Exact Interior Controllability of Magnetoelastic Plates by Means of Purely Magnetic Actuation

Buddhika Priyasad [*] Reinhard Racke [†]

January 10, 2026

Abstract

We establish exact interior controllability for a two-dimensional magnetoelastic plate system with control acting solely in the magnetic field equation. The main result shows that exact controllability of the coupled system is achievable in arbitrarily small time $T > 0$, despite the control acting only the magnetic dynamics. This extends the principle of indirect control - previously demonstrated for thermoelastic systems [6] - to the magnetoelastic regime, revealing that steering the mechanical plate displacement through magnetic actuation alone is possible. The analysis employs the operator-theoretic multiplier method adapted to handle vectorial fields with divergence-free constraints and non-self-adjoint coupling. The approach requires several technical components: norm equivalences for divergence-free vector fields, analysis of non-self-adjoint coupling operators, integration by parts identities for rot rot systems, and trace regularity results. The proof follows three steps: establishing a trace regularity result for the adjoint system, deriving an energy estimate via the multiplier method, and using a compactness-uniqueness argument to eliminate lower-order terms. This work provides the first controllability result for magnetoelastic systems and extends the indirect control framework from thermoelasticity to this setting. The techniques developed here are applicable to the control-theoretic investigation of magnetically-coupled elastic systems, with potential applications in smart materials, damping devices, and electromagnetic actuators.

Keywords: Magnetoelastic plates, exact controllability, multiplier method, indirect control, operator-theoretic multipliers, divergence-free fields, observability inequality.

1 Introduction

1.1 Statement of the problem.

Let Ω be a bounded domain in $\mathbb{R}^2$ with a sufficiently smooth boundary $\Gamma \equiv \partial\Omega$ and let the terminal time $T > 0$. In this study, we investigate the problem of exact controllability for a magnetoelastic

[*]Department of Mathematics and Statistics, University of Konstanz, 78457 Konstanz, Germany. (priyasad@uni-konstanz.de)

[†]Department of Mathematics and Statistics, University of Konstanz, 78457 Konstanz, Germany. (reinhard.racke@uni-konstanz.de)

plate system, subject to clamped (1d) boundary conditions, and controlled via interior magnetic input $\boldsymbol{u}$.

1.1.1 Controlled magnetoelastic plate with clamped boundary conditions

We consider the following magnetoelastic plate equation with clamped boundary conditions (1d) subject to a control input $\boldsymbol{u} \in \mathrm{L}^2(0,T;\mathbf{H}^1(\Omega)')$:

$$w_{tt} - \gamma\Delta w_{tt} + \Delta^2 w - \alpha(\operatorname{rot}\operatorname{rot}\boldsymbol{h})\cdot\overrightarrow{H} = 0 \quad \text{in } Q, \tag{1a}$$

$$\boldsymbol{h}_t + \operatorname{rot}\operatorname{rot}\boldsymbol{h} + \beta\operatorname{rot}\operatorname{rot}(\overrightarrow{H}w_t) = \boldsymbol{u} \quad \text{in } Q, \tag{1b}$$

$$\operatorname{div}\,\boldsymbol{h} = 0 \quad \text{in } \Omega, \tag{1c}$$

$$w = 0, \quad \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \Gamma, \tag{1d}$$

$$\boldsymbol{h}\cdot\nu = 0, \quad \nu\times\operatorname{rot}\boldsymbol{h} = 0 \quad \text{on } \Gamma, \tag{1e}$$

$$w(t=0) = w_0,\ w_t(t=0) = w_1,\ \boldsymbol{h}(t=0) = \boldsymbol{h}_0 \quad \text{on } \Omega. \tag{1f}$$

Here, w denotes the plate displacement, $\boldsymbol{h}$ is the magnetic field, and $\overrightarrow{H} \in \mathbb{R}^2$ is a constant applied magnetic field. The parameter $\gamma \geq 0$ is the Kelvin–Voigt damping coefficient, while $\alpha > 0$ and $\beta > 0$ are the magnetoelastic coupling and magnetic damping coefficients, respectively. The domain $\Omega \subset \mathbb{R}^2$ has boundary Γ with outward unit normal ν. The space–time cylinder is denoted by $Q = \Omega \times (0,T)$. For simplicity and wlog in the analysis throughout, we set the parameters $\alpha = \beta = 1$.

1.2 Main theorem

Before stating the main result of this paper, we briefly remark the results of well-posedness required for its formulation. A comprehensive set of results on existence, uniqueness, and regularity is provided later in Theorem 2.1. In particular, for any $\boldsymbol{u} \in \mathrm{L}^2(0,T;\mathbf{H}^1(\Omega)')$ and initial data $[w_0, w_1, \boldsymbol{h}_0] \in \mathrm{H}^2_0(\Omega) \times \mathrm{H}^1_{0,\gamma}(\Omega) \times \mathbf{L}^2_{\sigma,\nu}(\Omega)$ the theorem ensures the existence of a solution $[w, w_t, \boldsymbol{h}]$. With this preparation, the main theorem of the paper may now be stated as follows.

Theorem 1.1. *For all $\gamma \geq 0$, the system (1) is exactly controllable in arbitrary time $T > 0$. That means for any $T > 0$, and data $[w_0, w_1, \boldsymbol{h}_0]$, $[w_0^T, w_1^T, \boldsymbol{h}_0^T]$ in the space $\mathrm{H}^2_0(\Omega) \times \mathrm{H}^1_{0,\gamma}(\Omega) \times \mathbf{L}^2_{\sigma,\nu}(\Omega)$, there exists a control $\boldsymbol{u} \in \mathrm{L}^2(0,T;\mathbf{H}^1(\Omega)')$ such that the corresponding solution $[w, w_t, \boldsymbol{h}]$ of (1) satisfies $[w(T), w_t(T), \boldsymbol{h}(T)] = [w_0^T, w_1^T, \boldsymbol{h}_0^T]$.*

The exact controllability result holds for both $\gamma > 0$ and $\gamma = 0$. The methodology employed to prove Theorem 1.1 is based on the classical approach of establishing the surjectivity of the control-to-terminal-state map $\mathcal{L}_T$ (see (18) below for its explicit definition). For this purpose, it suffices to establish the following observability inequality:

$$\int_0^T \|\operatorname{rot}\boldsymbol{\psi}\|^2_{\mathbf{L}^2(\Omega)} \geq C_T \,\|[\phi_0, \phi_1, \boldsymbol{\psi}_0]\|^2_{\mathrm{H}^2_0(\Omega)\times\mathrm{H}^1_{0,\gamma}(\Omega)\times\mathbf{L}^2_{\sigma,\nu}(\Omega)}\,, \tag{2}$$

where $\boldsymbol{\psi}$ is the magnetic component of the solution $[\phi, \phi_t, \boldsymbol{\psi}]$ to the following backwards magnetoelastic system, adjoint with respect to (1):

$$\phi_{tt} - \gamma\Delta\phi_{tt} + \Delta^2\phi + (\operatorname{rot}\operatorname{rot}\boldsymbol{\psi}) \cdot \overrightarrow{H} = 0 \qquad \text{in } Q, \tag{3a}$$

$$-\psi_t + \operatorname{rot}\operatorname{rot}\boldsymbol{\psi} - \operatorname{rot}\operatorname{rot}(\overrightarrow{H}\phi_t) = 0 \qquad \text{in } Q, \tag{3b}$$

$$\operatorname{div}\boldsymbol{\psi} = 0 \qquad \text{in } \Omega, \tag{3c}$$

$$\phi = \frac{\partial\phi}{\partial\nu} = 0, \text{ and } \boldsymbol{\psi}\cdot\nu = \nu\times\operatorname{rot}\boldsymbol{\psi} = 0 \qquad \text{on } \Gamma, \tag{3d}$$

$$\phi(T) = \phi_0, \quad \phi_t(T) = \phi_1, \quad \boldsymbol{\psi}(T) = \boldsymbol{\psi}_0 \qquad \text{on } \Omega. \tag{3e}$$

A multiplier method is employed to derive the observability inequality (2) (see [15] for a comprehensive treatment of multiplier techniques in PDE control theory). Specifically, we choose a multiplier of operator-theoretic type, which is given by $\mathrm{A}_D^{-1}\boldsymbol{\psi}$ where A_D denotes the Laplacian with homogeneous Dirichlet boundary conditions. The choice of this multiplier is motivated by its role in controlling the energy associated with the adjoint system and ensuring the coercivity of the resulting estimates. Multipliers of this nature have been previously employed in similar PDE control problems, such as in the thermoelastic setting (see [6]).

1.3 Overview of Existing Literature and Organization of the Paper

The controllability of coupled PDE systems has been a central topic in control theory over the past several decades. This work contributes to this field by establishing exact interior controllability for magnetoelastic plate systems, extending classical ideas from thermoelasticity to a new regime with fundamentally different mathematical structure. We organize this section as follows: we first discuss the foundational theory of exact controllability and the multiplier method; we then review the extensive work on thermoelastic plate controllability, paying particular attention to the technical strategies and principles that motivate our approach; we explain why magnetoelastic systems present qualitatively new challenges compared to thermoelastic systems; we survey the existing mathematical theory of magnetoelastic systems; and finally we identify the gap in the literature that this work addresses.

The structure of the paper is as follows. In Section 1, we state the problem and main result (Theorem 1.1). In Section 2, we reformulate the magnetoelastic system as an abstract evolution equation in a Hilbert space, define the control operator, the control-to-terminal-state map, and establish well-posedness of the semigroup. The proof of Theorem 1.1 reduces to proving an observability inequality for the adjoint system (inequality (2)). In Section 3, we prove this observability inequality using the multiplier method, employing an operator-theoretic multiplier adapted to the vectorial setting. This section is divided into three steps: deriving a trace regularity result (Lemma 3.1), establishing a tainted observability inequality or a preliminary energy estimate (Lemma 3.2), and removing lower-order error terms using a compactness-uniqueness argument (Proposition 3.3). The appendix provides technical lemmas on vector calculus, norm equivalences, and specialized estimates needed for the multiplier argument.

Foundational Framework. The problem of exact controllability for coupled PDE systems rests on the framework established by [30], where exact controllability is equivalent to surjectivity of the control-to-terminal-state map, which in turn reduces to establishing an observability inequality for the adjoint system. For PDE systems, the multiplier method pioneered by [24] and systematically developed in [15] provides the most effective technique for deriving observability inequalities by constructing test functions that couple with adjoint dynamics to isolate the observable quantity.

Thermoelastic Plate Controllability. The exact controllability problem for thermoelastic plate systems has been extensively investigated. [16] initiated this line of work with boundary control on the mechanical component, obtaining partial exact controllability when the coupling parameter α is sufficiently small. [6] achieved the breakthrough result: control acting *solely in the thermal equation* suffices for exact controllability of the full thermoelastic system (both displacement and temperature) for all $\gamma \geq 0$, and remarkably, controllability is achieved in arbitrarily small time. This counterintuitive result reveals that indirect control through a diffusive component can be as effective as direct mechanical control. The proof employs an operator-theoretic multiplier $\mathrm{A}_D^{-1}\theta$ combined with a compactness-uniqueness argument to eliminate lower-order terms from a preliminary energy estimate. This strategy has become the standard approach for such coupled systems. [7, 8] developed complementary exponential stability results using similar multiplier techniques. [22] extended these results to the analytic case $\gamma = 0$, proving exact null controllability.

The methodology for thermoelastic plate controllability was significantly refined through the foundational work of Lasiecka and Triggiani. [18] established exact controllability and uniform stabilization of Kirchhoff plates with boundary control only on $\Delta\omega|_\Sigma$ and homogeneous boundary displacement, demonstrating that control of the Laplacian provides sufficient leverage. This work required developing sharp regularity results for mixed Neumann-type hyperbolic systems. [17, 21] proved sharp regularity for elastic and thermoelastic Kirchhoff equations with free boundary conditions, establishing that these regularity properties are essential technical tools for controllability analysis. [11, 12] proved simultaneous exact/approximate boundary controllability of thermoelastic plates with variable thermal coefficients and moment control, showing how regularity and unique continuation properties interact with controllability under varying parameter regimes. These refinements demonstrated the deep connection between sharp regularity results, unique continuation principles, and exact controllability, see [20].

[10] addressed interior controllability of thermoelastic plates, proving exact controllability of the displacement and approximate controllability of the temperature when control acts in the plate domain. Their results demonstrate that the location of control (interior versus boundary) and the component being controlled (mechanical versus thermal) are independent design parameters that affect the structure of controllability results. [19] proved exact null controllability of structurally damped and thermoelastic parabolic models, establishing controllability in the limiting case where hyperbolic and parabolic dynamics blend.

Technical Strategy and Magnetoelasticity. In order to prove exact controllability at time T, one establishes that the control-to-terminal-state map $\mathcal{L}_T$ is surjective. By duality, this is equivalent to proving an observability inequality for the adjoint system; more precisely, the surjectivity of $\mathcal{L}_T$ is equivalent to the boundedness from below of its adjoint operator $\mathcal{L}_T^*$, see [13, 33]. For thermoelasticity, one must show that the energy of the adjoint system is controlled by the time-integral of the thermal gradient. The multiplier method accomplishes this by strategically choosing test functions and integrating by parts, extracting positive control terms while carefully handling boundary integrals. The key steps are: integrating by parts to expose the observable quantity; employing standard inequalities and trace regularity to bound error terms; obtaining a preliminary estimate containing lower-order norm terms; and using compactness arguments to remove these error terms, yielding the clean observability inequality needed for controllability. Thermoelastic systems feature scalar-scalar coupling: the temperature couples to displacement through the thermal conductivity term appearing linearly in the plate equation. The adjoint thermal equation is a scalar parabolic equation, and multiplication by a scalar multiplier produces boundary integrals involving standard derivatives and function values on the boundary. The mathematical structure operates within standard Sobolev spaces with well-understood integration-by-parts properties.

Magnetoelastic systems present qualitatively different mathematical structure. The magnetic field $\boldsymbol{h}$ is vector-valued and satisfies the divergence-free constraint $\operatorname{div} \boldsymbol{h} = 0$. The coupling involves "rot rot" (also denoted by "curl curl") operators: $(\operatorname{rot}\operatorname{rot}\boldsymbol{h}) \cdot \overrightarrow{H}$, where $\overrightarrow{H}$ is a fixed magnetic field vector. The well-posedness of magnetoelastic plate equations was established by [29], who provided the foundational semigroup theory and abstract operator formulations. The long-time behavior and stability of magnetoelastic systems have been studied extensively. [25] and [26] proved exponential stability of magneto-thermo-elastic systems and polynomial stability for two-dimensional magneto-elastic systems. [28] established polynomial stability for three-dimensional magnetoelastic waves. [34] investigated Mindlin-Timoshenko plates with magnetic interactions, analyzing the dissipative effect of the magnetic field on coupled mechanical-magnetic dynamics. [27] established that magnetoelastic plate systems generate analytic semigroups and provided explicit decay rate estimates showing dependence on the magnetic field configuration.

Recent work on coupled piezoelectric-magnetic systems demonstrates the emerging importance of magnetic effects in multiphysics coupling. [1, 2, 4] established stability results for piezoelectric beams with magnetic effects. [3] proved stabilization of magnetizable piezoelectric and elastic systems, revealing that magnetic coupling alters system dynamics compared to classical elastic systems.

Despite this rich body of work on thermoelastic controllability and magnetoelastic stability, no prior work addresses controllability (exact, approximate, or null) of magnetoelastic systems. The natural question arises: Can one steer a magnetoelastic plate system by acting only on the magnetic equation? The present work answers this affirmatively, establishing exact interior controllability of magnetoelastic plates with control in the magnetic field dynamics alone. The key challenge is that the vectorial and divergence-free nature of the magnetic field introduces mathematical obstacles fundamentally absent in scalar-scalar thermoelasticity: (1) the magnetic field lives in the constrained space $\mathbf{L}^2_{\sigma,\nu}(\Omega)$ rather than standard Sobolev spaces, requiring specialized norm equivalences (Lemma A.3); (2) the coupling operators are non-self-adjoint (Lemma A.2 establishes duality, not equality), requiring precise domain analysis (Proposition 2.2); (3) integration by parts for curl-curl operators produces boundary terms with fundamentally different structure than scalar operations, necessitating new trace regularity results (Lemma 3.1); and (4) the magnetic boundary conditions $\boldsymbol{h}\cdot\nu = 0$ and $\nu \times \operatorname{rot}\boldsymbol{h} = 0$ couple normal and tangential components in ways that affect the entire multiplier argument. Our proof adapts Avalos's operator-theoretic multiplier method to this vectorial regime, employing tools from vector calculus and rot (curl) operator theory [23, 31] to overcome these new technical barriers while maintaining the compactness-uniqueness strategy for eliminating lower-order terms.

2 Abstract formulation of the PDE problem (1)

In this section, we reformulate the control problem (1) and its adjoint system (3) as abstract evolution equations within an appropriate Hilbert space framework. To develop the corresponding operator models, we first introduce the necessary definitions and notations that will be used throughout this paper. We have extracted the following definitions and notations from [6, 9, 29] and modified to our convenience. In particular, we define the state space, the relevant differential operators, and their domains, ensuring that the problem is embedded in a functional analytic setting.

(A.1) We introduce the following realization of elliptic operators. First, the elastic operator $\mathbb{A}$ with clamped boundary conditions (1d) is given by,

$$\mathbb{A} : \mathrm{L}^2(\Omega) \supset \mathcal{D}(\mathbb{A}) \to \mathrm{L}^2(\Omega),\ \mathbb{A}w := \Delta^2 w,\ \mathcal{D}(\mathbb{A}) := \mathrm{H}^4(\Omega) \cap \mathrm{H}^2_0(\Omega). \tag{4}$$

In order to define the magnetic operator, we first introduce the spaces

$$\mathbf{L}^2_\sigma(\Omega) := \left\{\boldsymbol{\psi} \in \mathbf{L}^2(\Omega) : \operatorname{div}\boldsymbol{\psi} = 0 \text{ in } \Omega\right\}, \tag{5}$$

$$\mathbf{L}^2_{\sigma,\nu}(\Omega) := \left\{\boldsymbol{\psi} \in \mathbf{L}^2(\Omega) : \operatorname{div}\boldsymbol{\psi} = 0 \text{ in } \Omega, \text{ and } \boldsymbol{\psi}\cdot\nu = 0 \text{ on } \Gamma\right\}. \tag{6}$$

Then the magnetic operator $\mathrm{B} : \mathbf{L}^2_{\sigma,\nu}(\Omega) \supset \mathcal{D}(\mathrm{B}) \to \mathbf{L}^2_{\sigma,\nu}(\Omega)$ is given by

$$\mathrm{B}\,\boldsymbol{h} := \operatorname{rot}\operatorname{rot}\boldsymbol{h},\ \text{with } \mathcal{D}(\mathrm{B}) := \left\{\boldsymbol{h} \in \mathbf{H}^2(\Omega) \cap \mathbf{L}^2_{\sigma,\nu}(\Omega) : \nu \times \operatorname{rot}\boldsymbol{h} = 0 \text{ on } \Gamma\right\}. \tag{7}$$

It is easy to see that $\mathcal{D}(\mathrm{B})$ is dense in $\mathbf{L}^2_{\sigma,\nu}(\Omega)$. In the model equations (1) and (3), we observe the occurrence of the operator "rot rot" acting on $\boldsymbol{h}$ and $\boldsymbol{\psi}$ respectively, followed by a scalar product with the fixed vector $\overrightarrow{H}$. To emphasize this dot-product with $\overrightarrow{H}$, we introduce the auxiliary operator $\mathrm{B}_{\overrightarrow{H}} : \mathbf{L}^2_{\sigma,\nu}(\Omega) \supset \mathcal{D}(\mathrm{B}_{\overrightarrow{H}}) \to \mathrm{L}^2(\Omega)$ defined by

$$\mathrm{B}_{\overrightarrow{H}}\,\boldsymbol{h} := \operatorname{rot}\operatorname{rot}\boldsymbol{h}\cdot\overrightarrow{H}, \quad \mathcal{D}(\mathrm{B}_{\overrightarrow{H}}) := \mathcal{D}(\mathrm{B}) \text{ in } \mathbf{L}^2_{\sigma,\nu}(\Omega). \tag{8}$$

We moreover define another magnetic operator $\mathrm{G}_{\overrightarrow{H}}$ by

$$\mathrm{G}_{\overrightarrow{H}}(\zeta) := \operatorname{rot}\operatorname{rot}(\zeta\overrightarrow{H}) \text{ with } \mathcal{D}(\mathrm{G}_{\overrightarrow{H}}) := \left\{\zeta \in \mathrm{H}^2_0(\Omega) : \mathrm{G}_{\overrightarrow{H}}(\zeta) \in \mathbf{L}^2_{\sigma,\nu}(\Omega)\right\}. \tag{9}$$

We remark here that $\mathrm{H}^2_0(\Omega) \subset \mathcal{D}(\mathrm{G}_{\overrightarrow{H}})$ and hence $\mathcal{D}(\mathbb{A}^{1/2}) \subset \mathcal{D}(\mathrm{G}_{\overrightarrow{H}})$.

To facilitate the multiplier technique, we introduce the homogeneous Dirichlet Laplacian which serves as a critical tool in obtaining the required observability inequality (2),

$$\mathrm{A}_D\,w := -\Delta w \text{ with } \mathcal{D}(\mathrm{A}_D) := \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega) \text{ in } \mathrm{L}^2(\Omega). \tag{10}$$

(A.2) For $\gamma \geq 0$, we define the operator P_γ by

$$\mathrm{P}_\gamma := \mathrm{I} + \gamma\, \mathrm{A}_D, \quad \mathcal{D}(\mathrm{P}_\gamma) := \mathcal{D}(\mathrm{A}_D) \text{ in } \mathrm{L}^2(\Omega) \tag{11}$$

with "I" being the identity operator on $\mathrm{L}^2(\Omega)$. We define a space

$$\mathrm{H}^1_{0,\gamma}(\Omega) := \begin{cases} \mathrm{H}^1_0(\Omega) & \text{if } \gamma > 0, \\ \mathrm{L}^2(\Omega) & \text{if } \gamma = 0 \end{cases} \tag{12a}$$

with its inner product being defined as

$$\big(\varphi, \varphi'\big)_{\mathrm{H}^1_{0,\gamma}(\Omega)} := (w, \widetilde{w})_{\mathrm{L}^2(\Omega)} + \gamma\, (\nabla w, \nabla \widetilde{w})_{\mathrm{L}^2(\Omega)}, \quad \forall w, \widetilde{w} \in \mathrm{H}^1_0(\Omega). \tag{12b}$$

The dual of $\mathrm{H}^1_{0,\gamma}(\Omega)$ is denoted by $\mathrm{H}^{-1}_{0,\gamma}(\Omega)$. When $\gamma = 0$ we have $\mathrm{P}_\gamma = \mathrm{I}$ and set $\mathrm{H}^1_{0,0}(\Omega) = \mathrm{H}^{-1}_{0,0}(\Omega) = \mathrm{L}^2(\Omega)$. The operator P_γ is clearly $\mathrm{H}^1_{0,\gamma}(\Omega)$-elliptic and by the Lax-Milgram theorem, it is boundedly invertible, i.e. $\mathrm{P}_\gamma^{-1} \in \mathcal{L}(\mathrm{H}^{-1}_{0,\gamma}(\Omega), \mathrm{H}^1_{0,\gamma}(\Omega))$.

(A.3) We denote the Hilbert space $\mathbf{H}_\gamma$ to be

$$\mathbf{H}_\gamma := \mathrm{H}^2_0(\Omega) \times \mathrm{H}^1_{0,\gamma}(\Omega) \times \mathbf{L}^2_{\sigma,\nu}(\Omega), \tag{13}$$

with the inner product

$$\left\langle \begin{bmatrix} w_0 \\ w_1 \\ \boldsymbol{h}_0 \end{bmatrix}, \begin{bmatrix} \widetilde{\phi}_0 \\ \widetilde{\phi}_1 \\ \widetilde{\psi}_0 \end{bmatrix} \right\rangle_{\mathbf{H}_\gamma} := \left(\Delta \phi_0, \Delta \widetilde{\phi}_0\right)_{\mathrm{L}^2(\Omega)} + \left(\phi_1, \widetilde{\phi}_1\right)_{\mathrm{L}^2(\Omega)} + \gamma \left(\nabla \phi_1, \nabla \widetilde{\phi}_1\right)_{\mathrm{L}^2(\Omega)} + \left(\psi_0, \widetilde{\psi}_0\right)_{\mathbf{L}^2(\Omega)}. \tag{14}$$

(A.4) We define $\mathcal{A}_\gamma : \mathbf{H}_\gamma \supset \mathcal{D}(\mathcal{A}_\gamma) \to \mathbf{H}_\gamma$ to be

$$\mathcal{A}_\gamma := \begin{pmatrix} 0 & \mathrm{I} & 0 \\ -\mathrm{P}_\gamma^{-1}\, \mathbb{A} & 0 & \mathrm{P}_\gamma^{-1}\, \mathrm{B}_{\overrightarrow{H}} \\ 0 & -\mathrm{G}_{\overrightarrow{H}} & -\mathrm{B} \end{pmatrix} \quad \text{with}$$
$$\mathcal{D}(\mathcal{A}_\gamma) := \left\{ [\phi_0, \phi_1, \boldsymbol{\psi}_0] \in \mathrm{H}^2_0(\Omega) \times \mathrm{H}^2_0(\Omega) \times \mathcal{D}(\mathrm{B}) :\ \mathbb{A}\phi_0 \in \mathrm{H}^{-1}_{0,\gamma}(\Omega) \right\}. \tag{15}$$

Alternatively we can now write (3) for initial data $[\phi_0, \phi_1, \boldsymbol{\psi}_0] \in \mathcal{D}(\mathcal{A}_\gamma)$ as

$$\mathrm{P}_\gamma\, \phi_{tt} = -\mathbb{A}\phi - \mathrm{B}_{\overrightarrow{H}}\, \boldsymbol{\psi} \qquad \text{in } \mathrm{H}^{-1}_{0,\gamma}(\Omega), \tag{16a}$$

$$\boldsymbol{\psi}_t = -\mathrm{B}\, \boldsymbol{\psi} + \mathrm{G}_{\overrightarrow{H}}\, \phi_t \qquad \text{in } \mathbf{L}^2_{\sigma,\nu}(\Omega), \tag{16b}$$

$$[\phi(T), \phi_t(T), \boldsymbol{\psi}(T)] = [\phi_0, \phi_1, \boldsymbol{\psi}_0]. \tag{16c}$$

(A.5) We define the control operator $\mathcal{B} \in \mathcal{L}(\mathbf{H}^1(\Omega)', \mathrm{H}_0^2(\Omega) \times \mathrm{L}^2(\Omega) \times \mathbf{H}^1(\Omega)')$ for $\boldsymbol{u} \in \mathbf{H}^1(\Omega)'$ by

$$\mathcal{B}\boldsymbol{u} := \begin{bmatrix} 0 \\ 0 \\ \boldsymbol{u} \end{bmatrix}. \tag{17}$$

With this definition, we define the *input* $\to$ *terminal state* map $\mathcal{L}_T : \mathrm{L}^2(0,T;\mathbf{H}^1(\Omega)') \to \mathbf{H}_\gamma$, possibly unbounded, for all $\boldsymbol{u} \in \mathrm{L}^2(0,T;\mathbf{H}^1(\Omega)')$ such that

$$\mathcal{L}_T\boldsymbol{u} := \int_0^T e^{\mathcal{A}_\gamma(T-t)}\mathcal{B}\boldsymbol{u}(t)\mathrm{d}t. \tag{18}$$

This will be made precise later.

(A.6) Throughout the paper, we use the symbol $\lesssim$ to denote an inequality up to a positive constant; that is, $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \leq CB$. The constant C is independent of the quantities under consideration.The L^2-inner-products are denoted by $(\cdot,\cdot)$, while other inner products are denoted by $\langle \cdot,\cdot \rangle$.

Taking initial data $[w_0, w_1, \boldsymbol{h}_0] \in \mathbf{H}_\gamma$ and control $\boldsymbol{u} \in \mathrm{L}^2(0,T;\mathbf{H}^1(\Omega)')$, the the system (1) can be written abstractly

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} w(t) \\ w_t(t) \\ \boldsymbol{h}(t) \end{bmatrix} = \mathcal{A}_\gamma \begin{bmatrix} w(t) \\ w_t(t) \\ \boldsymbol{h}(t) \end{bmatrix} + \mathcal{B}\boldsymbol{u}(t), \qquad \begin{bmatrix} w(0) \\ w_t(0) \\ \boldsymbol{h}(0) \end{bmatrix} = \begin{bmatrix} w_0 \\ w_1 \\ \boldsymbol{h}_0 \end{bmatrix}, \tag{19}$$

which has an a priori meaning in $[\mathcal{D}(\mathcal{A}_\gamma^*)]' \supset \mathbf{H}_\gamma$.

We have the following well-posedness result due to Theorems 3.1, Theorem 3.2 of [29].

Theorem 2.1 (Well-posedness)**.** *With the parameter $\gamma \geq 0$, $\mathcal{A}_\gamma$ as defined in (15) generates a C_0-semigroup of contractions $\{e^{\mathcal{A}_\gamma t}\}_{t\geq 0}$ on the energy space $\mathbf{H}_\gamma$. Let $[w(t), w_t(t), \boldsymbol{h}(t)] = e^{\mathcal{A}_\gamma t}[w_0, w_1, \boldsymbol{h}_0]$ denote the solution trajectory.*

(i) If initial data $[w_0, w_1, \boldsymbol{h}_0] \in \mathbf{H}_\gamma$, then problem (1) is globally well-posed in energy space $\mathbf{H}_\gamma$. The unique weak solution $[w, w_t, \boldsymbol{h}]$ belongs to $\mathrm{C}([0,\infty);\mathbf{H}_\gamma)$.

(ii) if initial data $[w_0, w_1, \boldsymbol{h}_0] \in \mathcal{D}(\mathcal{A}_\gamma)$, then the solution $[w, w_t, \boldsymbol{h}]$ belongs to $\mathrm{C}([0,\infty);\mathcal{A}_\gamma) \cap \mathrm{C}^1([0,\infty);\mathbf{H}_\gamma)$.

With these dynamics in hand, the solution $[w, w_t, \boldsymbol{h}]$ to (1) may be written explicitly as

$$\begin{bmatrix} w(t) \\ w_t(t) \\ \boldsymbol{h}(t) \end{bmatrix} = e^{\mathcal{A}_\gamma t}\begin{bmatrix} w_0 \\ w_1 \\ \boldsymbol{h}_0 \end{bmatrix} + \int_0^t e^{\mathcal{A}_\gamma(t-s)}\mathcal{B}\boldsymbol{u}(s)\mathrm{d}s \tag{20}$$

A fortiori, $\mathcal{A}_\gamma^{-1}\mathcal{B} \in \mathcal{L}(\mathbf{H}^1(\Omega)', \mathbf{H}_\gamma)$, or equivalently

$$\mathcal{B} \in \mathcal{L}(\mathbf{H}^1(\Omega)', [\mathcal{D}(\mathcal{A}_\gamma^*)]'). \tag{21}$$

The input to state map above thus gives that

$$[w, w_t, \boldsymbol{h}] \in \mathrm{C}([0,T]; [\mathcal{D}(\mathcal{A}_\gamma^*)]'). \tag{22}$$

Given the representation (20) for the solution $[w, w_t, \boldsymbol{h}]$, establishing the exact controllability at a given time $T > 0$ (Theorem 1.1) is equivalent to proving the surjectivity of the operator $\mathcal{L}_T$, where $\mathcal{L}_T$ is defined in (18) (see, for instance, [30, 35]). It is important to note that $\mathcal{L}_T$ is, a priori, well-defined as follows

$$\mathcal{L}_T \in \mathcal{L}(\mathrm{L}^2(0,T;\mathbf{H}^1(\Omega)'), [\mathcal{D}(\mathcal{A}_\gamma^*)]').$$

Consequently, the control-to-state operator $\mathcal{L}_T$, when viewed as a mapping into the state space $\mathbf{H}_\gamma$, is initially well-posed only as an unbounded operator, closed and densely defined with a specified domain of definition:

$$\mathcal{D}(\mathcal{L}_T) := \{\boldsymbol{u} \in \mathrm{L}^2(0,T;\mathbf{H}^1(\Omega)') : \mathcal{L}_T\boldsymbol{u} \in \mathbf{H}_\gamma\}. \tag{23}$$

To further analyze its properties, one can compute the adjoint

$$\mathcal{L}_T^* \in \mathcal{L}(\mathcal{D}(\mathcal{A}_\gamma^*), \mathrm{L}^2(0,T;\mathbf{H}^1(\Omega))), \tag{24}$$

and with the domain of definition:

$$\mathcal{L}_T^* : \mathcal{D}(\mathcal{L}_T^*) \subset \mathbf{H}_\gamma \to \mathrm{L}^2(0,T;\mathbf{H}^1(\Omega)), \tag{25}$$

$$\mathcal{D}(\mathcal{L}_T^*) := \left\{ \begin{bmatrix} \phi_0 \\ \phi_1 \\ \boldsymbol{\psi} \end{bmatrix} \in \mathbf{H}_\gamma : \mathcal{L}_T^* \begin{bmatrix} \phi_0 \\ \phi_1 \\ \boldsymbol{\psi} \end{bmatrix} \in \mathrm{L}^2(0,T;\mathbf{H}^1(\Omega)) \right\} \tag{26}$$

and takes the classical form (see, for instance, [35]),

$$\mathcal{L}_T^* \begin{bmatrix} \phi_0 \\ \phi_1 \\ \boldsymbol{\psi} \end{bmatrix}(t) = \mathcal{B}^* e^{\mathcal{A}_\gamma^*(T-t)} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \boldsymbol{\psi} \end{bmatrix}. \tag{27}$$

In the analysis, it is necessary to derive the PDE formulation of the adjoint operator $\mathcal{L}_T^*$, as the desired observability inequality (2) is intrinsically linked to it. To achieve this, one can explicitly compute the adjoint operator $\mathcal{A}_\gamma^*$. The following Proposition gives the characterization of $\mathcal{A}_\gamma^*$.

Proposition 2.2. *The adjoint $\mathcal{A}_\gamma^*$ of $\mathcal{A}_\gamma$ is given by*

$$\mathcal{A}_\gamma^* = \begin{pmatrix} 0 & -\mathrm{I} & 0 \\ \mathrm{P}_\gamma^{-1}\,\mathbb{A} & 0 & -\mathrm{P}_\gamma^{-1}\,\mathrm{B}_{\overrightarrow{H}} \\ 0 & \mathrm{G}_{\overrightarrow{H}} & -\mathrm{B} \end{pmatrix} \quad \textit{where } \mathcal{A}_\gamma = \begin{pmatrix} 0 & \mathrm{I} & 0 \\ -\mathrm{P}_\gamma^{-1}\,\mathbb{A} & 0 & \mathrm{P}_\gamma^{-1}\,\mathrm{B}_{\overrightarrow{H}} \\ 0 & -\mathrm{G}_{\overrightarrow{H}} & -\mathrm{B} \end{pmatrix} \tag{28}$$

with $\mathcal{D}(\mathcal{A}_\gamma^*) = \mathcal{D}(\mathcal{A}_\gamma)$.

Proof. We define the operator $\mathcal{T}_\gamma : \mathbf{H}_\gamma \to \mathbf{H}_\gamma$ as

$$\mathcal{T}_\gamma := \begin{pmatrix} 0 & -\mathrm{I} & 0 \\ \mathrm{P}_\gamma^{-1}\,\mathbb{A} & 0 & -\mathrm{P}_\gamma^{-1}\,\mathrm{B}_{\overrightarrow{H}} \\ 0 & \mathrm{G}_{\overrightarrow{H}} & -\mathrm{B} \end{pmatrix}, \quad \text{with } \mathcal{D}(\mathcal{T}_\gamma) = \mathcal{D}(\mathcal{A}_\gamma). \tag{29}$$

Let $[\phi_0, \phi_1, \boldsymbol{\psi}_0]$, $[\widetilde{\phi}_0, \widetilde{\phi}_1, \widetilde{\boldsymbol{\psi}}_0] \in \mathcal{D}(\mathcal{A}_\gamma)$. Then we compute

$$\begin{aligned}
\left\langle \mathcal{A}_\gamma \begin{bmatrix} \phi_0 \\ \phi_1 \\ \boldsymbol{\psi}_0 \end{bmatrix}, \begin{bmatrix} \widetilde{\phi}_0 \\ \widetilde{\phi}_1 \\ \widetilde{\boldsymbol{\psi}}_0 \end{bmatrix} \right\rangle_{\mathbf{H}_\gamma} &= \left(\mathbb{A}^{1/2}\phi_1, \mathbb{A}^{1/2}\widetilde{\phi}_0\right)_{\mathrm{L}^2(\Omega)} + \left(\mathrm{P}_\gamma^{1/2}\left(-\mathrm{P}_\gamma^{-1}\,\mathbb{A}\phi_0 + \mathrm{P}_\gamma^{-1}\,\mathrm{B}(\boldsymbol{\psi}_0)\cdot\overrightarrow{H}\right), \mathrm{P}_\gamma^{1/2}\,\widetilde{\phi}_1\right)_{\mathrm{L}^2(\Omega)} \\
&\qquad - \left(\mathrm{G}(\phi_1\overrightarrow{H}) + \mathrm{B}\,\boldsymbol{\psi}_0, \widetilde{\boldsymbol{\psi}}_0\right)_{\mathbf{L}^2(\Omega)} \\
&= \left(\phi_1, \mathbb{A}\widetilde{\phi}_0\right) - \left(\mathrm{P}_\gamma^{1/2}\,\mathrm{P}_\gamma^{-1}\,\mathbb{A}\phi_0, \mathrm{P}_\gamma^{1/2}\,\widetilde{\phi}_1\right) + \left(\mathrm{P}_\gamma^{1/2}\,\mathrm{P}_\gamma^{-1}\,\mathrm{B}(\boldsymbol{\psi}_0)\cdot\overrightarrow{H}, \mathrm{P}_\gamma^{1/2}\,\widetilde{\phi}_1\right) \\
&\qquad - \left(\mathrm{G}(\phi_1\overrightarrow{H}), \widetilde{\boldsymbol{\psi}}_0\right) - \left(\mathrm{B}\,\boldsymbol{\psi}_0, \widetilde{\boldsymbol{\psi}}_0\right).
\end{aligned} \tag{30}$$

Now we apply Lemma A.2 to handle the terms $\left(\mathrm{P}_\gamma^{1/2}\,\mathrm{P}_\gamma^{-1}\,\mathrm{B}(\boldsymbol{\psi}_0)\cdot\overrightarrow{H}, \mathrm{P}_\gamma^{1/2}\,\widetilde{\phi}_1\right)$, $\left(\mathrm{G}(\phi_1\overrightarrow{H}), \widetilde{\boldsymbol{\psi}}_0\right)$ and continue as follows.

$$\begin{aligned}
(30) &= \left(\mathbb{A}^{1/2}\phi_0, -\mathbb{A}^{1/2}\widetilde{\phi}_1\right) + \left(\mathrm{P}_\gamma^{1/2}\,\phi_1, \mathrm{P}_\gamma^{1/2}\,\mathrm{P}_\gamma^{-1}\,\mathbb{A}\widetilde{\phi}_2\right) + \left(\boldsymbol{\psi}_0, \mathrm{G}(\widetilde{\phi}_1\overrightarrow{H})\right) \\
&\qquad - \left(\mathrm{P}_\gamma^{1/2}\,\phi_1, \mathrm{P}_\gamma^{1/2}\,\mathrm{P}_\gamma^{-1}\,\mathrm{B}(\widetilde{\boldsymbol{\psi}}_0)\cdot\overrightarrow{H}\right) - \left(\boldsymbol{\psi}_0, \mathrm{B}\,\widetilde{\boldsymbol{\psi}}_0\right) \\
&= \left(\mathbb{A}^{1/2}\phi_0, -\mathbb{A}^{1/2}\widetilde{\phi}_1\right) + \left(\mathrm{P}_\gamma^{1/2}\,\phi_1, \mathrm{P}_\gamma^{1/2}\,\mathrm{P}_\gamma^{-1}\,\mathbb{A}\widetilde{\phi}_2 - \mathrm{P}_\gamma^{1/2}\,\mathrm{P}_\gamma^{-1}\,\mathrm{B}(\widetilde{\boldsymbol{\psi}}_0)\cdot\overrightarrow{H}\right) \\
&\qquad + \left(\boldsymbol{\psi}_0, \mathrm{G}(\widetilde{\phi}_1\overrightarrow{H}) - \mathrm{B}\,\widetilde{\boldsymbol{\psi}}_0\right) \\
&= \left\langle \begin{bmatrix} \phi_0 \\ \phi_1 \\ \boldsymbol{\psi}_0 \end{bmatrix}, \mathcal{T}_\gamma \begin{bmatrix} \widetilde{\phi}_0 \\ \widetilde{\phi}_1 \\ \widetilde{\boldsymbol{\psi}}_0 \end{bmatrix} \right\rangle_{\mathbf{H}_\gamma}.
\end{aligned} \tag{31}$$

This shows that

$$\mathcal{D}(\mathcal{T}_\gamma) \subset \mathcal{D}(\mathcal{A}_\gamma^*) \text{ and } \mathcal{A}_\gamma^*|_{\mathcal{D}(\mathcal{T}_\gamma)} = \mathcal{T}_\gamma. \tag{32}$$

Next, we determine the inverse $\mathcal{A}_\gamma^{-1}$ of $\mathcal{A}_\gamma$ and then compute the adjoint $(\mathcal{A}_\gamma^{-1})^*$. This approach is employed because $\mathcal{A}_\gamma^{-1}$ is a bounded operator on $\mathbf{H}_\gamma$, allowing us to calculate $(\mathcal{A}_\gamma^{-1})^* = (\mathcal{A}_\gamma^*)^{-1}$ without the need to specify domain conditions. Specifically, the adjoint of a bounded linear operator on the Hilbert space $\mathbf{H}_\gamma$ is itself bounded and defined on the entire space, enabling straightforward computation via the conjugate transpose of its matrix representation.

For this purpose we explicitly compute the inverse of $\mathcal{A}_\gamma \in \mathcal{L}(\mathbf{H}_\gamma; \mathcal{D}(\mathcal{A}_\gamma))$

$$\mathcal{A}_\gamma^{-1} = \begin{pmatrix} -\mathbb{A}^{-1}\,\mathrm{B}_{\overrightarrow{H}}\,\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}} & -\mathbb{A}^{-1}\,\mathrm{P}_\gamma & -\mathbb{A}^{-1}\,\mathrm{B}_{\overrightarrow{H}}\,\mathrm{B}^{-1} \\ \mathrm{I} & 0 & 0 \\ -\,\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}} & 0 & -\,\mathrm{B}^{-1} \end{pmatrix}. \tag{33}$$

Now we compute its Hilbert space adjoint $(\mathcal{A}_\gamma^*)^{-1} \in \mathcal{L}(\mathcal{D}(\mathcal{A}_\gamma^*); \mathbf{H}_\gamma)$ as follows:

$$\begin{aligned}
\left\langle \mathcal{A}_\gamma^{-1} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \boldsymbol{\psi}_0 \end{bmatrix}, \begin{bmatrix} \widetilde{\phi}_0 \\ \widetilde{\phi}_1 \\ \widetilde{\boldsymbol{\psi}}_0 \end{bmatrix} \right\rangle_{\mathbf{H}_\gamma} &= \left\langle \begin{pmatrix} -\mathbb{A}^{-1}\,\mathrm{B}_{\overrightarrow{H}}\,\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\phi_0 - \mathbb{A}^{-1}\,\mathrm{P}_\gamma\,\phi_1 - \mathbb{A}^{-1}\,\mathrm{B}_{\overrightarrow{H}}\,\mathrm{B}^{-1}\,\boldsymbol{\psi}_0 \\ \phi_0 \\ -\,\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\phi_0 - \mathrm{B}^{-1}\,\boldsymbol{\psi}_0 \end{pmatrix}, \begin{bmatrix} \widetilde{\phi}_0 \\ \widetilde{\phi}_1 \\ \widetilde{\boldsymbol{\psi}}_0 \end{bmatrix} \right\rangle_{\mathbf{H}_\gamma} \\
&= \left(\mathbb{A}^{1/2}\left(-\mathbb{A}^{-1}\,\mathrm{B}_{\overrightarrow{H}}\,\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\phi_0 - \mathbb{A}^{-1}\,\mathrm{P}_\gamma\,\phi_1 - \mathbb{A}^{-1}\,\mathrm{B}_{\overrightarrow{H}}\,\mathrm{B}^{-1}\,\boldsymbol{\psi}_0\right), \mathbb{A}^{1/2}\widetilde{\phi}_0\right) \\
&\quad + \left(\mathrm{P}_\gamma^{1/2}\,\phi_0, \mathrm{P}_\gamma^{1/2}\,\widetilde{\phi}_1\right) - \left(\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\phi_0 + \mathrm{B}^{-1}\,\boldsymbol{\psi}_0, \widetilde{\boldsymbol{\psi}}_0\right) \\
&= -\left(\mathrm{B}_{\overrightarrow{H}}\,\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\phi_0, \widetilde{\phi}_0\right) - \left(\mathrm{P}_\gamma\,\phi_1, \widetilde{\phi}_0\right) - \left(\mathrm{B}_{\overrightarrow{H}}\,\mathrm{B}^{-1}\,\boldsymbol{\psi}_0, \widetilde{\phi}_0\right) \\
&\quad + \left(\mathrm{P}_\gamma^{1/2}\,\phi_0, \mathrm{P}_\gamma^{1/2}\,\widetilde{\phi}_1\right) - \left(\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\phi_0, \widetilde{\boldsymbol{\psi}}_0\right) - \left(\mathrm{B}^{-1}\,\boldsymbol{\psi}_0, \widetilde{\boldsymbol{\psi}}_0\right).
\end{aligned} \tag{34}$$

The term $\left(\mathrm{B}_{\overrightarrow{H}}\,\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\phi_0, \widetilde{\phi}_0\right)$: To handle this term we successively apply integration by parts as shown in Lemma A.1. This the following computations, we drop the L^2-part from the underlying spaces $\mathrm{L}^2(\Omega), \mathrm{L}^2(\Gamma)$ since the context is clear.

$$\begin{aligned}
\left(\mathrm{B}_{\overrightarrow{H}}\,\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\phi_0, \widetilde{\phi}_0\right)_\Omega &= \left(\mathrm{rot}\,\mathrm{rot}\left(\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\phi_0\right)\cdot \overrightarrow{H}, \widetilde{\phi}_0\right)_\Omega \\
&= \left(\mathrm{rot}\,\mathrm{rot}\left(\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\phi_0\right), \left(\widetilde{\phi}_0\cdot \overrightarrow{H}\right)\right)_\Omega \\
&= \left(\mathrm{rot}\left(\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\phi_0\right), \mathrm{rot}\left(\widetilde{\phi}_0\cdot \overrightarrow{H}\right)\right)_\Omega + \left(\nu\times\mathrm{rot}\left(\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\phi_0\right), \left(\widetilde{\phi}_0\cdot \overrightarrow{H}\right)\right)_\Gamma.
\end{aligned}$$

The boundary term $\nu\times\mathrm{rot}\left(\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\phi_0\right) = 0$ since $\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\phi_0 \in \mathcal{D}(\mathrm{B})$. Then we again apply integration by parts to obtain

$$\left(\mathrm{B}_{\overrightarrow{H}}\,\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\phi_0, \widetilde{\phi}_0\right)_\Omega = \left(\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\phi_0, \mathrm{rot}\,\mathrm{rot}\left(\widetilde{\phi}_0\cdot \overrightarrow{H}\right)\right)_\Omega - \left(\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\phi_0, \nu\times\mathrm{rot}\left(\widetilde{\phi}_0\cdot \overrightarrow{H}\right)\right)_\Gamma.$$

Again the boundary term $\mathrm{rot}\left(\widetilde{\phi}_0\cdot \overrightarrow{H}\right)$ vanishes as shown in Lemma A.2. Then we have

$$\begin{aligned}
\left(\mathrm{B}_{\overrightarrow{H}}\,\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\phi_0, \widetilde{\phi}_0\right)_\Omega &= \left(\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\phi_0, \mathrm{G}_{\overrightarrow{H}}\,\widetilde{\phi}_0\right)_\Omega \\
&= \left(\mathrm{G}_{\overrightarrow{H}}\,\phi_0, \mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\widetilde{\phi}_0\right)_\Omega \qquad \text{(by the boundedness of } \mathrm{B}^{-1}\text{)} \\
&= \left(\mathrm{rot}\,\mathrm{rot}\left(\phi_0\overrightarrow{H}\right), \mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\widetilde{\phi}_0\right)_\Omega
\end{aligned}$$

$$
\begin{aligned}
&= \left(\operatorname{rot}\left(\phi_0 \overrightarrow{H}\right), \operatorname{rot}\left(\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\widetilde{\phi}_0\right)\right)_\Omega + \left(\nu \times \operatorname{rot}\left(\phi_0 \overrightarrow{H}\right), \left(\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\widetilde{\phi}_0\right)\right)_\Gamma \\
&= \left(\left(\phi_0 \overrightarrow{H}\right), \operatorname{rot}\operatorname{rot}\left(\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\widetilde{\phi}_0\right)\right)_\Omega - \left(\phi_0 \overrightarrow{H}, \nu \times \operatorname{rot}\left(\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\widetilde{\phi}_0\right)\right)_\Gamma \\
&= \left(\phi_0, \mathrm{B}_{\overrightarrow{H}}\,\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\widetilde{\phi}_0\right)_\Omega. \qquad (35)
\end{aligned}
$$

The boundary terms from integration by parts vanish due to the same reasoning as above. Arguing in a similar manner and using Lemma A.2, we get

$$
\left(\mathrm{B}_{\overrightarrow{H}}\,\mathrm{B}^{-1}\,\boldsymbol{\psi}_0, \widetilde{\phi}_0\right) = \left(\boldsymbol{\psi}_0, \mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\widetilde{\phi}_0\right) \text{ and } \left(\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\phi_0, \widetilde{\boldsymbol{\psi}}_0\right) = \left(\phi_0, \mathrm{B}_{\overrightarrow{H}}\,\mathrm{B}^{-1}\,\widetilde{\boldsymbol{\psi}}_0\right). \qquad (36)
$$

Then by combining (34), (35) and (36), we obtain

$$
(\mathcal{A}_\gamma^*)^{-1} = \begin{pmatrix} -\mathbb{A}^{-1}\,\mathrm{B}_{\overrightarrow{H}}\,\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}} & \mathbb{A}^{-1}\,\mathrm{P}_\gamma & -\mathbb{A}^{-1}\,\mathrm{B}_{\overrightarrow{H}}\,\mathrm{B}^{-1} \\ -\,\mathrm{I} & 0 & 0 \\ -\,\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}} & 0 & -\,\mathrm{B}^{-1} \end{pmatrix}. \qquad (37)
$$

Thus for $[\phi_0, \phi_1, \boldsymbol{\psi}_0] \in \mathbf{H}_\gamma$, we have that

$$
\begin{aligned}
(\mathcal{A}_\gamma^*)^{-1} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \boldsymbol{\psi}_0 \end{bmatrix} &= \begin{pmatrix} -\mathbb{A}^{-1}\,\mathrm{B}_{\overrightarrow{H}}\,\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}} & \mathbb{A}^{-1}\,\mathrm{P}_\gamma & -\mathbb{A}^{-1}\,\mathrm{B}_{\overrightarrow{H}}\,\mathrm{B}^{-1} \\ -\,\mathrm{I} & 0 & 0 \\ -\,\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}} & 0 & -\,\mathrm{B}^{-1} \end{pmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \boldsymbol{\psi}_0 \end{bmatrix} \\
&= \begin{pmatrix} -\mathbb{A}^{-1}\,\mathrm{B}_{\overrightarrow{H}}\,\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\phi_0 + \mathbb{A}^{-1}\,\mathrm{P}_\gamma\,\phi_1 - \mathbb{A}^{-1}\,\mathrm{B}_{\overrightarrow{H}}\,\mathrm{B}^{-1}\,\boldsymbol{\psi}_0 \\ -\phi_1 \\ -\,\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\phi_0 - \mathrm{B}^{-1}\,\boldsymbol{\psi}_0 \end{pmatrix}. \qquad (38)
\end{aligned}
$$

This shows that

$$
\mathcal{D}(\mathcal{A}_\gamma^*) \subset \mathrm{H}_0^2(\Omega) \times \mathrm{H}_0^2(\Omega) \times \mathcal{D}(\mathrm{B}). \qquad (39)
$$

Additionally, we have

$$
-\mathbb{A}^{-1}\,\mathrm{B}_{\overrightarrow{H}}\,\mathrm{B}^{-1}\,\mathrm{G}_{\overrightarrow{H}}\,\phi_0 + \mathbb{A}^{-1}\,\mathrm{P}_\gamma\,\phi_1 - \mathbb{A}^{-1}\,\mathrm{B}_{\overrightarrow{H}}\,\mathrm{B}^{-1}\,\boldsymbol{\psi}_0 \in \mathrm{H}_{0,\gamma}^1(\Omega). \qquad (40)
$$

Combining the inclusions (32), (39), and by the definition of $\mathcal{D}(\mathcal{T}_\gamma)$ in (29), we conclude that the adjoint operator is precisely the one given in (28). □

From the explicit characterization of $\mathcal{A}_\gamma^*$ established above, we can now state the following corollary to the Theorem 2.1.

Corollary 2.3. *For the parameter $\gamma \geq 0$, let $\mathcal{A}_\gamma^*$ be the adjoint operator to $\mathcal{A}_\gamma$ as defined in* (28)*. Then $\mathcal{A}_\gamma^*$ generates a C_0-semigroup of contractions $\{e^{\mathcal{A}_\gamma^* t}\}_{t\geq 0}$ on the energy space $\mathbf{H}_\gamma$. Let $[\phi(t), \phi_t(t), \boldsymbol{\psi}(t)] = e^{\mathcal{A}_\gamma^*(T-t)}[\phi_0, \phi_1, \boldsymbol{\psi}_0]$ denote the adjoint solution trajectory.*

1. *If terminal data* $[\phi_0, \phi_1, \boldsymbol{\psi}_0] \in \mathbf{H}_\gamma$*, then the adjoint problem is well-posed in the energy space* $\mathbf{H}_\gamma$*. The unique weak solution* $[\phi, \phi_t, \boldsymbol{\psi}]$ *satisfies*

$$[\phi, \phi_t, \boldsymbol{\psi}] \in \mathrm{C}([0,T]; \mathbf{H}_\gamma). \tag{41}$$

2. *If terminal data* $[\phi_0, \phi_1, \boldsymbol{\psi}_0] \in \mathcal{D}(\mathcal{A}_\gamma)$*, then the solution* $[\phi, \phi_t, \boldsymbol{\psi}]$ *satisfies*

$$[\phi, \phi_t, \boldsymbol{\psi}] \in \mathrm{C}([0,T]; \mathcal{D}(\mathcal{A}_\gamma)) \cap \mathrm{C}^1([0,T]; \mathbf{H}_\gamma). \tag{42}$$

Now, using the form of the control operator given in (17), one has the adjoint $\mathcal{B}^* \in \mathcal{L}(\mathcal{D}(\mathcal{A}_\gamma), \mathbf{H}^1(\Omega))$ taking the form

$$\mathcal{B}^* \begin{bmatrix} \phi_0 \\ \phi_1 \\ \boldsymbol{\psi}_0 \end{bmatrix} = \boldsymbol{\psi}_0. \tag{43}$$

With the inequality (27), for of the adjoints in (28) and (43), and the definitions (A.1), (A.2) for the components of $\mathcal{A}_\gamma$ and $\mathcal{A}_\gamma^*$, one has then that the solution $[\phi, \phi_t, \boldsymbol{\psi}] \in \mathrm{C}([0,T]; \mathbf{H}_\gamma)$ to the PDE system (3) is given by

$$e^{\mathcal{A}_\gamma^*(T-t)} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \boldsymbol{\psi}_0 \end{bmatrix} = \begin{bmatrix} \phi(t) \\ \phi_t(t) \\ \boldsymbol{\psi}(t) \end{bmatrix}. \tag{44}$$

Moreover, we have explicitly

$$\mathcal{L}_T^* \begin{bmatrix} \phi_0 \\ \phi_1 \\ \boldsymbol{\psi}_0 \end{bmatrix} (t) = \mathcal{B}^* e^{\mathcal{A}_\gamma^*(T-t)} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \boldsymbol{\psi}_0 \end{bmatrix} = \boldsymbol{\psi}(t). \tag{45}$$

With these definitions, to establish the surjectivity of $\mathcal{L}_T$, it suffices to show the existence of a constant $C_T > 0$ such that the following injectivity condition holds for all $[\phi_0, \phi_1, \psi_0] \in \mathbf{H}_\gamma$:

$$\left\| \mathcal{L}_T^* \begin{bmatrix} \phi_0 \\ \phi_1 \\ \boldsymbol{\psi}_0 \end{bmatrix} \right\|_{\mathrm{L}^2(0,T;\mathbf{H}^1(\Omega))} \geq C_T \left\| \begin{bmatrix} \phi_0 \\ \phi_1 \\ \boldsymbol{\psi}_0 \end{bmatrix} \right\|_{\mathbf{H}_\gamma}. \tag{46}$$

Using equation (27), this is equivalent to proving the observability estimate (2) for arbitrary terminal data $[\phi_0, \phi_1, \psi_0] \in \mathbf{H}_\gamma$. The next section is devoted to establishing this inequality.

Now we define the energy of the system (3) as follows:

$$E_\gamma(t) \equiv \frac{1}{2} \left\| e^{\mathcal{A}_\gamma^*(T-t)} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \boldsymbol{\psi}_0 \end{bmatrix} \right\|_{\mathbf{H}_\gamma}^2, \quad \text{for } 0 \leq t \leq T, \tag{47a}$$

or

$$E_\gamma(t) \equiv \frac{1}{2} \left(\|\phi(t)\|^2_{\mathrm{H}_0^2(\Omega)} + \|\phi_t(t)\|^2_{\mathrm{H}^1_{0,\gamma}(\Omega)} + \|\boldsymbol{\psi}(t)\|^2_{\mathbf{L}^2(\Omega)} \right) \tag{47b}$$

so that in particular

$$E_\gamma(T) = \frac{1}{2} \|[\phi_0, \phi_1, \boldsymbol{\psi}_0]\|_{\mathbf{H}_\gamma}, \text{ and } E_\gamma(0) = \frac{1}{2} \|[\phi(0), \phi_t(0), \boldsymbol{\psi}(0)]\|_{\mathbf{H}_\gamma}.$$

Furthermore, we can obtain the following energy relation

Proposition 2.4. *The component $\boldsymbol{\psi}$ of the weak solution $[\phi, \phi_t, \boldsymbol{\psi}]$ to the backward system (3) satisfies $\boldsymbol{\psi} \in \mathrm{L}^2(0, T; \mathbf{H}^1(\Omega))$. Moreover, for all initial data $[\phi_0, \phi_1, \boldsymbol{\psi}_0] \in \mathbf{H}_\gamma$ and for all $0 \le t \le T$, the following identity holds:*

$$E_\gamma(t) + \int_t^T \|\mathrm{rot}\, \boldsymbol{\psi}(\tau)\|_{\mathbf{L}^2(\Omega)} \, \mathrm{d}\tau = E_\gamma(T). \tag{48}$$

Proof. By computing the energy of the dual system (3) we obtain

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left[\|\Delta\phi\|^2_{\mathrm{L}^2(\Omega)} + \|\phi_t\|^2_{\mathrm{L}^2(\Omega)} + \gamma \|\nabla\phi\|^2_{\mathrm{L}^2(\Omega)} + \|\boldsymbol{\psi}\|^2_{\mathbf{L}^2(\Omega)} \right] = \|\mathrm{rot}\, \boldsymbol{\psi}(t)\|^2_{\mathbf{L}^2(\Omega)}.$$

This gives (48). □

Remark 1. *Recall the operator $\mathcal{L}_T^*$ from (45). For $\mathcal{L}_T^* \in \mathcal{L}(\mathbf{H}_\gamma, \mathrm{L}^2(0, T; \mathbf{H}^1(\Omega)))$ to hold, it is necessary that the following condition is satisfied*

$$\|\boldsymbol{\psi}\|^2_{\mathrm{L}^2(0,T;\mathbf{H}^1(\Omega))} \le C \|[\phi_0, \phi_1, \boldsymbol{\psi}_0]\|_{\mathbf{H}_\gamma}.$$

Then one can observe that the Proposition 2.4 implies this condition. This implies $\mathcal{L}_T \in \mathcal{L}(\mathrm{L}^2(0, T; \mathbf{H}^1(\Omega)'), \mathbf{H}_\gamma)$ by duality.

3 Proof of Theorem 1.1

3.1 Orientation

We now proceed to construct a multiplier argument in order to establish the following lemma. To this end, we consider a suitable multiplier of operator type, namely $\mathrm{A}_D^{-1} \boldsymbol{\psi}$, and take the L^2-inner product of this multiplier with equation (3a) from the system (3). One may also view such multiplier as a pseudo-differential type; similar multiplier techniques were used in [6, Lemma 3.2] and [9, Lemma 2.2]. Upon performing integration by parts, we obtain a set of terms that will be estimated using Lemmas 3.1, A.4 together with standard Sobolev embedding theorems. These estimates yield the desired inequality (52) in the lemma.

However, the resulting estimate, as formulated in (52), includes residual or "polluting" terms that prevent a direct conclusion. To address this, we employ a compactness–uniqueness argument (see Proposition 3.3) to eliminate these undesired contributions. This argument then yields the clean estimate required for the conclusion of Theorem 1.1, thereby completing the proof.

Remark 2. *To rigorously justify the integration-by-parts and the estimates used in the proof of Lemma 3.2 (and consequently in the proof of Theorem 1.1), we first establish the tainted observability inequality under a stronger regularity assumption on the initial data of the adjoint system. The terms evaluated on the boundary such as*

$$\gamma\left(\nabla\phi_t, \nabla\, \mathrm{A}_D^{-1}\, \psi_{i,t}\right)\Big|_0^T, \text{ and } \left.\frac{\partial\phi_{tt}}{\partial\nu}\right|_\Gamma = 0 \text{ in } (54)$$

require additional regularity. More specifically in the above cases we need $\nabla\phi_t \in \mathrm{C}([0,T];\mathrm{L}^2(\Omega))$ and $\phi_{tt} \in \mathrm{C}([0,T];\mathcal{D}(\mathbb{A}^{1/2}))$. Therefore, we temporarily assume

$$[\phi_0, \phi_1, \boldsymbol{\psi}_0] \in \mathcal{D}(\mathcal{A}_\gamma^2).$$

For such data the corresponding (backward) solution $[\phi, \phi_t, \boldsymbol{\psi}]$ is classical and enjoys the regularity

$$\begin{aligned} &\phi \in \mathrm{C}([0,T];\mathcal{D}(\mathcal{A}_\gamma)),\\ &\phi_t \in \mathrm{C}([0,T];\mathrm{H}^3(\Omega)\cap\mathrm{H}_0^2(\Omega)),\\ &\boldsymbol{\psi} \in \mathrm{C}([0,T];\mathbf{H}^3(\Omega)\cap\mathbf{L}^2_{\sigma,\nu}(\Omega)). \end{aligned} \tag{49}$$

With this regularity all formal computations performed in Lemmas 3.1–3.2 are fully justified. Since $\mathcal{D}(\mathcal{A}_\gamma^2)$ is dense in the finite-energy space $\mathbf{H}_\gamma$, a standard density argument (see for example [6, 9]) extends the observability inequality, and hence the exact controllability result of Theorem 1.1, to arbitrary initial and terminal data in $\mathbf{H}_\gamma$.

3.2 Main ingredients of the proof

First, we sketch the proof of the following trace regularity result for the adjoint system (3). This result does not follow from the standard Sobolev trace theory but is instead derived in a similar spirit to the trace results obtained for Euler-Bernoulli plates [24] and Kirchhoff plates [18]. Notably, this result is essential for deriving the observability estimate (2).

Lemma 3.1 (Sharp Trace Regularity)**.** *The component ϕ of the solution $[\phi(t), \phi_t(t), \boldsymbol{\psi}(t)]$ of (3) satisfies $\Delta\phi|_\Gamma \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma))$, with the estimate*

$$\int_0^T \|\Delta\phi\|_{\mathrm{L}^2(\Gamma)}\,\mathrm{d}t \le C\left(\int_0^T E_\gamma(t)\mathrm{d}t + \int_0^T \|\mathrm{rot}\,\boldsymbol{\psi}\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t + E_\gamma(T)\right), \tag{50}$$

where C is independent of the parameter γ.

Proof. Following the methodology of [8, Lemma 2.3] (see also [7, Lemma 2]), we sketch the proof. We multiply the first equation of (3) by $h\cdot\nabla\phi$, where $h(x,y) \equiv [h_1(x,y), h_2(x,y)]$ is a smooth vector field in $\mathrm{C}^2(\bar{\Omega})$ satisfying $h|_\Gamma = [\nu_1, \nu_2]$ on Γ. Integrating over $[0,T]\times\Omega$, we obtain

$$\int_0^T \left(\phi_{tt} - \gamma\Delta\phi_{tt} + \Delta^2\phi - (\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{\psi})\cdot\overrightarrow{H}, h\cdot\nabla\phi\right)_{\mathrm{L}^2(\Omega)}\mathrm{d}t = 0. \tag{51}$$

The crucial term requiring special treatment is the $\int_0^T \left((\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{\psi})\cdot\overrightarrow{H}, h\cdot\nabla\phi\right)_{\mathrm{L}^2(\Omega)} \mathrm{d}t$ term. Using Lemma A.1, we decompose this as

$$
\begin{aligned}
&\int_0^T \left((\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{\psi})\cdot\overrightarrow{H}, h\cdot\nabla\phi\right)_{\mathrm{L}^2(\Omega)} \mathrm{d}t \\
&\qquad = \int_0^T \left(\nu\times\mathrm{rot}\,\boldsymbol{\psi}, (h\cdot\nabla\phi)\overrightarrow{H}\right)_{\mathrm{L}^2(\Gamma)} \mathrm{d}t + \int_0^T \left(\mathrm{rot}\,\boldsymbol{\psi}, \mathrm{rot}\left[(h\cdot\nabla\phi)\overrightarrow{H}\right]\right)_{\mathbf{L}^2(\Omega)} \mathrm{d}t \\
&\qquad = \int_0^T \left(\mathrm{rot}\,\boldsymbol{\psi}, \mathrm{rot}\left[(h\cdot\nabla\phi)\overrightarrow{H}\right]\right)_{\mathbf{L}^2(\Omega)} \mathrm{d}t.
\end{aligned}
$$

The boundary integral in the penultimate step vanishes due to $\nu\times\mathrm{rot}\,\boldsymbol{\psi}=0$ on Γ. The remaining interior term, combined with the other terms arising from integration by parts, yields the estimate (50) through standard arguments (see [7, Lemma 2] for the detailed computation). □

We then obtain the following lemma, which provides the observability inequality, albeit tainted by certain lower-order terms, as indicated in the orientation.

Lemma 3.2 (Tainted Observability Inequality). *For $\gamma \geq 0$ and $T>0$, the solution $[\phi,\phi_t,\boldsymbol{\psi}]$ to (3) satisfies the following estimate*

$$
E_\gamma(T) \leq C_T \left(\int_0^T \|\mathrm{rot}\,\boldsymbol{\psi}\|_{\mathrm{L}^2(\Omega)} \mathrm{d}t + \|\phi\|^2_{\mathrm{C}([0,T];\mathrm{H}^1_{0,\gamma}(\Omega))} + \gamma\,\|\phi_t\|^2_{\mathrm{C}([0,T];\mathrm{L}^2(\Omega))} + \|\boldsymbol{\psi}\|^2_{\mathrm{C}([0,T];\mathbf{H}^1(\Omega)')} \right). \tag{52}
$$

Proof. We multiply the first equation of (3) by $\mathrm{A}_D^{-1}\psi_i$ where ψ_i is the i-th component of the magnetic vector field $\boldsymbol{\psi}$. Then, integrating over both space and time, we obtain

$$
\int_0^T \left(\phi_{tt} - \gamma\Delta\phi_{tt} + \Delta^2\phi - (\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{\psi})\cdot\overrightarrow{H}, \mathrm{A}_D^{-1}\psi_i\right)_{\mathrm{L}^2(\Omega)} \mathrm{d}t = 0. \tag{53}
$$

We then proceed to estimate the individual terms of this expression separately.

(i) *Estimating the term $\int_0^T (\phi_{tt}-\gamma\Delta\phi_{tt}, \mathrm{A}_D^{-1}\psi_i)_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t$.* We use integration by parts.

$$
\begin{aligned}
&\int_0^T (\phi_{tt}-\gamma\Delta\phi_{tt}, \mathrm{A}_D^{-1}\psi_i)_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t \\
&\qquad = \int_0^T (\phi_{tt}, \mathrm{A}_D^{-1}\psi_i)_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t - \gamma\int_0^T (\Delta\phi_{tt}, \mathrm{A}_D^{-1}\psi_i)_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t \\
&\qquad = (\phi_t, \mathrm{A}_D^{-1}\psi_i)\Big|_0^T - \int_0^T (\phi_t, \mathrm{A}_D^{-1}\psi_{i,t})_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t - \gamma\int_0^T (\Delta\phi_{tt}, \mathrm{A}_D^{-1}\psi_i)_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t \\
&\qquad = (\phi_t, \mathrm{A}_D^{-1}\psi_i)\Big|_0^T - \int_0^T (\phi_t, \mathrm{A}_D^{-1}\psi_{i,t})_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t - \gamma\int_0^T \left(\frac{\partial\phi_{tt}}{\partial\nu}, \mathrm{A}_D^{-1}\psi_i\right)_{\mathrm{L}^2(\Gamma)} \mathrm{d}t
\end{aligned}
$$

$$+\gamma\int_0^T \left(\nabla\phi_{tt}, \nabla\, \mathrm{A}_D^{-1}\,\psi_i\right)_{\mathrm{L}^2(\Omega)} \mathrm{d}t$$

By using $\left.\dfrac{\partial\phi_{tt}}{\partial\nu}\right|_{\Gamma} = 0$, we further obtain

$$\begin{aligned}
&\int_0^T \left(\phi_{tt} - \gamma\Delta\phi_{tt}, \mathrm{A}_D^{-1}\,\psi_i\right)_{\mathrm{L}^2(\Omega)} \mathrm{d}t \\
&\quad = \left(\phi_t, \mathrm{A}_D^{-1}\,\psi_i\right)\Big|_0^T - \int_0^T \left(\phi_t, \mathrm{A}_D^{-1}\,\psi_i\right)_{\mathrm{L}^2(\Omega)} \mathrm{d}t + \gamma\int_0^T \left(\nabla\phi_{tt}, \nabla\, \mathrm{A}_D^{-1}\,\psi_i\right)_{\mathrm{L}^2(\Omega)} \mathrm{d}t \\
&\quad = \left(\phi_t, \mathrm{A}_D^{-1}\,\psi_i\right)\Big|_0^T + \gamma\left(\nabla\phi_t, \nabla\, \mathrm{A}_D^{-1}\,\psi_{i,t}\right)\Big|_0^T \\
&\qquad - \int_0^T \left[\left(\phi_t, \mathrm{A}_D^{-1}\,\psi_{i,t}\right)_{\mathrm{L}^2(\Omega)} + \gamma\left(\nabla\phi_t, \nabla\, \mathrm{A}_D^{-1}\,\psi_{i,t}\right)_{\mathrm{L}^2(\Omega)}\right] \mathrm{d}t. \qquad (54)
\end{aligned}$$

Recalling (16b), we derive an explicit expression for $\mathrm{A}_D^{-1}\,\psi_{i,t}$ as follows. Consider the equation (16b) in the following abstract formulation

$$\boldsymbol{\psi}_t = \mathrm{G}(\overrightarrow{H}\phi_t) - \mathrm{B}\,\boldsymbol{\psi}.$$

Then we consider the component-wise version of the above vector equation. Note that the component-wise action on G on $\overrightarrow{H}\phi_t$ is denoted by G_i. We need to emphasize that $\mathrm{G}_i \neq \mathrm{G}_j$ if $i \neq j$ necessarily. Then we obtain the component-wise version of the above equation as follows:

$$\psi_{i,t} = \mathrm{G}_i(\overrightarrow{H}\phi_t) - \Delta\psi_i \quad \text{for } i = 1, 2.$$

Then we apply A_D^{-1} to obtain

$$\mathrm{A}_D^{-1}\,\psi_{i,t} = \mathrm{A}_D^{-1}\,\mathrm{G}_i(\overrightarrow{H}\phi_t) - \mathrm{A}_D^{-1}\,\Delta\psi_i \quad \text{for } i = 1, 2.$$

Then we rewrite (54) and proceed as follows.

$$\begin{aligned}
&\int_0^T \left(\phi_{tt} - \gamma\Delta\phi_{tt}, \mathrm{A}_D^{-1}\,\psi_i\right)_{\mathrm{L}^2(\Omega)} \mathrm{d}t \\
&\quad = \left(\phi_t, \mathrm{A}_D^{-1}\,\psi_i\right)\Big|_0^T + \gamma\left(\nabla\phi_t, \nabla\, \mathrm{A}_D^{-1}\,\psi_i\right)\Big|_0^T \\
&\qquad + \int_0^T \Bigg[\left(\phi_t, \mathrm{A}_D^{-1}\,\mathrm{G}_i(\overrightarrow{H}\phi_t) - \mathrm{A}_D^{-1}\,\Delta\psi_i\right)_{\mathrm{L}^2(\Omega)} \\
&\qquad\qquad + \gamma\left(\nabla\phi_t, \nabla\left(\mathrm{A}_D^{-1}\,\mathrm{G}_i(\overrightarrow{H}\phi_t) - \mathrm{A}_D^{-1}\,\Delta\psi_i\right)\right)_{\mathrm{L}^2(\Omega)}\Bigg] \mathrm{d}t. \qquad (55)
\end{aligned}$$

Consider the first two terms of (55). Applying Green's theorem and utilizing the fact that $\phi_t \in \mathrm{H}_0^1(\Omega)$ for $\gamma > 0$ and $t \in [0, T]$, we obtain

$$\sum_{i=1}^{2} \gamma \left(\nabla \phi_t(t), \nabla \mathrm{A}_D^{-1} \psi_i(t)\right)_{\mathrm{L}^2(\Omega)} \leq \sum_{i=1}^{2} \gamma \left(\phi_t(t), \psi_i(t)\right)$$
$$\leq \gamma C_\varepsilon \left\|\phi_t\right\|^2_{\mathrm{C}([0,T];\mathrm{L}^2(\Omega))} + \gamma \frac{\varepsilon}{16} \left\|\boldsymbol{\psi}(t)\right\|^2_{\mathbf{L}^2(\Omega)}$$
$$\leq \frac{\varepsilon}{8} E_\gamma(T) + \gamma C_\varepsilon \left\|\phi_t\right\|^2_{\mathrm{C}([0,T];\mathrm{L}^2(\Omega))} \quad (56)$$

and

$$\sum_{i=1}^{2} \left(\phi_t(t), \mathrm{A}_D^{-1} \psi_i(t)\right)_{\mathrm{L}^2(\Omega)} \leq \frac{\varepsilon}{16} \left\|\phi_t\right\|^2_{\mathrm{H}^1_{0,\gamma}(\Omega)} + C_\varepsilon \sum_{i=1}^{2} \left\|\mathrm{A}_D^{-1} \psi_i\right\|^2_{\mathbf{L}^2(\Omega)}$$
$$\leq \frac{\varepsilon}{16} \left\|\phi_t\right\|^2_{\mathrm{H}^1_{0,\gamma}(\Omega)} + C_\varepsilon \left\|\boldsymbol{\psi}\right\|^2_{\mathbf{H}^1(\Omega)'} \leq \frac{\varepsilon}{8} E_\gamma(T) + C_\varepsilon \left\|\boldsymbol{\psi}\right\|^2_{\mathrm{C}([0,T];\mathbf{H}^1(\Omega)')}, \quad (57)$$

where we have used the contraction of the semigroup $\{e^{\mathcal{A}_\gamma^* t}\}_{t\geq 0}$. Then we estimate the remaining integral terms of (55) by utilizing Lemma A.4.

$$\int_0^T \left[\left(\phi_t, \mathrm{A}_D^{-1} \mathrm{G}_i(\overrightarrow{H}\phi_t)\right)_{\mathrm{L}^2(\Omega)} - \gamma \left(\nabla \phi_t, \nabla \left(\mathrm{A}_D^{-1} \mathrm{G}_i(\overrightarrow{H}\phi_t)\right)\right)_{\mathrm{L}^2(\Omega)}\right] \mathrm{d}t$$
$$\lesssim \int_0^T \left[\left\|\phi_t\right\|_{\mathrm{L}^2(\Omega)} \left\|\nabla \phi_t\right\|_{\mathrm{L}^2(\Omega)} + \gamma \left\|\nabla \phi_t\right\|^2_{\mathrm{L}^2(\Omega)}\right] \mathrm{d}t$$
$$\leq C_1 \int_0^T \left[\left\|\phi_t\right\|^2_{\mathrm{L}^2(\Omega)} + \gamma \left\|\nabla \phi_t\right\|^2_{\mathrm{L}^2(\Omega)}\right] \mathrm{d}t + \frac{\varepsilon}{16} \int_0^T \gamma \left\|\nabla \phi_t\right\|^2_{\mathrm{L}^2(\Omega)} \mathrm{d}t. \quad (58)$$

In the previous step, the constant $C_1 > 0$ is taken sufficiently large, and the parameter $\varepsilon > 0$ is chosen small enough so that the second integral appears with coefficients ε and the (fixed) parameter $\gamma > 0$, allowing it to be absorbed into the left-hand side of the energy inequality. We again estimate

$$\int_0^T \left[\left(\phi_t, \mathrm{A}_D^{-1} \Delta\psi_i\right)_{\mathrm{L}^2(\Omega)} - \gamma \left(\nabla \phi_t, \nabla \left(\mathrm{A}_D^{-1} \Delta\psi_i\right)\right)_{\mathrm{L}^2(\Omega)}\right] \mathrm{d}t$$
$$\lesssim \int_0^T \left[\left\|\phi_t\right\|_{\mathrm{L}^2(\Omega)} \left\|\nabla \psi_i\right\|_{\mathrm{L}^2(\Omega)} + \gamma \left\|\nabla \phi_t\right\|_{\mathrm{L}^2(\Omega)} \left\|\nabla \psi_i\right\|_{\mathrm{L}^2(\Omega)}\right] \mathrm{d}t. \quad (59)$$

Then by taking sum over components ψ_i for $i = 1, 2$,

$$\sum_{i=1}^{2} \int_0^T \left[\left(\phi_t, \mathrm{A}_D^{-1} \Delta\psi_i\right)_{\mathrm{L}^2(\Omega)} - \gamma \left(\nabla \phi_t, \nabla \left(\mathrm{A}_D^{-1} \Delta\psi_i\right)\right)_{\mathrm{L}^2(\Omega)}\right] \mathrm{d}t$$
$$\leq C_2 \int_0^T \left[\left\|\phi_t\right\|_{\mathrm{L}^2(\Omega)} \left\|\boldsymbol{\psi}\right\|_{\mathbf{H}^1(\Omega)} + \gamma \left\|\nabla \phi_t\right\|_{\mathrm{L}^2(\Omega)} \left\|\boldsymbol{\psi}\right\|_{\mathbf{H}^1(\Omega)}\right] \mathrm{d}t. \quad (60)$$

Now we combine the estimates (55)-(59) to obtain

$$\left|\sum_{i=1}^{2} \int_0^T \left(\phi_{tt} - \gamma\Delta\phi_{tt}, \mathrm{A}_D^{-1} \psi_i\right)_{\mathrm{L}^2(\Omega)} \mathrm{d}t - C_1 \int_0^T \left[\left\|\phi_t\right\|^2_{\mathrm{L}^2(\Omega)} + \gamma \left\|\nabla \phi_t\right\|^2_{\mathrm{L}^2(\Omega)}\right] \mathrm{d}t\right|$$

$$\leq C_2 \int_0^T \Big[\|\phi_t\|_{\mathrm{L}^2(\Omega)} \|\boldsymbol{\psi}\|_{\mathbf{H}^1(\Omega)} + \gamma \|\nabla \phi_t\|_{\mathrm{L}^2(\Omega)} \|\boldsymbol{\psi}\|_{\mathbf{H}^1(\Omega)} \Big] \, \mathrm{d}t$$

$$+ \frac{\varepsilon}{16} \int_0^T \gamma \|\nabla \phi_t\|^2_{\mathrm{L}^2(\Omega)} \, \mathrm{d}t + \frac{\varepsilon}{4} E_\gamma(T) + C_\varepsilon \left(\gamma \|\phi_t\|^2_{\mathrm{C}([0,T];\mathrm{L}^2(\Omega))} + \|\boldsymbol{\psi}\|^2_{\mathrm{C}([0,T];\mathbf{H}^1(\Omega)')} \right)$$

$$\leq \frac{\varepsilon}{16} \int_0^T \Big[\|\phi_t\|^2_{\mathrm{L}^2(\Omega)} + \gamma \|\nabla \phi_t\|^2_{\mathrm{L}^2(\Omega)} \Big] \, \mathrm{d}t + \hat{C}_\varepsilon \int_0^T \|\boldsymbol{\psi}\|^2_{\mathbf{H}^1(\Omega)} \, \mathrm{d}t + \frac{\varepsilon}{16} \int_0^T \gamma \|\nabla \phi_t\|^2_{\mathrm{L}^2(\Omega)} \, \mathrm{d}t$$

$$+ \frac{\varepsilon}{4} E_\gamma(T) + C_\varepsilon \left(\gamma \|\phi_t\|^2_{\mathrm{C}([0,T];\mathrm{L}^2(\Omega))} + \|\psi\|^2_{\mathrm{C}([0,T];\mathbf{H}^1(\Omega)')} \right) \tag{61}$$

$$\leq C \left(\int_0^T \|\mathrm{rot}\, \boldsymbol{\psi}\|^2_{\mathbf{L}^2(\Omega)} \, \mathrm{d}t + \gamma \|\phi_t\|^2_{\mathrm{C}([0,T];\mathrm{L}^2(\Omega))} + \|\boldsymbol{\psi}\|^2_{\mathrm{C}([0,T];\mathbf{H}^1(\Omega)')} \right)$$

$$+ \frac{\varepsilon}{4} \left(\int_0^T E_\gamma(t) \mathrm{d}t + E_\gamma(T) \right) \tag{62}$$

where the constant C is independent of γ, where $0 \leq \gamma \leq M$.

(ii) *Estimating the term* $\int_0^T \big(\Delta^2 \phi, \mathrm{A}_D^{-1} \psi_i\big) \, \mathrm{d}t$. We compute using Green's second identity and the fact that $\mathrm{A}_D^{-1} \psi_i\big|_\Gamma = 0$:

$$\int_0^T \big(\Delta^2 \phi, \mathrm{A}_D^{-1} \psi_i\big) \, \mathrm{d}t$$

$$= \int_0^T \big(\Delta \phi, \Delta\, \mathrm{A}_D^{-1} \psi_i\big)_{\mathrm{L}^2(\Omega)} \, \mathrm{d}t + \int_0^T \left(\frac{\partial \Delta \phi}{\partial \nu}, \mathrm{A}_D^{-1} \psi_i \right)_{\mathrm{L}^2(\Gamma)} \mathrm{d}t - \int_0^T \left(\Delta \phi, \frac{\partial\, \mathrm{A}_D^{-1} \psi_i}{\partial \nu} \right)_{\mathrm{L}^2(\Gamma)} \mathrm{d}t$$

$$= \int_0^T (\Delta \phi, \psi_i)_{\mathrm{L}^2(\Omega)} \, \mathrm{d}t + \int_0^T \left(\frac{\partial \Delta \phi}{\partial \nu}, \mathrm{A}_D^{-1} \psi_i \right)_{\mathrm{L}^2(\Gamma)} \mathrm{d}t - \int_0^T \left(\Delta \phi, \frac{\partial\, \mathrm{A}_D^{-1} \psi_i}{\partial \nu} \right)_{\mathrm{L}^2(\Gamma)} \mathrm{d}t.$$

Since $\mathrm{A}_D^{-1} \psi_i|_\Gamma = 0$, we have

$$\int_0^T \big(\Delta^2 \phi, \mathrm{A}_D^{-1} \psi_i\big) \, \mathrm{d}t = \int_0^T (\Delta \phi, \psi_i)_{\mathrm{L}^2(\Omega)} \, \mathrm{d}t - \int_0^T \left(\Delta \phi, \frac{\partial\, \mathrm{A}_D^{-1} \psi_i}{\partial \nu} \right)_{\mathrm{L}^2(\Gamma)} \mathrm{d}t. \tag{63}$$

Now we estimate the second integral of (63) by using elliptic regularity.

$$\left| \int_0^T \left(\Delta \phi, \frac{\partial\, \mathrm{A}_D^{-1} \psi_i}{\partial \nu} \right)_{\mathrm{L}^2(\Gamma)} \mathrm{d}t \right| \leq \frac{\varepsilon}{8C} \int_0^T \|\Delta \phi\|^2_{\mathrm{L}^2(\Gamma)} \, \mathrm{d}t + C_\varepsilon \int_0^T \left\| \frac{\partial\, \mathrm{A}_D^{-1} \psi_i}{\partial \nu} \right\|^2_{\mathrm{L}^2(\Gamma)} \mathrm{d}t$$

$$\leq \frac{\varepsilon}{8C} \int_0^T \|\Delta \phi\|^2_{\mathrm{L}^2(\Gamma)} \, \mathrm{d}t + C_\varepsilon \int_0^T \big\|\mathrm{A}_D^{-1} \psi_i\big\|^2_{\mathrm{H}^{3/2}(\Omega)} \, \mathrm{d}t$$

$$\leq \frac{\varepsilon}{8C} \int_0^T \|\Delta \phi\|^2_{\mathrm{L}^2(\Gamma)} \, \mathrm{d}t + C_\varepsilon \int_0^T \|\psi_i\|^2_{\mathrm{H}^{-1/2}(\Omega)} \, \mathrm{d}t. \tag{64}$$

The preceding estimate relied on the sharp trace regularity result in Lemma 3.1 and classical trace theory, with the constant C_ε reused throughout. To further estimate the final inequality, we invoke the norm equivalence provided in Lemma A.3, which is explained below.

$$\begin{aligned}\left|\sum_{i=1}^{2}\int_0^T \left(\Delta^2\phi, \mathrm{A}_D^{-1}\psi_i\right)\mathrm{d}t\right| &\le C_0\int_0^T \|\Delta\phi\|_{\mathrm{L}^2(\Omega)}\|\boldsymbol{\psi}\|_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t\\ &\quad+\frac{\varepsilon}{8C}\int_0^T\|\Delta\phi\|^2_{\mathrm{L}^2(\Gamma)}\,\mathrm{d}t+\hat{C}_\varepsilon\int_0^T\|\mathrm{rot}\,\boldsymbol{\psi}\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t\\ &\le C_0\int_0^T \|\Delta\phi\|_{\mathrm{L}^2(\Omega)}\|\boldsymbol{\psi}\|_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t\\ &\quad+C_\varepsilon\int_0^T\|\mathrm{rot}\,\boldsymbol{\psi}\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t+\frac{\varepsilon}{8}\left(\int_0^T E_\gamma(t)\mathrm{d}t+E(T)\right)\\ &\le C_\varepsilon\int_0^T\|\mathrm{rot}\,\boldsymbol{\psi}\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t+\frac{\varepsilon}{4}\left(\int_0^T E_\gamma(t)\mathrm{d}t+E(T)\right).\end{aligned}\tag{65}$$

(iii) *Estimating the term* $\int_0^T \left(\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{\psi}\cdot\overrightarrow{H}, \mathrm{A}_D^{-1}\psi_i\right)_{\mathbf{L}^2(\Omega)}\mathrm{d}t$: By invoking Lemmas A.1 and A.3 we estimate

$$\begin{aligned}\int_0^T \left(\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{\psi}\cdot\overrightarrow{H}, \mathrm{A}_D^{-1}\psi_i\right)_{\mathbf{L}^2(\Omega)}\mathrm{d}t &= \int_0^T\left(\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{\psi}, (\mathrm{A}_D^{-1}\psi_i)\overrightarrow{H}\right)_{\mathbf{L}^2(\Omega)}\mathrm{d}t\\ &= \int_0^T\left(\mathrm{rot}\,\boldsymbol{\psi}, \mathrm{rot}\left[(\mathrm{A}_D^{-1}\psi_i)\overrightarrow{H}\right]\right)_{\mathbf{L}^2(\Omega)}\mathrm{d}t+\int_0^T\left(\nu\times\mathrm{rot}\,\psi, (\mathrm{A}_D^{-1}\psi_i)\overrightarrow{H}\right)_{\mathbf{L}^2(\Gamma)}\mathrm{d}t\\ &= \int_0^T\left(\mathrm{rot}\,\boldsymbol{\psi}, \mathrm{rot}\left[(\mathrm{A}_D^{-1}\psi_i)\overrightarrow{H}\right]\right)_{\mathbf{L}^2(\Omega)}\mathrm{d}t\\ &\lesssim \int_0^T\|\boldsymbol{\psi}\|_{\mathbf{H}^1(\Omega)}\|\boldsymbol{\psi}\|_{\mathbf{H}^{-1}(\Omega)}\,\mathrm{d}t\\ &\le C_\varepsilon\int_0^T\|\mathrm{rot}\,\boldsymbol{\psi}\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t+\frac{\varepsilon}{2}\int_0^T\|\boldsymbol{\psi}\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t.\end{aligned}\tag{66}$$

(iv) Combining (53), (62), (65), and (66), we obtain the following estimate: For any arbitrary $\varepsilon > 0$ sufficiently small, there exists a constant $C > 0$, independent of γ, such that the solution $[\phi, \phi_t, \boldsymbol{\psi}]$ of (3) satisfies

$$\begin{aligned}&\int_0^T\left[\|\phi_t\|^2_{\mathrm{L}^2(\Omega)}+\gamma\|\nabla\phi_t\|^2_{\mathrm{L}^2(\Omega)}\right]\mathrm{d}t\\ &\quad\le C\left(\int_0^T\|\mathrm{rot}\,\boldsymbol{\psi}\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t+\gamma\|\phi_t\|^2_{\mathrm{C}([0,T];\mathrm{L}^2(\Omega))}+\|\boldsymbol{\psi}\|^2_{\mathrm{C}([0,T];\mathbf{H}^{-1}(\Omega))}\right)\\ &\qquad+\varepsilon\left(\int_0^T E_\gamma(t)\mathrm{d}t+E_\gamma(T)\right)\end{aligned}\tag{67}$$

where the dependence of C on ε has been omitted, as it is not crucial to the analysis.

(v) Now, we multiply the first equation of (3) by ϕ and apply integrate by parts, and integrate from 0 to T with respect to time to obtain the following relation

$$
\begin{aligned}
(\phi_t,\phi)_{\mathrm{H}^1_{0,\gamma}(\Omega)} - \int_0^T \left[\|\phi_t\|^2_{\mathrm{L}^2(\Omega)} + \gamma\,\|\nabla\phi\|^2_{\mathrm{L}^2(\Omega)}\right]\mathrm{d}t \\
= -\int_0^T \|\Delta\phi\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t + \int_0^T \left((\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{\psi})\cdot\overrightarrow{H},\phi\right)_{\mathbf{L}^2(\Omega)}\mathrm{d}t \\
= -\int_0^T \|\Delta\phi\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t + \int_0^T \left(\mathrm{rot}\,\boldsymbol{\psi},\mathrm{rot}(\phi\overrightarrow{H})\right)_{\mathbf{L}^2(\Omega)}\mathrm{d}t \\
\leq -(1-\varepsilon)\int_0^T \|\Delta\phi\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t + C_\varepsilon\int_0^T \|\mathrm{rot}\,\boldsymbol{\psi}\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t. \quad (68)
\end{aligned}
$$

Now we estimate $(\phi_t,\phi)_{\mathrm{H}^1_{0,\gamma}(\Omega)}$ using the contraction of the underlying semigroup $e^{\mathcal{A}_\gamma t}$ for all $t\in[0,T]$:

$$
(\phi_t,\phi)_{\mathrm{H}^1_{0,\gamma}(\Omega)} \leq \frac{\varepsilon}{2}\,\|\phi_t(t)\|^2_{\mathrm{H}^1_{0,\gamma}(\Omega)} + C_\varepsilon\,\|\phi(t)\|^2_{\mathrm{H}^1_{0,\gamma}(\Omega)} \leq \varepsilon E_\gamma(T) + C_\varepsilon\,\|\phi(t)\|^2_{\mathrm{H}^1_0(\Omega)}. \quad (69)
$$

Then by combining (68), and (69), we obtain the following estimate for sufficiently small $\varepsilon>0$

$$
\begin{aligned}
\int_0^T \|\Delta\phi\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t \leq \int_0^T \left[\|\phi_t\|^2_{\mathrm{L}^2(\Omega)} + \gamma\,\|\nabla\phi\|^2_{\mathrm{L}^2(\Omega)}\right]\mathrm{d}t \\
+ C_T\left(\int_0^T \|\mathrm{rot}\,\boldsymbol{\psi}\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t + \|\phi\|^2_{\mathrm{C}([0,T];\mathrm{H}^1_0(\Omega))}\right) + \varepsilon E(T) \quad (70)
\end{aligned}
$$

where the ε-dependence of C_T is omitted.

Thus, for sufficiently small ε, by combining (67) and (70), and recalling the definition of the energy $E_\gamma(t)$ in (47), we obtain the existence of a constant $C_\varepsilon>0$ such that

$$
\begin{aligned}
\int_0^T \left[\|\Delta\phi\|^2_{\mathrm{L}^2(\Omega)} + \|\phi_t\|^2_{\mathrm{L}^2(\Omega)} + \gamma\,\|\nabla\phi_t\|^2_{\mathrm{L}^2(\Omega)} + \|\boldsymbol{\psi}\|^2_{\mathbf{L}^2(\Omega)}\right]\mathrm{d}t \leq C_\varepsilon E_\gamma(T) \\
+ C_T\left(\int_0^T \|\mathrm{rot}\,\boldsymbol{\psi}\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t + \|\phi\|^2_{\mathrm{C}([0,T];\mathrm{H}^1_0(\Omega))} + \gamma\,\|\phi_t\|^2_{\mathrm{C}([0,T];\mathrm{L}^2(\Omega))} + \|\boldsymbol{\psi}\|^2_{\mathrm{C}([0,T];\mathbf{H}^1(\Omega)')}\right). \quad (71)
\end{aligned}
$$

Applying the relation (48) and using the property that $E_\gamma(t)\geq E_\gamma(0)$ for all $t\in[0,T]$, we obtain

$$
\begin{aligned}
T\left(E_\gamma(T) - \int_0^T \|\mathrm{rot}\,\boldsymbol{\psi}\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t\right) = TE_\gamma(0) \leq \int_0^T E_\gamma(t)\mathrm{d}t \\
\leq C_\varepsilon E_\gamma(T) + C_T\left(\int_0^T \|\mathrm{rot}\,\boldsymbol{\psi}\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t\right.
\end{aligned}
$$

$$+\|\phi\|^2_{C([0,T];H^1_0(\Omega))}+\gamma\|\phi_t\|^2_{C([0,T];L^2(\Omega))}+\|\boldsymbol{\psi}\|^2_{C([0,T];\mathbf{H}^1(\Omega)')}\Big). \quad (72)$$

By further selecting $\varepsilon>0$ sufficiently small, we obtain the following estimate for all $\gamma\geq 0$ and $0<T<\infty$:

$$E_\gamma(T)\leq C_T\left(\int_0^T\|\mathrm{rot}\,\boldsymbol{\psi}\|_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t+\|\phi\|^2_{C([0,T];H^1_0(\Omega))}+\gamma\|\phi_t\|^2_{C([0,T];L^2(\Omega))}+\|\boldsymbol{\psi}\|^2_{C([0,T];\mathbf{H}^1(\Omega)')}\right). \quad (73)$$

This proves the Lemma. □

Remark 3. *In the concluding estimate of the preceding Lemma, the expression comprises three terms:* $\|\phi\|^2_{C([0,T];H^1_0(\Omega))}$, $\|\phi_t\|^2_{C([0,T];L^2(\Omega))}$, *and* $\|\boldsymbol{\psi}\|^2_{C([0,T];\mathbf{H}^1(\Omega)')}$. *These terms must be eliminated to establish the target observability inequality* (2)*. To achieve this, we employ a subsequent Proposition that leverages a compactness-uniqueness argument. This approach exploits the compact embedding properties of the relevant function spaces and the uniqueness of solutions to an associated homogeneous problem, thereby enabling the absorption of these terms into the desired inequality.*

Proposition 3.3 (Elimination of the Polluting Terms)**.** *The inequality* (73) *implies the existence of a constant* $C_T>0$ *such that the corresponding solution* $[\phi,\phi_t,\boldsymbol{\psi}]$ *of* (3) *satisfies*

$$\|\phi\|^2_{C([0,T];H^1_0(\Omega))}+\gamma\|\phi_t\|^2_{C([0,T];L^2(\Omega))}+\|\boldsymbol{\psi}\|^2_{C([0,T];\mathbf{H}^1(\Omega)')}\leq C_T\int_0^T\|\mathrm{rot}\,\boldsymbol{\psi}\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t. \quad (74)$$

Proof. The proof of this proposition follows the same argumentation as in [6, Prop 3.3]. But due to the magnetic boundary conditions at hand, several technicalities must be properly handled. Due to that reason, we give a full proof for this proposition here. We proceed by contradiction. If the proposition is false, then there exists a sequence $\left\{\left[\phi_0^{(n)},\phi_1^{(n)},\boldsymbol{\psi}_0^{(n)}\right]\right\}_{n=1}^\infty\subset\mathbf{H}_\gamma$, and a corresponding solution sequence $\left\{\left[\phi^{(n)},\phi_t^{(n)},\boldsymbol{\psi}^{(n)}\right]\right\}_{n=1}^\infty$ to (3) which satisfies

$$\|\phi\|^2_{C([0,T];H^1_0(\Omega))}+\gamma\|\phi_t\|^2_{C([0,T];L^2(\Omega))}+\|\psi\|^2_{C([0,T];\mathbf{H}^1(\Omega)')}=1 \text{ for all } n \quad (75)$$

and

$$\int_0^T\|\mathrm{rot}\,\boldsymbol{\psi}\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t\to 0 \text{ as } n\to\infty. \quad (76)$$

Equations (73), (75) and (76) imply that there exists a constant

$$E_\gamma^{(n)}(T)\leq C \text{ uniformly in } n. \quad (77)$$

Then, we can find a subsequence in $\mathbf{H}_\gamma$, still denoted by $\left\{\left[\phi^{(n)},\phi_t^{(n)},\boldsymbol{\psi}^{(n)}\right]\right\}_{n=1}^\infty$, and $\left\{\left[\widetilde{\phi}_0,\widetilde{\phi}_1,\widetilde{\boldsymbol{\psi}}_0\right]\right\}\in\mathbf{H}_\gamma$ such that

$$\left[\phi^{(n)},\phi_t^{(n)},\boldsymbol{\psi}^{(n)}\right]\to\left[\widetilde{\phi}_0,\widetilde{\phi}_1,\widetilde{\boldsymbol{\psi}}_0\right] \text{ weakly in } \mathbf{H}_\gamma. \quad (78)$$

Moreover, if we denote $\left[\widetilde{\phi}, \widetilde{\phi}_t, \widetilde{\boldsymbol{\psi}}\right]$ to be the solution to (3), corresponding to the terminal data $\left[\widetilde{\phi}_0, \widetilde{\phi}_1, \widetilde{\boldsymbol{\psi}}_0\right]$, then a fortiori,

$$\left[\phi^{(n)}, \phi_t^{(n)}, \boldsymbol{\psi}^{(n)}\right] \to \left[\widetilde{\phi}_0, \widetilde{\phi}_1, \widetilde{\boldsymbol{\psi}}_0\right] \text{ weak* in } \mathrm{L}^\infty(0,T;\mathbf{H}_\gamma). \tag{79}$$

Now we need to prove the following regularity estimate

$$\|\phi_{tt}\|_{\mathrm{L}^\infty(0,T;[\mathcal{D}(\mathbb{A}^{1/2}\,\mathrm{P}_\gamma^{-1})]')} \le C\,\|[\phi_0,\phi_1,\boldsymbol{\psi}_0]\|_{\mathbf{H}_\gamma}. \tag{80}$$

For terminal data $[\phi_0,\phi_1,\boldsymbol{\psi}_0] \in \mathcal{D}(\mathcal{A}_\gamma^*)$, for any test function $\hat{\phi} \in \mathrm{L}^1(0,T;\mathcal{D}(\mathbb{A}^{1/2}\,\mathrm{P}_\gamma^{-1}))$, satisfying $\mathrm{P}_\gamma^{-1}\,\hat{\phi} \in \mathrm{L}^1(0,T;\mathcal{D}(\mathbb{A}^{1/2}))$ with P_γ in (11), we employ the abstract equation (16a) and duality arguments to derive the required estimate, as detailed in the subsequent calculations.

$$\begin{aligned}
\int_0^T \left(\hat{\phi}, \phi_{tt}\right)_{\mathrm{L}^2(\Omega)} \mathrm{d}t &= \int_0^T \left(\hat{\phi}, \mathrm{P}_\gamma^{-1}\left(-\mathbb{A}\phi + \mathrm{B}_{\overrightarrow{H}}\,\boldsymbol{\psi}\right)\right)\mathrm{d}t \\
&= \int_0^T \left[-\left(\mathbb{A}^{1/2}\,\mathrm{P}_\gamma^{-1}\,\hat{\phi}, \mathbb{A}^{1/2}\phi\right) + \left(\mathrm{G}_{\overrightarrow{H}}(\mathrm{P}_\gamma^{-1}\,\hat{\phi}), \boldsymbol{\psi}\right)\right]\mathrm{d}t \\
&\le C\left(\|\phi\|_{\mathrm{C}([0,T];\mathcal{D}(\mathbb{A}^{1/2}))}\left\|\hat{\phi}\right\|_{\mathrm{L}^1(0,T;\mathcal{D}(\mathbb{A}^{1/2}\,\mathrm{P}_\gamma^{-1}))} + \|\boldsymbol{\psi}\|_{\mathrm{C}([0,T];\mathbf{L}^2(\Omega))}\left\|\hat{\phi}\right\|_{\mathrm{L}^1(0,T;\mathcal{D}(\mathrm{G}_{\overrightarrow{H}}\,\mathrm{P}_\gamma^{-1}))}\right) \\
&\le C\left(\|\phi\|_{\mathrm{C}([0,T];\mathcal{D}(\mathbb{A}^{1/2}))} + \|\boldsymbol{\psi}\|_{\mathrm{C}([0,T];\mathbf{L}^2(\Omega))}\right)\left\|\hat{\phi}\right\|_{\mathrm{L}^1(0,T;\mathcal{D}(\mathbb{A}^{1/2}\,\mathrm{P}_\gamma^{-1}))}.
\end{aligned}$$

In obtaining the last estimate, we have used the fact $\mathcal{D}(\mathbb{A}^{1/2}) \subset \mathcal{D}(\mathrm{G}_{\overrightarrow{H}})$. By leveraging the contractivity of the semigroup $\{e^{\mathcal{A}_\gamma^* t}\}_{t\ge 0}$, we derive

$$\int_0^T \left(\hat{\phi}, \phi_{tt}\right)_{\mathrm{L}^2(\Omega)} \mathrm{d}t \le C\,\|[\phi_0,\phi_1,\boldsymbol{\psi}_0]\|_{\mathbf{H}_\gamma}\left\|\hat{\phi}\right\|_{\mathrm{L}^1(0,T;\mathcal{D}(\mathbb{A}^{1/2}\,\mathrm{P}_\gamma^{-1}))}. \tag{81}$$

Then the density argument yields the bound (80). Arguing in a similar manner and utilizing the equation (16b), we obtain

$$\|\boldsymbol{\psi}_t\|_{\mathrm{L}^\infty(0,T;[\mathcal{D}(\mathrm{B})]')} \le C\,\|[\phi_0,\phi_1,\boldsymbol{\psi}_0]\|_{\mathbf{H}_\gamma}. \tag{82}$$

Then (80) and (82) yield

$$\left\|\phi_{tt}^{(n)}\right\|_{\mathrm{L}^\infty(0,T;[\mathcal{D}(\mathbb{A}^{1/2}\,\mathrm{P}_\gamma^{-1})]')} \le C, \text{ and } \left\|\boldsymbol{\psi}_t^{(n)}\right\|_{\mathrm{L}^\infty(0,T;[\mathcal{D}(\mathrm{B})]')} \le C. \tag{83}$$

The boundedness of $\{\left[\phi_{tt}^{(n)}, \boldsymbol{\psi}_t^{(n)}\right]\}$ and that of $\{\left[\phi^{(n)}, \phi_t^{(n)}, \boldsymbol{\psi}^{(n)}\right]\}$ allow the use of Simon's compactness result in [32] so as to have

$$\phi^{(n)} \to \widetilde{\phi} \quad \text{strongly in} \quad \mathrm{C}([0,T];\mathrm{H}_0^1(\Omega)), \tag{84}$$

$$\phi_t^{(n)} \to \widetilde{\phi}_t \quad \text{strongly in} \quad \mathrm{C}([0,T];\mathrm{L}^2(\Omega)) \quad \text{if } \gamma > 0, \tag{85}$$

$$\boldsymbol{\psi}^{(n)} \to \widetilde{\boldsymbol{\psi}} \quad \text{strongly in} \quad \mathrm{C}([0,T];\mathbf{H}^1(\Omega)'). \tag{86}$$

We then pass to the limit in (75) to get

$$\left\|\widetilde{\phi}\right\|^2_{\mathrm{C}([0,T];\mathrm{H}^1_0(\Omega))} + \gamma \left\|\widetilde{\phi}_t\right\|^2_{\mathrm{C}([0,T];\mathrm{L}^2(\Omega))} + \left\|\widetilde{\boldsymbol{\psi}}\right\|^2_{\mathrm{C}([0,T];\mathbf{H}^1(\Omega)')} = 1. \tag{87}$$

Conversely, the convergence established in (76) implies that $\widetilde{\boldsymbol{\psi}} = \mathbf{0}$. This, combined with the magnetic equation (16b), yields $\widetilde{\phi}_t = 0$. In turn, the coupled plate equation (16a) yields $\widetilde{\phi} = 0$. Thus, we obtain $\left[\widetilde{\phi}, \widetilde{\phi}_t, \widetilde{\boldsymbol{\psi}}\right] = [0, 0, \mathbf{0}]$, which contradicts the equality in (87). □

Remark 4 (Unique Continuation Property)**.** *At the conclusion of the proof of Proposition 3.3, we observe that the vanishing of $\widetilde{\boldsymbol{\psi}}$ ensures uniqueness through straightforward elliptic regularity arguments. This embodies the unique continuation property essential for controllability in our system. Since the control $\mathbf{u}$ acts across the entire interior in* (1b)*, these implications follow directly. However, if the control's support were restricted to a proper subset of the interior or confined to the boundary, the complexity of establishing controllability would increase significantly, as evident in related thermoelastic systems; see [14].*

3.3 Completion of the proof of Theorem 1.1

By combining the tainted observability inequality from Lemma 3.2 with the estimate of the polluting terms provided in Lemma 3.3, we obtain the desired bound, which completes the proof of Theorem 1.1. □

Appendix A Technical Lemmas

Lemma A.1 (Integration by parts)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and outward unit normal ν. For all vector fields $\mathbf{u}, \mathbf{v} \in \mathbf{L}^2(\Omega)$ and $\operatorname{rot}\mathbf{u}, \operatorname{rot}\mathbf{v} \in \mathbf{L}^2(\Omega)$, the following identities hold:*

$$\int_\Omega \operatorname{rot}\mathbf{u} \cdot \operatorname{rot}\mathbf{v}\,\mathrm{d}x = \int_\Omega \operatorname{rot}\operatorname{rot}\mathbf{u} \cdot \mathbf{v}\,\mathrm{d}x + \int_{\partial\Omega} (\nu \times \operatorname{rot}\mathbf{u}) \cdot \mathbf{v}\,\mathrm{d}S. \tag{88a}$$

$$\int_\Omega (\operatorname{rot}\operatorname{rot}\mathbf{u} \cdot \mathbf{v} - \operatorname{rot}\operatorname{rot}\mathbf{v} \cdot \mathbf{u})\,\mathrm{d}x = \int_{\partial\Omega} \big((\nu \times \operatorname{rot}\mathbf{u}) \cdot \mathbf{v} - (\nu \times \operatorname{rot}\mathbf{v}) \cdot \mathbf{u}\big)\,\mathrm{d}S. \tag{88b}$$

Proof. Both identities follow from the vector calculus identity

$$\nabla \cdot \big((\operatorname{rot}\mathbf{u}) \times \mathbf{v}\big) = (\operatorname{rot}\operatorname{rot}\mathbf{u}) \cdot \mathbf{v} - (\operatorname{rot}\mathbf{u}) \cdot (\operatorname{rot}\mathbf{v}),$$

together with the divergence theorem and exchanging $\mathbf{u}$ and $\mathbf{v}$ for symmetry. □

Lemma A.2. *The following duality relation holds:*

$$\left(\mathrm{G}_{\overrightarrow{H}}\,\phi,\boldsymbol{\psi}\right)=\left(\phi,\mathrm{B}_{\overrightarrow{H}}\,\boldsymbol{\psi}\right)\ \textit{for}\ \phi\in\mathcal{D}(\mathrm{G}_{\overrightarrow{H}}),\ \boldsymbol{\psi}\in\mathcal{D}(\mathrm{B}_{\overrightarrow{H}}).\tag{89}$$

Proof. Let us compute

$$\begin{aligned}\left\langle\mathrm{G}_{\overrightarrow{H}}\,\phi,\boldsymbol{\psi}\right\rangle&=\int_\Omega\mathrm{rot}\,\mathrm{rot}(\phi\overrightarrow{H})\cdot\boldsymbol{\psi}\,\mathrm{d}x\\&=\int_\Omega(\phi\overrightarrow{H})\cdot\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{\psi}\,\mathrm{d}x+\int_\Gamma\left[\left(\nu\times\mathrm{rot}(\phi\overrightarrow{H})\right)\cdot\boldsymbol{\psi}-(\nu\times\mathrm{rot}\,\boldsymbol{\psi})\cdot(\phi\overrightarrow{H})\right]\mathrm{d}S.\end{aligned}$$

Since $\boldsymbol{\psi}_0\in\mathcal{D}(\mathrm{B}_{\overrightarrow{H}})=\mathcal{D}(\mathrm{B})$, we have $(\nu\times\mathrm{rot}\,\boldsymbol{\psi})\cdot(\phi\overrightarrow{H})=0$.

Now we calculate $\int_\Gamma\left(\nu\times\mathrm{rot}(\phi\overrightarrow{H})\right)\cdot\boldsymbol{\psi}\,\mathrm{d}S$. For this purpose, let $\overrightarrow{H}=(H_1,H_2)\in\mathbb{R}^2$. Then

$$\mathrm{rot}(\phi_1 H)=\partial_1(\phi_1H_2)-\partial_2(\phi_1H_1)=(\partial_1u)H_2-(\partial_2u)H_1.$$

Writing $\overrightarrow{H}^\perp=(H_2,-H_1)$, this reads

$$\mathrm{rot}(\phi\overrightarrow{H})=\nabla\phi\cdot\overrightarrow{H}^\perp.$$

If $\phi=0$ and $\partial_\nu\phi=0$ on Γ, then $\nabla\phi=0$ on Γ, hence

$$\mathrm{rot}(\phi\overrightarrow{H})=0\text{ on }\Gamma\text{ implying }\int_\Gamma\left(\nu\times\mathrm{rot}(\phi\overrightarrow{H})\right)\cdot\boldsymbol{\psi}\,\mathrm{d}S=0.$$

This means

$$\left(\mathrm{G}(\phi_1\overrightarrow{H}),\boldsymbol{\psi}\right)=\left(\phi_1,\mathrm{G}^*(\boldsymbol{\psi})\cdot\overrightarrow{H}\right)=\left(\phi_1,\mathrm{B}(\boldsymbol{\psi})\cdot\overrightarrow{H}\right).\tag{90}$$

□

Lemma A.3. *[5, Theorem 2.1] Let $\Omega\subset\mathbb{R}^2$ be a simply connected domain with a C^2 boundary Γ. Let $\boldsymbol{\psi}\in\mathbf{H}^1(\Omega)$ satisfy $\mathrm{div}\,\boldsymbol{\psi}=0$ in Ω and either $\boldsymbol{\psi}\cdot\nu=0$ or $\nu\times\boldsymbol{\psi}=0$ on Γ, where ν is the outward unit normal. Then the $\mathbf{H}^1$ norm $\|\boldsymbol{\psi}\|_{\mathbf{H}^1(\Omega)}$ is equivalent to $\|\mathrm{rot}\,\boldsymbol{\psi}\|_{\mathbf{L}^2(\Omega)}$, i.e., there exist constants $c,C>0$ such that:*

$$c\,\|\boldsymbol{\psi}\|_{\mathbf{H}^1(\Omega)}\le\|\mathrm{rot}\,\boldsymbol{\psi}\|_{\mathbf{L}^2(\Omega)}\le C\,\|\boldsymbol{\psi}\|_{\mathbf{H}^1(\Omega)}.$$

Lemma A.4. *The following estimates hold true.*

$$\left\|\mathrm{A}_D^{-1}\,\mathrm{G}_i(\overrightarrow{H}\phi_t)\right\|_{\mathrm{L}^2(\Omega)}\lesssim\|\nabla\phi_t\|_{\mathrm{L}^2(\Omega)},\tag{91a}$$

$$\left\|\nabla\,\mathrm{A}_D^{-1}\,\mathrm{G}_i(\overrightarrow{H}\phi_t)\right\|_{\mathrm{L}^2(\Omega)}\lesssim\|\nabla\phi_t\|_{\mathrm{L}^2(\Omega)},\tag{91b}$$

$$\left\|\mathrm{A}_D^{-1}\,\Delta\psi_i\right\|_{\mathbf{L}^2(\Omega)}\lesssim\|\boldsymbol{\psi}\|_{\mathbf{H}^1(\Omega)},\tag{91c}$$

$$\left\|\nabla\,\mathrm{A}_D^{-1}\,\Delta\psi_i\right\|_{\mathbf{L}^2(\Omega)}\lesssim\|\boldsymbol{\psi}\|_{\mathbf{H}^1(\Omega)}.\tag{91d}$$

Proof. 1. To prove (91a), we estimate using elliptic regularity

$$\left\| \mathrm{B}_D^{-1}\, \mathrm{G}(\overrightarrow{H}\phi_t) \right\|_{\mathrm{L}^2(\Omega)} \lesssim \left\| \mathrm{G}(\overrightarrow{H}\phi_t) \right\|_{\mathrm{H}^{-1}(\Omega)} \lesssim \left\| \overrightarrow{H}\phi_t \right\|_{\mathrm{H}_0^1(\Omega)} \lesssim \|\nabla\phi_t\|_{\mathrm{L}^2(\Omega)} \,.$$

2. To prove (91b), we estimate as before

$$\left\| \nabla\, \mathrm{B}_D^{-1}\, \mathrm{G}(\overrightarrow{H}\phi_t) \right\|_{\mathrm{L}^2(\Omega)} \lesssim \left\| \mathrm{B}_D^{-1}\, \mathrm{G}(\overrightarrow{H}\phi_t) \right\|_{\mathbf{H}^1(\Omega)} \lesssim \left\| \mathrm{G}(\overrightarrow{H}\phi_t) \right\|_{\mathbf{H}^{-1}(\Omega)} \lesssim \|\nabla\phi_t\|_{\mathrm{L}^2(\Omega)} \,.$$

The estimates (91c) and (91d) follow from classical elliptic regularity theory. □

References

[1] M. Afilal, A. Soufyane, and M. L. de Lima Santos. Piezoelectric beams with magnetic effect and localized damping. *Mathematical Control and Related Fields*, 13(1):250–264, 2023.

[2] M. Akil. Stability of piezoelectric beam with magnetic effect under (coleman or pipkin)–gurtin thermal law. *Zeitschrift für angewandte Mathematik und Physik*, 73(6), 2022.

[3] M. Akil, S. Nicaise, A. Özer, Özkan, and V. Régnier. Stability results for novel serially-connected magnetizable piezoelectric and elastic smart-system designs. *Applied Mathematics & Optimization*, 89(3):64, 2024.

[4] M. Akil, A. Soufyane, and Y. Belhamadia. *Applied Mathematics and Optimization*, 87(2), apr 2023.

[5] C. Amrouche and N. E. H. Seloula. On the Stokes equations with the Navier–type boundary conditions. *Differential Equations & Applications*, 3(4):581–607, 2011.

[6] G. Avalos. Exact controllability of a thermoelastic system with control in the thermal component only. *Differential and Integral Equations*, 13(4-6):613 – 630, 2000.

[7] G. Avalos and I. Lasiecka. Exponential stability of a thermoelastic system without mechanical dissipation. *Rendiconti dell'Istituto di Matematica dell'Università di Trieste*, 28:1–28, 1997.

[8] G. Avalos and I. Lasiecka. Exponential stability of a thermoelastic system with free boundary conditions without mechanical dissipation. *SIAM Journal on Mathematical Analysis*, 29(1):155–182, 1998.

[9] G. Avalos and I. Lasiecka. Boundary controllability of thermoelastic plates via the free boundary conditions. *SIAM Journal on Control and Optimization*, 38(2):337–383, 2000.

[10] L. de Teresa and E. Zuazua. Controllability of the linear system of thermoelastic plates. *Advances in Differential Equations*, 1(4):369–402, 1996.

[11] M. Eller, I. Lasiecka, and R. Triggiani. Simultaneous exact/approximate boundary controllability of thermo-elastic plates with variable thermal coefficient and moment control. *Journal of Mathematical Analysis and Applications*, 251(2):452–478, 2000.

[12] M. Eller, I. Lasiecka, and R. Triggiani. Simultaneous exact/approximate boundary controllability of thermoelastic plates with variable thermal coefficient and clamped/dirichlet boundary controls. *Discrete and Continuous Dynamical Systems*, 7(2):283–302, 2001.

[13] R. Gulliver, W. Littman, and R. Triggiani. *Differential Geometric Methods in the Control of Partial Differential Equations*, volume 268 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 2000.

[14] V. Isakov. On the uniqueness of the continuation for a thermoelasticity system. *SIAM Journal on Mathematical Analysis*, 33(3):509–522, 2001. Unique continuation / Carleman estimates for thermoelastic systems.

[15] V. Komornik. *Exact Controllability and Stabilization, the Multiplier Method.* Research in Applied Mathematics. John Wiley & Sons, New York, 1994.

[16] J. E. Lagnese. The reachability problem for thermoelastic plates. *Archive for Rational Mechanics and Analysis*, 112(3):223–267, 1990.

[17] I. Lasiecka and R. Triggiani. Sharp regularity for mixed second order hyperbolic equations of Neumann type. Part I: The L^2-boundary case. *Annali di Matematica Pura ed Applicata*, 157:285–367, 1990.

[18] I. Lasiecka and R. Triggiani. Exact controllability and uniform stabilization of Kirchoff plates with boundary control only on $\Delta\omega|_\Sigma$ and homogeneous boundary displacement. *Journal of Differential Equations*, 93(1):62–101, 1991.

[19] I. Lasiecka and R. Triggiani. Exact null controllability of structurally damped and thermoelastic parabolic models. *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni*, 9(1):43–69, 1998.

[20] I. Lasiecka and R. Triggiani. *Control Theory for Partial Differential Equations: Continuous and Approximation Theories.* Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2000.

[21] I. Lasiecka and R. Triggiani. Sharp regularity for elastic and thermoelastic Kirchoff equations with free boundary conditions. *Rocky Mountain Journal of Mathematics*, 30(3):981–1024, 2000.

[22] I. Lasiecka and R. Triggiani. Exact null controllability of structurally damped and thermoelastic models. *Academia Nazionale dei Lincei. Atti. Rendiconti. Matematica e Applicazioni*, 23(4):417–472, 2015.

[23] R. Leis. *Initial Boundary Value Problems in Mathematical Physics.* John Wiley & Sons, New York, 1963.

[24] J.-L. Lions. *Contrôlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués*, volume 1. Masson, Paris, 1989.

[25] J. E. Muñoz-Rivera and R. Racke. Magneto-thermo-elasticity—large-time behavior for linear systems. *Advances in Differential Equations*, 6(4):359–384, 2001.

[26] J. E. Muñoz-Rivera and R. Racke. Polynomial stability in two-dimensional magneto-elasticity. *IMA Journal of Applied Mathematics*, 66(3):269–283, 2001.

[27] J. E. Muñoz-Rivera and R. Racke. Regularity and stabilization of magneto-elastic systems. *Applied Mathematics & Optimization*, 91(1):72, 2025.

[28] J. E. Muñoz-Rivera and M. L. Santos. Polynomial stability to three-dimensional magnetoelastic waves. *Acta Applicandae Mathematicae*, 76(3):265–281, 2003.

[29] S. A. Ochoa Quintanilla. *A System of Equations for Magnetoelastic Plates.* PhD thesis, Universität Konstanz, February 2004.

[30] D. L. Russell. Controllability and stabilizability theory for linear partial differential equations: Recent progress and open questions. *SIAM Review*, 20:639–739, 1978.

[31] M. Sermange and R. Temam. Some mathematical questions related to the MHD equations. *Communications on Pure and Applied Mathematics*, 36(5):635–664, 1984.

[32] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Annali di Matematica Pura ed Applicata*, 146(4):65–96, 1987.

[33] A. E. Taylor and D. C. Lay. *Introduction to Functional Analysis.* John Wiley & Sons, New York, 2 edition, 1980.

[34] M. G. Van Dalsen. On the dissipative effect of a magnetic field in a Mindlin-Timoshenko plate model. *Zeitschrift für Angewandte Mathematik und Physik*, 63(6):1047–1065, 2012.

[35] J. Zabczyk. *Mathematical Control Theory: An Introduction.* Birkhäuser, Boston, 1992.