rho\theta_r r^2u_t\mathrm{d}r\mathrm{d}t +\\
&\int_0^t\int_\Omega\left(\left(\frac{4}{3}\mu(\theta)+\lambda(\theta)\right)\left(u_{rr}+\frac{2}{r}u_r-\frac{2}{r^2}u\right)+\lambda'(\theta)\theta_r\left(u_r+\frac{2}{r}u\right)+\frac{4}{3}\mu'(\theta)\theta_r\left(u_r-\frac{u}{r}\right)\right)r^2u_t.
\end{aligned} \tag{3.44}
$$

We estimate each term in (3.44) as follows. First, it is easy to see that

$$\int_0^t \int_\Omega \rho u u_r r^2 u_t \mathrm{d}r\mathrm{d}t \le E^{\frac12}(t) \int_0^t \mathcal{D}(s)\mathrm{d}s.$$

Using the boundary condition $u|_{\partial\Omega}=0$, (3.42) and (3.43), one has

$$\begin{aligned}
&-\int_0^t \int_\Omega R\theta\rho_r r^2 u_t \mathrm{d}r\mathrm{d}t\\
&= -\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega R\theta\rho_r r^2 u \mathrm{d}r\mathrm{d}t + \int_0^t \int_\Omega R\theta_t\rho_r r^2 u \mathrm{d}t\mathrm{d}t + \int_0^t \int_\Omega R\theta\rho_{rt} r^2 u \mathrm{d}r\mathrm{d}t\\
&\le -\int_0^t \int_\Omega R\theta\rho_t (r^2u)_r \mathrm{d}r\mathrm{d}t + C(E_0 + E^{\frac12}(t)\int_0^t \mathcal{D}(s)\mathrm{d}s)\\
&\le C\eta \int_0^t \int_\Omega r^2 u_t^2 \mathrm{d}r\mathrm{d}t + C(E_0 + E^{\frac12}(t)\int_0^t \mathcal{D}(s)\mathrm{d}s).
\end{aligned}$$

Using (3.40), we have

$$\int_0^t \int_\Omega R\rho\theta_r r^2 u_t \mathrm{d}r\mathrm{d}t \le \frac12 \int \rho r^2 u_t^2 \mathrm{d}r\mathrm{d}t + C(E_0 + E^{\frac12}(t)\int_0^t \mathcal{D}(s)\mathrm{d}s).$$

For the last term of (3.44), one has

$$\begin{aligned}
&\int_0^t \int_\Omega \Bigg(\left(\frac43\mu(\theta)+\lambda(\theta)\right)\left(u_{rr} + \frac{2}{r}u_r - \frac{2}{r^2}u\right) + \lambda'(\theta)\theta_r\left(u_r + \frac{2}{r}u\right)\\
&\qquad\qquad + \frac43\mu'(\theta)\theta_r\left(u_r - \frac{u}{r}\right)\Bigg) r^2 u_t \mathrm{d}r\mathrm{d}t\\
&\le \int_0^t \int_\Omega \left(\frac43\mu(\theta)+\lambda(\theta)\right)\left((r^2u_r)_r - 2u\right) u_t \mathrm{d}r\mathrm{d}t + CE^{\frac12}(t)\int_0^t \mathcal{D}(s)\mathrm{d}s\\
&\le -\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac43\mu(\theta)+\lambda(\theta)\right)(\frac12 r^2u_r^2 + u^2)\mathrm{d}r\mathrm{d}t + CE^{\frac12}(t)\int_0^t \mathcal{D}(s)\mathrm{d}s\\
&\le C(E_0 + E^{\frac12}(t)\int_0^t \mathcal{D}(s)\mathrm{d}s).
\end{aligned}$$

Now, we can choose η sufficiently small and get

$$\int_0^t \int_\Omega r^2 u_t^2 \mathrm{d}r\mathrm{d}t \le C(E_0 + E^{\frac12}(t)\int_0^t \mathcal{D}(s)\mathrm{d}s). \tag{3.45}$$

This together with (3.42) and (3.43) imply

$$\int_0^t \int_\Omega ((r^2u_r^2 + 2u^2) + r^2\rho_t^2)\mathrm{d}r\mathrm{d}t \le C(E_0 + E^{\frac12}(t)\int_0^t \mathcal{D}(s)\mathrm{d}s). \tag{3.46}$$

Until now, we have get the dissipation estimates of $r(u_t, u_r, \rho_t, \theta_r)$ and u. Multiplying the momentum equation $(1.7)_2$ by $r^2\rho_r$, using (3.40), (3.45) and mass equation $(1.7)_1$, we derive that

$$\begin{aligned}
\int_0^t\int_\Omega R\theta r^2\rho_r^2\mathrm{d}r\mathrm{d}t =& -\int_0^t\int_\Omega (\rho u_t+\rho u u_r+R\rho\theta_r)\, r^2\rho_r\mathrm{d}r\mathrm{d}t\\
&+\int_0^t\int_\Omega\Big(\Big(\frac{4}{3}\mu(\theta)+\lambda(\theta)\Big)\Big(u_{rr}+\frac{2}{r}u_r-\frac{2}{r^2}u\Big)+\lambda'(\theta)\theta_r\Big(u_r+\frac{2}{r}u\Big)\\
&\qquad+\frac{4}{3}\mu'(\theta)\theta_r\Big(u_r-\frac{u}{r}\Big)\Big)\, r^2\rho_r\mathrm{d}r\mathrm{d}t\\
\le& \frac{1}{2}\int_0^t\int_\Omega R\theta r^2\rho_r^2\mathrm{d}r\mathrm{d}t+\int_0^t\int_\Omega\Big(\frac{4}{3}\mu(\theta)+\lambda(\theta)\Big)\Big(u_{rr}+\frac{2}{r}u_r-\frac{2}{r^2}u\Big)\, r^2\rho_r\mathrm{d}r\mathrm{d}t\\
&+C(E_0+E^{\frac{1}{2}}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s)\\
=& \frac{1}{2}\int_0^t\int_\Omega R\theta r^2\rho_r^2\mathrm{d}r\mathrm{d}t+\int_0^t\int_\Omega\Big(\frac{4}{3}\mu(\theta)+\lambda(\theta)\Big)\Big(-\frac{1}{\rho}\Big)\Big(\rho_{tr}+u\rho_{rr}+2\rho_r u_r+\frac{2}{r}u\rho_r\Big)\, r^2\rho_r\mathrm{d}r\mathrm{d}t\\
&+C(E_0+E^{\frac{1}{2}}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s)\\
\le& \frac{1}{2}\int_0^t\int_\Omega R\theta r^2\rho_r^2\mathrm{d}r\mathrm{d}t-\int_0^t\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\Big(\frac{4}{3}\mu(\theta)+\lambda(\theta)\Big)\frac{1}{2\rho}\rho_r^2\mathrm{d}r\mathrm{d}t+C(E_0+E^{\frac{1}{2}}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s)\\
\le& \frac{1}{2}\int_0^t\int_\Omega R\theta r^2\rho_r^2\mathrm{d}r\mathrm{d}t+C(E_0+E^{\frac{1}{2}}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s),
\end{aligned}$$

which gives

$$\int_0^t\int_\Omega r^2\rho_r^2\mathrm{d}r\mathrm{d}t\le C(E_0+E^{\frac{1}{2}}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s). \tag{3.47}$$

Finally, using the energy equation $(1.7)_3$, (3.46) and Lemma 3.4, we have

$$\begin{aligned}
\int_0^t\int_\Omega r^2\theta_t^2\mathrm{d}r\mathrm{d}t &\le \int_0^t\int_\Omega (r^2u_r^2+2u^2+r^2q_r^2+2q^2)\mathrm{d}r\mathrm{d}t+CE^{\frac{1}{2}}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s\\
&\le C(E_0+E^{\frac{1}{2}}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s).
\end{aligned}\tag{3.48}$$

This finishes the proof of Lemma 3.6. □

The following lemmas give the second-order a priori estimates.

Lemma 3.7. *There exists a constant C such that*

$$\begin{aligned}
&\int_\Omega\big(r^2\rho_{tr}^2+(r^2u_{tr}^2+2u_t^2)+r^2\theta_{tr}^2+\tau(r^2q_{tr}^2+2q_t^2)\big)\,\mathrm{d}r+\int_0^t\int_\Omega(r^2q_{tr}^2+2q_t^2)\mathrm{d}r\mathrm{d}t\\
&+\Big(\frac{4}{3}\mu+\lambda\Big)\int_0^t\int_\Omega(r^2u_{trr}^2+u_{tr}^2)\mathrm{d}r\mathrm{d}t\le C(E_0+E^{\frac{1}{2}}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s+E^2(t)).
\end{aligned}\tag{3.49}$$

Proof. Taking derivative with respect to (t,r), one gets

$$\begin{cases}
\rho_{ttr}+u\rho_{trr}+\rho u_{trr}+\frac{2}{r}\rho u_{rt}-\frac{2}{r^2}\rho u_t=H_1,\\
\rho u_{ttr}+\rho u u_{trr}+p_{trr}-\big(\big(\frac{4}{3}\mu(\theta)+\lambda(\theta)\big)\big(u_{rr}+\frac{2}{r}u_r-\frac{2}{r^2}u\big)\big)_{tr}=H_2,\\
\rho e_\theta\theta_{ttr}+B(\rho,u,\theta,q)\theta_{trr}+p\big(u_r+\frac{2}{r}u\big)_{tr}+\big(q_r+\frac{2}{r}q\big)_{tr}=H_3,\\
\tau(\theta)\rho(q_{ttr}+uq_{trr})+q_{tr}+\kappa(\theta)\theta_{trr}=H_4,
\end{cases}\tag{3.50}$$

where

$$
\begin{aligned}
H_1 :=& -\left(\rho_t u_{rr} + 2\rho_r u_{rt} + 2u_r \rho_{tr} + u_t \rho_{rr} - \frac{2}{r^2}\rho_t u + \frac{2}{r} u \rho_{tr} + \frac{2}{r}\rho_t u_r + \frac{2}{r}\rho_r u_t\right), \\
H_2 :=& -\left(\rho_t u_{tr} + \rho_r u_{tt} + u_t \rho_{tr} + (\rho u)_t u_{rr} + (\rho u)_r u_{tr} + (\rho u)_{tr} u_r\right) \\
& \qquad + \left(\lambda'(\theta)\theta_r\left(u_r + \frac{2}{r}u\right) + \frac{4}{3}\mu'(\theta)\theta_r\left(u_r - \frac{u}{r}\right)\right)_{tr}, \\
H_3 :=& -\Bigg((\rho e_\theta)_t \theta_{tr} + (\rho e_\theta)_r \theta_{tt} + (\rho e_\theta)_{tr}\theta_t + B_t \theta_{rr} + B_r \theta_{tr} + B_{tr}\theta_r + p_t (u_r + \frac{2}{r}u)_r \\
& \qquad + p_r (u_r + \frac{2}{r}u)_t + p_{tr}\left((u_r + \frac{2}{r}u)\right)\Bigg) + \left(\left(\frac{2}{\tau(\theta)} + \frac{4}{r}u\right)a(\theta)q^2\right)_{tr} \\
& \qquad + \left(\mu(\theta)\left(2u_r^2 + \frac{4}{r^2}u^2 - \frac{2}{3}\left(u_r + \frac{2}{r}u\right)^2\right) + \lambda(\theta)\left(u_r + \frac{2}{r}u\right)^2\right)_{tr}, \\
H_4 :=& -\Big((\tau(\theta)\rho)_t q_{tr} + (\tau(\theta)\rho)_r q_{tt} + (\tau(\theta)\rho)_{tr} q_t + (\tau(\theta)\rho u)_t q_{rr} + (\tau(\theta\rho u)_r q_{tr} + (\tau(\theta)\rho u)_{tr} q_r \\
& \qquad + \kappa'(\theta)(\theta_t\theta_{rr} + 2\theta_r\theta_{tr}) + \kappa''\theta_t\theta_r^2 + \left(\tau(\theta)\rho\frac{2}{r}uq\right)_{tr}\Big).
\end{aligned}
$$

Multiplying equation $(3.50)_2$ by $r^2 u_{tr}$, one gets

$$
\begin{aligned}
& \int_\Omega \rho u_{ttr} r^2 u_{tr} \mathrm{d}r + \int_\Omega \rho u u_{trr} r^2 u_{tr} \mathrm{d}r \\
& + \int_\Omega \left(p_{trr} - \left(\left(\frac{4}{3}\mu(\theta) + \lambda(\theta)\right)\left(u_{rr} + \frac{2}{r}u_r - \frac{2}{r^2}u\right)\right)_{tr}\right) r^2 u_{tr} \mathrm{d}r = \int_\Omega H_2 r^2 u_{tr} \mathrm{d}r.
\end{aligned}
\tag{3.51}
$$

We estimate each term in the above equation as follows.

First,

$$
\int_\Omega \rho r^2 (u_{ttr} u_{tr} + u u_{trr} u_{tr}) \mathrm{d}r = \int_\Omega \rho r^2 \left(\left(\frac{1}{2}u_{tr}^2\right)_t + u\left(\frac{1}{2}u_{tr}^2\right)_r\right) \mathrm{d}r = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\rho r^2 u_{tr}^2 \mathrm{d}r.
$$

Second, using the boundary condition

$$
\left(p_{rt} - \left(\left(\frac{4}{3}\mu(\theta) + \lambda(\theta)\right)\left(u_{rr} + \frac{2}{r}u_r - \frac{2}{r^2}u\right)\right)_t\right)|_{\partial\Omega} = 0,
$$

we have

$$
\begin{aligned}
& \int_\Omega \left(p_{trr} - \left(\left(\frac{4}{3}\mu(\theta) + \lambda(\theta)\right)\left(u_{rr} + \frac{2}{r}u_r - \frac{2}{r^2}u\right)\right)_{tr}\right) r^2 u_{tr} \mathrm{d}r \\
= & -\int_\Omega \left(p_{tr} - \left(\left(\frac{4}{3}\mu(\theta) + \lambda(\theta)\right)\left(u_{rr} + \frac{2}{r}u_r - \frac{2}{r^2}u\right)\right)_t\right) (r^2 u_{tr})_r \mathrm{d}r.
\end{aligned}
$$

Using equation $(3.50)_1$, we have

$$
\begin{aligned}
&-\int_\Omega p_{tr}(r^2u_{tr})_r \mathrm{d}r\\
&=-\int_\Omega R\left(\theta\rho_{tr}+\rho\theta_{tr}+\theta_t\rho_r+\rho_t\theta_r\right)(r^2u_{tr})_r\mathrm{d}r\\
&=-\int_\Omega R(\theta\rho_{tr}+\rho\theta_{tr})(r^2u_{tr})_r\mathrm{d}r-\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega R(\theta_t\rho_r+\rho_t\theta_r)r^2u_{rr}\mathrm{d}r+\int_\Omega R(\theta_t\rho_r+\rho_t\theta_r)_t r^2u_{rr}\mathrm{d}r\\
&\geq \int_\Omega R\theta\rho_{tr}\left(\frac{1}{\rho}r^2\rho_{ttr}+\frac{u}{\rho}r^2\rho_{trr}-2u_t-\frac{1}{\rho}r^2H_1\right)\mathrm{d}r-\int_\Omega R\rho\theta_{tr}(r^2u_{tr})_r\\
&\qquad\qquad -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega R(\theta_t\rho_r+\rho_t\theta_r)r^2u_{rr}\mathrm{d}r-CE^{\frac12}(t)\mathcal{D}(t)\\
&\geq \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{R\theta}{2\rho}r^2\rho_{tr}^2\mathrm{d}r-\int_\Omega R\left(\left(\frac{\theta}{\rho}r^2\right)_t+\left(\frac{\theta}{\rho}ur^2\right)_r\right)\frac12\rho_{tr}^2\mathrm{d}r-\int_\Omega R\rho\theta_{tr}(r^2u_{tr})_r\mathrm{d}r\\
&\qquad\qquad -\int_\Omega 2R\theta\rho_{tr}u_t\mathrm{d}r-\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega R(\theta_t\rho_r+\rho_t\theta_r)r^2u_{rr}\mathrm{d}r-CE^{\frac12}(t)\mathcal{D}(t)\\
&\geq \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{R\theta}{2\rho}r^2\rho_{tr}^2-R(\theta_t\rho_r+\rho_t\theta_r)r^2u_{rr}\right)\mathrm{d}r-\int_\Omega(R\rho\theta_{tr}(r^2u_{tr})_r+2R\theta\rho_{tr}u_t)\mathrm{d}r-CE^{\frac12}(t)\mathcal{D}(t).
\end{aligned}
$$

On the other hand, using the boundary condition $u_t|_{\partial\Omega}=0$, one has

$$
\begin{aligned}
&\int_\Omega\left(\left(\frac43\mu(\theta)+\lambda(\theta)\right)\left(u_{rr}+\frac2r u_r-\frac{2}{r^2}u\right)\right)_t(r^2u_{tr})_r\mathrm{d}r\\
&\geq\int_\Omega\left(\frac43\mu(\theta)+\lambda(\theta)\right)\left(u_{trr}+\frac2r u_{tr}-\frac{2}{r^2}u_t\right)(r^2u_{trr}+2ru_{tr})\mathrm{d}r-CE^{\frac12}(t)\mathcal{D}(t).\\
&=\int_\Omega\left(\frac43\mu(\theta)+\lambda(\theta)\right)\left(r^2u_{trr}^2+4u_{tr}^2+4ru_{trr}u_{tr}-2u_tu_{trr}-\frac4r u_tu_{tr}\right)\mathrm{d}r-CE^{\frac12}(t)\mathcal{D}(t).\\
&\geq\int_\Omega\left(\frac43\mu(\theta)+\lambda(\theta)\right)\left(\frac15 r^2u_{trr}^2+u_{tr}^2-\frac{2}{r^2}u_t^2\right)\mathrm{d}r-CE^{\frac12}(t)\mathcal{D}(t).
\end{aligned}
$$

Thus, we derive that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{R\theta}{2\rho}r^2\rho_{tr}^2-R(\theta_t\rho_r+\rho_t\theta_r)r^2u_{rr}+\frac12\rho r^2u_{tr}^2\right)\mathrm{d}r-\int_\Omega R\rho\theta_{tr}(r^2u_{tr})_r\mathrm{d}r\\
&-\int_\Omega 2R\theta\rho_{tr}u_t\mathrm{d}r+\int_\Omega\left(\frac43\mu(\theta)+\lambda(\theta)\right)\left(\frac15r^2u_{trr}^2+u_{tr}^2-\frac{2}{r^2}u_t^2\right)\mathrm{d}r\leq CE^{\frac12}(t)\mathcal{D}(t).
\end{aligned}
\tag{3.52}
$$

Multiplying equation $(3.50)_3$ by $\frac{1}{\theta}r^2\theta_{tr}$, and integrating the results, we obtain

$$
\begin{aligned}
&\int_\Omega\frac{\rho e_\theta}{\theta}\theta_{ttr}r^2\theta_{tr}\mathrm{d}r+\int_\Omega\frac{B}{\theta}\theta_{trr}r^2\theta_{tr}\mathrm{d}r+\int_\Omega R\rho\left(u_r+\frac2r u\right)_{rt}r^2\theta_{tr}\mathrm{d}r\\
&\qquad+\int_\Omega\frac1\theta\left(q_r+\frac2r q\right)_{rt}r^2\theta_{tr}\mathrm{d}r=\int_\Omega H_3\frac1\theta r^2\theta_{tr}\mathrm{d}r.
\end{aligned}
\tag{3.53}
$$

We estimate each term in the above equation as follows. First,

$$
\int_\Omega\frac{\rho e_\theta}{\theta}r^2\theta_{ttr}\theta_{tr}\mathrm{d}r\geq\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{\rho e_\theta}{2\theta}r^2\theta_{tr}^2\mathrm{d}r-CE^{\frac12}(t)\mathcal{D}(t).
$$

Using the fact that $B|_{\partial\Omega}=0$, one has

$$\int_\Omega \frac{B}{\theta}\theta_{trr}r^2\theta_{tr}\mathrm{d}r = -\int_\Omega \left(\frac{B}{\theta}r^2\right)_r \frac{1}{2}\theta_{tr}^2\mathrm{d}r \geq -CE^{\frac{1}{2}}(t)\mathcal{D}(t).$$

Second,

$$\begin{aligned}\int_\Omega R\rho\left(u_r+\frac{2}{r}u\right)_{rt} r^2\theta_{tr}\mathrm{d}r &= \int_\Omega R\rho\left(u_{trr}+\frac{2}{r}u_{tr}-\frac{2}{r^2}u_t\right)r^2\theta_{tr}\mathrm{d}r\\ &= \int_\Omega R\rho(r^2u_{rt})_r\theta_{tr}\mathrm{d}r - \int_\Omega 2R\rho u_t\theta_{tr}\mathrm{d}r.\end{aligned}$$

Similarly, one get

$$\int_\Omega \frac{1}{\theta}\left(q_r+\frac{2}{r}q\right)_{rt} r^2\theta_{tr}\mathrm{d}r = \int_\Omega \frac{1}{\theta}(r^2q_{tr})_r\theta_{tr}\mathrm{d}r - \int_\Omega \frac{2}{\theta}q_t\theta_{tr}\mathrm{d}r.$$

One should pay attention to the term

$$\int_\Omega H_3\frac{1}{\theta}r^2\theta_{tr}\mathrm{d}r,$$

since there will appear third-order terms. The typical third-order term can be dealt with as follows:

$$\begin{aligned}&\int_\Omega \left(\left(\frac{4}{3}\mu(\theta)+\lambda(\theta)\right)u_r^2\right)_{tr} r^2\theta_{tr}\mathrm{d}r\\ &\leq C\int_\Omega \left(\frac{4}{3}\mu(\theta)+\lambda(\theta)\right)(u_ru_{trr}+u_{tr}u_{rr})r^2\theta_{tr}\mathrm{d}r\\ &\leq \eta\int_\Omega \left(\frac{4}{3}\mu(\theta)+\lambda(\theta)\right)r^2u_{trr}^2\mathrm{d}r + C(\eta)E^{\frac{1}{2}}(t)\mathcal{D}(t),\end{aligned}$$

where we use Young's inequality and the following estimate,

$$\begin{aligned}\|\left(\frac{4}{3}\mu(\theta)+\lambda(\theta)\right)r^2u_{tr}u_{rr}\theta_{tr}\|_{L^1} &\leq C\|\sqrt{\left(\frac{4}{3}\mu(\theta)+\lambda(\theta)\right)}u_{tr}\|_{L^\infty}\|ru_{rr}\|_{L^2}\|r\theta_{tr}\|_{L^2}\\ &\leq CE^{\frac{1}{2}}\mathcal{D}(t).\end{aligned}$$

Thus, we conclude that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\rho e_\theta}{2\theta}r^2\theta_{tr}^2\mathrm{d}t + \int_\Omega R\rho(r^2u_{tr})_r\theta_{tr}\mathrm{d}r - \int_\Omega 2R\rho u_t\theta_{tr}\mathrm{d}r + \int_\Omega \frac{1}{\theta}(r^2q_{rt})_r\theta_{tr}\mathrm{d}r\\ -\eta\int_\Omega \left(\frac{4}{3}\mu(\theta)+\lambda(\theta)\right)r^2u_{trr}^2\mathrm{d}r - \int_\Omega \frac{2}{\theta}q_t\theta_{tr}\mathrm{d}r \leq CE^{\frac{1}{2}}(t)\mathcal{D}(t).\end{aligned} \tag{3.54}$$

Multiplying equation $(3.50)_4$ by $\frac{1}{\kappa(\theta)\theta}r^2q_{tr}$, one gets

$$\int_\Omega \frac{\tau(\theta)}{\kappa(\theta)\theta}r^2q_{tr}\rho(q_{ttr}+uq_{trr})\mathrm{d}r + \int_\Omega \frac{1}{\kappa(\theta)\theta}r^2q_{tr}^2\mathrm{d}r + \int_\Omega \frac{1}{\theta}r^2q_{tr}\theta_{trr}\mathrm{d}r = \int_\Omega \frac{1}{\kappa(\theta)\theta}r^2q_{tr}H_4. \tag{3.55}$$

We estimate again each term in the above equation separately. First, using the mass equation, one has

$$\begin{aligned}\int_\Omega \frac{\tau(\theta)}{\kappa(\theta)\theta} r^2 q_{tr}\rho(q_{ttr}+uq_{trr})\mathrm{d}r &= \int_\Omega \frac{\tau(\theta)}{\kappa(\theta)\theta}\rho r^2((\frac{1}{2}q_{tr}^2)_t + u(\frac{1}{2}q_{tr}^2)_r)\mathrm{d}r \\ &\geq \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\tau(\theta)}{\kappa(\theta)\theta}\rho r^2 \frac{1}{2} q_{tr}^2 \mathrm{d}r - CE^{\frac{1}{2}}(t)\mathcal{D}(t).\end{aligned}$$

Second, using the fact that $\theta_{tr}|_{\partial\Omega} = 0$, we have

$$\int_\Omega \frac{1}{\theta} r^2 q_{tr}\theta_{trr}\mathrm{d}r \geq -\int_\Omega \frac{1}{\theta}(r^2 q_{tr})_r \theta_{tr}\mathrm{d}r - CE^{\frac{1}{2}}(t)\mathcal{D}(t).$$

Thus, we derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\tau(\theta)}{2\kappa(\theta)\theta}\rho r^2 q_{tr}^2\mathrm{d}r - \int_\Omega \frac{1}{\theta}(r^2 q_{tr})_r\theta_{tr}\mathrm{d}r + \int_\Omega \frac{1}{\kappa(\theta)\theta} r^2 q_{tr}^2 \mathrm{d}r \leq CE^{\frac{1}{2}}(t)\mathcal{D}(t). \tag{3.56}$$

Combining (3.52), (3.54) and (3.56), and choosing η sufficiently small, we derive

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{R\theta}{2\rho} r^2\rho_{tr}^2 - R(\theta_t\rho_r + \rho_t\theta_r)r^2 u_{rr} + \frac{1}{2}\rho r^2 u_{tr}^2 + \frac{\rho e_\theta}{2\theta} r^2\theta_{tr}^2 + \frac{\tau(\theta)}{2\kappa(\theta)\theta}\rho r^2 q_{tr}^2\right)\mathrm{d}r \\ &- \int_\Omega \left(2Ru_t(\theta\rho_{tr} + \rho\theta_{tr}) + \frac{2}{\theta}q_t\theta_{tr}\right)\mathrm{d}r + \int_\Omega \frac{1}{\kappa(\theta)\theta} r^2 q_{tr}^2\mathrm{d}r \\ &+ \int_\Omega \left(\frac{4}{3}\mu(\theta)+\lambda(\theta)\right)\left(\frac{1}{10}r^2 u_{trr}^2 + u_{tr}^2 - \frac{2}{r^2}u_t^2\right)\mathrm{d}r \leq CE^{\frac{1}{2}}(t)\mathcal{D}(t). \end{aligned}\tag{3.57}$$

From the momentum equation, one has

$$\begin{aligned}&\rho u_{tt} + \rho u u_{tr} + p_{tr} = -(\rho_t u_t + (\rho u)_t u_r) \\ &+ \left(\left(\frac{4}{3}\mu(\theta)+\lambda(\theta)\right)\left(u_{rr} + \frac{2}{r}u_r - \frac{2}{r^2}u\right) + \lambda'(\theta)\theta_r\left(u_r + \frac{2}{r}u\right) + \frac{4}{3}\mu'(\theta)\theta_r\left(u_r - \frac{u}{r}\right)\right)_t.\end{aligned}$$

Multiplying the above equation by $2u_t$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho u_t^2\mathrm{d}r + \int_\Omega 2u_t R(\rho\theta_{tr} + \theta\rho_{tr})\mathrm{d}r + \int_\Omega \left(\frac{4}{3}\mu(\theta)+\lambda(\theta)\right)\left(2u_{tr}^2 + \frac{2}{r^2}u_t^2\right)\mathrm{d}r \leq CE^{\frac{1}{2}}(t)\mathcal{D}(t), \tag{3.58}$$

where we use the fact that

$$\begin{aligned}&\int_\Omega \left(\left(\frac{4}{3}\mu(\theta)+\lambda(\theta)\right)\left(u_{rr} + \frac{2}{r}u_r - \frac{2}{r^2}u\right) + \lambda'(\theta)\theta_r\left(u_r + \frac{2}{r}u\right) + \frac{4}{3}\mu'(\theta)\theta_r\left(u_r - \frac{u}{r}\right)\right)_t 2u_t\mathrm{d}r \\ &\leq -\int_\Omega \left(\frac{4}{3}\mu(\theta)+\lambda(\theta)\right)\left(2u_{tr}^2 + \frac{2}{r^2}u_t^2\right)\mathrm{d}r + CE^{\frac{1}{2}}(t)\mathcal{D}(t).\end{aligned}$$

Similarly, taking the derivative with respect to t in equation $(1.7)_4$, and multiplying the result by $\frac{2}{\kappa(\theta)\theta}q_t$, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\tau(\theta)}{\kappa(\theta)\theta}\rho q_t^2\mathrm{d}r + \int_\Omega \frac{2}{\kappa(\theta)\theta}q_t^2\mathrm{d}r + \int_\Omega \frac{2}{\theta}\theta_{rt}q_t\mathrm{d}r \leq CE^{\frac{1}{2}}(t)\mathcal{D}(t). \tag{3.59}$$

Therefore, by using (3.57), (3.58) and (3.59), one derives

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{R\theta}{2\rho}r^2\rho_{tr}^2-R(\theta_t\rho_r+\rho_t\theta_r)r^2u_{rr}+\frac{1}{2}\rho(r^2u_{tr}^2+2u_t^2)+\frac{\rho e_\theta}{2\theta}r^2\theta_{tr}^2+\frac{\tau(\theta)}{2\kappa(\theta)\theta}\rho(r^2q_{tr}^2+2q_t^2)\right)\mathrm{d}r\\&\quad+\int_\Omega\left(\frac{4}{3}\mu(\theta)+\lambda(\theta)\right)\left(\frac{1}{10}r^2u_{trr}^2+3u_{tr}^2\right)\mathrm{d}r+\int_\Omega\frac{1}{\kappa(\theta)\theta}(r^2q_{tr}^2+2q_t^2)\mathrm{d}r\le CE^{\frac{1}{2}}(t)\mathcal{D}(t).\end{aligned}\tag{3.60}$$

Integrating this over $(0,t)$, we get the desired result. The proof of Lemm 3.7 is finished. □

Similarly, we can derive

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{1}{2}\rho r^2u_{tt}^2+\frac{R\theta}{2\rho}r^2\rho_{tt}^2+\frac{\rho e_\theta}{2\theta}r^2\theta_{tt}^2+\frac{\tau(\theta)}{2\kappa(\theta)\theta}\rho r^2q_{tt}^2\right)\mathrm{d}r\\&+\int_\Omega\left(\frac{4}{3}\mu(\theta)+\lambda(\theta)\right)\left(r^2u_{ttr}^2\right)\mathrm{d}r+\int_\Omega\frac{1}{\kappa(\theta)\theta}r^2q_{tt}^2\mathrm{d}r\le\frac{C}{\tau}E^{\frac{1}{2}}(t)\mathcal{D}(t).\end{aligned}\tag{3.61}$$

Note that the factor $\frac{1}{\tau}$ in (3.61) arises from the τ^2 weight assigned to $\|rq_{tt}\|_{L^2}^2$ in the definition of $\mathcal{D}(t)$. Furthermore, motivated by the definition of $E(t)$, we integrate (3.61) and multiply the resulting equation by τ^2 to establish the following lemma.

Lemma 3.8. *There exists a constant C such that*

$$\begin{aligned}&\int_\Omega\tau^2\left(r^2\rho_{tt}^2+r^2u_{tt}^2+r^2\theta_{tt}^2+\tau r^2q_{tt}^2\right)\mathrm{d}r+\int_0^t\int_\Omega\tau^2r^2q_{tt}^2\mathrm{d}r\mathrm{d}t\\&+\left(\frac{4}{3}\mu+\lambda\right)\int_0^t\int_\Omega\left(r^2u_{ttr}^2\right)\mathrm{d}r\mathrm{d}t\le C(E_0+E^{\frac{1}{2}}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s).\end{aligned}\tag{3.62}$$

Furthermore, we have

Lemma 3.9. *There exists a constant C such that*

$$\int_\Omega\left(r^2\rho_{rr}^2+(r^2u_{rr}^2+2u_r^2)+r^2\theta_{rr}^2+\tau(r^2q_{rr}^2+2q_r^2)\right)\mathrm{d}r\le C(E_0+E^{\frac{1}{2}}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s+E^2(t)).\tag{3.63}$$

and

$$\int_\Omega\left(\frac{4}{3}\mu+\lambda\right)^2r^2(u_{rrr}^2+\tau^2u_{trr}^2)\mathrm{d}r\le C(E_0+E^{\frac{1}{2}}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s+E^2(t)).\tag{3.64}$$

Proof. From equation $(3.27)_1$, we have

$$\int_\Omega\rho^2r^2\left(u_{rr}+\frac{2}{r}u_r-\frac{2}{r^2}u\right)^2\mathrm{d}r=\int_\Omega r^2\left(-\rho_{tr}-u\rho_{rr}+G_1\right)^2\mathrm{d}r.\tag{3.65}$$

For the left-hand side of the above equation, one has

$$\begin{aligned}&\int_\Omega\rho^2r^2\left(u_{rr}+\frac{2}{r}u_r-\frac{2}{r^2}u\right)^2\mathrm{d}r\\&=\int_\Omega\rho^2r^2\left(u_{rr}^2+\frac{4}{r^2}u_r^2+\frac{4}{r^4}u^2+\frac{4}{r}u_ru_{rr}-\frac{4}{r^2}uu_{rr}-\frac{8}{r^3}uu_r\right)\mathrm{d}r\\&=\int_\Omega\rho^2\left(r^2u_{rr}^2+8u_r^2+4u_ru_{rr}\right)\mathrm{d}r\ge\frac{1}{4}\int_\Omega\left(\frac{1}{3}r^2u_{rr}^2+2u_r^2\right)\mathrm{d}r,\end{aligned}$$

where we used the ε-Young inequality

$$4ru_r u_{rr} \le \varepsilon (ru_{rr})^2 + \frac{1}{4\varepsilon}(4u_r)^2,$$

and take $\varepsilon := \frac{2}{3}$.

For the right hand side of (3.63), using Lemma 3.7, one has

$$\int_\Omega r^2 \left(-\rho_{tr} - u\rho_{rr} + G_1\right)^2 \mathrm{d}r \le C(E_0 + E^{\frac{1}{2}}(t)\int_0^t \mathcal{D}(s)\mathrm{d}s + E^2(t)),$$

where we use the fact

$$\int_\Omega r^2 G_1^2 \mathrm{d}r \le \int_\Omega r^2(-2\rho_r u_r - \frac{2}{r}\rho_r u)^2 \mathrm{d}r \le CE^2(t).$$

Thus, we derive that

$$\int_\Omega (r^2 u_{rr}^2 + u_r^2)\mathrm{d}r \le C(E_0 + E^{\frac{1}{2}}(t)\int_0^t \mathcal{D}(s)\mathrm{d}s + E^2(t)). \tag{3.66}$$

From equation $(3.27)_4$, using Lemmas 3.3-3.7, one immediately gets

$$\begin{aligned}\int_\Omega \kappa^2(\theta) r^2 \theta_{rr}^2 \mathrm{d}r &\le C\int_\Omega \left(\tau^2 r^2 q_{tr}^2 + \tau^2 u^2 r^2 q_{rr}^2 + r^2 q_r^2 + r^2 G_4^2\right)\mathrm{d}r \\ &\le C(E_0 + E^{\frac{1}{2}}(t)\int_0^t \mathcal{D}(s)\mathrm{d}s + E^2(t)). \end{aligned} \tag{3.67}$$

Multiplying the momentum equation $(3.27)_2$ by $r^2\rho_{rr}$, using (3.67), Lemma 3.7 and the mass equation $(1.7)_1$, we derive

$$\begin{aligned}
\int_\Omega R\theta r^2 \rho_{rr}^2 \mathrm{d}r =& -\int_\Omega \left(\rho u_{tr} + \rho u u_{rr} + R\rho\theta_{rr} + 2R\rho_r\theta_r - G_2\right) r^2\rho_{rr}\mathrm{d}r \\
&+ \int_\Omega \left((\frac{4}{3}\mu(\theta) + \lambda(\theta)) \left(u_{rr} + \frac{2}{r}u_r - \frac{2}{r^2}u\right)\right)_r r^2\rho_{rr}\mathrm{d}r \\
\le& \frac{1}{2}\int_\Omega R\theta r^2\rho_{rr}^2\mathrm{d}r + \int_\Omega \left(\frac{4}{3}\mu(\theta) + \lambda(\theta)\right)\left(u_{rr} + \frac{2}{r}u_r - \frac{2}{r^2}u\right)_r r^2\rho_{rr}\mathrm{d}r \\
&+ C(E_0 + E^{\frac{1}{2}}(t)\int_0^t \mathcal{D}(s)\mathrm{d}s + E^2(t)) \\
=& \frac{1}{2}\int_\Omega R\theta r^2\rho_{rr}^2\mathrm{d}r + \int_\Omega \left(\frac{4}{3}\mu(\theta) + \lambda(\theta)\right)\left(\left(-\frac{1}{\rho}\right)\left(\rho_{tr} + u\rho_{rr} + 2\rho_r u_r + \frac{2}{r}u\rho_r\right)\right)_r r^2\rho_{rr}\mathrm{d}r \\
&+ C(E_0 + E^{\frac{1}{2}}(t)\int_0^t \mathcal{D}(s)\mathrm{d}s + E^2(t)) \\
\le& \frac{1}{2}\int_\Omega R\theta r^2\rho_{rr}^2\mathrm{d}r - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{4}{3}\mu(\theta) + \lambda(\theta)\right)\frac{1}{2\rho}\rho_{rr}^2\mathrm{d}r + C(E_0 + E^{\frac{1}{2}}(t)\int_0^t \mathcal{D}(s)\mathrm{d}s + E^2(t)).
\end{aligned}$$

Therefore, Gronwall's inequality implies that

$$\int_\Omega r^2\rho_{rr}^2 \mathrm{d}r \le C(E_0 + E^{\frac{1}{2}}(t)\int_0^t \mathcal{D}(s)\mathrm{d}s + E^2(t)). \tag{3.68}$$

Finally, from equation $(3.27)_3$, one has

$$\int_\Omega r^2\left(q_r+\frac{2}{r}q\right)_r^2\mathrm{d}r=\int_\Omega r^2\left(G_3-\rho e_\theta\theta_{tr}-B\theta_{rr}-p\left(u_r+\frac{2}{r}u\right)_r\right)^2\mathrm{d}r. \tag{3.69}$$

Similar to (3.65), we have

$$\int_\Omega r^2\left(q_r+\frac{2}{r}q\right)_r^2\mathrm{d}r\geq\int_\Omega(\frac{1}{3}r^2q_{rr}^2+2q_r^2)\mathrm{d}r.$$

While, using (3.66) and Lemma 3.7, one has

$$\begin{aligned}&\int_\Omega r^2\left(G_3-\rho e_\theta\theta_{tr}-B\theta_{rr}-p\left(u_r+\frac{2}{r}u\right)_r\right)^2\mathrm{d}r\\&\leq C\int_\Omega\left(r^2(G_3^2+\theta_{tr}^2+B^2\theta_{rr}^2)+r^2u_{rr}^2+u_r^2\right)\mathrm{d}r\\&\leq C(E_0+E^{\frac{1}{2}}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s+E^2(t)).\end{aligned}$$

So, we derive that

$$\int_\Omega(r^2q_{rr}^2+q_r^2)\mathrm{d}r\leq C(E_0+E^{\frac{1}{2}}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s+E^2(t)). \tag{3.70}$$

By virtue of (3.66)-(3.70), the estimate (3.63) in Lemma 3.9 is verified. Furthermore, combining the second equations of (3.11) and (3.27) with Lemmas 3.3-3.7, we directly derive inequality (3.64). This completes the proof of Lemma 3.9. □

Lemma 3.10. *There exists a constant C such that*

$$\int_0^t\int_\Omega\left(r^2|D^2(\rho,u,\theta)|^2+r^2q_{rr}^2+(u_r^2+u_t^2+q_r^2)\right)\mathrm{d}r\mathrm{d}t\leq C(E_0+E^{\frac{1}{2}}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s+E^2(t)). \tag{3.71}$$

Proof. From $(3.27)_4$ and using Lemma 3.7 and 3.4, we have

$$\begin{aligned}\int_0^t\int_\Omega\kappa^2(\theta)r^2\theta_{rr}^2\mathrm{d}r\mathrm{d}t&=\int_0^t\int_\Omega r^2(G_4-\tau(\theta)\rho(q_{tr}+uq_{rr})-q_r)^2\mathrm{d}r\\&\leq C(E_0+E^{\frac{1}{2}}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s+E^2(t)).\end{aligned} \tag{3.72}$$

Similarly, from equation $(3.11)_4$ and using Lemmas 3.3, 3.7 and 3.8, one gets

$$\int_0^t\int_\Omega r^2\theta_{rt}^2\mathrm{d}r\mathrm{d}t\leq C(E_0+E^{\frac{1}{2}}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s+E^2(t)). \tag{3.73}$$

Multiplying $(3.11)_3$ by $r^2(u_r+\frac{2}{r}u)_t$, using Lemma 3.7 and (3.73), one has

$$\begin{aligned}&\int_0^t\int_\Omega pr^2\left(u_r+\frac{2}{r}u\right)_t^2\mathrm{d}r\mathrm{d}t\\&=-\int_0^t\int_\Omega\left(\rho e_\theta\theta_{tt}+B\theta_{rt}+\left(q_r+\frac{2}{r}q\right)_t-F_3\right)r^2\left(u_r+\frac{2}{r}u\right)_t\mathrm{d}r\mathrm{d}t\\&\leq-\int_0^t\int_\Omega\rho e_\theta\theta_{tt}(r^2u)_{rt}\mathrm{d}r\mathrm{d}t+\frac{1}{2}\int_0^t\int_\Omega pr^2\left(u_r+\frac{2}{r}u\right)_t^2\mathrm{d}r\mathrm{d}t+C(E_0+E^{\frac{1}{2}}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s+E^2(t)).\end{aligned}$$

Using the boundary condition $u|_{\partial\Omega} = 0$, (3.73) and Lemmas 3.3, 3.7, one has

$$
\begin{aligned}
&-\int_0^t\int_\Omega \rho e_\theta \theta_{tt}(r^2u)_{rt}\mathrm{d}r\mathrm{d}t\\
&= -\int_0^t\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho e_\theta\theta_t(r^2u)_{tr}\mathrm{d}r\right)\mathrm{d}t + \int_0^t\int_\Omega (\rho e_\theta)_t\theta_t(r^2u)_{rt}\mathrm{d}r\mathrm{d}t + \int_0^t\int_\Omega \rho e_\theta\theta_t(r^2u)_{ttr}\mathrm{d}r\mathrm{d}t\\
&\le -\int_0^t\int_\Omega \rho e_\theta\theta_{tr}r^2u_{tt}\mathrm{d}r\mathrm{d}t + C(E_0 + E^{\frac12}(t)\int_0^t \mathcal{D}(s)\mathrm{d}s + E^2(t))\\
&\le \eta\int_0^t\int_\Omega r^2u_{tt}^2\mathrm{d}r\mathrm{d}t + C(\eta)(E_0 + E^{\frac12}(t)\int_0^t \mathcal{D}(s)\mathrm{d}s + E^2(t)).
\end{aligned}
$$

On the other hand, using the boundary condition $u_t|_{\partial\Omega} = 0$, we obtain

$$
\begin{aligned}
\int_0^t\int_\Omega pr^2\left(u_r + \frac{2}{r}u\right)_t^2\mathrm{d}r\mathrm{d}t &= \int_0^t\int_\Omega pr^2(u_{rt}^2 + \frac{4}{r}u_tu_{tr} + \frac{4}{r^2}u_t^2)\mathrm{d}r\mathrm{d}t\\
&\ge \int_0^t\int_\Omega p(r^2u_{tr}^2 + 2u_t^2)\mathrm{d}x\mathrm{d}t - CE^{\frac12}(t)\mathcal{D}(t).
\end{aligned}
$$

Therefore, we derived that

$$
\int_0^t\int_\Omega (r^2u_{tr}^2 + 2u_t^2)\mathrm{d}x\mathrm{d}t \le \eta\int_0^t\int_\Omega r^2u_{tt}^2\mathrm{d}r\mathrm{d}t + C(\eta)(E_0 + E^{\frac12}(t)\int_0^t \mathcal{D}(s)\mathrm{d}s + E^2(t)). \tag{3.74}
$$

Multiplying $(3.11)_2$ by r^2u_{tt}, using (3.73), Lemma 3.7, and the boundary condition $u_{tt}|_{\partial\Omega} = 0$, one gets

$$
\begin{aligned}
\int_0^t\int_\Omega \rho r^2u_{tt}^2\mathrm{d}r\mathrm{d}t &= -\int_0^t\int_\Omega (\rho u u_{tr} + R\rho\theta_{tr} + R\theta\rho_{tr} - F_2)\, r^2u_{tt}\mathrm{d}r\mathrm{d}t\\
&\quad + \int_0^t\int_\Omega (\frac{4}{3}\mu(\theta) + \lambda(\theta))\left(u_{rr} + \frac{2}{r}u_r - \frac{2}{r^2}u\right)_t r^2u_{tt}\mathrm{d}r\mathrm{d}t\\
&\le -\int_0^t\int_\Omega R\theta\rho_{rt}r^2u_{tt}\mathrm{d}r\mathrm{d}t + \frac12\int_0^t\int_\Omega \rho r^2u_{tt}^2\mathrm{d}r\mathrm{d}t + CE^{\frac12}(t)\int_0^t \mathcal{D}(s)\mathrm{d}s\\
&\quad - \int_0^t\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{4}{3}\mu(\theta) + \lambda(\theta)\right)\left(\frac12 r^2u_{rt}^2 + u_t^2\right)\mathrm{d}r\mathrm{d}t\\
&\le -\int_0^t\int_\Omega R\theta\rho_{rt}r^2u_{tt}\mathrm{d}r\mathrm{d}t + \frac12\int_0^t\int_\Omega \rho r^2u_{tt}^2\mathrm{d}r\mathrm{d}t + C(E_0 + E^{\frac12}(t)\int_0^t \mathcal{D}(s)\mathrm{d}s + E^2(t)).
\end{aligned}
$$

Using equation $(3.11)_1$, Lemmas 3.7-3.8 and (3.74), we have

$$
\begin{aligned}
&-\int_0^t\int_\Omega R\theta\rho_{rt}r^2u_{tt}\mathrm{d}r\mathrm{d}t=\int_0^t\int_\Omega R\theta\left((\rho u)_r+\frac{2}{r}\rho u\right)_r r^2u_{tt}\mathrm{d}r\mathrm{d}t\\
&\le\int_0^t\int_\Omega R\theta\rho(u_{rr}+2u_r)r^2u_{tt}\mathrm{d}r\mathrm{d}t+CE^{\frac12}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s\\
&\le\int_0^t\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega R\theta\rho(r^2u_r)_r u_t\mathrm{d}r\mathrm{d}t-\int_0^t\int_\Omega R\theta\rho(r^2u_r)_{rt}u_t\mathrm{d}r\mathrm{d}t+CE^{\frac12}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s\\
&\le\int_0^t\int_\Omega R\theta\rho(r^2u_r)_t u_{rt}\mathrm{d}r\mathrm{d}t+C(E_0+E^{\frac12}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s+E^2(t))\\
&\le C\eta\int_0^t\int_\Omega r^2u_{tt}^2\mathrm{d}r\mathrm{d}t+C(\eta)(E_0+E^{\frac12}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s+E^2(t)).
\end{aligned}
$$

Therefore, by choosing η sufficiently small, we get

$$
\int_0^t\int_\Omega(r^2(u_{tr}^2+u_{tt}^2)+u_t^2)\mathrm{d}r\mathrm{d}t\le C(E_0+E^{\frac12}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s+E^2(t)). \tag{3.75}
$$

Multiplying the momentum equation $(3.11)_2$ by $r^2\rho_{tr}$, using (3.73), (3.75) and Lemma 3.7, one has

$$
\begin{aligned}
\int_0^t\int_\Omega R\theta r^2\rho_{tr}^2\mathrm{d}r\mathrm{d}t=&-\int_0^t\int_\Omega(\rho u_{tt}+\rho u u_{tr}+R\rho\theta_{tr}+F_2)r^2\rho_{tr}\mathrm{d}r\mathrm{d}t\\
&+\int_0^t\int_\Omega(\frac43\mu(\theta)+\lambda(\theta))\left(u_{rr}+\frac{2}{r}u_r-\frac{2}{r^2}u\right)_t r^2\rho_{tr}\mathrm{d}r\mathrm{d}t\\
\le&\frac12\int_0^t\int_\Omega R\theta r^2\rho_{tr}^2\mathrm{d}r\mathrm{d}t+C(E_0+E^{\frac12}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s+E^2(t))\\
&-\int_0^t\int_\Omega\left(\frac43\mu(\theta)+\lambda(\theta)\right)\frac{1}{\rho}(\rho_{tr}+u\rho_{rr}+2\rho_r u_r+\frac{2}{r}\rho_r u)_t r^2\rho_{tr}\mathrm{d}r\mathrm{d}t\\
\le&\frac12\int_0^t\int_\Omega R\theta r^2\rho_{tr}^2\mathrm{d}r\mathrm{d}t+C(E_0+E^{\frac12}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s+E^2(t))\\
&-\int_0^t\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac43\mu(\theta)+\lambda(\theta)\right)\frac{1}{2\rho}r^2\rho_{tr}^2\mathrm{d}r\mathrm{d}t\\
\le&\frac12\int_0^t\int_\Omega R\theta r^2\rho_{tr}^2\mathrm{d}r\mathrm{d}t+C(E_0+E^{\frac12}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s+E^2(t)),
\end{aligned}
$$

which gives

$$
\int_0^t\int_\Omega R\theta r^2\rho_{tr}^2\mathrm{d}r\mathrm{d}t\le C(E_0+E^{\frac12}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s+E^2(t)). \tag{3.76}
$$

In a similar way, we can derive that

$$
\int_0^t\int_\Omega R\theta r^2\rho_{rr}^2\mathrm{d}r\mathrm{d}t\le C(E_0+E^{\frac12}(t)\int_0^t\mathcal{D}(s)\mathrm{d}s+E^2(t)). \tag{3.77}
$$

Similarly, $(3.11)_1$, $(3.27)_1$ together with (3.75) and (3.76) imply

$$\int_0^t \int_\Omega (r^2(u_{rr}^2 + \rho_{tt}^2) + u_r^2) \mathrm{d}r\mathrm{d}t \le C(E_0 + E^{\frac{1}{2}}(t) \int_0^t \mathcal{D}(s)\mathrm{d}s + E^2(t)). \tag{3.78}$$

Finally, using the energy equation $(3.27)_3$ and the above estimates, we have

$$\begin{aligned}\int_0^t \int_\Omega (r^2 q_{rr}^2 + q_r^2)\mathrm{d}r\mathrm{d}t &\le \int_0^t \int_\Omega (r^2(\theta_{tr}^2 + u_{rr}^2) + u_r^2)\mathrm{d}r\mathrm{d}t + CE^{\frac{1}{2}}(t)\int_0^t \mathcal{D}(s)\mathrm{d}s \\ &\le C(E_0 + E^{\frac{1}{2}}(t)\int_0^t \mathcal{D}(s)\mathrm{d}s + E^2(t)). \end{aligned} \tag{3.79}$$

Thus, the proof of Lemma 3.10 is finished. □

Now, combining Lemmas 3.2 - 3.10 with the choice

$$\delta = \min\left\{\delta_1,\ \left(\frac{1}{2C}\right)^2,\ 1\right\}$$

where C is the universal constant from these lemmas, we complete the proof of Proposition 3.1.

4. Proof of the main theorems

Proof of Theorem 1.2: By the definition of $E(0)$, we know that

$$E(0) \le C_1 \left(\|r(\rho_0 - 1, \theta_0 - 1, q_0)\|_{H^3}^2 + \|ru_0\|_{H^4}^2\right) \le C_1\epsilon_0$$

with C_1 a universal constant independent of τ, λ, μ. Now, choosing ϵ_0 sufficiently small such that

$$C_1 C\epsilon_0 < \frac{\delta}{2},$$

then, by virtue of Proposition 3.1, $E(t) < \frac{\delta}{2}$ which close the a priori assumption $E(t) \le \delta$. Therefore, the estimate (3.1) holds for the local solution (ρ, u, θ, q). By usual continuation methods, the local solution can be extended uniquely to the global one satisfying the estimate (3.1). Thus, the proof of Theorem 1.2 is finished.

Proof of Theorem 1.3 : Fix $\tau > 0$. Let $\epsilon = (\mu, \lambda)$ and $(\rho_\tau^\epsilon, u_\tau^\epsilon, \theta_\tau^\epsilon, q_\tau^\epsilon)$ be the global solutions obtained in Theorem 1.2. Then, we have

$$\sup_{0\le t<\infty} \|(\rho_\tau^\epsilon - 1, u_\tau^\epsilon, \theta_\tau^\epsilon - 1, q_\tau^\epsilon)\|_{H^2}^2 + \int_0^\infty \|D(\rho_\tau^\epsilon, u_\tau^\epsilon, \theta_\tau^\epsilon, q_\tau^\epsilon)\|_{H^1}^2 \mathrm{d}t \le CE(0), \tag{4.1}$$

where C is a constant independent of ϵ. Thus, there exists $(\rho_\tau^0, u_\tau^0, \theta_\tau^0, q_\tau^0) \in L^\infty([0,\infty), H^2)$ such that

$$(\rho_\tau^\epsilon, u_\tau^\epsilon, \theta_\tau^\epsilon, q_\tau^\epsilon) \to (\rho_\tau^0, u_\tau^0, \theta_\tau^0, q_\tau^0) \quad \text{weak} - * \quad \text{in } L^\infty([0,\infty), H^2)$$

Furthermore, since $\partial_t(\rho_\tau^\epsilon, u_\tau^\epsilon, \theta_\tau^\epsilon, q_\tau^\epsilon)$ are bounded in $L^2([0,\infty); H^1)$, by classical compactness argument, $(\rho_\tau^\epsilon, u_\tau^\epsilon, \theta_\tau^\epsilon, q_\tau^\epsilon)$ are relatively compact in $C([0,T], H_{loc}^{2-\delta_0})$ for any $T > 0$ and $0 < \delta_0 < 2$. As a consequence, as $\epsilon \to 0$ and up to subsequences,

$$(\rho_\tau^\epsilon, u_\tau^\epsilon, \theta_\tau^\epsilon, q_\tau^\epsilon) \to (\rho_\tau^0, u_\tau^0, \theta_\tau^0, q_\tau^0) \quad \text{strongly} \quad \text{in } C^0([0,T], H^{2-\delta_0})$$

On the other hand, quantities involving $\epsilon = (\mu, \lambda)$, such as $\left(\left(\frac{4}{3}\mu(\theta) + \lambda(\theta)\right)\left(u_r + \frac{2}{r}u\right)\right)_r$, converge to zero almost everywhere as $\epsilon \to 0$. Therefore, it is sufficient to pass to the limit in (1.7) and $(\rho_\tau^0, u_\tau^0, \theta_\tau^0, q_\tau^0)$ satisfies the following hyperbolized Euler-Cattaneo-Christov equations

$$\begin{cases} \rho_t + (\rho u)_r + \frac{2}{r}\rho u = 0, \\ \rho u_t + \rho u u_r + p_r = 0, \\ \rho e_\theta \theta_t + (\rho u e_\theta - \frac{2a(\theta)}{Z(\theta)}q)\theta_r + p\left(u_r + \frac{2}{r}u\right) + q_r + \frac{2}{r}q = \left(\frac{2}{\tau(\theta)} + \frac{4}{r}u\right)a(\theta)q^2, \\ \tau(\theta)\rho\left(q_t + uq_r + \frac{2}{r}uq\right) + q + \kappa(\theta)\theta_r = 0. \end{cases} \tag{4.2}$$

This finishes the proof of Theorem 1.3.

Proof of Theorem1.4: Fix $\epsilon = (\mu, \lambda)$. Let $(\rho_\tau^\epsilon, u_\tau^\epsilon, \theta_\tau^\epsilon, q_\tau^\epsilon)$ be the global solutions obtained in Theorem 1.2. Then, we have

$$\sup_{0\le t<\infty} \|(\rho_\tau^\epsilon - 1, \theta_\tau^\epsilon - 1, q_\tau^\epsilon)\|_{H^2}^2 + \int_0^\infty \|D(\rho_\tau^\epsilon, \theta_\tau^\epsilon, q_\tau^\epsilon)\|_{H^1}^2 \mathrm{d}t \le CE(0), \tag{4.3}$$

and

$$\|u_\tau^\epsilon\|_{H^3}^2 + \int_0^\infty \|Du_\tau^\epsilon\|_{H^2}^2 \mathrm{d}t \le CE(0), \tag{4.4}$$

where C is a constant independent of τ but possible depends on ϵ. Thus, there exists $(\rho_0^\epsilon - 1, \theta_0^\epsilon - 1, q_0^\epsilon) \in L^\infty([0,\infty), H^2)$ and $u_0^\epsilon \in L^\infty([0,\infty), H^3)$ such that

$$(\rho_\tau^\epsilon, \theta_\tau^\epsilon, q_\tau^\epsilon) \to (\rho_0^\epsilon, \theta_0^\epsilon, q_0^\epsilon) \quad \text{weak} - * \quad \text{in } L^\infty([0,\infty), H^2)$$

and

$$u_\tau^\epsilon \to u_0^\epsilon \quad \text{weak} - * \quad \text{in } L^\infty([0,\infty), H^3).$$

Furthermore, since $\partial_t(\rho_\tau^\epsilon, \theta_\tau^\epsilon, q_\tau^\epsilon)$ and $\partial_t u_\tau^\epsilon$ are bounded in $L^2([0,\infty); H^1)$ and $L^2([0,\infty); H^2)$, respectively, by classical compactness argument, $(\rho_\tau^\epsilon, \theta_\tau^\epsilon, q_\tau^\epsilon)$ and u_τ^ϵ are relatively compact in $C([0,T], H_{loc}^{2-\delta_0})$ and $C([0,T], H_{loc}^{3-\delta_0})$, respectively, for any $T > 0$ and $0 < \delta_0 < 2$. As a consequence, as $\tau \to 0$ and up to subsequences,

$$(\rho_\tau^\epsilon, \theta_\tau^\epsilon, q_\tau^\epsilon) \to (\rho_0^\epsilon, \theta_0^\epsilon, q_0^\epsilon) \quad \text{strongly} \quad \text{in } C^0([0,T], H_{loc}^{2-\delta_0})$$

and

$$u_\tau^\epsilon \to u_0^\epsilon \quad \text{strongly} \quad \text{in } C^0([0,T], H_{loc}^{3-\delta_0})$$

On the other hand, as $\tau \to 0$, we have

$$\tau(\theta_\tau^\epsilon)\rho_\tau^\epsilon(\partial_t q_\tau^\epsilon + u_\tau^\epsilon \partial_r q_\tau^\epsilon + \frac{2}{r}u_\tau^\epsilon q_\tau^\epsilon) \rightharpoonup 0 \quad \text{in } \mathcal{D}'((0,\infty)\times\Omega),$$

which gives $q_0^\epsilon = -\kappa(\theta_0^\epsilon)\partial_r\theta_0^\epsilon, a.e.$. Therefore, it is sufficient to pass to the limit in (1.7) and $(\rho_0^\epsilon, u_0^\epsilon, \theta_0^\epsilon)$ satisfies the compressible Navier-Stokes-Fourier equations in spherical symmetry as

$$\begin{cases} \rho_t + (\rho u)_r + \frac{2}{r}\rho u = 0, \\ \rho u_t + \rho u u_r + p_r = \left(\left(\frac{4}{3}\mu(\theta) + \lambda(\theta)\right)\left(u_r + \frac{2}{r}u\right)\right)_r, \\ \rho C_v \theta_t + \rho u C_v \theta_r + p\left(u_r + \frac{2}{r}u\right) - \frac{1}{r^2}(r^2\kappa(\theta)\theta_r)_r \\ \qquad = \mu(\theta)\left(2u_r^2 + \frac{4}{r^2}u^2 - \frac{2}{3}\left(u_r + \frac{2}{r}u\right)^2\right) + \lambda(\theta)\left(u_r + \frac{2}{r}u\right)^2. \end{cases} \tag{4.5}$$

This finishes the proof of Theorem 1.4.

References

[1] C. Cattaneo, Sulla coduzione del calore, *Atti Sem. Mat. Fis. Univ. Modena* **3** (1948), 83-101.
[2] Y. Cho and B.J. Jin, Blow-up of viscous heat-conducting compressible flows, *J. Math. Anal. Appl.* **320** (2) (2006), 819-826.
[3] C.I. Christov, On frame indifferent formulation of the Maxwell-Cattaneo model of finite-speed heat conduction, *Mech. Res. Comm.* **36** (2009), 481-486.
[4] B.D. Coleman, W.J. Hrusa and D.R. Owen, Stability of equilibrium for a nonlinear hyperbolic system describing heat propagation by second sound in solids, *Arch. Rational Mech. Anal.* **94** (1986), 267-289.
[5] T. Crin-Barat, S. Kawashima and J. Xu, The Cattaneo-Christov approximation of Fourier heat-conductive compressible fluids, *arXiv:2404.07809*, 2024.
[6] E. Feireisl, A. Novotny and H. Petzeltová, On the existence of globally defined weak solutions to the Navier-Stokes equations, *J. Math. Fluid Mech.* **3** (2001), 358-392.
[7] K.O. Friedrichs, Symmetric hyperbolic linear differential equations, *Comm. Pure Appl. Math.* **7** (1954), 345-392.
[8] Z.Y. Guo and Z.X. Li, Size effect on microscale single-phase flow and heat transfer, *Int. J. Heat Mass Transf.* **46**(1) (2003), 149-159.
[9] D. Hoff, Global existence for 1D, compressible, isentropic Navier-Stokes equations with large initial data, *Trans. Amer. Math. Soc.* **303** (1) (1987), 169-181.
[10] D. Hoff, Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data, *J. Differential Equations* **120** (1) (1995), 215-254.
[11] Y. Hu and Y.C. Li, Initial boundary value problem for one-dimensional hyperbolic compressible Navier-Stokes equations, *arXiv:2508.01634*, 2025.
[12] Y. Hu and R. Racke, Compressible Navier-Stokes equations with hyperbolic heat conduction, *J. Hyper. Diff. Equ.* **13** (2016), 233-247.
[13] Y. Hu and R. Racke, Hyperbolic compressible Navier-Stokes equations, *J. Differential Equations* **269** (2020), 3196-3220.
[14] Y. Hu and R. Racke, Blow-up of solutions for relaxed compressible Navier-Stokes equations, *J. Hyper. Diff. Equ.* **21** (2024), 129-141.
[15] Y. Hu and M.R Yuan, Global spherically symmetric solutions and relaxation limit for the relaxed compressible Navier-Stokes equations, *arXiv:2507.15179v1*, 2025.
[16] Y. Hu and X.N. Zhao, Global well-posedness and relaxation limit for relaxed compressible Navier-Stokes-Fourier equations in bounded domain, *arXiv:2502.00544*, 2025.
[17] N. Itaya,, The existence and uniqueness of the solution of the equations describing compressible viscous fluid flow, *Proc. Japan Acad.*, **46** (1970), 379-382.
112. Chapman & Hall/CRC, Boca Raton (2000).
[18] S. Jiang and P. Zhang, Global spherically symmetry solutions of the compressible isentropic Navier-Stokes equations, *Comm. Math. Phys.* **215** (2001), 559-581.
[19] S. Jiang and P. Zhang, Axisymmetric solutions of the 3-D Navier-Stokes equations for compressible isentropic fluids, *J. Math. Pures. Appl.* **82** (2003), 949-973.
[20] D. Jou, A. Sellitto and F.X. Alvarez, Heat waves and phonon-wall collisions in nanowires, *Proc. R. Soc. A* **467** (2011), 2520-2533.
[21] R.E. Khayat, et al., Non-Fourier effects in macro- and micro-scale non-isothermal flow of liquids and gases. Review, *International Journal of Thermal Sciences*, **97** (2015), 163-177.
[22] R.R. Letfullin, et al., Ultrashort laser pulse heating of nanoparticles: comparison of theoretical approaches, *Adv. Opt. Technol.* **2008**(1) (2008), 251718.
[23] P.L. Lions, *Mathematical Topics in Fluid Mechanics*, Vol.II, Compressible Models. Clarendon Press, Oxford (1998).
[24] M.Q. Liu and Z.G. Wu, Space-time behavior of the compressible Navier-Stokes equations with hyperbolic heat conduction, *J. Math. Phys.* **64**(10) (2023), 103101.
[25] A. Matsumura and T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases, *J. Math. Kyoto Univ.* **20** (1) (1980), 67-104.

[26] A. Morro, Thermodynamic consistency of objective rate equations, *Mech. Res. Comm.* **84** (2017), 72-76.

[27] J. Nash, Le problème de Cauchy pour les équations différentielles d'un fluide général, *Bull. Soc. Math. France* **90** (1962), 487-497.

[28] R. Racke, Blow-up for hyperbolized compressible Navier-Stokes equations, *In: Special Issues on Hyperbolic PDEs and Applications in Honor of Professor Thomas C. Sideris on the Occasion of His 70th Birthday. Commun. Analysis Mech.* **17** (2025), 550-581.

[29] J. Rauch and F.J. Massey, Differentiability of solutions to hyperbolic initial boundary value problems, *Trans. Amer. Math. Soc.* **189** (1974), 303-318.

[30] J. Serrin, On the uniqueness of compressible fluid motion, *Arch. Rational Mech. Anal.* **3** (1959), 271-288.

[31] H.Z. Tang, S.X. Zhang and W.Y. Zou, Decay of the compressible Navier-Stokes equations with hyperbolic heat conduction, *J. Differential Equations* **388** (2024), 1-33.

[32] Z.P. Xin, Blowup of smooth solutions to the compressible Navier-Stokes equation with compact density, *Comm. Pure. Appl. Math.* **51** (1998), 229-240.

[33] S. Zheng, Nonlinear parabolic equations and hyperbolic-parabolic coupled system, Hu, Pitman Monographs and Surveys in Pure and Applied Mathematics **76**, 1995.