Sharp decay rates in parabolic and hyperbolic thermoelasticity

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Abstract: We present a discussion of the exponential decay — in particular giving explicit sharp decay rates — for solutions to the system of classical thermoelasticity as well as for that of thermoelasticity with second sound. The relevance of the latter model in contrast to the first one with respect to the asymptotic behavior is investigated. Different real materials are considered. First, solution representations are used that allow for a numerical calculation of the corresponding characteristic roots. Second, the Hurwitz criterion is applied both on the qualitative and on the quantitative level. It is demonstrated that it is possible to obtain accurate decay rates with this fast approach.
The paper may also serve as collection of material data for future investigations.

1 Introduction

For different thermoelastic systems like that of classical thermoelasticity or that of thermoelasticity with second sound, there are many results on the exponential stability of the solutions, for a variety of boundary conditions on bounded domains cp. [6] for an extensive discussion and further references. Those are mostly of qualitative nature proving that the associated energy decays exponentially at a certain rate, but the rate is not given explicitly, only in some cases rough estimates are presented. Moreover, in most cases there are no real materials considered in the sense that the appearing constants are given explicitly.

Having recently proved an exponential stability result in thermoelasticity with second sound in [13] together with a very coarse estimate for the material silicon, we wish to give a precise decay rate, both for this case as well as for other materials and in particular also for classical thermoelasticity. This will also allow to compare these two thermoelastic systems and the impact of hyperbolic versus parabolic heat conduction models. This will be first done by computing the roots of the relevant fourth-order polynomial numerically.

Afterwards, the Hurwitz criterion for the existence of roots of polynomials with negative real part will be shown to apply not only quantitatively, as done e.g. for another thermoelastic system in [12], but also qualitatively. This way, by rather simple but material dependent and adapted calculations, we also obtain decay rates with the desired accuracy. It is really surprising how these estimates efficiently lead to explicit and sharp decay rates. This observation may be also useful for other more complicated coupled systems, where one cannot any longer compute the roots of the possibly higher-than-fifth-order polynomial explicitly.

Altogether we present a discussion of exponential stability and of the determination of explicit and sharp decay rates with different methods for different thermoelastic systems, including their comparison, both on the analytical and on the numerical level.

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The collection of material data presented here may also serve in future investigations.

In section 2 we investigate the system of classical thermoelasticity, also proving the optimality of the decay rates obtained numerically. In section 3 we present the results for thermoelasticity with second sound. In these sections a comparison of the systems as well as a comparison with pure heat conduction is given. In the last section 4 we use the Hurwitz criterion to find out that this can be exploited to both quantitatively and qualitatively calculate the decay rates. It turns out that this approach efficiently yields the optimal decay rates, in conclusion being a useful tool also for more complicated systems where explicit calculations of roots of polynomials are impossible.

Throughout the paper we shall consider eight different materials: silicon, aluminum alloy, steel, germanium, gallium arsenid, indium arsenid, copper, and diamond.

2 Classical thermoelasticity

We first consider the following initial-boundary value problem from classical thermoelasticity in the bounded domain $(0, L)$, $L > 0$:

$$u_{tt} - \alpha u_{xx} + \beta\theta_x = 0 \quad \text{in } [0,\infty) \times (0,L), \tag{2.1}$$

$$\theta_t - \gamma\kappa\theta_{xx} + \delta u_{tx} = 0 \quad \text{in } [0,\infty) \times (0,L), \tag{2.2}$$

where u and θ are functions of $(t,x) \in [0,\infty) \times (0,L)$ describing the displacement and the temperature difference of the thermoelastic material under consideration, respectively; the positive constants $\alpha, \beta, \gamma, \kappa, \delta$ are material constants given below for different materials.

Additionally, one has initial conditions

$$u(t=0) = u^0, \quad u_t(t=0) = u^1, \quad \theta(t=0) = \theta^0, \tag{2.3}$$

and the boundary conditions

$$u(x=0) = u(x=L) = \theta_x(x=0) = \theta_x(x=L) = 0. \tag{2.4}$$

The well-posedness for (2.1)–(2.4) is well-known as well as the fact that the associated energy tends to zero exponentially for initial data satisfying the natural condition

$$m_{\theta_0} := \int_0^L \theta^0(x)dx = 0 \tag{2.5}$$

(otherwise $\theta - m_\theta$ has to replace θ), cp. [17, 6]. The ansatz

$$u(t,x) = \sum_{j=1}^{\infty} a_j(t)s_j(x) \quad \theta(t,x) = \sum_{j=1}^{\infty} b_j(t)c_j(x) \tag{2.6}$$

where

$$s_j(x) := \sqrt{\frac{L}{2}}\sin(\sqrt{\lambda_j}x), \quad c_j(x) := \sqrt{\frac{L}{2}}\cos(\sqrt{\lambda_j}x), \quad \lambda_j := \frac{j^2\pi^2}{L^2},$$

is compatible with the boundary conditions (2.4), and the coefficients $a_j(t), b_j(t)$ satisfy

$$a_j(t) = \sum_{k=1}^{3} u_k(j)\mathrm{e}^{-\rho_k(j)t}, \quad b_j(t) = \sum_{k=1}^{3} \theta_k(j)\mathrm{e}^{-\rho_k(j)t} \tag{2.7}$$

where u_k, θ_k are determined by the initial values u_0, u_1, θ_0, and $\rho_1(j), \rho_2(j), \rho_3(j)$ are the roots of the characteristic polynomial satisfying

$$-\rho^3 + \gamma\kappa\lambda_j\rho^2 - (\alpha + \beta\delta)\lambda_j\rho + \alpha\gamma\kappa\lambda_j^2 = 0. \tag{2.8}$$

The latter can be seen for example, assuming sufficiently smooth solutions, by differentiating the differential equations (2.1), (2.2) to eliminate u or θ and conclude that u and θ both satisfy the following differential equation:

$$w_{ttt} - \gamma\kappa w_{ttxx} - (\alpha + \beta\delta)w_{txx} + \alpha\gamma\kappa w_{xxxx} = 0. \tag{2.9}$$

The characteristic polynomial in (2.8) is well known in classical thermoelasticity and has been studied from a qualitative point of view in different works, see e.g. [7, 23, 6].

Remark: The same polynomial governs the behavior of the curl-free component for the Cauchy problem in three space dimensions, cp. [6].

Knowing the exponential decay of solutions, we define as optimal decay rate

$$d_{ct} := \inf_{j\in\mathbb{N}, k=1,2,3} \Re\rho_k(j). \tag{2.10}$$

This definition correponds — up to a factor two — to the decay rates discussed in the literature for exponential decay results of the L^2-energy, cp. [6] and the references there. Below we present a proof of this for classical thermoelasticity.

We see from the numerical data below — also known from the qualitative behavior — that the infimum is obtained for $j = 1$. One has one root ρ_3 with imaginary part zero (arising from pure heat conduction) and two complex conjugate roots ρ_1, ρ_2 mainly arising from elasticity — for the coupling constants β, δ set to zero, these two roots would have real part zero typical for pure elasticity — with real part approaching a limit:

$$\mathcal{L} := \lim_{j\to\infty} \Re\rho_{2,3}(j) = \frac{\beta\delta}{2\gamma\kappa}, \tag{2.11}$$

cp. [6, Lemma 2.17] or the work of [7, 23].

We consider eight different materials: silicon, 6061-T6 aluminum alloy, steel, germanium, gallium arsenid, indium arsenid, copper, and diamond. The values of the material constants for silicon were provided in [13], the data for aluminum alloy and for steel are based on [19] and [18], respectively; the medium temperature T_0 for [18] is assumed to be $T_0 := 295K$. The values for germanium, gallium arsenid, indium arsenid, copper, and diamond are taken from the internet sources [4]. The value for the characteristic length L of the specimen is taken to be

$$L := 6.25\ 10^{-4}\ [m]$$

which was typical for the silicon example from laser cleaning considered in [13], and hence is taken for all materials here, allowing for a better comparison of the results.

material constants					
parameter	silicon	aluminum alloy	steel	germanium	unit
α	$9.622 \cdot 10^7$	$3.782 \cdot 10^7$	$2.544 \cdot 10^7$	$2.37 \cdot 10^7$	m^2/s^2
β	$3.9162 \cdot 10^2$	$1.75647 \cdot 10^3$	$4.2711 \cdot 10^2$	$2.34 \cdot 10^2$	m^2/s^2K
γ	$5.99 \cdot 10^{-7}$	$3.84 \cdot 10^{-7}$		$6.06 \cdot 10^{-7}$	s^2Km/kg
δ	$1.6382 \cdot 10^2$	$5.33865 \cdot 10^2$	$2.7331 \cdot 10^2$	$1.32 \cdot 10^1$	K
κ	$1.48 \cdot 10^2$	$1.55 \cdot 10^2$		$5.80 \cdot 10^1$	W/mK
$\gamma\kappa$			$4.715 \cdot 10^{-6}$		

material constants					
parameter	gallium arsenid	indium arsenid	copper	diamond	unit
α	$2.24 \cdot 10^7$	$1.45 \cdot 10^7$	$1.99 \cdot 10^7$	$3.05 \cdot 10^8$	m^2/s^2
β	$2.21 \cdot 10^2$	$1.16 \cdot 10^2$	$5.94 \cdot 10^2$	$3.79 \cdot 10^2$	m^2/s^2K
γ	$5.70 \cdot 10^{-7}$	$7.04 \cdot 10^{-7}$	$2.96 \cdot 10^{-7}$	$5.46 \cdot 10^{-7}$	s^2Km/kg
δ	$1.25 \cdot 10^1$	$6.13 \cdot 10^0$	$4.69 \cdot 10^2$	$3.23 \cdot 10^1$	K
κ	$5.50 \cdot 10^1$	$2.70 \cdot 10^1$	$4.10 \cdot 10^2$	$1.30 \cdot 10^3$	W/mK

The following numerical results for the values of the roots $\rho_k(j)$ as well as those numerical results in later sections were obtained using MAPLE[1] V Release 5.1.

$\rho_k(j)$			
	silicon	aluminum alloy	steel
$\rho_{1/2}(j)$			
$j = 1$	$7.462 \cdot 10^{-1} \pm 4.932 \cdot 10^7 i$	$1.819 \cdot 10^1 \pm 3.129 \cdot 10^7 i$	$2.721 \cdot 10^{-1} \pm 2.541 \cdot 10^7 i$
$j = 5$	$1.865 \cdot 10^1 \pm 2.466 \cdot 10^8 i$	$4.548 \cdot 10^2 \pm 1.565 \cdot 10^8 i$	$6.802 \cdot 10^0 \pm 1.270 \cdot 10^8 i$
$j = 10$	$7.462 \cdot 10^1 \pm 4.932 \cdot 10^9 i$	$1.819 \cdot 10^3 \pm 3.129 \cdot 10^8 i$	$2.721 \cdot 10^1 \pm 2.541 \cdot 10^8 i$
$j = 50$	$1.865 \cdot 10^3 \pm 2.466 \cdot 10^9 i$	$4.548 \cdot 10^4 \pm 1.565 \cdot 10^9 i$	$6.802 \cdot 10^2 \pm 1.271 \cdot 10^9 i$
$j = 10^4$	$6.188 \cdot 10^7 \pm 4.932 \cdot 10^{11} i$	$1.489 \cdot 10^9 \pm 3.122 \cdot 10^{11} i$	$2.715 \cdot 10^7 \pm 2.541 \cdot 10^{11} i$
$j = 10^{50}$	$3.618 \cdot 10^8 \pm 4.931 \cdot 10^{57} i$	$7.877 \cdot 10^9 \pm 3.091 \cdot 10^{57} i$	$1.238 \cdot 10^{10} \pm 2.535 \cdot 10^{57} i$
$j = 10^{100}$	$3.618 \cdot 10^8 \pm 4.931 \cdot 10^{107} i$	$7.877 \cdot 10^9 \pm 3.091 \cdot 10^{107} i$	$1.238 \cdot 10^{10} \pm 2.535 \cdot 10^{107} i$
$\rho_3(j)$			
$j = 1$	$2.238 \cdot 10^3$	$1.467 \cdot 10^3$	$1.186 \cdot 10^2$
$j = 5$	$5.596 \cdot 10^4$	$3.669 \cdot 10^4$	$2.965 \cdot 10^3$
$j = 10$	$2.238 \cdot 10^5$	$1.467 \cdot 10^5$	$1.186 \cdot 10^4$
$j = 50$	$5.596 \cdot 10^6$	$3.669 \cdot 10^6$	$2.965 \cdot 10^5$
$j = 10^4$	$2.239 \cdot 10^{11}$	$1.474 \cdot 10^{11}$	$1.186 \cdot 10^{10}$
$j = 10^{50}$	$2.240 \cdot 10^{103}$	$1.504 \cdot 10^{103}$	$1.191 \cdot 10^{102}$
$j = 10^{100}$	$2.240 \cdot 10^{203}$	$1.504 \cdot 10^{203}$	$1.191 \cdot 10^{202}$
theoretical value for $\mathcal{L}$ by formula (2.11):			
$\mathcal{L}$	$3.618 \cdot 10^8$	$7.877 \cdot 10^9$	$1.238 \cdot 10^{10}$
decay rate by definition (2.10):			
d_{ct}	$7.462 \cdot 10^{-1}$	$1.819 \cdot 10^1$	$2.721 \cdot 10^{-1}$

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$\rho_k(j)$			
	germanium	gallium arsenid	indium arsenid
$\rho_{1/2}(j)$			
$j=1$	$5.786 \cdot 10^{-2} \pm 2.447 \cdot 10^{7} i$	$4.884 \cdot 10^{-2} \pm 2.379 \cdot 10^{7} i$	$1.178 \cdot 10^{-2} \pm 1.914 \cdot 10^{7} i$
$j=5$	$1.447 \cdot 10^{0} \pm 1.224 \cdot 10^{8} i$	$1.221 \cdot 10^{0} \pm 1.190 \cdot 10^{8} i$	$2.944 \cdot 10^{-1} \pm 9.570 \cdot 10^{7} i$
$j=10$	$5.786 \cdot 10^{0} \pm 2.447 \cdot 10^{8} i$	$4.884 \cdot 10^{0} \pm 2.379 \cdot 10^{8} i$	$1.178 \cdot 10^{0} \pm 1.914 \cdot 10^{8} i$
$j=50$	$1.447 \cdot 10^{2} \pm 1.223 \cdot 10^{9} i$	$1.221 \cdot 10^{2} \pm 1.190 \cdot 10^{9} i$	$2.944 \cdot 10^{1} \pm 9.570 \cdot 10^{8} i$
$j=10^4$	$5.113 \cdot 10^{6} \pm 2.447 \cdot 10^{11} i$	$4.396 \cdot 10^{6} \pm 2.379 \cdot 10^{11} i$	$1.108 \cdot 10^{6} \pm 1.914 \cdot 10^{11} i$
$j=10^{50}$	$4.394 \cdot 10^{7} \pm 2.447 \cdot 10^{57} i$	$4.406 \cdot 10^{7} \pm 2.379 \cdot 10^{57} i$	$1.870 \cdot 10^{7} \pm 1.914 \cdot 10^{57} i$
$j=10^{100}$	$4.394 \cdot 10^{7} \pm 2.447 \cdot 10^{107} i$	$4.406 \cdot 10^{7} \pm 2.379 \cdot 10^{107} i$	$1.870 \cdot 10^{7} \pm 1.914 \cdot 10^{107} i$
$\rho_3(j)$			
$j=1$	$8.879 \cdot 10^{2}$	$7.920 \cdot 10^{2}$	$4.802 \cdot 10^{2}$
$j=5$	$2.220 \cdot 10^{4}$	$1.980 \cdot 10^{4}$	$1.201 \cdot 10^{4}$
$j=10$	$8.879 \cdot 10^{4}$	$7.920 \cdot 10^{4}$	$4.802 \cdot 10^{4}$
$j=50$	$2.220 \cdot 10^{6}$	$1.980 \cdot 10^{6}$	$1.201 \cdot 10^{6}$
$j=10^4$	$8.879 \cdot 10^{10}$	$7.920 \cdot 10^{10}$	$4.802 \cdot 10^{10}$
$j=10^{50}$	$8.881 \cdot 10^{102}$	$7.921 \cdot 10^{102}$	$4.803 \cdot 10^{102}$
$j=10^{100}$	$8.881 \cdot 10^{202}$	$7.921 \cdot 10^{202}$	$4.803 \cdot 10^{202}$
theoretical value for $\mathcal{L}$ by formula (2.11):			
$\mathcal{L}$	$4.394 \cdot 10^{7}$	$4.406 \cdot 10^{7}$	$1.870 \cdot 10^{7}$
decay rate by definition (2.10):			
d_{ct}	$5.786 \cdot 10^{-2}$	$4.884 \cdot 10^{-2}$	$1.178 \cdot 10^{-2}$

$\rho_k(j)$		
	copper	diamond
$\rho_{1/2}(j)$		
$j=1$	$2.117 \cdot 10^1 \pm 2.258 \cdot 10^7 i$	$3.599 \cdot 10^{-1} \pm 8.779 \cdot 10^7 i$
$j=5$	$5.292 \cdot 10^2 \pm 1.129 \cdot 10^8 i$	$8.997 \cdot 10^0 \pm 4.389 \cdot 10^8 i$
$j=10$	$2.117 \cdot 10^3 \pm 2.258 \cdot 10^8 i$	$3.599 \cdot 10^1 \pm 8.779 \cdot 10^8 i$
$j=50$	$5.291 \cdot 10^4 \pm 1.129 \cdot 10^9 i$	$8.996 \cdot 10^2 \pm 4.389 \cdot 10^9 i$
$j=10^4$	$7.489 \cdot 10^8 \pm 2.248 \cdot 10^{11} i$	$6.957 \cdot 10^6 \pm 8.779 \cdot 10^{11} i$
$j=10^{50}$	$1.148 \cdot 10^9 \pm 2.242 \cdot 10^{57} i$	$8.623 \cdot 10^6 \pm 8.778 \cdot 10^{57} i$
$j=10^{100}$	$1.148 \cdot 10^9 \pm 2.242 \cdot 10^{107} i$	$8.623 \cdot 10^6 \pm 8.778 \cdot 10^{107} i$
$\rho_3(j)$		
$j=1$	$3.024 \cdot 10^3$	$1.793 \cdot 10^4$
$j=5$	$7.560 \cdot 10^4$	$4.483 \cdot 10^5$
$j=10$	$3.024 \cdot 10^5$	$1.793 \cdot 10^6$
$j=50$	$7.560 \cdot 10^6$	$4.483 \cdot 10^7$
$j=10^4$	$3.051 \cdot 10^{11}$	$1.793 \cdot 10^{12}$
$j=10^{50}$	$3.066 \cdot 10^{103}$	$1.793 \cdot 10^{104}$
$j=10^{100}$	$3.066 \cdot 10^{203}$	$1.793 \cdot 10^{204}$
theoretical value for $\mathcal{L}$ by formula (2.11):		
$\mathcal{L}$	$1.148 \cdot 10^9$	$8.623 \cdot 10^6$
decay rate by definition (2.10):		
d_{ct}	$2.117 \cdot 10^1$	$3.599 \cdot 10^{-1}$

We observe that the theoretical value for $\mathcal{L}$ is clearly visible in the numerical data in $\Re\rho_{1,2}(j)$, The decay rate is determined by the first two eigenvalues, these being mainly determined by the hyperbolic part of the system, while in the third eigenvalue, the parabolic part from heat conduction is predominant.

In [13] an estimate for the decay rate in classical thermoelasticity for silicon was given using rather coarse estimates in the energy method used there, the computed value $2\tilde{d}_{ct}$ was $2\tilde{d}_{ct} = 1.59 \cdot 10^{-12}$ which is much lower than the now obtained optimal value $d_{ct}(\text{silicon}) = 7.462 \cdot 10^{-1}$ shows.

It is interesting to compare the result for classical thermoelasticity just obtained with pure heat conduction. By this we mean the following system:

$$v_t - \gamma\kappa v_{xx} = 0, \quad v(t=0) = v_0, \quad v_x(x=0) = v_x(x=L) = 0. \tag{2.12}$$

The decay rate for this system — for v with $\int_0^L v_0(x)dx = 0$ of course — is obviously

$$d_h := \frac{\gamma\kappa\pi^2}{L^2} \tag{2.13}$$

which gives the following values:

decay rate for pure heat conduction				
	silicon	aluminum alloy	steel	germanium
d_h	$2.240 \cdot 10^3$	$1.504 \cdot 10^3$	$1.191 \cdot 10^2$	$8.881 \cdot 10^2$

decay rate for pure heat conduction				
	gallium arsenid	indium arsenid	copper	diamond
d_h	$7.921 \cdot 10^2$	$4.803 \cdot 10^2$	$3.066 \cdot 10^3$	$1.793 \cdot 10^4$

(The corresponding value for silicon in [13] is misprinted.)

These values are very close to those obtained for the (real) third eigenvalue $\rho_3(1)$ in classical thermolasticity. This reflects the fact that in $\rho_3(j)$ the impact of heat conduction predominates the hyperbolic over all.

The decay rate for pure heat conduction is much higher than for classical thermoelasticity:

$$d_{ct} < d_h. \tag{2.14}$$

ratio of decay rates				
	silicon	aluminum alloy	steel	germanium
d_h/d_{ct}	$3.002 \cdot 10^3$	$8.268 \cdot 10^1$	$4.377 \cdot 10^2$	$1.535 \cdot 10^4$

ratio of decay rates				
	gallium arsenid	indium arsenid	copper	diamond
d_h/d_{ct}	$1.621 \cdot 10^4$	$4.077 \cdot 10^4$	$1.448 \cdot 10^2$	$4.982 \cdot 10^4$

Of course this does not imply that heat itself is dissipated so much slower if the elastic effects are not neglected, since the lower numbers there may (and will) arise from the lower decay rate for the displacements u, while we expect that the heat itself may be dissipated almost as fast as in pure heat conduction.

We also observe that the decay rate for pure heat conduction is best for silicon, then for aluminun alloy, and then for steel while the decay rate in classical thermoelasticity is best for aluminum alloy, then for silicon and then for steel. The interpretation of this is that the impact of the elastic waves on the lowering of the decay rate (with respect to pure heat conduction) is stronger for silicon than for aluminum.

Having discussed the values of d_{ct} and d_h we will now show that d_{ct} is indeed the optimal decay rate. Therefore we define

$$E(t) := \int_0^L u_t^2(t,x) + u_x^2(t,x) + \theta^2(t,x)dx$$

as usual as the energy and we shall prove

Theorem 2.1 *Assuming (2.14), we have exponential stability with decay rate* $2d_{ct}$ *for the energy* $E(\cdot)$*:*

$$\exists C > 0 \quad \forall t \geq 0 : \quad E(t) \leq Ce^{-2d_{ct}t}E(0)$$

Remark: It is to be expected naturally, and it should be possible to show it rigorously that (2.14) is satisfied for *any* material. Therefore Theorem 2.1 holds in general.

PROOF: From (2.6) and (2.1)–(2.3) we get

$$\begin{aligned}
a_j(0) &= \frac{L}{\pi j}\,\langle u_x^0, c_j\rangle, \\
a_j'(0) &= \langle u^1, s_j\rangle, \\
a_j''(0) &= -\alpha\frac{\pi j}{L}\langle u_x^0, c_j\rangle + \beta\frac{\pi j}{L}\langle \theta^0, c_j\rangle,
\end{aligned} \tag{2.15}$$

where we use $\langle f, g\rangle := \int\limits_0^L fg\, dx$. From (2.7) we obtain

$$\begin{pmatrix} a_j(0) \\ a_j'(0) \\ a_j''(0) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ -\rho_1(j) & -\rho_2(j) & -\rho_3(j) \\ (\rho_1(j))^2 & (\rho_2(j))^2 & (\rho_3(j))^2 \end{pmatrix}}_{=:B(j)} \begin{pmatrix} u_1(j) \\ u_2(j) \\ u_3(j) \end{pmatrix} \tag{2.16}$$

with the Vandermonde determinant

$$\det\ B(j) = (\rho_1(j) - \rho_2(j))(\rho_1(j) - \rho_3(j))(\rho_2(j) - \rho_3(j)) \tag{2.17}$$

The behavior of the roots $\rho_k(j)$ is well known, see e.g. [6]:

$$\begin{aligned}
& \rho_1(j) && \sim j^2, \\
\Re\rho_2(j) &= \Re\rho_3(j) && \sim 1, \\
\Im\rho_2(j) &= -\Im\rho_3(j) && \sim j,
\end{aligned} \tag{2.18}$$

where we use the definition

$$f(j) \sim g(j) :\Leftrightarrow \exists\, c \neq 0 : \lim_{j\to\infty} \frac{f(j)}{g(j)} = c.$$

Hence we can conclude:

$$\det\ B(j) \sim j^5. \tag{2.19}$$

Applying Cramer's rule to (2.16) we get:

$$u_1(j) = \frac{1}{\det\ B(j)}\ \det \begin{pmatrix} a_j(0) & 1 & 1 \\ a_j'(0) & -\rho_2(j) & -\rho_3(j) \\ a_j''(0) & (\rho_2(j))^2 & (\rho_3(j))^2 \end{pmatrix}$$

and after an easy calculation observing (2.18) and (2.19):

$$u_1(j) \sim a_j(0) j^{-2} + a_j'(0) j^{-3} + a_j''(0) j^{-4}. \tag{2.20}$$

We can conclude the same way:

$$u_{2,3}(j) \sim a_j(0) + a_j'(0) j^{-1} + a_j''(0) j^{-3}. \tag{2.21}$$

From (2.6), (2.7), (2.15), (2.18), (2.20), and (2.21), we deduce:

$$\begin{aligned}
\int_0^L u_t^2 dx &= \sum_{j=1}^{\infty}\left|\sum_{k=1}^{3}\rho_k(j)u_k(j)e^{-\rho_k(j)t}\right|^2 \\
&\le Ce^{-2d_{ct}t}\sum_{j=1}^{\infty}\sum_{k=1}^{3} \mid \rho_k(j)u_k(j)\mid^2 \\
&\le Ce^{-2d_{ct}t}\sum_{j=1}^{\infty}\langle u_x^0, c_j\rangle^2 + \langle u^1, s_j\rangle^2 + j^{-1}\langle\theta^0, c_j\rangle^2 \\
&\le Ce^{-2d_{ct}t}E(0),
\end{aligned} \tag{2.22}$$

where $C > 0$ is a constant that may vary from line to line. In this manner it also follows:

$$\begin{aligned}
\int_0^L u_x^2 dx &= \sum_{j=1}^{\infty}\left|\sum_{k=1}^{3}\frac{\pi j}{L}\, u_k(j)e^{-\rho_k(j)t}\right|^2 \\
&\le Ce^{-2d_{ct}t}E(0).
\end{aligned} \tag{2.23}$$

Now we will treat the term $\int\theta^2 dx$. In analogy to (2.21) and (2.15) we get:

$$\theta_2(j) \sim b_j(0) + b_j'(0)j^{-1} + b_j''(0)j^{-3}$$

with

$$\begin{aligned}
b_j(0) &= \langle\theta^0, c_j\rangle \\
b_j'(0) &= -\gamma\kappa(\frac{\pi j}{L})^2\,\langle\theta^0, c_j\rangle - \delta\frac{\pi j}{L}\,\langle u^1, s_j\rangle, \\
b_j''(0) &= (\gamma\kappa)^2(\frac{\pi j}{L})^4\,\langle\theta^0, c_j\rangle + \gamma\kappa\delta\,(\frac{\pi j}{L})^3\langle u^1, s_j\rangle + \alpha\delta(\frac{\pi j}{L})^2\langle u_x^0, c_j\rangle - \beta\delta(\frac{\pi j}{L})^2\langle\theta^0, c_j\rangle,
\end{aligned}$$

therefore:

$$\theta_2(j) \sim j,$$

and the integral $\int_0^L \theta^2 dx$ cannot be estimated as above. This seems to be an inherent problem due to the hyperbolic-parabolic coupling, and is not expected for similar calculations in hyperbolic thermoelasticity with second sound below. To overcome this problem here we deduce from (2.2) with Poincaré's

$$\begin{aligned}
\frac{d}{dt}\int_0^L \theta^2 dx &= -2\gamma\kappa\int_0^L \theta_x^2 dx + 2\delta\int_0^L u_t\theta_x dx \\
&\le -2\,\underbrace{(\frac{\gamma\kappa\pi^2}{L^2} - \varepsilon)}_{=:d_h^\varepsilon}\int_0^L \theta^2 dx + C\int_0^L u_t^2 dx.
\end{aligned}$$

With the Gronwall lemma we conclude:

$$\int_0^L \theta^2 dx \le e^{-2d_h^\varepsilon t}\int_0^L \theta^2(0)dx + Ce^{-2d_{ct}t}E(0)$$

Now we choose ε so small that

$$d_{ct} < d_h^{\varepsilon}.$$

Together with (2.22) and (2.23) we finally get

$$\int_0^L u_t^2 + u_x^2 + \theta^2 dx \leq C e^{-2d_{ct}t} E(0).$$

Q.e.d.

3 Thermoelasticity with second sound

Now we turn to the system of thermoelasticity with second sound. Here thermal disturbances are modeled propagating as wave-like pulses traveling at finite speed. This removal of the physical paradox of infinite propagation speed in the classical theory of thermoelasticity within Fourier's law is achieved using Cattaneo's law for heat conduction. It arises for example in pulsed laser heating of solids, cp. [13] and the references therein. In that paper there was also given an explicit estimate for the decay rate using (necessarily?) coarse estimates within an energy method (mulitiplier technique). The estimate, giving a very small value for the decay rate, seemed to be much too pessimistic, and it was the main motivation for the investigations presented here to get information on the optimal values.

The underlying initial-boundary value problem is the following:

$$\begin{aligned} u_{tt} - \alpha u_{xx} + \beta\theta_x &= 0 \quad \text{in } [0,\infty)\times(0,L), && (3.1)\\ \theta_t + \gamma q_x + \delta u_{tx} &= 0 \quad \text{in } [0,\infty)\times(0,L), && (3.2)\\ \tau_0 q_t + q + \kappa\theta_x &= 0 \quad \text{in } [0,\infty)\times(0,L), && (3.3)\end{aligned}$$

with the initial conditions

$$u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1, \quad \theta(0,\cdot) = \theta_0, \quad q(0,\cdot) = q_0, \tag{3.4}$$

and the boundary conditions

$$u(x=0) = u(x=L) = q(x=0) = q(x=L) = 0. \tag{3.5}$$

Here $u, \theta, L, \alpha, \beta, \gamma, \delta, \kappa$ are the same as in the previous section. The new variable $q = q(t,x)$ now models the heat flux as additional, from θ not immediately determined function, and the positive constant τ_0 is the relaxation time, a parameter being small in comparison to the others and responsible, through the term $\tau_0 q_t$ in (3.3), that we have a (damped) purely hyperbolic system. The equation (3.3) represents Cattaneo's law of heat conduction modeling thermal disturbances as wave-like pulses traveling at finite speed. This way, the well-known physical paradox of infinite propagation speed in the classical thermoelastic theory, or the classical pure theory of heat conduction, where instead of Cattaneo's law (3.3) Fourier's law of heat conduction is used, is removed. Fourier's law corresponds to $\tau_0 = 0$ in (3.3), i.e.

$$q + \kappa\theta_x = 0, \tag{3.6}$$

which leads together with (3.1) and (3.2) to the system of classical thermoelasticity, (2.1) and (2.2), studied in section 2.

The well-posedness for (3.1)–(3.5) is well-known as well as the fact that the associated energy tends to zero exponentially for initial data satisfying again the natural condition

$$m_{\theta_0} := \int_0^L \theta_0(x)dx = 0 \tag{3.7}$$

(otherwise $\theta - m_\theta$ has to replace θ), cp. [13].

An ansatz analogous to that in (2.6) leads to the characteristic polynomial satisfying

$$\tau_0\rho^4 - \rho^3 + (\tau_0\alpha + \tau_0\beta\delta + \gamma\kappa)\lambda_j\rho^2 - (\alpha + \beta\delta)\lambda_j\rho + \alpha\gamma\kappa\lambda_j^2 = 0. \tag{3.8}$$

where

$$\lambda_j = \frac{j^2\pi^2}{L^2},$$

again.

Remark: If we set $\tau_0 = 0$ then the polynomial turns into that for classical thermoelasticity obtained in (2.8). Consequently, putting $\tau_0 = 0$ in the calculation of the roots below, one recovers the results from the previous section.

Observe that, for sufficiently smooth solutions, we conclude from the differential equations (3.1)–(3.3) that u, θ, and q satisfy the same differential equation,

$$\tau_0 w_{tttt} + w_{ttt} - (\tau_0\alpha + \tau_0\beta\delta + \gamma\kappa)w_{ttxx} - (\alpha + \beta\delta)w_{txx} + \alpha\gamma\kappa w_{xxxx} = 0. \tag{3.9}$$

which turns into equation (2.9) if $\tau_0 = 0$.

As optimal decay rate we now define similar to section 2

$$d_{tess} := \inf_{j\in\mathbb{N},k=1,2,3,4} \Re\rho_k(j). \tag{3.10}$$

With the parameter values

relaxation time				
	silicon	aluminum alloy	steel	germanium
τ_0	$1.0 \cdot 10^{-12}$	$3.0 \cdot 10^{-13}$	$1.0 \cdot 10^{-14}$	$1.0 \cdot 10^{-12}$

relaxation time				
	gallium arsenid	indium arsenid	copper	diamond
τ_0	$1.0 \cdot 10^{-12}$	$1.0 \cdot 10^{-12}$	$1.0 \cdot 10^{-12}$	$1.0 \cdot 10^{-12}$

we have obtained the following numerical results:

$\rho_k(j)$			
	silicon	aluminum alloy	steel
$\rho_{1,2}(j)$			
$j=1$	$7.462 \cdot 10^{-1} \pm 4.932 \cdot 10^{7} i$	$1.819 \cdot 10^{1} \pm 3.129 \cdot 10^{7} i$	$2.721 \cdot 10^{-1} \pm 2.541 \cdot 10^{7} i$
$j=5$	$1.866 \cdot 10^{1} \pm 2.466 \cdot 10^{8} i$	$4.548 \cdot 10^{2} \pm 1.565 \cdot 10^{8} i$	$6.802 \cdot 10^{0} \pm 1.270 \cdot 10^{8} i$
$j=10$	$7.462 \cdot 10^{1} \pm 4.932 \cdot 10^{8} i$	$1.819 \cdot 10^{3} \pm 3.129 \cdot 10^{8} i$	$2.721 \cdot 10^{1} \pm 2.541 \cdot 10^{8} i$
$j=50$	$1.866 \cdot 10^{3} \pm 2.466 \cdot 10^{9} i$	$4.548 \cdot 10^{4} \pm 1.565 \cdot 10^{9} i$	$6.802 \cdot 10^{2} \pm 1.271 \cdot 10^{9} i$
$j=10^4$	$7.452 \cdot 10^{7} \pm 4.932 \cdot 10^{11} i$	$1.594 \cdot 10^{9} \pm 3.123 \cdot 10^{11} i$	$2.715 \cdot 10^{7} \pm 2.541 \cdot 10^{11} i$
$j=10^{50}$	$3.800 \cdot 10^{10} \pm 4.950 \cdot 10^{57} i$	$1.191 \cdot 10^{10} \pm 3.082 \cdot 10^{57} i$	$1.383 \cdot 10^{10} \pm 2.535 \cdot 10^{57} i$
$j=10^{100}$	$3.800 \cdot 10^{10} \pm 4.950 \cdot 10^{107} i$	$1.191 \cdot 10^{10} \pm 3.082 \cdot 10^{107} i$	$1.383 \cdot 10^{10} \pm 2.535 \cdot 10^{107} i$
$\rho_3(j)$			
$j=1$	$2.238 \cdot 10^{3}$	$1.467 \cdot 10^{3}$	$1.186 \cdot 10^{2}$
$j=5$	$5.596 \cdot 10^{4}$	$3.669 \cdot 10^{4}$	$2.965 \cdot 10^{3}$
$j=10$	$2.238 \cdot 10^{5}$	$1.467 \cdot 10^{5}$	$1.186 \cdot 10^{4}$
$j=50$	$5.596 \cdot 10^{6}$	$3.669 \cdot 10^{6}$	$2.965 \cdot 10^{5}$
$j=10^4$	$3.384 \cdot 10^{11}$	$1.546 \cdot 10^{11}$	$1.186 \cdot 10^{10}$
$j=10^{50}$	$4.620 \cdot 10^{11} - 4.714 \cdot 10^{57} i$	$1.655 \cdot 10^{12} - 7.101 \cdot 10^{57} i$	$4.999 \cdot 10^{13} - 1.092 \cdot 10^{58} i$
$j=10^{100}$	$4.620 \cdot 10^{11} - 4.714 \cdot 10^{107} i$	$1.655 \cdot 10^{12} - 7.101 \cdot 10^{107} i$	$4.999 \cdot 10^{13} - 1.092 \cdot 10^{108} i$
$\rho_4(j)$			
$j=1$	$1.000 \cdot 10^{12}$	$3.333 \cdot 10^{12}$	$1.000 \cdot 10^{14}$
$j=5$	$1.000 \cdot 10^{12}$	$3.333 \cdot 10^{12}$	$1.000 \cdot 10^{14}$
$j=10$	$1.000 \cdot 10^{12}$	$3.333 \cdot 10^{12}$	$1.000 \cdot 10^{14}$
$j=50$	$1.000 \cdot 10^{12}$	$3.333 \cdot 10^{12}$	$1.000 \cdot 10^{14}$
$j=10^4$	$6.614 \cdot 10^{11}$	$3.176 \cdot 10^{12}$	$9.999 \cdot 10^{13}$
$j=10^{50}$	$4.620 \cdot 10^{11} + 4.714 \cdot 10^{57} i$	$1.655 \cdot 10^{12} + 7.101 \cdot 10^{57} i$	$4.999 \cdot 10^{13} + 1.092 \cdot 10^{58} i$
$j=10^{100}$	$4.620 \cdot 10^{11} + 4.714 \cdot 10^{107} i$	$1.655 \cdot 10^{12} + 7.101 \cdot 10^{107} i$	$4.999 \cdot 10^{13} + 1.092 \cdot 10^{108} i$
decay rate by definition (3.10):			
d_{tess}	$7.462 \cdot 10^{-1}$	$1.819 \cdot 10^{1}$	$2.721 \cdot 10^{-1}$

$\rho_k(j)$			
	germanium	gallium arsenid	indium arsenid
$\rho_{1,2}(j)$			
$j = 1$	$5.786 \cdot 10^{-2} \pm 2.447 \cdot 10^{7} i$	$4.884 \cdot 10^{-2} \pm 2.379 \cdot 10^{7} i$	$1.178 \cdot 10^{-2} \pm 1.914 \cdot 10^{7} i$
$j = 5$	$1.447 \cdot 10^{0} \pm 1.224 \cdot 10^{8} i$	$1.221 \cdot 10^{0} \pm 1.190 \cdot 10^{8} i$	$2.944 \cdot 10^{-1} \pm 9.570 \cdot 10^{7} i$
$j = 10$	$5.786 \cdot 10^{0} \pm 2.447 \cdot 10^{8} i$	$4.884 \cdot 10^{0} \pm 2.379 \cdot 10^{8} i$	$1.178 \cdot 10^{0} \pm 1.914 \cdot 10^{8} i$
$j = 50$	$1.447 \cdot 10^{2} \pm 1.224 \cdot 10^{9} i$	$1.221 \cdot 10^{2} \pm 1.190 \cdot 10^{9} i$	$2.944 \cdot 10^{1} \pm 9.570 \cdot 10^{8} i$
$j = 10^4$	$5.707 \cdot 10^{6} \pm 2.447 \cdot 10^{11} i$	$4.840 \cdot 10^{6} \pm 2.379 \cdot 10^{11} i$	$1.173 \cdot 10^{6} \pm 1.914 \cdot 10^{11} i$
$j = 10^{50}$	$4.134 \cdot 10^{8} \pm 2.447 \cdot 10^{57} i$	$5.392 \cdot 10^{8} \pm 2.379 \cdot 10^{57} i$	$3.320 \cdot 10^{8} \pm 1.914 \cdot 10^{57} i$
$j = 10^{100}$	$4.134 \cdot 10^{8} \pm 2.447 \cdot 10^{107} i$	$5.392 \cdot 10^{8} \pm 2.379 \cdot 10^{107} i$	$3.320 \cdot 10^{8} \pm 1.914 \cdot 10^{107} i$
$\rho_3(j)$			
$j = 1$	$8.879 \cdot 10^{2}$	$7.920 \cdot 10^{2}$	$4.802 \cdot 10^{2}$
$j = 5$	$2.220 \cdot 10^{4}$	$1.980 \cdot 10^{4}$	$1.201 \cdot 10^{4}$
$j = 10$	$8.879 \cdot 10^{4}$	$7.920 \cdot 10^{4}$	$4.802 \cdot 10^{4}$
$j = 50$	$2.220 \cdot 10^{6}$	$1.980 \cdot 10^{6}$	$1.201 \cdot 10^{6}$
$j = 10^4$	$9.850 \cdot 10^{10}$	$8.672 \cdot 10^{10}$	$5.058 \cdot 10^{10}$
$j = 10^{50}$	$4.996 \cdot 10^{11} - 2.980 \cdot 10^{57} i$	$4.995 \cdot 10^{11} - 2.815 \cdot 10^{57} i$	$4.997 \cdot 10^{12} - 2.192 \cdot 10^{57} i$
$j = 10^{100}$	$4.996 \cdot 10^{11} - 2.980 \cdot 10^{107} i$	$4.995 \cdot 10^{11} - 2.815 \cdot 10^{107} i$	$4.997 \cdot 10^{11} - 2.192 \cdot 10^{107} i$
$\rho_4(j)$			
$j = 1$	$1.000 \cdot 10^{12}$	$1.000 \cdot 10^{12}$	$1.000 \cdot 10^{12}$
$j = 5$	$1.000 \cdot 10^{12}$	$1.000 \cdot 10^{12}$	$1.000 \cdot 10^{12}$
$j = 10$	$1.000 \cdot 10^{12}$	$1.000 \cdot 10^{12}$	$1.000 \cdot 10^{12}$
$j = 50$	$1.000 \cdot 10^{12}$	$1.000 \cdot 10^{12}$	$1.000 \cdot 10^{12}$
$j = 10^4$	$9.015 \cdot 10^{11}$	$9.133 \cdot 10^{11}$	$9.484 \cdot 10^{11}$
$j = 10^{50}$	$4.996 \cdot 10^{11} + 2.980 \cdot 10^{57} i$	$4.995 \cdot 10^{11} + 2.815 \cdot 10^{57} i$	$4.997 \cdot 10^{11} + 2.192 \cdot 10^{57} i$
$j = 10^{100}$	$4.996 \cdot 10^{11} + 2.980 \cdot 10^{107} i$	$4.995 \cdot 10^{11} + 2.815 \cdot 10^{107} i$	$4.997 \cdot 10^{11} + 2.192 \cdot 10^{107} i$
decay rate by definition (3.10):			
d_{tess}	$5.786 \cdot 10^{-2}$	$4.884 \cdot 10^{-2}$	$1.178 \cdot 10^{-2}$

$\rho_k(j)$		
	copper	diamond
$\rho_{1,2}(j)$		
$j=1$	$2.117 \cdot 10^1 \pm 2.258 \cdot 10^7 i$	$3.599 \cdot 10^{-1} \pm 8.779 \cdot 10^7 i$
$j=5$	$5.292 \cdot 10^2 \pm 1.129 \cdot 10^8 i$	$8.997 \cdot 10^1 \pm 4.389 \cdot 10^8 i$
$j=10$	$2.117 \cdot 10^3 \pm 2.258 \cdot 10^8 i$	$3.599 \cdot 10^1 \pm 8.779 \cdot 10^8 i$
$j=50$	$5.292 \cdot 10^4 \pm 1.129 \cdot 10^9 i$	$8.998 \cdot 10^2 \pm 4.389 \cdot 10^9 i$
$j=10^4$	$9.331 \cdot 10^8 \pm 2.247 \cdot 10^{11} i$	$1.527 \cdot 10^7 \pm 8.778 \cdot 10^{11} i$
$j=10^{50}$	$1.635 \cdot 10^9 \pm 2.239 \cdot 10^{57} i$	$2.651 \cdot 10^7 \pm 8.778 \cdot 10^{57} i$
$j=10^{100}$	$1.635 \cdot 10^9 \pm 2.239 \cdot 10^{107} i$	$2.651 \cdot 10^7 \pm 8.778 \cdot 10^{107} i$
$\rho_3(j)$		
$j=1$	$3.024 \cdot 10^3$	$1.793 \cdot 10^4$
$j=5$	$7.560 \cdot 10^4$	$4.483 \cdot 10^5$
$j=10$	$3.024 \cdot 10^5$	$1.793 \cdot 10^6$
$j=50$	$7.560 \cdot 10^6$	$4.483 \cdot 10^7$
$j=10^4$	$4.991 \cdot 10^{11} - 2.370 \cdot 10^{11} i$	$5.000 \cdot 10^{11} - 1.242 \cdot 10^{12} i$
$j=10^{50}$	$4.984 \cdot 10^{11} - 5.545 \cdot 10^{57} i$	$5.000 \cdot 10^{11} - 1.339 \cdot 10^{58} i$
$j=10^{100}$	$4.984 \cdot 10^{11} - 5.545 \cdot 10^{107} i$	$5.000 \cdot 10^{11} - 1.339 \cdot 10^{108} i$
$\rho_4(j)$		
$j=1$	$1.000 \cdot 10^{12}$	$1.000 \cdot 10^{12}$
$j=5$	$1.000 \cdot 10^{12}$	$1.000 \cdot 10^{12}$
$j=10$	$1.000 \cdot 10^{12}$	$1.000 \cdot 10^{12}$
$j=50$	$1.000 \cdot 10^{12}$	$1.000 \cdot 10^{12}$
$j=10^4$	$4.991 \cdot 10^{11} + 2.370 \cdot 10^{11} i$	$5.000 \cdot 10^{11} + 1.242 \cdot 10^{12} i$
$j=10^{50}$	$4.984 \cdot 10^{11} + 5.545 \cdot 10^{57} i$	$5.000 \cdot 10^{11} + 1.339 \cdot 10^{58} i$
$j=10^{100}$	$4.984 \cdot 10^{11} + 5.545 \cdot 10^{107} i$	$5.000 \cdot 10^{11} + 1.339 \cdot 10^{108} i$
decay rate by definition (3.10):		
d_{tess}	$2.117 \cdot 10^1$	$3.599 \cdot 10^{-1}$

The first and most important observation is that the optimal decay rates are exactly the same as in classical thermoelasticity.

Also the behavior of the eigenvalues ρ_1, ρ_2 is qualitatively, and even quantitatively until $j = 10^4$, quite the same. The same holds for $\rho_3(j)$ until $j = 10^4$, then a new effect occurs, namely the appearance of nonzero values for the imaginary part of ρ_3.

We can regard the fourth eigenvalue ρ_4 as the real new one. But this one does affect the large-time behavior essentially. In conclusion, the large-time behavior of the two systems classical thermoelasticity and thermoelasticity with second sound, respectively, is essentially the same.

The hyperbolic character of the system of thermoelasticity with second sound becomes apparent in the fact that the real parts of all four eigenvalues remain bounded as j tends to infinity (in contrast to classical thermoelasticity where this does not hold for the third eigenvalue).

The estimate computed in [13] for for the decay rate in thermoelasticity with second sound for silicon was given using rather coarse estimates in the energy method used there, the computed value $2\tilde{d}_{ct}$ was $2\tilde{d}_{ct} = 1.07 \cdot 10^{-50}$ which is much too pessimistic as the now obtained optimal value $d_{tess}(\text{silicon}) = 7.462 \cdot 10^{-1}$ shows.

In contrast to the case of classical thermoelasticity, the numerical /graphical output is not yet fully validated analytically, but in the next section we confirm the decay rate using the Hurwitz criterion and subtle estimates observing the seize of the parameters.

Looking once more at the two systems considered in section 2 and 3, respectively, also keeping in mind the results for pure heat conduction from section 2, we can summarize further comparisons of the numerical results obtained for the two systems:

The two systems behave quite similar for large times. Also the comparison between thermoelasticity with second sound and pure heat conduction is now similar to the correponding comparison between classical thermoelasticity and pure heat conduction given in section 2.

Plots of the eigenvalues in thermoelasticity with second sound for large j show an oscillating behavior (like $x \mapsto x\sin(x)$ for large x). But this might be due to the numerical difficulties MAPLE may have with the parameter τ_0. In any case it would be interesting to have more knowledge on the eigenvalues like knowing $\lim_{j\to\infty} \Re\rho_k(j)$ exactly, investigating a possible oscillatory behavior , investigating the systems for varying parameters like $L \to 0$ or $L \to \infty$, or $\tau_0 \to 0$, and so on.

The analysis should also be carried out for other thermoelastic systems like those of thermoelasticity of type III cp. [12] and the further thermoelastic models mentioned there. This in particular, since the experimental data and the relevance of these systems in practice is yet under discussion.

4 The Hurwitz criterion

Finally we present a different approach to obtain exponential stability, both on the qualitative and the quantitative level. We shall apply the Hurwitz criterion to describe the real parts of the characteristic roots. On the qualitative level this has been done for other thermoelastic systems before, see e.g. [12], while it turns out to yield the optimal decay rates also numerically, a new computation and a useful observation for other systems, as we believe. This observation has been first made in [5].

We start with thermoelasticity with second sound, i.e. with the system (3.1)–(3.5) together with the normalization (3.7).

Defining

$$\xi := -\rho, \qquad A := \frac{1}{\tau_0}, \qquad B := \frac{\pi^2}{L^2}\frac{\tau_0\alpha + \tau_0\beta\delta + \gamma\kappa}{\tau_0},$$

$$C := \frac{\pi^2}{L^2}\frac{\alpha + \beta\delta}{\tau_0}, \qquad D := \frac{\pi^4}{L^4}\frac{\alpha\gamma\kappa}{\tau_0}.$$

We can rewrite the characteristic polynomial given in (3.8) as polynomial $\chi_j(\xi)$, where

$$\chi_j(\xi) := \xi^4 + A\xi^3 + Bj^2\xi^2 + Cj^2\xi + Dj^4$$

and we are looking for a negative supremum for the real parts of the zeros φ_j satisfying

$$\chi_j(\varphi_j^k) = 0, \quad k = 1, 2, 3, 4,$$

i.e. we wish to prove

$$\exists\, d > 0 \quad \forall j \in \mathbb{N} \quad \forall k = 1,2,3,4: \quad \Re\, \varphi_j^k \leq -d. \tag{4.1}$$

Moreover, we wish to determine d numerically as sharp as possible.
To do so we use the Hurwitz criterion (cp. e.g. [1]) saying in our situation that the polynomial p,

$$p(\xi) = \xi^4 + A_1\xi^3 + B_1\xi^2 + C_1\xi + D_1,$$

with real coefficients $A_1, ..., D_1$ has zeros with negative real parts if and only if

$$A_1 > 0, \;\; B_1 > 0, \;\; C_1 > 0, \;\; D_1 > 0 \;\text{ and }\; A_1B_1C_1 > C_1^2 + A_1^2 D_1. \tag{4.2}$$

To apply this criterion we define

$$p_j(\xi) := \chi_j(\xi - d)$$

where $d > 0$ will have to be determined such that (4.2) holds uniformly in j.

Since

$$\begin{aligned} p_j(\xi) \;&= \xi^4 + (-4d + A)\xi^3 + (6d^2 - 3Ad + Bj^2)\xi^2 \\ &\quad +(-4d^3 + 3Ad^2 - 2Bj^2d + Cj^2)\xi \\ &\quad +(d^4 - Ad^3 + Bj^2d^2 - Cj^2d + Dj^4) \end{aligned}$$

according to the Hurwitz criterion, all roots possess negative real parts if and only if the following conditions $(i) - (v)$ are satisfied:

(i) $-4d + A > 0,$

(ii) $6d^2 - 3Ad + Bj^2 > 0,$

(iii) $-4d^3 + 3Ad^2 - 2Bj^2d + Cj^2 > 0,$

(iv) $d^4 - Ad^3 + Bj^2d^2 - Cj^2d + Dj^4 > 0,$

(v) $(-4d + A)(6d^2 - 3Ad + Bj^2)(-4d^3 + 3Ad^2 - 2Bj^2d + Cj^2)$
$> (-4d^3 + 3Ad^2 - 2Bj^2d + Cj^2)^2 + (-4d + A)^2\,(d^4 - Ad^3 + Bj^2d^2 - Cj^2d + Dj^4).$

Condition (i) is equivalent to

$$d < A/4. \tag{4.3}$$

Sufficient for (ii) is to have

$$6d^2 - 3Ad + B > 0$$

or, equivalently,

$$d < \frac{3A - \sqrt{9A^2 - 24B}}{12} =: h_{AB}.$$

Computing the values h_{AB} for the different materials, we obtain (approximately, but for later purposes sufficiently sharp)

material	h_{AB}
silicon	1558
aluminum alloy	599
steel	42
germanium	496
gallium arsenid	453
indium arsenid	282
copper	1150
diamond	8550

Hence

$$d < 40 \tag{4.4}$$

is sufficient to satisfy the conditions (i) and (ii). This is also sufficient for (iii) and (iv) as we shall prove now: Sufficient for (iii) is to have

$$-2.56 \cdot 10^5 j^2 - 2Bj^2 d + Cj^2 > 0$$

or, equivalently,

$$d < \frac{C - 2.56 \cdot 10^5}{2B}$$

which is satisfied with (4.3) since we have

material	$A/10^{12}$	$B/10^{15}$	$C/10^{27}$	$D/10^{30}$
silicon	1	4.67	2.43	5.45
aluminum alloy	3.3	5.99	3.26	4.79
steel	100	12.56	64.57	7.66
germanium	1	1.49	0.60	0.53
gallium arsenid	1	1.36	0.57	0.45
indium arsenid	1	0.85	0.37	0.18
copper	1	3.58	0.51	1.54
diamond	1	25.7	7.71	138

To satisfy (iv) we require

$$-6.4 \cdot 10^4 A j^4 - C j^4 d + D j^4 > 0$$

or, equivalently,

$$d < \frac{D - 6.4 \cdot 10^4 A}{C}$$

which is again already satisfied if (4.4) holds.
It remains to satisfy condition (v) which is equivalent to

$$\begin{aligned} 0 \ < \ & 64d^6 - 96Ad^5 + (32Bj^2 + 48A^2)d^4 \\ & -(32ABj^2 + 8A^3)d^3 + (8A^2Bj^2 + 4B^2j^4 + 4ACj^2 - 16Dj^4)d^2 \\ & -(2AB^2j^2 + 2A^2Cj^2 - 8ADj^4)d + (ABCj^4 - C^2j^4 - A^2Dj^4). \end{aligned} \tag{4.5}$$

Observing that (4.3) implies

$$-16\ Dj^4d^2 + 8\ ADj^4d > 0$$

we get a sufficient condition for (4.5) by requiring

$$(2AB^2j^2 + 2A^2Cj^2)\ d < (ABC - C^2 - A^2D - 9.84 \cdot 10^9 A - 2.05 \cdot 10^6 AB - 5.12 \cdot 10^5 A^3)j^4.$$

Remark: The missing terms, e.g. $64d^6$, could be dropped without loosing optimality because of the sizes of the other terms.
The final sufficient condition now is to take

$$d < \tilde{d} := \frac{ABC - C^2 - A^2D - 5.12 \cdot 10^5 A^3 - 9.84 \cdot 10^9 A - 2.05 \cdot 10^6 AB}{2\,(AB^2 + A^2C)}$$

or in terms of the coefficients $\alpha, \beta, \ldots$:

$$\begin{aligned} \tilde{d} \ = \ & \frac{1}{2}\,\frac{\pi^2}{L^2}\,\frac{\beta\gamma\delta\kappa \ - \ 5.12 \cdot 10^5\,\frac{L^4}{\pi^4}}{\alpha + \beta\delta + \frac{\pi^2}{L^2}\,(\tau_0\,(\alpha + \beta\delta)\ + \gamma\kappa)^2} \\ & -\tau_0\,\frac{\left(10.25 \cdot 10^5\,(\alpha + \beta\delta) + 4.92 \cdot 10^9\,\frac{L^2}{\pi^2}\right)\tau_0 + 10.25 \cdot 10^5 \gamma\kappa}{\alpha + \beta\delta + \frac{\pi^2}{L^2}\,(\tau_0\,(\alpha + \beta\delta) + \gamma\kappa)^2} \equiv \tilde{d}(\tau_0) \end{aligned} \tag{4.6}$$

We observe that $\tilde{d}$ is continuously depending on τ_0, and

$$\lim_{\tau_0 \to 0} \tilde{d}(\tau_0) = \frac{1}{2} \frac{\pi^2}{L^2} \frac{\beta\gamma\delta\kappa - 5.12 \cdot 10^5 \frac{L^4}{\pi^4}}{\alpha + \beta\delta + \frac{\pi^2}{L^2} (\gamma\kappa)^2} =: \tilde{d}_0 \tag{4.7}$$

and we shall compare $\tilde{d}_0$ to the decay rate $\tilde{d}^0$ obtained with the Hurwitz criterion for the system where $\tau_0 = 0$ (classical thermoelasticity) later on.
Computing $\tilde{d}$ and $\tilde{d}_0$ we get

material	$\tilde{d} = \tilde{d}_0$
silicon	0.746
aluminum alloy	1.81
steel	2.72
germanium	0.058
gallium arsenid	0.049
indium arsenid	0.011
copper	21.1
diamond	0.359

These are in this accuracy exactly the values we obtained in the previous section for the exact decay rates for thermoelasticity with second sound ($\tilde{d} \approx d_{tess}$) and classical thermoelasticity ($\tilde{d}_0 \approx d_{ct}$), respectively. Hence we have proved

Theorem 4.1 *The decay rate, obtained with the Hurwitz criterion, for thermoelasticity with second sound (3.1)–(3.5) is given by*

$$\tilde{d} = \tilde{d}(\tau_0) = \frac{1}{2} \frac{\pi^2}{L^2} \frac{\beta\gamma\delta\kappa - 5.12 \cdot 10^5 \frac{L^4}{\pi^4}}{\alpha + \beta\delta + \frac{\pi^2}{L^2} (\tau_0 (\alpha + \beta\delta) + \gamma\kappa)^2}$$

$$-\tau_0 \frac{\left(10.25 \cdot 10^5 (\alpha + \beta\delta) + 4.92 \cdot 10^9 \frac{L^2}{\pi^2}\right) \tau_0 + 10.25 \cdot 10^5 \gamma\kappa}{\alpha + \beta\delta + \frac{\pi^2}{L^2} (\tau_0 (\alpha + \beta\delta) + \gamma\kappa)^2} \approx d_{tess}.$$

It equals

$$\tilde{d}_0 = \lim_{\tau_0 \to 0} \tilde{d}(\tau_0) = \frac{1}{2} \frac{\pi^2}{L^2} \frac{\beta\gamma\delta\kappa - 5.12 \cdot 10^5 \frac{L^4}{\pi^4}}{\alpha + \beta\delta + \frac{\pi^2}{L^2} (\gamma\kappa)^2}.$$

Remark: The formula for $\tilde{d}(\tau_0)$ and for $\tilde{d}_0$ also clearly show the dependence on the other parameters $\alpha, \beta, ...$, but it must be observed that the proof above used the size ratios of certain coefficients. On the other hand, the proof remains valid as long as these ratios are reasonable, the absolute values of the parameters are then of less importance.
Finally we shortly demonstrate that the Hurwitz criterion also applies and gives the exact decay rate $\tilde{d}^0 = d_{ct}$ for classical thermoelasticity ($\tau_0 = 0$), hence also $\tilde{d}_0 = \tilde{d}^0$.

In analogy to thermoelasticity with second sound we get for the system of classical thermoelasticity (2.1)–(2.5) the characteristic polynomial

$$\chi_j(\xi) := \xi^3 + B_0 j^2 \xi^2 + C_0 j^2 \xi + D_0 j^4$$

where

$$B_0 := \frac{\pi^2}{L^2}\ \gamma\kappa, \quad C_0 := \frac{\pi^2}{L^2}\ (\alpha + \beta\delta), \quad D_0 := \frac{\pi^4}{L^4}\ \alpha\gamma\kappa,$$

and

$$\begin{aligned} p_j(\xi) \quad := \quad & \chi_j(\xi - d) = \ \xi^3 + (-3d + B_0 j^2)\ \xi^2 + (3d^2 - 2B_0 j^2 d + C_0 j^2)\ \xi \\ & +(-d^3 + B_0 j^2 d^2 - C_0 j^2 d + D_0 j^4). \end{aligned} \tag{4.8}$$

and the Hurwitz criterion for the polynomial

$$p(\xi) = \xi^3 + B_1 \xi^2 + C_1 \xi + D_1$$

says that the roots have negative real parts if and only if

$$B_1 > 0, \;\; C_1 > 0, \;\; D_1 > 0 \;\text{ and }\; B_1 C_1 > D_1. \tag{4.9}$$

To satisfy the first three conditions in (4.9) for our polynomial p_j given in (4.8) it is again sufficient to have

$$d < 40.$$

Therefore, it is sufficient for the last condition to require

$$-(2B_0^2 + 2C_0)\ j^4 d - 5.12 \cdot 10^5 j^4 + (B_0 C_0 - D_0) j^4 > 0$$

or

$$d < \tilde{d}^0 := \frac{B_0 C_0 - D_0 - 5.12 \cdot 10^5}{2(B_0^2 + C_0)}$$

or

$$\tilde{d}^0 = \frac{1}{2}\ \frac{\pi^2}{L^2}\ \frac{\beta\gamma\delta\kappa - 5.12 \cdot 10^5 \frac{L^4}{\pi^4}}{\alpha + \beta\delta + \frac{\pi^2}{L^2}(\gamma\kappa)^2}.$$

Hence we have proved

$$\tilde{d}_0 = \tilde{d}^0$$

summarized in

Theorem 4.2 *The decay rate obtained with the Hurwitz criterion, for classical thermoelasticity (2.1)-(2.4) is given by*

$$\tilde{d}^0 = \frac{1}{2}\ \frac{\pi^2}{L^2}\ \frac{\beta\gamma\delta\kappa - 5.12 \cdot 10^5 \frac{L^4}{\pi^4}}{\alpha + \beta\delta + \frac{\pi^2}{L^2}(\gamma\kappa)^2} = \tilde{d}_0 \approx d_{ct}.$$

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