Mathematical Analysis of Thermoplasticity with Linear Kinematic Hardening

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Abstract

We study thermoplasticity with the Prandtl-Reuss flow rule and with a linear evolution equation for the kinematic hardening. The yield function associated with the system under consideration depends explicitly on the temperature. To have a control on the temperature, we slightly modify the heat equation and prove that an approximation process, based on the Yosida approximation, converges to a global in time solution of the (modified) system of thermoplasticity.

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1 Introduction and Formulation of the Problem

The theory of inelastic and infinitesimal deformations is mathematically very well understood assuming that the temperature is constant during the deformation process. In the kinematic setting of such problems we refer for example to [Alb98], [AL87], [Che97], [Che01], [Suq80] and in the quasistatic case to [ACa], [ACb], [Tem86]. This article presents a mathematical analysis of a quasistatic model from thermoplasticity. The model under consideration is a modification of the Prandtl-Reuss model for elasto-plasticity with a yield function depending explicitly on the temperature. This function is monotone decreasing in the temperature, and for a critical value of the temperature the set of admissible stresses becomes the empty set. However, the system of equations which we are going to study is constructed in a way that the temperature does not take values greater as the critical value. Moreover, we assume that the considered material possesses the kinematic hardening property and the evolution equation for the backstress (new internal variable enlarging locally the set of admissible stresses) is linear. From the mathematical point of view this assumption regularizes the considered system of equations and yields a very important property: coerciveness of the energy associated with the system. In the theory of inelastic deformations the following approach, using for example in [CG00], works

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well: a maximal monotone multifunction defining the flow rule was replaced by its Yosida approximation and solutions of the approximated system were tending to the solutions of the original problem. We are going to follow this idea in the article. Unfortunately, the heat equation coupled with the Prandtl-Reuss system violates the monotonity of the flow rule and $\mathbb{L}^\infty(\mathbb{L}^2)$-estimates for the time derivatives, working well in the case without thermical effects, are very complicated in thermoplasticity. To obtain a control for the temperature we modify the right-hand side of the heat equation slightly.
Using this control trick we present an approximation procedure for the considered problem based on the Yosida approximation. The existence of global in time strong $\mathbb{L}^2$-solutions for the approximated system is proved in the Appendix. The main result of the article is the $\mathbb{L}^2(\mathbb{L}^2)$-boundedness of the time derivatives for the approximated system. This estimate is obtained using the gradient structure of the flow rule (for similar estimates we refer to [Che03]). This result and the strong $\mathbb{L}^2$-convergence of the stresses and the strains allow us to pass to the limit in the approximation procedure and to prove existence of global in time strong $\mathbb{L}^2$-solutions of our thermoplasticity model with kinematic hardening.
Let us formulate the boundary-value problem considered in this article. Assume that an inelastic body occupies a bounded domain $\Omega \subset \mathbb{R}^3$ with the smooth boundary $\partial\Omega$. The deformation process of the material is determined by the displacement vector $u : \Omega \times [0,\infty) \to \mathbb{R}^3$, the stress tensor $T : \Omega \times [0,\infty) \to \mathbb{R}^{3\times 3}_{sym} =: \mathcal{S}^3$ and the temperature $\theta : \Omega \times [0,\infty) \to \mathbb{R}$ (here we assume that θ is the difference between the temperature and the reference temperature $\theta^* = const > 0$). These three unknown functions satisfy the following system of linear partial differential equations

$$\begin{aligned} \operatorname{div}_x T(x,t) &= -F(x,t) \qquad \text{(balance of momentum)}, && (1.1)\\ \theta_t(x,t) &= \kappa\Delta\theta(x,t) - \gamma \operatorname{div}_x u_t(x,t) \qquad \text{(heat equation)} && (1.2) \end{aligned}$$

where the subscripts t and x denote the partial derivatives with respect to t or x respectively $F : \Omega \times [0,\infty) \to \mathbb{R}^3$ describes the external forces acting on the body, κ and γ are positive material parameters, cp. [JR00] for classical thermoelasticity. System (1.1) - (1.2) with the elastic constitutive equation for the stress tensor

$$T(x,t) = \mathcal{D}\varepsilon(x,t) = \mathcal{D}\left(\frac{1}{2}(\nabla u(x,t) + \nabla^T u(x,t))\right) \tag{1.3}$$

describes infinitesimal thermoelastic, quasistatic deformations. (Here $\mathcal{D} : S^3 \to S^3$ is the elasticity tensor, which we assume to be constant, symmetric and positive definite). To model inelastic effects it is assumed that the infinitesimal strain tensor is additively splitted into the pure elastic and the inelastic part

$$\varepsilon(x,t) = \frac{1}{2}(\nabla u(x,t) + \nabla^T u(x,t)) = \varepsilon(x,t) - \varepsilon^p(x,t) + \varepsilon^p(x,t) \tag{1.4}$$

where the inelastic strain tensor $\varepsilon^p : \Omega \times [0,\infty) \to \mathcal{S}^3$ is an internal variable for which a new equation is needed to close the entire system. In infinitesimal thermoplasticity the elastic constitutive equation (an analogoue to (1.3)) is in the form (see for example [Hau02])

$$T(x,t) = \mathcal{D}(\varepsilon(x,t) - \varepsilon^p(x,t)) - c\theta(x,t)\,\mathbb{I} \tag{1.5}$$

where $\mathbb{I}$ is the identity matrix and $c > 0$ is a material parameter. Hence we can eliminate the stress tensor from system (1.1) - (1.2) and obtain

$$\mathrm{div}_x \mathcal{D}(\varepsilon(x,t) - \varepsilon^p(x,t)) - c\nabla\theta(x,t) = -F(x,t), \tag{1.6}$$

$$\theta_t(x,t) = \kappa\Delta\theta(x,t) - \gamma \mathrm{div} u_t(x,t) \tag{1.7}$$

where the dependence $\varepsilon(u)$ is given by (1.4). In thermoplasticity system (1.6) - (1.7) is closed by the so called inelastic constitutive equation yielding the evolution in time of the inelastic strain tensor ε^p. In this article we assume that the Prandtl-Reuss inelastic flow rule with the von Mises yield function holds, this means that

$$\varepsilon^p_t(x,t) \in \partial I_{K(\theta(x,t))}(T(x,t)) \tag{1.8}$$

where $I_{K(\theta)}$ is the indicator function of the closed and convex set $K(\theta) = \{T \in \mathcal{S}^3 : |PT| = |T - \frac{1}{3}\mathrm{tr}\,(T)\mathbb{I}| \leq k - \theta\}$ and $k > 0$ is a material parameter. Moreover, $\partial I_{K(\theta)}$ is the subgradient of the convex function $\partial I_{K(\theta)}$. In other words (1.8) forces the stress tensor $T(x,t)$ to stay in the convex set $K(\theta(x,t))$ during the deformation process and an inelastic response of the material take place if and only if $T(x,t) \in \partial K(\theta(x,t))$. From the definition of the set $K(\theta)$ follows that $\mathrm{tr}\,\varepsilon^p_t = 0$ and if additionally the initial value $\varepsilon^{p,0}$ of the inelastic strain tensor is trace free, then during the whole deformation process we have $\mathrm{tr}\,\varepsilon^p = 0$. System (1.6) - (1.8) describes thermoplastic deformations in the quasistatic setting. Unfortunately, we are not yet able to study this system of equations. Therefore, we regularize the inelastic constitutive equation and assume that the considered material possesses the linear kinematic hardening property. Mathematically this means that a new internal variable is introduced, the so called backstress $b : \Omega \times [0,\infty) \to P\mathcal{S}^3$, which moves the set of admissible stresses $T(x,t) \in b(x,t) + K(\theta(x,t))$ and which evolution in time depends linearly of the inelastic strain rate. Hence, the set of inelastic constitutive equations is in the form

$$\varepsilon^p_t(x,t) \in \ \partial I_{K(\theta(x,t))}(T(x,t) - b(x,t))\,, \tag{1.9}$$

$$b_t(x,t) = \alpha\, \varepsilon^p_t(x,t)\,, \tag{1.10}$$

where $\alpha > 0$ is a material parameter. If we integrate (1.10) and assume that at $t = 0$ $b(x,0) = \alpha\varepsilon^p(x,0)$ then system (1.9) -(1.10) can be reduced to one differential inclusion

$$\varepsilon^p_t(x,t) \in \ \partial I_{K(\theta(x,t))}(T(x,t) - \alpha\varepsilon^p(x,t)) \tag{1.11}$$

System (1.6), (1.7), (1.11) models thermoplastic deformation processes, provided that the considered material possesses the kinematic hardening property. If we define a "mathematical" free energy function by

$$\rho\psi(\varepsilon, \varepsilon^p, \theta) = \frac{1}{2}\,\mathcal{D}(\varepsilon - \varepsilon^p)\cdot(\varepsilon - \varepsilon^p) + \frac{\alpha}{2}\,\varepsilon^p\cdot\varepsilon^p + \frac{c}{2\gamma}\,\theta^2 \tag{1.12}$$

and regard θ as internal variable, then (1.11) takes the form

$$\varepsilon_t^p \in g(-\rho \nabla_{(\varepsilon^p,\theta)} \psi(\varepsilon, \varepsilon^p, \theta)) \tag{1.13}$$

where $g\ (T - \alpha\varepsilon^p, \theta) = \partial\ I_{K(\theta)}(T - \alpha\varepsilon^p)$.
Moreover, $0 \in g\ (0,0)$ and $g\ (T - \alpha\varepsilon^p, \theta) \cdot (T - \alpha\varepsilon^p) \geq 0$ which imply that the considered inelastic constitutive equation is of premonotone type (for the definition we refer to the monograph [Alb98]. Let us define the total energy associated with system (1.6), (1.7), (1.11) by

$$\mathcal{E}(t) = \int_\Omega \rho\psi(\varepsilon(x,t), \varepsilon^p(x,t), \theta(x,t))\, dx. \tag{1.14}$$

A simple calculation shows that in the homogeneous case, this means for vanishing external forces and vanishing boundary data, the following equality

$$\mathcal{E}(t) + \int_0^t \int_\Omega T \cdot \varepsilon_t^p dx d\tau + \frac{c\kappa}{\gamma} \int_0^t \int_\Omega |\nabla\theta|^2 dx d\tau = \mathcal{E}(0) \tag{1.15}$$

holds, providing that the integral in (1.14) is well defined during the deformation process. Hence, the considered system in the homogeneous case possesses a natural semi-invariant and this allows us to formulate the problem in the strong $\mathbb{L}^2$-sense.
The inelastic constitutive relation implies that the temperature difference function θ is bounded from above by k. Thus, to have a control for the function θ we modify slightly the right-hand side of the heat equation

$$\theta_t(x,t) = \kappa\Delta\theta(x,t) - \gamma \mathrm{div} u_t(x,t)\, \beta(\theta(x,t)) \tag{1.16}$$

where β is defined by

$$\beta(\theta) = \begin{cases} 1 & \text{for} \quad \theta < k, \\ [0,1] & \text{for} \quad \theta = k, \\ 0 & \text{for} \quad \theta > k. \end{cases} \tag{1.17}$$

Thus, for physically relevant deformation processes $\beta(\theta) = 1$ and (1.16) becomes (1.17). Hence, the system of equations considered in this article is in the form

$$\begin{aligned} \mathrm{div}_x \mathcal{D}(\varepsilon(x,t) - \varepsilon^p(x,t)) - c\nabla\theta(x,t) &= -F(x,t) \\ \theta_t(x,t) &= \kappa\Delta\theta(x,t) - \gamma \mathrm{div} u_t(x,t)\beta(\theta(x,t)) \\ \varepsilon_t^p(x,t) &\in \partial I_{K(\theta)(x,t)}(T(x,t) - \alpha\varepsilon^p(x,t)) \end{aligned} \tag{MS}$$

where $\varepsilon(u)$ and T are defined by (1.4) and (1.5). System (MS) will be study with the following boundary conditions of the mixed type

$$u(x,t) = g_D(x,t) \text{ for } (x,t) \in \Gamma_1 \times (0,\infty) \tag{1.18}$$

$$T(x,t)\cdot n(x) = g_N(x,t) \text{ for } (x,t)\in \Gamma_2\times(0,\infty) \tag{1.19}$$

where Γ_1 and Γ_2 are open and smooth subsets of the boundary $\partial\Omega$ and satisfy $\Gamma_1\cup\Gamma_2=\partial\Omega$ and $\Gamma_1\cap\Gamma_2=\emptyset$. Additionally, we suppose that the two-dimensional measure of Γ_1 is positive. (Here $n(x)$ denotes the unit normal vector to the boundary $\partial\Omega$ at the point $x\in\partial\Omega$.) Moreover, for the temperature we consider the Dirichlet boundary condition

$$\theta(x,t) = g_\theta(x,t) \text{ for } (x,t)\in\partial\Omega\times(0,+\infty)\,, \tag{1.20}$$

where functions g_D, g_N and g_θ are given boundary data. Finally, the initial conditions are in the form

$$\theta(x,0)=\theta^0(x)\,,\quad \varepsilon^p(x,0)=\varepsilon^{p,0}(x)\,, \tag{1.21}$$

where $\theta^0:\Omega\to\mathbb{R}$ and $\varepsilon^{p,0}:\Omega\to P\mathcal{S}^3$ are given initial data.

The article is organized as follows: In the next section we present the main result of the article (global in time existence theorem for system (MS)). Section 3 presents the approximation procedure, which we propose to use for system (MS). An existence theorem for the approximated problem is proved at the end of the article (in the Appendix). In Section 4 we prove the crucial $\mathbb{L}^2(\mathbb{L}^2)$-estimate for the time derivatives of the approximation sequence. Section 5 presents a proof of the Main Theorem and in Section 6 we formulate some open problems associated with the considered system of equations.

In the whole article we use the following standard notation: for an open set $\Omega\subset\mathbb{R}^n$ the symbol $\mathbb{W}^{k,p}(\Omega;\mathbb{R}^N)$ denotes the usual Sobolev space of vector-valued $\mathbb{L}^p$-functions possessing $\mathbb{L}^p$-weak derivatives up to the order k. For $p=2$ we write $\mathbb{W}^{k,2}(\Omega;\mathbb{R}^N)=\mathbb{H}^k(\Omega;\mathbb{R}^N)$.

2 The Main Result

In this section we formulate an existence result of global in time $\mathbb{L}^2$-solutions for system (MS) assuming that the given data satisfy some very natural conditions and that the material parameters c,γ and α satisfy the technical inequality $\alpha>\gamma/4c$ and assumption (A) from Theorem 3.1. In the theory of inelastic deformations, for models of monotone type all nonlinearities arising in such models are monotone and the Yosida approximation can be used in this case. In system (MS) the control function β is maximal monotone and the constitutive function $\partial I_{K(\theta)}$ is maximal monotone for all fixed values θ. Our idea is to use the Yosida approximation for the function $\partial I_{K(\theta)}$ and therefore we write the inelastic constitutive equation in the following equivalent form

$$\varepsilon^p_t(x,t)\in\Pi_2\Big(\partial I_{\mathcal{K}}(\theta(x,t),T(x,t)-\alpha\varepsilon^p(x,t))\Big) \tag{2.1}$$

where the set $\mathcal{K}$ contains admissible stresses and temperatures for our model, this means that $\mathcal{K}=\{(\theta,T)\in\mathbb{R}\times\mathcal{S}^3 : |PT|+\theta\le k\}$. Moreover, the operator Π_2 is a projection defined by $\Pi_2(\theta,T)=T$. Note, that the set $\mathcal{K}$ is a closed and convex set in the space

$\mathbb{R} \times \mathcal{S}^3$ and consequently the multifunction $\partial I_{\mathcal{K}}$ is maximal monotone. Using this new notation the figure below presents geometrically the flow rule from system (MS).

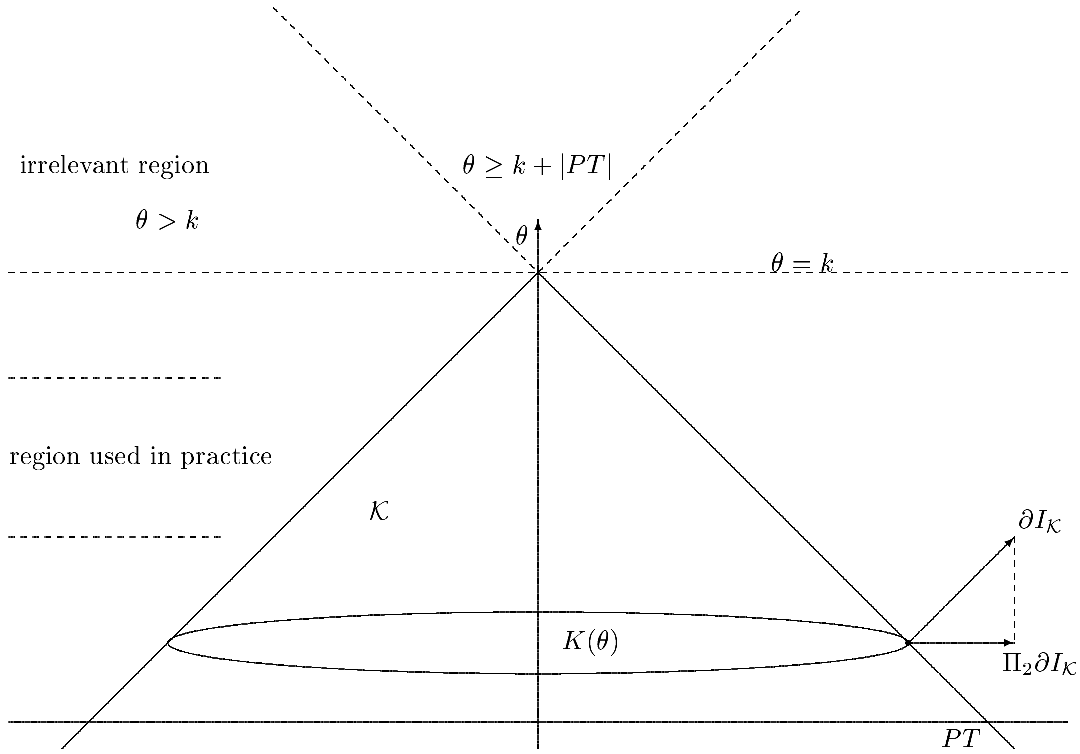


Figure 1: Flow rule of the model

Hence, the boundary-value problem studied in this article can be written in the form

$$\operatorname{div}_x D(\varepsilon(x,t) - \varepsilon^p(x,t)) - c\nabla\theta(x,t) = -F(x,t), \tag{2.2}$$

$$\theta_t(x,t) = \kappa\Delta\theta(x,t) - \gamma \operatorname{div}_x u_t(x,t)\ \beta(\theta(x,t)), \tag{2.3}$$

$$\varepsilon^p_t(x,t) \in \Pi_2\Big(\partial I_{\mathcal{K}}(\theta(x,t), T(x,t) - \alpha\varepsilon^p(x,t))\Big), \tag{2.4}$$

$$u(x,t) = g_D(x,t) \text{ on } \Gamma_1 \times (0,+\infty),\ T(x,t)\cdot n(x) = g_N(x,t) \text{ on } \Gamma_2 \times (0,+\infty), \tag{2.5}$$

$$\theta(x,t) = g_\theta(x,t) \text{ on } \partial\Omega \times (0,+\infty), \tag{2.6}$$

$$\varepsilon^p(x,0) = \varepsilon^{p,0}(x),\ \theta(x,0) = \theta^0(x) \tag{2.7}$$

with given functions $F, g_D, g_N, g_\theta, \varepsilon^{p,0}$ and θ^0.

Main Theorem

Let us suppose that the given data have the following regularity:
for all $T > 0$

$$F \in \mathbb{L}^\infty((0,T);\mathbb{L}^2(\Omega;\mathbb{R}^3)),\ F_t \in \mathbb{L}^2((0,T);\mathbb{L}^2(\Omega;\mathbb{R}^3)), \tag{2.8}$$

$$g_D \in \mathbb{L}^\infty((0,T);\mathbb{H}^{1/2}(\Gamma_1;\mathbb{R}^3)),\ g_{D,t} \in \mathbb{L}^2((0,T);\mathbb{H}^{1/2}(\Gamma_1;\mathbb{R}^3)), \tag{2.9}$$

$$g_N \in \mathbb{L}^\infty((0,T);\mathbb{H}^{-1/2}(\Gamma_2;\mathbb{R}^3)),\ g_{N,t} \in \mathbb{L}^2((0,T);\mathbb{H}^{-1/2}(\Gamma_2;\mathbb{R}^3)), \tag{2.10}$$

$$g_\theta \in \mathbb{L}^\infty((0,T);\mathbb{H}^{3/2}(\partial\Omega;\mathbb{R})),\ g_{\theta,t} \in \mathbb{L}^2((0,T);\mathbb{H}^{-1/2}(\partial\Omega;\mathbb{R})), \tag{2.11}$$

$$\varepsilon^{p,0} \in \mathbb{L}^2(\Omega;PS^3),\ \theta^0 \in \mathbb{H}^1(\Omega;\mathbb{R}). \tag{2.12}$$

For the initial data we assume that $|PT^0 - \alpha\varepsilon^{p,0}| + \theta^0 \leq k$ a.e. in Ω, where the initial stress is determinated by the initial data. In addition, we assume that the temperature data g_θ and θ^0 are not larger than the critical temperature k, this means

$$g_\theta(x,t) \leq k \textit{ for a.a. } (x,t) \in \partial\Omega \times (0,\infty) \textit{ and } \theta^0(x) \leq k \textit{ for a.a. } x \in \Omega. \tag{2.13}$$

If the material parameters c, γ and α satisfy

$$\alpha > \frac{\gamma}{4c} \tag{2.14}$$

and assumption (A) from Theorem 3.1, then boundary-value problem (2.2) - (2.7) possesses global in time strong $\mathbb{L}^2$-solutions. This means that there exist functions

$$(u,\varepsilon,\varepsilon^p,\theta) \in \mathbb{L}^\infty((0,T);\mathbb{H}^1(\Omega;\mathbb{R}^3) \times \mathbb{L}^2(\Omega;\mathcal{S}^3)^2 \times \mathbb{H}^1(\Omega;\mathbb{R}))$$

with $(u_t,\varepsilon_t,\varepsilon^p_t,\theta_t) \in \mathbb{L}^2((0,T);\mathbb{L}^2(\Omega;\mathbb{R}^3) \times \mathbb{L}^2(\Omega;\mathcal{S}^3)^2 \times \mathbb{L}^2(\Omega;\mathbb{R}))$ such that (2.2) - (2.6) are satisfied pointwise for almost all $(x,t) \in \Omega \times (0,T)$ and (2.7) for almost all $x \in \Omega$.

3 Approximation Procedure

To prove the Main Theorem we are going to approximate the boundary-value problem (2.2) - (2.7). The idea of our approximation is very simple. We want to use the maximal monotonity of the multifunction $\partial I_{\mathcal{K}}$ and approximate this function by its Yosida approximation Y_λ. For maximal monotone operators of gradient type their Yosida approximation can be calculated very simply (for such result we refer for example to [AC84]). In our case we obtain

$$Y_\lambda(\theta,T) = \begin{cases} \left(\dfrac{(|PT|+\theta-k)_+}{2\lambda}, \dfrac{(|PT|+\theta-k)_+}{2\lambda}\dfrac{PT}{|PT|}\right) & \text{for} \quad \theta \leq |PT|+k, \\ \left(\dfrac{1}{\lambda}(\theta-k), \dfrac{1}{\lambda}PT\right) & \text{for} \quad \theta \geq |PT|+k\,, \end{cases} \tag{3.1}$$

where $\lambda > 0$. Here, the symbol $(\xi)_+$ denotes the nonnegative part of the real number ξ and for $PT = 0$ we set the value 0 for the quotient $PT/|PT|$. Hence, for all $\lambda > 0$ we study the following system

$$\mathrm{div}_x\mathcal{D}(\varepsilon^\lambda(x,t) - \varepsilon^{p,\lambda}(x,t)) - c\nabla\theta^\lambda(x,t) = -F(x,t), \tag{3.2}$$

$$\theta_t^\lambda(x,t) = \kappa\Delta\theta^\lambda(x,t) - \gamma\mathrm{div}_x u_t^\lambda(x,t)\beta(\theta^\lambda(x,t)), \tag{3.3}$$

$$\varepsilon_t^p(x,t) = \Pi_2\Big(Y_\lambda(\theta^\lambda(x,t), T^\lambda(x,t) - \alpha\varepsilon^{p,\lambda}(x,t))\Big) \tag{3.4}$$

where $\varepsilon^\lambda = \frac{1}{2}(\nabla u^\lambda + \nabla^T u^\lambda)$ and $T^\lambda = \mathcal{D}(\varepsilon^\lambda - \varepsilon^{p,\lambda}) - c\ \theta^\lambda\mathbb{I}$. We consider system (3.2) - (3.4) with the same boundary and initial conditions as for system (MS). System (3.2) - (3.4) contains the second component $\Pi_2 Y_\lambda$ of the Yosida approximation only. In the set $\{\theta \le |PT| + k\}$ this component controls the entire function Y_λ. In the set $\{\theta \le |PT| + k\}$ it is not possible to control Y_λ by the second component of this vector field. Fortunately, the modificated heat equation forces the temperature to stay below the critical temperature k if the data g_θ and θ^0 do the same.

Lemma 3.1 *(Maximum principle for the temperature)*
Let us fix $T > 0$ and assume that $(\theta^\lambda, u^\lambda) \in \mathbb{L}^\infty((0,T); \mathbb{H}^1(\Omega;\mathbb{R}) \times \mathbb{L}^2(\Omega;\mathbb{R}^3))$ satisfy equation (3.3) in the $\mathbb{L}^2$-sense, this means that $\theta_t^\lambda, \Delta\theta^\lambda \in \mathbb{L}^\infty((0,T);\mathbb{L}^2(\Omega;\mathbb{R}))$, $u_t^\lambda \in \mathbb{L}^\infty((0,T);\mathbb{L}^2(\Omega;\mathbb{R}^3))$ and (3.3) is satisfied for almost all $(x,t) \in \Omega \times (0,T)$. If the temperature data g_θ and θ^0 satisfy

$$g_\theta(x,t) \le k \textit{ for a.a. } (x,t) \in \Omega \times (0,T),\ \theta^0(x) \le k \textit{ for a.a. } x \in \Omega$$

then the temperature θ^λ cannot cross the critical value k:

$$\theta^\lambda(x,t) \le k \textit{ for almost all } (x,t) \in \Omega \times (0,T).$$

Proof: The proof is standard, but for convenience of the reader we present it here. The trick is to show that the function $(\theta^\lambda - k)_+(x,t) = 0$ a.e. in $\Omega \times (0,T)$.

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{1}{2}(\theta^\lambda - k)_+^2 dx &= \int_\Omega (\theta^\lambda - k)_+ \theta_t^\lambda dx \\
&= \int_\Omega (\theta^\lambda - k)_+ \Delta\theta^\lambda dx - \gamma\int_\Omega \mathrm{div}_x u_t^\lambda \beta(\theta^\lambda)(\theta^\lambda - k)_+ dx \\
&= -\int_\Omega \mathrm{sgn}\,(\theta^\lambda - k)_+ |\nabla\theta^\lambda|^2 dx + \int_{\partial\Omega} (\theta^\lambda - k)_+ \frac{\partial\theta^\lambda}{\partial n}\, ds \le 0\,.
\end{aligned}$$

Here, we have used that the product $\beta(\theta^\lambda)(\theta^\lambda - k)_+ \equiv 0$ and that $(g_\theta - k)_+ = 0$. Integrating the last inequality in time we conclude that

$$\frac{1}{2}\int_\Omega (\theta^\lambda - k)_+^2 dx \le \frac{1}{2}\int_\Omega (\theta^0 - k)_+^2 dx = 0.$$

Hence, assuming that the temperature data stay below the critical value equation (3.4) can be rewritten in the form

$$\varepsilon_t^{p,\lambda}(x,t) = \frac{\Big(|PT^\lambda(x,t)| + \theta^\lambda(x,t) - k\Big)_+}{2\lambda}\frac{PT^\lambda(x,t)}{|PT^\lambda(x,t)|}\,. \tag{3.5}$$

Theorem 3.1 *(existence theorem for the approximated system)*
Let us fix $T > 0$ and assume that the data $F, g_D, g_N, g_\theta, \varepsilon^{p,0}, \theta^0$ have the regularity required in the Main Theorem and the temperature data g_θ and θ^0 are not greater as the critical temperature k. Moreover, suppose that the material parameters $\mathcal{D}, c$ and γ satisfy

$$\forall\, S \in \mathcal{S}^3 \;\; \forall\, \theta \in (-\infty, k] \qquad \mathcal{D}S \cdot S + \frac{c}{\gamma}\theta^2 - c\theta \operatorname{tr} S > 0\,. \tag{A}$$

Then for all $\lambda > 0$ system (3.2), (3.3), (3.5) with boundary conditions (2.5), (2.6) and with initial conditions (2.7) possesses global in time $\mathbb{L}^2$-solutions $(u^\lambda, \varepsilon^\lambda, \varepsilon^{p,\lambda}, \theta^\lambda) \in \mathbb{L}^\infty(0,T); \mathbb{H}^1(\Omega; \mathbb{R}^3) \times \mathbb{L}^2(\Omega; \mathcal{S}^3)^2 \times \mathbb{H}^1(\Omega; \mathbb{R})$ such that $\theta^\lambda(x,t) \le k$ a.e. in $\Omega \times (0,T)$ and $(u^\lambda_t, \varepsilon^\lambda_t, \varepsilon^{p,\lambda}_t, \theta^\lambda_t) \in \mathbb{L}^\infty((0,T); \mathbb{L}^2(\Omega; \mathbb{R}^3) \times \mathbb{L}^2(\Omega; \mathcal{S}^3)^2 \times \mathbb{L}^2(\Omega; \mathbb{R}))$.

A proof of this theorem is presented in the Appendix.

4 $\mathbb{L}^2(\mathbb{L}^2)$-estimates for time derivatives

In this section we prove that the time derivatives of the approximate sequence are uniformly $\mathbb{L}^2(\mathbb{L}^2)$-bounded with respect to $\lambda > 0$. This result is very important for the entire article and allows us to pass to the limit $\lambda \to 0^+$ in the $\mathbb{L}^2$ space equipped with the weak topology in the approximate system. We will also mention, that the method used in the theorem below do not use an upper bound for the temperature and works well for system (3.2), (3.3), (3.5) without the control function β. In this non-standard energy-type estimate we use the gradient structure of the inelastic flow rule.

Theorem 4.1 *($\mathbb{L}^2(\mathbb{L}^2)$ boundedness of time derivatives)*
Let us fix $T > 0$ and suppose that the given data satisfy all assumptions from the Main Theorem. If the material parameters α, c and γ satisfy the inequality $4c\alpha > \gamma$ then the sequence $(u^\lambda_t, \varepsilon^\lambda_t, \varepsilon^{p,\lambda}_t, \theta^\lambda_t)$ of time derivatives of solutions of approximate system (3.2), (3.3), (3.5) is bounded in the space $\mathbb{L}^2\Big((0,T); \mathbb{L}^2(\Omega; \mathbb{R}^3 \times \mathcal{S}^3 \times \mathcal{S}^3 \times \mathbb{R})\Big)$. Moreover, the sequence $\{\nabla\theta^\lambda\}$ is bounded in the space $\mathbb{L}^\infty\Big((0,T); \mathbb{L}^2(\Omega; \mathbb{R})\Big)$.

Proof: Let us define a "potential" of the right-hand side of (3.5) by

$$M_\lambda(x,t) := \frac{(|PT^\lambda(x,t) - \alpha\varepsilon^{p,\lambda}(x,t)| + \theta^\lambda(x,t) - k)^2_+}{4\lambda} \tag{4.1}$$

For this function we obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega M_\lambda dx &= \int_\Omega \frac{(|PT^\lambda - \alpha\varepsilon^{p,\lambda}| + \theta^\lambda - k)_+}{2\lambda}\left[\frac{PT^\lambda - \alpha\varepsilon^{p,\lambda}}{|PT^\lambda - \alpha\varepsilon^{p,\lambda}|}(PT^\lambda_t - \alpha\varepsilon^{p,\lambda}_t) + \theta^\lambda_t\right] dx \\
&= \int_\Omega (\varepsilon^{p,\lambda}_t \cdot T^\lambda_t - \alpha|\varepsilon^{p,\lambda}_t|^2 + |\varepsilon^{p,\lambda}_t|\theta^\lambda_t) dx = (\text{using that } \operatorname{tr}\varepsilon^{p,\lambda}_t = 0)
\end{aligned}$$

$$= \int_\Omega (\varepsilon_t^{p,\lambda} \cdot \mathcal{D}(\varepsilon_t^\lambda - \varepsilon_t^{p,\lambda}) + |\varepsilon_t^{p,\lambda}|\theta_t^\lambda - \alpha|\varepsilon_t^{p,\lambda}|^2)dx$$

$$= \int_\Omega (-\mathcal{D}(\varepsilon_t^\lambda - \varepsilon_t^{p,\lambda})(\varepsilon_t^\lambda - \varepsilon_t^{p,\lambda}) + \varepsilon_t^\lambda \mathcal{D}(\varepsilon_t^\lambda - \varepsilon_t^{p,\lambda}) + |\varepsilon_t^{p,\lambda}|\theta_t^\lambda - \alpha|\varepsilon_t^{p,\lambda}|^2)dx$$

$$= \int_\Omega (-\mathcal{D}(\varepsilon_t^\lambda - \varepsilon_t^{p,\lambda})(\varepsilon_t^\lambda - \varepsilon_t^{p,\lambda}) + \varepsilon_t^\lambda (T_t^\lambda + c\theta_t^\lambda \cdot I) + |\varepsilon_t^{p,\lambda}|\theta_t^\lambda - \alpha|\varepsilon_t^{p,\lambda}|^2)dx \tag{4.2}$$

On the set $\{(x,t) \in \Omega \times (0,T) : \theta^\lambda(x,t) = k\}$ is the time derivative θ_t^λ equal to zero and we can rewrite the right-hand side of (4.2) in the form

$$\begin{aligned}
\frac{d}{dt}\int_\Omega M_\lambda dx &= \int_\Omega (-\mathcal{D}(\varepsilon_t^\lambda - \varepsilon_t^{p,\lambda})(\varepsilon_t^\lambda - \varepsilon_t^{p,\lambda}) - \alpha|\varepsilon_t^{p,\lambda}|^2)\, dx \\
&\quad + \int_\Omega (\varepsilon_t^\lambda T_t^\lambda + c\chi_{\{\theta^\lambda < k\}}\theta_t^\lambda \mathrm{div} u_t^\lambda + |\varepsilon_t^{p,\lambda}|\theta_t^\lambda)\, dx \\
&= \int_\Omega (-\mathcal{D}(\varepsilon_t^\lambda - \varepsilon_t^{p,\lambda})(\varepsilon_t^\lambda - \varepsilon_t^{p,\lambda}) - \alpha|\varepsilon_t^{p,\lambda}|^2)\, dx \\
&\quad + \int_\Omega (\varepsilon_t^\lambda T_t^\lambda + \frac{c}{\gamma}(\kappa\Delta\theta_t^\lambda - \theta_t^\lambda)\chi_{\{\theta^\lambda < k\}}\theta_t^\lambda + |\varepsilon_t^{p,\lambda}|\theta_t^\lambda)\, dx\,.
\end{aligned}$$

Putting the nonpositive terms from the right-hand side on the left we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega M_\lambda dx + \int_\Omega \mathcal{D}(\varepsilon_t^\lambda - \varepsilon_t^{p,\lambda})(\varepsilon_t^\lambda - \varepsilon_t^{p,\lambda})dx + \alpha\int_\Omega |\varepsilon_t^{p,\lambda}|^2 dx + \frac{c}{\gamma}\int_\Omega |\theta_t^\lambda|^2 dx \leq \\
\leq \int_\Omega \varepsilon_t^\lambda T_t^\lambda dx + \frac{c\kappa}{\gamma}\int_\Omega \Delta\theta^\lambda\theta_t^\lambda dx + \int_\Omega |\varepsilon_t^{p,\lambda}|\theta_t^\lambda dx\,.
\end{aligned} \tag{4.3}$$

Let us denote by I_i $i = 1, 2, 3$ the three integrals from the right-hand side of (4.3). We have to estimate these integrals by norms of the given data and by the "good terms" from the left-hand side of (4.3). Let us start with an estimation for I_1. Using the symmetry of the stress tensor we write I_1 in the form

$$I_1 = \int_\Omega \nabla u_t^\lambda T_t^\lambda dx = -\int_\Omega u_t^\lambda \mathrm{div} T_t^\lambda dx + \int_{\partial\Omega} u_t^\lambda T_t^\lambda \cdot n\ ds = \int_\Omega u_t^\lambda F_t dx + \int_{\partial\Omega} u_t^\lambda, T_t^\lambda \cdot n\ ds$$

(Here, the surface integral $\int_{\partial\Omega} u_t^\lambda T_t^\lambda \cdot n\ ds$ is to understand as the value of a functional from the space $\mathbb{H}^{-1/2}(\partial\Omega)$ on an element from the space $\mathbb{H}^{1/2}(\partial\Omega)$.)
Using the continuity of the duality mapping we conclude that

$$I_1 \leq \|u_t^\lambda\|\ \|F_t\| + \|g_{D,t}\|_{1/2}\|T_t^\lambda \cdot n\|_{-1/2} + \|u_t^\lambda\|_{1/2}\|g_{N,t}\|_{-1/2}$$

where the symbol $\|\cdot\|_s$ denotes the norm from the Sobolev space $\mathbb{H}^s$ and the symbol $\|\cdot\|$ denotes the $\mathbb{L}^2$-norm. By the boundedness of the trace operator in the space $\mathbb{H}^1$ and in the space $\mathbb{L}^2(\mathrm{div}) = \{f \in \mathbb{L}^2 : \mathrm{div} f \in \mathbb{L}^2\}$ there exists a positive constant C, which do not depend on λ, such that

$$I_1 \leq \|u_t^\lambda\|_1(\|F_t\| + C\|g_{N,t}\|_{-1/2}) + (\|T_t^\lambda\| + \|F_t\|)C\|g_{D,t}\|_{1/2}. \tag{4.4}$$

To finish the estimate for I_1 we only need an estimate for the norm $\|u_t^\lambda\|_1$. The vector u_t^λ is a solution of the following boundary-value problem

$$\begin{aligned} \mathrm{div}_x \mathcal{D}\Big(\frac{1}{2}(\nabla u_t^\lambda(x,t) + \nabla^T u_t^\lambda(x,t)) - \varepsilon_t^{p,\lambda}(x,t)\Big) &= -F_t(x,t) + c\nabla\theta_t^\lambda(x,t), \\ u_t^\lambda(x,t)_{|\Gamma_1} &= g_{D,t}(x,t), \\ T_t^\lambda(x,t)\cdot u(x)_{|\Gamma_2} &= g_{N,t}(x,t), \end{aligned}$$

where $T_t^\lambda = \mathcal{D}\Big(\frac{1}{2}(\nabla u_t^\lambda + \nabla^T u_t^\lambda) - \varepsilon_t^{p,\lambda}\Big) - c\theta_t^\lambda \cdot \mathbb{I}$. Assuming that $\varepsilon_t^{p,\lambda}$ and θ_t^λ are given functions we see that u_t^λ satisfies an elliptic system with constant coefficients and with a mixed boundary condition. Using the regularity theory for such systems we obtain that there exists a constant $C(\Omega) > 0$ such that for all $\lambda > 0$ the following inequality holds

$$\|u_t^\lambda\|_1 \leq C(\Omega)(\|F_t\| + \|\varepsilon_t^{p,\lambda}\| + \|\theta_t^\lambda\| + \|g_{D,t}\|_{1/2} + \|g_{N,t}\|_{-1/2}). \tag{4.5}$$

Inserting (4.5) into (4.4) yields : There exists a constant $C > 0$, independent of λ such that

$$I_1 \leq C(\|\varepsilon_t^{p,\lambda}\| + \|\theta_t^\lambda\|)(\|F_t\| + \|g_{N,t}\|_{-1/2}) + C(\|\mathcal{D}(\varepsilon_t^\lambda - \varepsilon_t^{p,\lambda})\| + \|\theta_t^\lambda\|)\|g_{D,t}\|_{1/2} + \mathcal{F}(t)$$

where $\mathcal{F}(t)$ depends only on the given data. Next, we estimate the integral I_2.

$$\begin{aligned} \frac{\gamma}{c\kappa} I_2 &= \int_\Omega \Delta\theta^\lambda \theta_t^\lambda dx = -\frac{d}{dt}\int_\Omega |\nabla\theta^\lambda|^2 dx + \int_{\partial\Omega} \frac{\partial\theta^\lambda}{\partial n}\theta_t^\lambda dx \\ &= -\frac{d}{dt}\int_\Omega |\nabla\theta^\lambda|^2 dx + \int_{\partial\Omega} \frac{\partial\theta^\lambda}{\partial n}\cdot g_{\theta,t} dx \leq -\frac{d}{dt}\int_\Omega |\nabla\theta^\lambda|^2 dx + C\|\theta^\lambda\|_2\|g_{\theta,t}\|_{-1/2}, \end{aligned} \tag{4.6}$$

where the nonnegative constant C does not depend on λ, and which follows from the continuity of the trace operator. The function θ^λ is a solution of the following boundary-value problem

$$\begin{aligned} \Delta\theta^\lambda(x,t) &= \frac{1}{\kappa}\theta_t^\lambda(x,t) + \frac{\gamma}{\kappa}\mathrm{div} u_t^\lambda(x,t)\beta(\theta^\lambda(x,t)), \\ \theta^\lambda(x,t)_{|\partial\Omega} &= g_\theta(x,t). \end{aligned} \tag{4.7}$$

Assuming that the right-hand side of (4.7) is given we conclude that there exist a positive constant $C(\Omega)$ such that

$$\|\theta^\lambda\|_2 \leq C(\Omega)\ (\|\theta_t^\lambda\| + \|\mathrm{div} u_t^\lambda\| + \|g_\theta\|_{3/2})\,. \tag{4.8}$$

Inserting (4.8) into (4.7) and using that $\|\mathrm{tr}\,\varepsilon_t^\lambda\| = \|\mathrm{tr}\,(\varepsilon_t^\lambda - \varepsilon_t^{p,\lambda})\|$ can be estimated by a "good term" from the left-hand side of (4.3) we obtain

$$I_2 + \frac{c\kappa}{\gamma}\int_\Omega |\nabla\theta^\lambda|^2 dx \le C\|g_{\theta,t}\|_{-1/2}\;(\|\theta_t^\lambda\| + \|\mathcal{D}(\varepsilon_t^\lambda - \varepsilon_t^{p,\lambda})\| + \|g_\theta\|_{3/2}). \tag{4.9}$$

To estimate I_3 we use the required inequality between the material parameters $4c\alpha > \gamma$. In this case there exists a positive constant c^* such that

$$-\frac{c}{\gamma}\;|\theta_t^\lambda|^2 + |\varepsilon_t^{p,\lambda}|\theta_t^\lambda - \alpha|\varepsilon_\lambda^{p,t}|^2 \le -\;c^*(|\theta_t^\lambda|^2 + |\varepsilon_t^{p,\lambda}|^2).$$

Hence, I_3 can be absorbed by the "good terms". Adding all estimates together we rewrite (4.3) in the form

$$\begin{aligned}\frac{d}{dt}\int_\Omega M_\lambda dx + c^*\int_\Omega(|\theta_t^\lambda|^2 + |\varepsilon_t^{p,\lambda}|^2)dx + \int_\Omega \mathcal{D}(\varepsilon_t^\lambda - \varepsilon_t^{p,\lambda})(\varepsilon_t^\lambda - \varepsilon_t^{p,\lambda})dx\\ +\frac{d}{dt}\,\frac{c\kappa}{\gamma}\int_\Omega |\nabla\theta^\lambda|^2 dx \le C(\|\varepsilon_t^{p,\lambda}\| + \|\theta_t^\lambda\|)\;(\|F_t\| + \|g_N,t\|_{-1/2})\\ +C\;(\|\mathcal{D}(\varepsilon_t^\lambda - \varepsilon_t^{p,\lambda})\| + \|\theta_t^\lambda\|)\;(\|g_{D,t}\|_{1/2} + \|g_{\theta,t}\|_{-1/2}) + \mathcal{F}(t)\end{aligned} \tag{4.10}$$

where $C > 0$ does not depend on λ and $\mathcal{F}(t)$ contains the given data only. Splitting the two products from the right-hand side of (4.10) such that the norms on the left-hand side will be multiplied by a small factor we conclude that

$$\begin{aligned}\frac{d}{dt}\int_\Omega M_\lambda\,dx + \frac{d}{dt}\,\frac{c\kappa}{\gamma}\int_\Omega |\nabla\theta^\lambda|^2\,dx\\ +\frac{c^*}{2}\int_\Omega(|\theta_t^\lambda|^2 + |\varepsilon_t^{p,\lambda}|^2)\,dx + \frac{1}{2}\int_\Omega \mathcal{D}(\varepsilon_t^\lambda - \varepsilon_t^{p,\lambda})(\varepsilon_t^\lambda - \varepsilon_t^{p,\lambda})\,dx \le \mathcal{F}^*(t)\end{aligned}$$

where $\mathcal{F}^*(t)$ depends on the given data only. Integrating the last inequality over the interval $(0, T)$ we have

$$\begin{aligned}\int_\Omega M_\lambda dx + \|\nabla\theta^\lambda\|^2 + \frac{c^*}{2}\int_0^T(\|\theta_t^\lambda\|^2 + \|\varepsilon_t^{p,\lambda}\|^2)dt + \frac{1}{2}\int_0^T\int_\Omega \mathcal{D}(\varepsilon_t^\lambda - \varepsilon_t^{p,\lambda})(\varepsilon_t^\lambda - \varepsilon_t^{p,\lambda})dx\;dt\\ \le \int_0^T \mathcal{F}^*(t)dt + \|\nabla\theta^0\|^2 + \int_\Omega M_\lambda(0)dx.\end{aligned}$$

Finally, using that the initial data $(T^0 - \alpha\varepsilon^{p,0}, \theta^0)$ belong to the set $\mathcal{K}$ and consequently $\int_\Omega M_\lambda(0)dx = 0$, we conclude the theorem. □

At the end of this subsection we note the following simple consequence of the last theorem.

Corollary 4.2 *Under assumptions of Theorem 4.1 we have that for all $T > 0$ the sequence $\{(\varepsilon^\lambda, \varepsilon^{p,\lambda})\}$ is bounded in the space $\mathbb{L}^\infty((0,T);\mathbb{L}^2(\Omega;\mathcal{S}^3)^2)$ the sequence $\{u_t^\lambda\}$ is bounded in $\mathbb{L}^\infty((0,T);\mathbb{H}^1(\Omega;\mathbb{R}^3))$ and the sequence $\{\theta^\lambda\}$ is relatively compact in the space $\mathbb{L}^2((0,T);\mathbb{L}^2(\Omega,\mathbb{R}))$ and bounded in $\mathbb{L}^\infty((0,T);\mathbb{H}^1(\Omega;\mathbb{R}))$.*

5 Proof of the Main Theorem

In this section we are going to prove the Main Theorem. The idea is to show that the approximation sequence defined in Section 3 possesses an accummulation point, which satisfies our boundary-value problem (2.2) - (2.7). Let us start with the following Lemma, yielding estimates for differences of two approximation steps.

Lemma 5.1 *(strong convergence of stresses and strains)*
Let us denote by $\mathcal{E}^{\lambda,\mu}(t)$ the following part of the free energy evaluated on the differences of stresses and strains for two approximation steps

$$\mathcal{E}^{\lambda,\mu}(t) = \frac{1}{2}\int_\Omega \mathcal{D}(\varepsilon^\lambda - \varepsilon^\mu - \varepsilon^{p,\lambda} + \varepsilon^{p,\mu})(\varepsilon^\lambda - \varepsilon^\mu - \varepsilon^{p,\lambda} + \varepsilon^{p,\mu})dx + \frac{\alpha}{2}\int_\Omega |\varepsilon^{p,\lambda} - \varepsilon^{p,\mu}|^2 dx.$$

Let us fix $T > 0$. If the sequence $\{\theta^\lambda\}$ is a Cauchy sequence in $\mathbb{L}^2((0,T);\mathbb{L}^2(\Omega;\mathbb{R}))$ and sequences of strains time derivatives $\{\varepsilon_t^\lambda\}, \{\varepsilon_t^{p,\lambda}\}$ are bounded in $\mathbb{L}^2((0,T);\mathbb{L}^2(\Omega;\mathcal{S}^3))$, then $\mathcal{E}^{\lambda,\mu}(t)$ tends to zero for $\lambda, \mu \to 0^+$ uniformly on bounded time intervals.

Proof. For the time derivative of the function $\mathcal{E}^{\lambda,\mu}(t)$ we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}^{\lambda,\mu} &= \int_\Omega \mathcal{D}(\varepsilon^\lambda - \varepsilon^\mu - \varepsilon^{p,\lambda} + \varepsilon^{p,\mu})(\varepsilon_t^\lambda - \varepsilon_t^\mu - \varepsilon_t^{p,\lambda} + \varepsilon_t^{p,\mu})\, dx \\
&\quad + \alpha\int_\Omega (\varepsilon^{p,\lambda} - \varepsilon^{p,\mu})(\varepsilon_t^{p,\lambda} - \varepsilon_t^{p,\mu})\, dx = \\
&= \int_\Omega (T^\lambda - T^\mu)(\varepsilon_t^\lambda - \varepsilon_t^\mu)dx + c\int_\Omega (\theta^\lambda - \theta^\mu)\cdot I(\varepsilon_t^\lambda - \varepsilon_t^\mu)\, dx \\
&\quad - \int_\Omega (T^\lambda - \alpha\varepsilon^{p,\lambda} - T^\mu + \alpha\varepsilon^{p,\mu})(\varepsilon_t^{p,\lambda} - \varepsilon_t^{p,\mu})\, dx - c\int_\Omega (\theta^\lambda - \theta^\mu)\cdot I(\varepsilon_t^{p,\lambda} - \varepsilon_t^{p,\mu})\, dx
\end{aligned}$$

where $T^\lambda = \mathcal{D}(\varepsilon^\lambda - \varepsilon^{p,\lambda}) - c\theta^\lambda \cdot I$ and $T^\mu = \mathcal{D}(\varepsilon^\mu - \varepsilon^{p,\mu}) - c\theta^\mu \cdot I$. The given data are the same in both approximation steps, $\mathrm{tr}\,\varepsilon_t^{p,\lambda} = \mathrm{tr}\,\varepsilon_t^{p,\mu} = 0$ and we conclude that

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}^{\lambda,\mu} &= c\int_\Omega (\theta^\lambda - \theta^\mu)(\mathrm{tr}\,\varepsilon_t^\lambda - \mathrm{tr}\varepsilon_t^\mu)\, dx \\
&\quad - \int_\Omega (T^\lambda - \alpha\varepsilon^{p,\lambda} - T^\mu + \alpha\varepsilon^{p,\mu})(\varepsilon_t^{p,\lambda} - \varepsilon_t^{p,\mu})\, dx = I_1 + I_2. \quad (5.1)
\end{aligned}$$

Next, we are going to estimate both integrals from the right-hand side of (5.1).
An estimate for I_1 is very simple:

$$I_1 \le c \int_\Omega (|\varepsilon_t^\lambda| + |\varepsilon_t^\mu|)|\theta^\lambda - \theta^\mu| dx. \tag{5.2}$$

To estimate I_2 we insert the right-hand side of the ordinary differential equation for the inelastic strain and obtain

$$\begin{aligned}
I_2 = &- \int_\Omega (T^\lambda - \alpha\varepsilon^{p,\lambda} - T^\mu + \alpha\varepsilon^{p,\mu}) \cdot \Big(\frac{(|PT^\lambda - \alpha\varepsilon^{p,\lambda}| + \theta^\lambda - k)_+}{2\lambda} \frac{PT^\lambda - \alpha\varepsilon^{p,\lambda}}{|PT^\lambda - \alpha\varepsilon^{p,\lambda}|} \\
&\qquad - \frac{(|PT^\mu - \alpha\varepsilon^{p,\mu}| + \theta^\mu - k)_+}{2\mu} \frac{PT^\mu - \alpha\varepsilon^{p,\mu}}{|PT^\mu - \alpha\varepsilon^{p,\mu}|} \Big)\, dx \\
&\le - \int_\Omega \frac{(|PT^\lambda - \alpha\varepsilon^{p,\lambda}| + \theta^\lambda - k)_+}{2\lambda} |PT^\lambda - \alpha\varepsilon^{p,\lambda}|\, dx \\
&\quad - \int_\Omega \frac{(|PT^\mu - \alpha\varepsilon^{p,\mu}| + \theta^\mu - k)_+}{2\mu} |PT^\mu - \alpha\varepsilon^{p,\mu}| dx \\
&\quad + \int_\Omega \frac{(|PT^\lambda - \alpha\varepsilon^{p,\lambda}| + \theta^\lambda - k)_+}{2\lambda} |PT^\mu - \alpha\varepsilon^{p,\mu}|\, dx \\
&\quad + \int_\Omega \frac{(|PT^\mu - \alpha\varepsilon^{p,\mu}| + \theta^\mu - k)_+}{2\mu} |PT^\lambda - \alpha\varepsilon^{p,\lambda}|\, dx \\
= &- \int_\Omega (|\varepsilon_t^{p,\lambda}| - |\varepsilon_t^{p,\mu}|)(|PT^\lambda - \alpha\varepsilon^{p,\lambda}| - |PT^\mu - \alpha\varepsilon^{p,\mu}|) dx \\
= &- \int_\Omega (|\varepsilon_t^{p,\lambda}| - |\varepsilon_t^{p,\mu}|)(|PT^\lambda - \alpha\varepsilon^{p,\lambda}| + \theta^\lambda - k - |PT^\mu - \alpha\varepsilon^{p,\mu}| - \theta^\mu + k - \theta^\lambda + \theta^\mu)\, dx\,.
\end{aligned}$$

Using the simple facts: $xx_+ = x_+^2$ and $y \le y_+$ for all real numbers x, y we obtain

$$\begin{aligned}
I_2 \ &\le\ - \int_\Omega (\lambda|\varepsilon_t^{p,\lambda}|^2 + \mu|\varepsilon_t^{p,\mu}|^2 - (\lambda + \mu)|\varepsilon_t^{p,\lambda}||\varepsilon_t^{p,\mu}|) dx + \int_\Omega (|\varepsilon_t^{p,\lambda}| - |\varepsilon_t^{p,\mu}|)(\theta^\lambda - \theta^\mu)\, dx \\
&\le\ \frac{\lambda}{4} \int_\Omega |\varepsilon_t^{p,\mu}|^2\, dx + \frac{\mu}{4} \int_\Omega |\varepsilon_t^{p,\lambda}|^2 dx + \int_\Omega (|\varepsilon_t^{p,\lambda}| + |\varepsilon_t^{p,\mu}|)|\theta^\lambda - \theta^\mu|\, dx\,.
\end{aligned} \tag{5.3}$$

Inserting (5.2) into (5.1) we arrive at the inequality

$$\frac{d}{dt}\mathcal{E}^{\mu,\lambda} \le \frac{\lambda}{4} \int_\Omega |\varepsilon_t^{p,\mu}|^2 dx + \frac{\mu}{4} \int_\Omega |\varepsilon_t^{p,\lambda}|^2 dx + (c+1) \int_\Omega (|\varepsilon_t^{p,\lambda}| + |\varepsilon_t^{p,\mu}| + |\varepsilon_t^\lambda| + |\varepsilon_t^\mu|)|\theta^\lambda - \theta^\mu|\, dx\,. \tag{5.4}$$

After integration with respect to t over the intervall $(0, T)$ we conclude that

$$\mathcal{E}^{\mu,\lambda}(t) \le \frac{\lambda}{4}\|\varepsilon_t^{p,\mu}\|^2 + \frac{\mu}{4}\|\varepsilon_t^{p,\lambda}\|^2 + (c+1)(\|\varepsilon_t^{p,\lambda}\| + \|\varepsilon_t^{p,\mu}\| + \|\varepsilon_t^\lambda\| + \|\varepsilon_t^\mu\|)\|\theta^\lambda - \theta^\mu\|$$

which implies the statement of the lemma. $\square$

Proof of the Main Theorem. Using the estimate from Theorem 4.1 we can assume (passing eventually to a subsequence) that the sequences $\{\varepsilon_t^\lambda\}, \{\varepsilon_t^{p,\lambda}\}, \{\theta_t^\lambda\}$ converge weakly in $\mathbb{L}^2(\mathbb{L}^2), \{\varepsilon^\lambda\}, \{\varepsilon^{p,\lambda}\}, \{\nabla\theta^\lambda\}$ converge weakly-* in $\mathbb{L}^\infty(\mathbb{L}^2)$ and the sequence $\{\theta^\lambda\}$ converges strongly in $\mathbb{L}^2(\mathbb{L}^2)$ and pointwise a.e. in $\Omega \times (0,T)$. Let us denote the limit functions by $\varepsilon, \varepsilon^p$ and θ, respectively. Moreover, we see that the sequence of stresses $\{T^\lambda = \mathcal{D}(\varepsilon^\lambda - \varepsilon^{p,\lambda}) - c\theta^\lambda \cdot I\}$ converges weakly-* in $\mathbb{L}^\infty(\mathbb{L}^2)$ to the function $T = \mathcal{D}(\varepsilon - \varepsilon^p) - c\theta \cdot I$. It is also easy to see that the limit functions satisfy boundary conditions (2.5) - (2.6) and initial condition (2.7). Passing to the limit $\lambda \to 0^+$ in system (3.2) - (3.4) we obtain

$$\begin{aligned} \mathrm{div}_x \mathcal{D}(\varepsilon(x,t) - \varepsilon^p(x,t)) - c\nabla\theta(x,t) &= -F(x,t) \quad \text{in the space} \quad \mathbb{L}^\infty((0,T);\mathbb{L}^2(\Omega;\mathbb{R}^3)),\\ \theta_t(x,t) &= \kappa\Delta\theta(x,t) - Q(x,t) \quad \text{in the space} \quad \mathbb{L}^2((0,T);\mathbb{L}^2(\Omega;\mathbb{R})),\\ \varepsilon_t^p(x,t) &= R(x,t) \quad \text{in the space} \quad \mathbb{L}^2((0,T);\mathbb{L}^2(\Omega;\mathcal{S}^3)), \end{aligned}$$

where $Q = \mathrm{w-lim}\ \gamma \mathrm{div}_x u_t^\lambda \beta(\theta^\lambda)$ and $R = \mathrm{w-lim}\ \Pi_2 Y_\lambda(\theta^\lambda, T^\lambda - \alpha\varepsilon^{p,\lambda})$. To end the proof we have to show that $R \in \Pi_2 \partial I_{\mathcal{K}}(\theta, T - \alpha\varepsilon^p)$ and $Q = \gamma \mathrm{div}_x u_t \beta(\theta)$. On the set $\{\theta \leq k\}$ the $\mathbb{L}^2(\mathbb{L}^2)$ boundedness of the second component of the Yosida approximation Y_λ implies the $\mathbb{L}^2(\mathbb{L}^2)$ boundedness of the entire vector Y_λ. The graph of the maximal monotone multifunction $\partial I_{\mathcal{K}}$ is weakly-strongly closed. By Lemma 5.1 we conclude that the sequence $\{T^\lambda - \alpha\varepsilon^{p,\lambda}\}$ converges strongly in $\mathbb{L}^2(\mathbb{L}^2)$ to the limit $T - \alpha\varepsilon^p$. Moreover, $\{\theta^\lambda\}$ converges strongly in $\mathbb{L}^2(\mathbb{L}^2)$ to θ and the projector Π_2 is a linear and continuous mapping. Hence, we obtain that $R \in \Pi_2 \partial I_{\mathcal{K}}(\theta, T - \alpha\varepsilon^p)$. Next, it is easy to see that

$$\beta(\theta^\lambda)\chi_{\{\theta<k\}} \to 1 \text{ and } \beta(\theta^\lambda)\chi_{\{\theta>k\}} \to 0 \text{ pointwise a.e. on } \Omega \times (0,T).$$

From the $\mathbb{L}^\infty$-boundedness of these two sequences we have that

$$\beta(\theta^\lambda)\chi_{\{\theta<k\}} \to 1 \ \text{ and } \ \beta(\theta^\lambda)\chi_{\{\theta<k\}} \to 0 \ \text{ strongly in } \ \mathbb{L}^2(\mathbb{L}^2)\,.$$

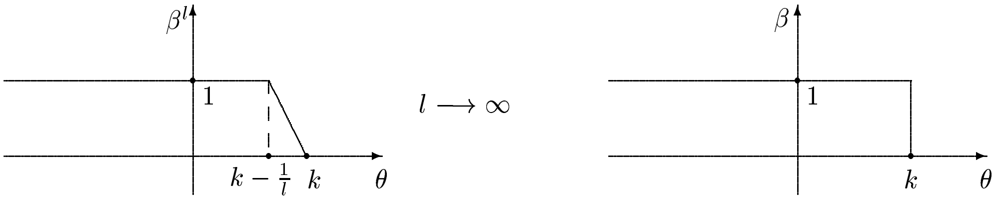
Consequently, we obtain that $Q = \gamma \mathrm{div}_x u_t \beta(\theta)$ and the proof is complete.

6 Open problems

This contribution is the first step in the mathematical analysis of the boundary-value problem (2.2) - (2.7). In this section we want to point out on some interesting open problems, associated with this model of thermoplasticity. The first one is the uniqueness of solutions of the considered system of equations. Uniqueness provided, it can be shown that the entire approximation procedure converges to a solution, and not only a subsequence. The second one is the necessity of the control trick. The cut-off function β implies the maximum principle for the temperature and enables to estimate the entire Yosida approximation by the second component of this vector, only. This was the main idea of the presented proof. The next open problem is the $\mathbb{L}^\infty(\mathbb{L}^2)$-boundedness of time derivatives of the approximation sequence. With such an estimate it can be proved that for small given data the temperature stays below the critical value k and the control trick is not necessary. The statement of Theorem 2 was proved under the assumption that $\alpha > \frac{\gamma}{4c}$. This inequality violates the possibility to study the limit case $\alpha \to 0^+$. Hence, the next open problem is an $\mathbb{L}^2(\mathbb{L}^2)$ estimate for the time derivatives of the approximation sequence for all $\alpha > 0$. Finally, assuming that the last problem is solved, we want to point out on very interesting limit procedure $\alpha \to 0^+$, which is connected with loss of regularity for the strains (compare with the perfect elasto-plasticity [Che01],[Suq88],[Tem83]).

7 Appendix

Proof of Theorem 3.1 The proof presented here is based on the methods from Theorem 4.1 and Lemma 5.1 and therefore is placed at the end of the article. The main idea of the proof is illustrated on the figure below.



Hence, for fixed $\lambda > 0$ we regularize system (3.2)-(3.4) by inserting the function β^l (see the figure) instead of β, where $l > 0$ is a natural number. Thus, we study the sytem

$$\mathrm{div}_x \mathcal{D}(\varepsilon^l(x,t) - \varepsilon^{p,l}(x,t)) - c\nabla\theta^l(x,t) = -F(x,t), \tag{7.1}$$

$$\theta^l_t(x,t) = \kappa\Delta\theta^l(x,t) - \gamma \mathrm{div}_x u^l_t(x,t)\beta^l(\theta^l(x,t)), \tag{7.2}$$

$$\varepsilon^p_t(x,t) = \Pi_2\Big(Y_\lambda(\theta^l(x,t), T^l(x,t) - \alpha\varepsilon^{p,l}(x,t))\Big) \tag{7.3}$$

For each l this system of equations possesses global Lipschitz nonlinearities only. Hence, existence and uniquness of strong solutions can be done using the Banach Fixed Point Theorem for example (compare with [CN03]). Next, we want to prove some estimates for the sequence $\{(u^l, \varepsilon^l, \varepsilon^{p,l}, \theta^l)\}$. Estimate for the time derivatives will be done following the idea of Theorem 4.1. Using the potential function

$$M^l_\lambda(x,t) = \frac{(|PT^l(x,t) - \alpha\varepsilon^{p,l}(x,t)| + \theta^l(x,t) - k)^2_+}{4\lambda}$$

we arrive at the equality (compare with (4.2))

$$\frac{d}{dt}\int_\Omega M^l_\lambda dx = \int_\Omega \Big(-\mathcal{D}(\varepsilon^l_t - \varepsilon^{p,l}_t)(\varepsilon^l_t - \varepsilon^{p,l}_t) + \varepsilon^l_t(T^l_t + c\theta^l_t\beta^l(\theta^l)\cdot I)$$

$$+ c\varepsilon^l_t(1-\beta^l(\theta^l))\theta^l_t \cdot I + |\varepsilon^{p,l}_t|\theta^l_t - \alpha|\varepsilon^{p,\lambda}_t|^2\Big) dx$$

The term $c\varepsilon_t^l\theta_t^l\beta^l(\theta^l)\cdot I$ plays now the same role as the term $c\varepsilon_t^l\theta_t^l\cdot I$ in the proof of Theorem 4.1. The next term $c\varepsilon_t^l(1-\beta^l(\theta^l))\theta_t^l\cdot I$ will be estimated by $c|\mathrm{tr}\,\varepsilon_t^l||\theta_t^l|$. For fixed $\lambda>0$ the crucial integral $\int_\Omega|\varepsilon_t^l|\theta_t^l\,dx$ in the proof of Theorem 4.1 can be now estimated as follows,

$$\begin{aligned}\int_\Omega|\varepsilon_t^l|\theta_t^l\,dx &= \int_\Omega\frac{(|PT^l|+\theta^l-k)_+}{2\lambda}\theta_t^l\,dx \leq \frac{1}{2\lambda}\|T^l\|\|\theta_t^l\|\\ &\leq \frac{1}{\lambda}\|\theta_t^l\|\Big(\|T^l(0)\|^2+t\int_0^t\|T_t^l\|^2\,d\tau\Big)^{\frac12}\\ &\leq \frac{C}{\lambda}\|\theta_t^l\|\Big(\|T^l(0)\|^2+t\int_0^t(\mathcal{D}(\varepsilon_t^l-\varepsilon_t^{p,l})(\varepsilon_t^l-\varepsilon_t^{p,l})+|\theta_t^l|^2)\,d\tau\Big)^{\frac12}\end{aligned}$$

where the nonnegative constant C does not depend on l. Using assumption (A) we absorb the term $-c|\mathrm{tr}\,\varepsilon_t^l||\theta_t^l|$ by the "good terms" $\mathcal{D}(\varepsilon_t^l-\varepsilon_t^{p,l})(\varepsilon_t^l-\varepsilon_t^{p,l})$ and $|\theta_t^l|^2$. The estimate for the time derivatives now follows by the Gronwall inequality. Next, we prove an analogon to Lemma 5.1 and obtain that the sequences of strains and stresses converge strongly in $\mathbb{L}^\infty(\mathbb{L}^2)$. (Note that the proof of Lemma 5.1 does not use the right-hand side of equation (3.3).) At the end we pass to the limit $l\to\infty$ and show that the limit functions are strong solutions of system (3.2)-(3.4).

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