Exponential Stability for Wave Equations with Non-Dissipative Damping[1]

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Abstract: We consider the nonlinear wave equation $u_{tt} - \sigma(u_x)_x + a(x)u_t = 0$ in a bounded interval $(0, L) \subset \mathbb{R}^1$. The function a is allowed to change sign, but has to satisfy $\overline{a} = \frac{1}{L}\int_0^L a(x)dx > 0$. For this non-dissipative situation we prove the exponential stability of the corresponding linearized system for: (I) possibly large $\|a\|_{L^\infty}$ with small $\|a(\cdot)-\overline{a}\|_{L^2}$, and (II) a class of pairs (a, L) with possibly negative moment $\int_0^L a(x)\sin^2(\pi x/L)\,dx$. Estimates for the decay rate are also given in terms of $\overline{a}$. Moreover, we show the global existence of smooth, small solutions to the corresponding nonlinear system if, additionally, the negative part of a is small enough.

1 Introduction

We consider the following nonlinear wave equation

$$u_{tt} - \sigma(u_x)_x + a(x)u_t = 0 \tag{1.1}$$

for a function $u = u(t, x)$, $\quad t \geq 0$, $\quad x \in (0, L) \subset \mathbb{R}^1$, $\quad L > 0$ fixed, with initial conditions

$$u(t = 0) = u_0, \quad u_t(t = 0) = u_1 \tag{1.2}$$

and Dirichlet type boundary conditions

$$u(\cdot, 0) = u(\cdot, L) = 0. \tag{1.3}$$

We assume that $a \in L^\infty((0, L))$ for the part on the exponential stability of the associated semigroup, and $a \in C^3([0, L])$ for the discussion of the nonlinear system, as well as

$$\overline{a} := \frac{1}{L}\int_0^L a(x)dx > 0, \tag{1.4}$$

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in particular a may change sign in $[0, L]$ or be zero in open subsets. The nonlinear function σ is assumed to satisfy

$$\sigma \in C^3(\mathbb{R}), \qquad d_0 := \sigma'(0) > 0, \text{ and } \sigma''(0) = 0. \tag{1.5}$$

Remark: This is, for instance, satisfied for σ corresponding to a vibrating string,

$$\sigma(y) = \frac{y}{\sqrt{1+y^2}}.$$

Rewriting (1.1) as

$$u_{tt} - d_0 u_{xx} + a u_t = b(u_x) u_{xx} \tag{1.6}$$

with

$$b(u_x) := \sigma'(u_x) - d_0 = \sigma'(u_x) - \sigma'(0) \tag{1.7}$$

the associated linearized system is

$$u_{tt} - d_0 u_{xx} + a u_t = 0 \tag{1.8}$$

together with the initial conditions (1.2) and the boundary conditions (1.3). Since a may change sign we have a non-dissipative system still regarding au_t to be a non-local but indefinite damping. There are many papers on solutions to (1.1) or on decay rates for (1.1) or (1.8) if $a \geq 0$ i.e. if a does not change sign, see for example the papers of Cox and Overton [3], Cox and Zuazua [5], Kawashima, Nakao and Ono [9], Nakao [13, 14], da Silva Ferreira [19] or Zuazua [21] and the references therein. If $a(x) \geq a_0 > 0$ is strictly positive, the exponential decay of solutions to (1.8) and also to (1.1), for small data, easily follows.
The non-dissipative case with indefinite a seems to have been posed first by Chen, Fulling, Narcovich and Sun [2] where it was conjectured that the energy

$$E_0(t) = \int_0^L (u_t^2 + u_x^2)(t, x) dx \tag{1.9}$$

decays exponentially if

$$\exists\, \gamma > 0\ \forall\, n = 1, 2, \ldots : \quad \int_0^L a(x) \sin^2(n\pi x/L) dx \geq \gamma \tag{1.10}$$

holds. Later Freitas [6] found that (1.10) is not sufficient to guarantee exponential stability when $\|a\|_{L^\infty}$ is large. Replacing a by εa, Freitas and Zuazua [7] proved that when a is of bounded variation and (1.10) holds, then there is $\varepsilon^* = \varepsilon^*(a)$ such that for all $\varepsilon \in (0, \varepsilon^*)$ the energy decays indeed exponentially. This result was extended to a differential equation of the type

$$u_{tt} - u_{xx} + \varepsilon a(x) u_t + b(x) u = 0$$

by Benaddi and Rao [1]. K. Liu, Z. Liu and Rao [10] gave an abstract treatment of these results under certain conditions on the abstract damping operator. An extension to higher

space dimensions was presented by Liu, Rao and Zhang [11].
Here we show that solutions to the linearized system (1.8), (1.2), (1.3) decay exponentially for

(I) possibly large $\|a\|_{L^\infty}$ with small $\|a(\cdot)-\overline{a}\|_{L^2}$, and

(II) a class of pairs (a,L) with possibly negative moment $\int_0^L a(x)\sin^2(\pi x/L)\,dx$.

Part (II) is not a contradiction to a result of Freitas in [6] saying that if (1.10) is not valid, then the solution is not exponentially decaying for sufficiently small $\|a\|_{L^\infty}$, because in our examples of admissible pairs (a,L), leading to exponential decay, a, resp. $\|a\|_{L^\infty}$, and L are not independent. Estimates for the decay rate are also given in terms of $\overline{a}$. Moreover, we show the global existence of smooth, small solutions to the corresponding nonlinear system if, additionally, the negative part of a is small enough. More precisely: If α_0 denotes the decay rate for the linear system,

$$\int_0^L (u_t^2+u_x^2)(t,x) \le c_1 e^{-2\alpha_0 t}, \quad c_1>0,$$

then a^- has to satisfy in particular

$$||a^-||_{L^\infty} < \alpha_0, \tag{1.11}$$

see Section 3.
The paper is organized as follows: In Section 2 we shall prove the exponential stability for the linearized system under either of the situations (I) or (II) above. This is the crucial part, and the method will be the spectral one characterizing exponentially stable semigroups in terms of the spectrum of the associated generator of the semigroup. It is possible to give an explicit lower bound on the decay rate which, in turn, is necessary to make (1.11) a reasonable condition in the nonlinear case.
In Section 3 the global existence of small solutions to the nonlinear system is investigated. Using the result from Section 2 and pertubation arguments the condition (1.11) is shown to be sufficient to guarantee the global existence and also the exponential stability of the nonlinear system.
Summarizing the contributions of our paper we present results on exponential stability for the wave equation when the function a may change sign under conditions that extend the existing results to cases with indfinite damping a with possibly large L^∞-norm, and give examples of pairs (a,L) for which exponential stability holds but the moment $\int_0^L a(x)\sin^2(\pi x/L)\,dx$ is negative. We also present an explicit description of the decay rate and of the type of the associated semigroup, and also a discussion of a corresponding nonlinear problem with global existence and stability. Finally, our approach can be applied to other one-dimensional models.
We use standard notations. e.g. for Sobolev spaces.

2 Exponential stability for the linearized system

We first consider the linearized system. Without loss of generality we assume $d_0=\sigma'(0)=1$.

$$u_{tt}-u_{xx}+a(x)u_t=0 \quad \text{in } (0,\infty)\times(0,L) \tag{2.1}$$

$$u(0,\cdot)=u_0, \quad u_t(0,\cdot)=u_1 \quad \text{in } (0,L) \tag{2.2}$$

$$u(\cdot,0) = u(\cdot,L) = 0 \quad \text{in } (0,\infty). \tag{2.3}$$

We assume that $a \in L^\infty((0,L))$ and satisfies (1.4). The aim is to prove that the energy given in (1.9) decays to zero exponentially as time t tends to infinity in either

- Case (I): $\|a(\cdot) - \overline{a}\|_{L^2}$ is sufficiently small — but $\|a\|_{L^\infty}$ may be large —, or
- Case (II): $\|a\|_{L^\infty}$ is small, (a,L) satisfy certain relations — but the moment $\int_0^L a(x)\sin^2(\pi x/L)\,dx$ may be negative.

We introduce the variables

$$p := u_t - u_x, \qquad q := u_t + u_x$$

such that

$$p_t + p_x = -a(x)u_t, \qquad q_t - q_x = -a(x)u_t. \tag{2.4}$$

Let

$$U := \begin{pmatrix} p \\ q \end{pmatrix}, \ K := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ B := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

and let $\mathcal{A}$ denote the operator given by

$$\mathcal{A}U := -KU_x - \frac{a}{2}BU$$

with domain

$$D(\mathcal{A}) := \left\{ \begin{pmatrix} p \\ q \end{pmatrix} \in H^1((0,L)) \times H^1((0,L)) \mid p(0)+q(0) = p(L)+q(L) = 0, \int\limits_0^L p(s) - q(s)ds = 0 \right\}$$

in the Hilbert space

$$\mathcal{H} := \left\{ \begin{pmatrix} p \\ q \end{pmatrix} \in L^2((0,L)) \times L^2((0,L)) \mid \quad \int\limits_0^L p(s) - q(s)ds = 0 \right\}$$

with the $L^2((0,L))$ inner product. $D(\mathcal{A})$ is dense in $\mathcal{H}$ and $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$. We can rewrite (2.1) as

$$U_t = \mathcal{A}U, \quad U(0) = U_0, \quad U \in D(\mathcal{A}) \tag{2.5}$$

To verify the equivalence of the systems (2.1)–(2.3) and (2.5), let U solve (2.5) for appropriate initial conditions. Let

$$w := \frac{p+q}{2}, \qquad v := \frac{q-p}{2}.$$

Then

$$w_x - v_t = 0$$

hence there exists a function u with

$$\begin{pmatrix} u_x \\ u_t \end{pmatrix} = \begin{pmatrix} v \\ w \end{pmatrix}.$$

We can choose u such that $u(0,0)=0$. It is not difficult to see that u satisfies (2.1). Moreover,

$$0=w(t,0)=u_t(t,0), \qquad 0=w(t,L)=u_t(t,L).$$

Using $u(0,0)=0$ we obtain

$$u(t,0)=0, \qquad t\geq 0.$$

Since $U\in D(\mathcal{A})$ we conclude

$$0=\int_0^L u_x(t,s)ds=u(t,L)-u(t,0) \quad\Rightarrow\quad u(t,L)=0, \qquad t\geq 0,$$

therefore the system (2.1)–(2.3) is equivalent to (2.5).

Lemma 2.1 *$\mathcal{A}^{-1}$ is compact.*

Proof: $\mathcal{A}U=-F$ is solvable for $F\in\mathcal{H}$, $\mathcal{A}^{-1}$ is bounded:
With

$$M_0(x):=\frac{a(x)}{2}KB=\frac{a(x)}{2}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}$$

$\mathcal{A}U=-F$ is equivalent to

$$U_x+M_0(x)U=KF$$

or

$$U(x)=e^{-\int\limits_0^x M_0(s)ds}U_0+\int_0^x e^{-\int\limits_s^x M_0(t)dt}KF(s)ds$$

where U_0 has to be determined appropriately. Since $M_0^2=0$, the corresponding series $e^{-\int_0^x M_0(s)ds}$ has only two terms and therefore we have for $U=(p,q)'$, $\quad U_0=(p_0,q_0)'$, $\quad F=(f,g)'$ that

$$\begin{aligned}\begin{pmatrix}p\\ q\end{pmatrix}(x) &= \left(Id-\frac{1}{2}\int_0^x a(s)ds\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\right)\begin{pmatrix}p_0\\ q_0\end{pmatrix}\\ &\quad+\int_0^x\left(Id-\frac{1}{2}\int_s^x a(t)dt\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\right)\begin{pmatrix}f(s)\\ -g(s)\end{pmatrix}ds.\end{aligned}$$

The boundary conditions require $p_0+q_0=0$, hence

$$\begin{pmatrix}p\\ q\end{pmatrix}(x)=\begin{pmatrix}p_0\\ -p_0\end{pmatrix}+\int_0^x\begin{pmatrix}f(s)\\ -g(s)\end{pmatrix}ds-\int_0^x\begin{pmatrix}f(s)-g(s)\\ -f(s)+g(s)\end{pmatrix}\frac{1}{2}\int_s^x a(t)dt\,ds.$$

Then $p(L)+q(L)=0$ is satisfied since $F\in\mathcal{H}$. Finally the condition $\int\limits_0^L p(s)-q(s)ds=0$ determines p_0 uniquely by

$$p_0=\frac{1}{2L}\left\{\int_0^L\int_0^x (f(s)-g(s))\int_s^x a(t)dt\,ds\,dx+\int_0^L\int_0^x f(s)+g(s)ds\,dx\right\}.$$

Hence $\mathcal{A}U = -F$ is uniquely solvable and

$$\|U\| \leq c\|F\|,$$

where $\|\cdot\|$ denotes the $L^2((0,L))$-norm. Thus $0 \in \varrho(\mathcal{A})$ (resolvent set).
Now we prove $\mathcal{A}^{-1}$ is compact:
Let $(F_n)_n \subset \mathcal{H}$ be bounded, let $U_n := \mathcal{A}^{-1}F_n$. Then $(U_n)_n$ is bounded according to (ii). This implies that $(p_n, q_n)_n$ is bounded in $H^1((0,L))$ and hence has a convergent subsequence in $L^2((0,L))$. Q.E.D.

Lemma 2.1 implies that the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ consists of eigenvalues $(\lambda_n)_n$ only, without any finite accumulation point.
First we consider Case (I), where finally $\|a(\cdot) - \overline{a}\|_{L^2}$ will be chosen sufficiently small, and we will show below, by using fixed point arguments, that for $\varepsilon > 0$

$$\Gamma_\varepsilon^I := \left\{ \varepsilon + \alpha + i\beta; \quad \alpha > \Re\left(-\frac{\overline{a}}{2} + \sqrt{(\frac{\overline{a}}{2})^2 - \frac{\pi^2}{L^2}} \right) \quad \text{and} \quad \beta \in \mathbb{R} \right\} \subset \varrho(\mathcal{A}),$$

and also that

$$\sup_{\lambda \in \Gamma_\varepsilon^I} \|(\lambda I - \mathcal{A})^{-1}\| < \infty.$$

This will imply, see e.g. the results by Prüss [17] or [12, Thm 1.3.1], that the corresponding semigroup decays exponentially.
We shall regard $\mathcal{A}$ as a perturbation of $\mathcal{A}_0^I$ defined by

$$\mathcal{A}_0^I U := -KU_x - \frac{\overline{a}}{2}BU$$

on

$$D(\mathcal{A}_0^I) := \left\{ \begin{pmatrix} p \\ q \end{pmatrix} \in H^1((0,L)) \times H^1((0,L)) \mid p(0) + q(0) = p(L) + q(L) = 0 \right\}$$

Lemma 2.2 *Let $\sigma(\mathcal{A}_0^I)$ denote the spectrum of $\mathcal{A}_0^I$. Then we have that*

$$\sigma(\mathcal{A}_0^I) = \left\{ -\frac{1}{2}\overline{a} \pm \sqrt{\left(\frac{1}{2}\overline{a}\right)^2 - \frac{k^2\pi^2}{L^2}} \;\mid\; k \in \mathbb{N} \right\}$$

Proof: As for $\mathcal{A}$, one can show that $\mathcal{A}_0^I$ has a compact inverse. We now consider the equation

$$\lambda U - \mathcal{A}_0^I U = F \tag{2.6}$$

where $F \in \mathcal{H}$, which is equivalent to finding a function U such that

$$U_x + MU = KF$$

where $M := K(\lambda I + \frac{\overline{a}}{2}B)$. The solution to this equation is given by

$$U(x) = e^{-Mx}U_0 + \int_0^x e^{-M(x-s)}KF(s)\, ds \tag{2.7}$$

This is the solution of an initial value problem. To find the set of λ belonging to the resolvent set, is equivalent to the problem of finding λ such that U satisfies the boundary conditions of the problem and can be estimated appropriately by F. Denoting

$$U = \begin{pmatrix} p \\ q \end{pmatrix}, \quad U_0 = \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

we have to find $U \in D(\mathcal{A}_0^I)$. To get this we have to satisfy first $p_0 + q_0 = 0$ which implies

$$p_0 = -q_0$$

Now the problem reduces to find p_0 for which we have

$$p(L) + q(L) = 0. \tag{2.8}$$

Let us denote by

$$E(x,s) := e^{-M(x-s)} =: \begin{pmatrix} e_{11}(x,s) & e_{12}(x,s) \\ e_{21}(x,s) & e_{22}(x,s) \end{pmatrix}$$

Note that

$$\begin{pmatrix} p \\ q \end{pmatrix} = E(x,0) \begin{pmatrix} p_0 \\ -p_0 \end{pmatrix} + \int_0^x E(x,s) \begin{pmatrix} f(s) \\ -g(s) \end{pmatrix} ds \tag{2.9}$$

Using the expression above to verify relation (2.8) we conclude that p_0 should satisfy

$$\begin{aligned} &\{e_{11}(L,0) - e_{12}(L,0) + e_{21}(L,0) - e_{22}(L,0)\}\, p_0 \\ &\qquad = \int_0^L \{e_{11}(L,s) + e_{21}(L,s)\}\, f(s) - \{e_{12}(L,s) + e_{22}(L,s)\}\, g(s)\, ds \end{aligned} \tag{2.10}$$

It is not difficult to see that the above problem has a solution if and only if

$$e_{11}(L,0) - e_{12}(L,0) + e_{21}(L,0) - e_{22}(L,0) \neq 0 \tag{2.11}$$

and that therefore $\lambda \in \varrho(\mathcal{A})$ holds if and only if condition (2.11) holds. To characterize the spectrum precisely, we need to calculate the matrix $E(x,s)$ explicitly. To do this we note that

$$M = \begin{pmatrix} \alpha & \beta \\ -\beta & -\alpha \end{pmatrix} \quad \text{where} \quad \alpha := \lambda + \frac{1}{2}\overline{a}, \qquad \beta := \frac{1}{2}\overline{a} \tag{2.12}$$

To get the explicit representation of the exponential matrix E we will use the eigenvector representation. Therefore our next step is to calculate the eigenvalues and eigenvectors. Since

$$\det \begin{pmatrix} \mu - \alpha & -\beta \\ +\beta & \mu + \alpha \end{pmatrix} = \mu^2 - (\alpha^2 - \beta^2)$$

it follows that the eigenvalues are given by $\mu = \pm\sqrt{\alpha^2 - \beta^2}$. Let us denote by $\mu_0 := \sqrt{\alpha^2 - \beta^2}$ with non-negative real part. Then we have that the eigenvector are given by

$$w_1 = \begin{pmatrix} \mu_0 + \alpha \\ -\beta \end{pmatrix}, \quad w_2 = \begin{pmatrix} -\mu_0 + \alpha \\ -\beta \end{pmatrix}$$

Letting

$$D := \begin{pmatrix} \mu_0 + \alpha & -\mu_0 + \alpha \\ -\beta & -\beta \end{pmatrix} \quad \Rightarrow \quad D^{-1} = -\frac{1}{2\beta\mu_0} \begin{pmatrix} -\beta & \mu_0 - \alpha \\ \beta & \mu_0 + \alpha \end{pmatrix}.$$

we have the representation

$$E(x,0) = B \begin{pmatrix} e^{\mu_0 x} & 0 \\ 0 & e^{-\mu_0 x} \end{pmatrix} B^{-1}.$$

which implies

$$E(x,0) = \begin{pmatrix} \cosh(\mu_0 x) + \frac{\alpha}{\mu_0} \sinh(\mu_0 x) & \frac{\beta}{\mu_0} \sinh(\mu_0 x) \\ \frac{\beta}{\mu_0} \sinh(\mu_0 x) & \cosh(\mu_0 x) - \frac{\alpha}{\mu_0} \sinh(\mu_0 x) \end{pmatrix}. \tag{2.13}$$

The condition (2.11) now turns into

$$2\frac{\alpha}{\mu_0} \sinh(\mu_0 L) = 0 \quad \Rightarrow \quad e^{2\mu_0 L} = 1$$

Therefore we conclude that $\mu_0 L = k\pi i$, for integers k, or

$$\mu_0^2 L^2 = -k^2\pi^2.$$

Recalling the definition of μ_0 we get and using (2.12) we get

$$\lambda^2 L^2 + L\lambda\overline{a} + k^2\pi^2 = 0 \quad \Rightarrow \quad \lambda = -\frac{1}{2}\overline{a} \pm \sqrt{\left(\frac{1}{2}\overline{a}\right)^2 - \frac{k^2\pi^2}{L^2}}$$

Finally, note that if $k = 0$ then we will have $\mu_0 = 0$ and therefore $\lambda = 0$. But $\lambda = 0 \in \varrho(\mathcal{A}_0)$.

Q.E.D.

Lemma 2.3 *Let $\lambda = \gamma + i\eta$ with $\gamma > -\frac{\overline{a}}{2}$, and let*

$$\mu_0 = A(\eta) + iB(\eta)$$

define A and B. Then we have

$$A^2 + B^2 = \sqrt{\eta^4 + [2\gamma^2 + 2\gamma\overline{a} + \overline{a}^2]\eta^2 + (\gamma^2 + \gamma\overline{a})^2}. \tag{2.14}$$

$$\lim_{\eta\to\infty} |\frac{\alpha}{\mu_0}| = 1, \quad \lim_{\eta\to\infty} |\frac{\beta}{\mu_0}| = 0, \quad \lim_{\eta\to\infty} |\frac{\lambda}{\mu_0}| = 1, \quad \lim_{\eta\to\infty} A = \gamma + \frac{\overline{a}}{2}. \tag{2.15}$$

$$\lim_{(\gamma,\eta)\to(0,0)} \frac{|\sinh(\mu_0 x)|}{|\mu_0|} = x, \qquad \limsup_{\eta\to\infty} |\sinh(\mu_0 x)| \le \sqrt{1 + \sinh^2(\gamma + \frac{\overline{a}}{2})L} \tag{2.16}$$

Proof: Recalling that

$$\mu_0 = \sqrt{\alpha^2 - \beta^2} = \sqrt{\lambda^2 + \lambda\overline{a}}.$$

we have

$$\mu_0 = \sqrt{\alpha^2 - \beta^2} = \sqrt{\gamma^2 + \overline{a}\gamma + i\eta(\overline{a} + 2\gamma) - \eta^2} = A + iB$$

Squaring the above expression we get

$$\gamma^2 + \overline{a}\gamma - \eta^2 = A^2 - B^2$$

$$\eta(\overline{a} + 2\gamma) = 2AB$$

Solving the equation for A we conclude that

$$A^2 = \frac{\gamma^2 + \overline{a}\gamma}{2} + \frac{-\eta^2 + \sqrt{\eta^4 + [2\gamma^2 + 2\gamma\overline{a} + \overline{a}^2]\eta^2 + (\gamma^2 + \gamma\overline{a})^2}}{2}$$

$$B^2 = -\frac{\gamma^2 + \overline{a}\gamma}{2} + \frac{\eta^2 + \sqrt{\eta^4 + [2\gamma^2 + 2\gamma\overline{a} + \overline{a}^2]\eta^2 + (\gamma^2 + \gamma\overline{a})^2}}{2}$$

Summing up the above identities we get (2.14). Note that

$$\begin{aligned}
\frac{\sqrt{\eta^4 + [2\gamma^2 + 2\gamma\overline{a} + \overline{a}^2]\eta^2 + (\gamma^2 + \gamma\overline{a})^2}}{2} &\leq \frac{\sqrt{(\eta^2 + \gamma^2 + \gamma\overline{a} + \frac{\overline{a}^2}{2})^2 - \overline{a}^2(\gamma + \frac{\overline{a}}{2})^2}}{2} \\
&= \frac{\eta^2 + \gamma^2 + \gamma\overline{a} + \frac{\overline{a}^2}{2}}{2}
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\frac{\sqrt{\eta^4 + [2\gamma^2 + 2\gamma\overline{a} + \overline{a}^2]\eta^2 + (\gamma^2 + \gamma\overline{a})^2}}{2} &\geq \frac{\sqrt{\eta^4 + [2\gamma^2 + 2\gamma\overline{a}]\eta^2 + (\gamma^2 + \gamma\overline{a})^2}}{2} \\
&= \frac{\eta^2 + \gamma^2 + \gamma\overline{a}}{2}
\end{aligned}$$

Therefore we obtain

$$\gamma^2 + \overline{a}\gamma \leq A(\eta)^2 \leq (\gamma + \frac{\overline{a}}{2})^2.$$

Similarly we have

$$\eta^2 \leq B(\eta)^2 \leq \eta^2 + \frac{\overline{a}^2}{4}.$$

Thus we conclude

$$\lim_{\eta\to\infty} |\frac{\alpha}{\mu_0}|^2 = \lim_{\eta\to\infty} \frac{(\gamma + \frac{1}{2}\overline{a})^2 + \eta^2}{A(\eta)^2 + B(\eta)^2} = 1.$$

proving (2.15). Finally,

$$\begin{aligned}
\sinh(\mu_0 x) &= \frac{1}{2}\left\{e^{Ax}(\cos(Bx) + i\sin(Bx)) - e^{-Ax}(\cos(Bx) + i\sin(Bx))\right\} \\
&= \cos(Bx)\sinh(Ax) + i\sin(Bx)\cosh(Ax).
\end{aligned}$$

implies

$$\begin{aligned}
|\sinh(\mu_0 x)| &= \sqrt{\cos^2(Bx)\sinh^2(Ax) + \sin^2(Bx)\cosh^2(Ax)} \\
&= \sqrt{\sinh^2(Ax) + \sin^2(Bx)}
\end{aligned}$$

similarly we have

$$|\cosh(\mu_0 x)| = \sqrt{\sinh^2(Ax) + \cos^2(Bx).)}$$

yielding (2.16).

Q.E.D.

Lemma 2.4 *Let $F = (f_1, f_2)$ and $\lambda = \gamma + i\eta$ as above, and let*

$$\begin{aligned} I(\lambda, F) \quad := \quad & \frac{1}{2} \frac{\int_0^L (a - \overline{a})(\cosh((L-s)\mu_0) + \frac{\lambda + \overline{a}}{\mu_0} \sinh(\mu_0(L-s)) f_1 \, ds}{-\frac{\alpha}{\mu_0} \sinh(\mu_0 L)} \\ & - \frac{1}{2} \frac{\int_0^L (a - \overline{a})(\cosh(\mu_0(L-s)) + \frac{\lambda}{\mu_0} \sinh(\mu_0(L-s))) f_2 \, dx}{-\frac{\alpha}{\mu_0} \sinh(\mu_0 L)} \end{aligned}$$

Then we have

$$\begin{aligned} \limsup_{(\gamma,\eta)\to(0,0)} |I(\lambda, F)| \quad & \leq \quad \|a - \overline{a}\|_{L^2} \|F\|_{L^2} \left(\frac{1 + \overline{a}L}{\overline{a}L} \right) \\ \limsup_{\eta\to\infty} |I(\lambda, F)| \quad & \leq \quad \|a - \overline{a}\|_{L^2} \|F\|_{L^2} \left(\frac{1 + \sinh(\gamma + \frac{\overline{a}}{2})L}{\sinh((\gamma + \frac{\overline{a}}{2})L)} \right) \\ \limsup_{(\lambda,\eta)\to(0,0)} |E(x, s)| \quad & \leq \quad 1 + \frac{\overline{a}L}{2} \\ \limsup_{\eta\to\infty} |E(x, s)| \quad & \leq \quad 2\sqrt{1 + \sinh^2((\gamma + \frac{\overline{a}}{2})L)} \end{aligned}$$

Proof: We have

$$|I(\lambda, F)| \quad \leq \quad \frac{1}{2} \frac{\|a - \overline{a}\|_{L^2} (2\|\cosh(\mu_0 x)\|_\infty + \frac{|\lambda + \overline{a}| + |\lambda|}{|\mu_0|} \|\sinh(\mu_0 x)\|_\infty) \|F\|_{L^2}}{|\frac{\alpha}{\mu_0}| |\sinh(\mu_0 L)|}$$

where $\|\cdot\|_\infty$ denotes the sup-norm with respect to x. Our conclusion now follows from Lemma 2.3 using

$$|\sinh(\mu_0 L)| \geq |\sinh(A(\eta)L)| \to \sinh((\gamma + \frac{\overline{a}}{2})L) \qquad \text{as } \eta \to \infty.$$

Q.E.D.

Corollary 2.5 *There exists a positive constant C_0, depending essentially only on $|\gamma - \frac{\overline{a}}{2}|$, such that for any $\eta \in \mathbb{R}$ and any (x, s) we have*

$$|I(\lambda, F)| \leq C_0 \|a - \overline{a}\|_{L^2} \|F\|_{L^2},$$

$$|E(x, s)| \leq C_0.$$

Remark: For the interval $(0,1)$ and $0<\mu<\frac{1}{2}$, let the function $a_\mu:[0,1]\longrightarrow\mathbb{R}$ be given by

$$a_\mu(x) := \begin{cases} -1 & 0\le x<\mu \\ 1 & \mu\le x<1 \end{cases}$$

Then

$$\overline{a_\mu} = 1-2\mu \qquad \text{and} \qquad \|a_\mu-\overline{a_\mu}\|_{L^2} = 2\sqrt{(1-\mu)\mu}<1,$$

hence, as $\mu\to 0$,

$$\|a_\mu\|_{L^\infty}=1 \qquad \text{and} \qquad \|a_\mu-\overline{a_\mu}\|_{L^2}\to 0.$$

Lemma 2.6 *There exists $\tau>0$ such that when $\|a-\overline{a}\|_{L^2}<\tau$, we have*

(i) $\Gamma^I_\varepsilon\subset\varrho(\mathcal{A})$,

(ii) $\sup_{\lambda\in\Gamma^I_\varepsilon}\|(\lambda-\mathcal{A})^{-1}\|<\infty$.

Proof: It suffices to show that for sufficiently small $\tau>0$ and for $\lambda\in\Gamma^I_\varepsilon$ the equation $(\lambda-\mathcal{A})U=F$ is solvable for any $F\in\mathcal{H}$, and $\|U\|\le C\|F\|$ with a constant C at most depending on ε and τ. We shall use a fixed point argument to prove this. Now let $F=(f,g)'\in\mathcal{H}$ be given as well as $\lambda\in\Gamma^I_\varepsilon$. Let

$$\Phi:\ \tilde{\mathcal{H}}\to\tilde{\mathcal{H}},\quad V\mapsto U=\Phi V$$

be defined as solution $U=(U_1,U_2)'$ to

$$(\lambda-\mathcal{A}_0)U = F-\frac{a-\overline{a}}{2}BV$$

which is well defined since $\lambda\in\varrho(\mathcal{A}_0)$. Using the explicit representation of U we have

$$U(x)=(\Phi V)(x)=E_0(x)U_0+\int_0^x E_0(x-s)\left(KF(s)-\frac{a(s)-\overline{a}}{2}KBV(s)\right)ds \tag{2.17}$$

where $U_0=(p_0,-p_0)$ with $p_0=I(\lambda,V)$, cp. (2.10),

$$\begin{aligned} p_0 \quad = \quad & \left\{\int_0^L (e_{11}+e_{21})(L-s)\left(f(s)-\frac{a(s)-\overline{a}}{2}(V_1(s)+V_2(s)\right)\right. \\ & \left.-(e_{12}+e_{22})(L-s)\left(g(s)-\frac{a(s)-\overline{a}}{2}(V_1(s)+V_2(s))\right)ds\right\}/ \\ & (e_{11}(L)-e_{12}(L)+e_{21}(L)-e_{22}(L)), \end{aligned} \tag{2.18}$$

where $V=(V_1,V_2)'$. Let

$$U^j := \Phi V^j,\quad j=1,2.$$

Then

$$U^1(x)-U^2(x)=E_0(x)(U_0^1-U_0^2)+\int_0^x E_0(x-s)\left(\frac{a(s)-\overline{a}}{2}KB(V^2(s)-V^1(s))\right)ds.$$

This implies

$$|U^1(x)-U^2(x)|\le|E_0(x,0)(U_0^1-U_0^2)|+c_1\int_0^x|(a-\overline{a})E_0(x,s)(V^1(s)-V^2(s))|ds \tag{2.19}$$

and c_1 denotes here and in the sequel a positive constant at most depending on τ and ε. We conclude from Corollary 2.5 that

$$|U_0^1 - U_0^2| \leq c_1 \|a - \overline{a}\|_{L^2} \|V^1 - V^2\|_{L^2}. \tag{2.20}$$

The last two inequalities yield and using Corollary 2.5 once more we have

$$\|U^1 - U^2\|^2 \leq C_2 \|a - \overline{a}\|_{L^2} \|V^1 - V^2\|_{L^2}.$$

Hence Φ is a contraction mapping if

$$c_1 \|a - \overline{a}\|_{L^2}^2 < 1$$

determining τ as $\tau = \frac{1}{c_1}$. Let $U \equiv (p,q)'$ be the unique fixed point. It satisfies

$$\lambda U + KU_x + \frac{a}{2} BU = F. \tag{2.21}$$

By definition we have $U \in D(\mathcal{A}_0)$. Since $F \in \mathcal{H}$ we obtain by integration of (2.21)

$$\begin{aligned} 0 = \int_0^L f(s) - g(s) ds &= \lambda \int_0^L p(s) - q(s) ds + \int_0^L (p+q)_x(s) ds \\ &= \lambda \int_0^L p(s) - q(s) ds. \end{aligned}$$

Without loss of generality we can assume that $\lambda \neq 0$. Then we conclude

$$U \in D(\mathcal{A}) \quad \text{ and } (\lambda - \mathcal{A})U = F.$$

Finally, we estimate the inverse $(\lambda - \mathcal{A})^{-1}$. Let U be still the fixed point, and let

$$\tilde{U} := \Phi(0)$$

or, in other words,

$$(\lambda - \mathcal{A}_0)\tilde{U} = F.$$

Then we get

$$\|U\| - \|\tilde{U}\| \leq \|U - \tilde{U}\| = \|\Phi U - \Phi(0)\| \leq d\|U\|$$

where $d < 1$ describes the contraction mapping property. It follows

$$\|U\| \leq \frac{1}{1-d} \|\tilde{U}\|.$$

On the other hand we obtain from (2.17), (2.18) (cp. (2.19), (2.20))

$$\|\tilde{U}\| \leq c_1 \|F\|$$

Hence we have proved

$$\|(\lambda - \mathcal{A})^{-1}\| \leq c_1 \frac{1}{1-d}$$

which proves the Lemma.

Q.E.D.

As a consequence we obtain the following Theorem on the exponential decay.

Theorem 2.7 *There exists $\tau > 0$ such that when $\|a - \overline{a}\|_{L^2} < \tau$, we have that the solution to the linearized system decays exponentially, that is to say*

$$\exists\, c_0 > 0 \;\exists\, \alpha_0 > 0 \;\forall t \geq 0 : \quad E_0(t) \leq c_0 e^{-2\alpha_0 t} E_0(0).$$

In particular we can take any

$$\alpha_0 > \Re\left(-\frac{\overline{a}}{2} + \sqrt{(\frac{\overline{a}}{2})^2 - \frac{\pi^2}{L^2}} \right),$$

e.g. $\alpha_0 = -\frac{1}{4L}\int_0^L a(x)\, dx$ *when* $\int_0^L a(x)\, dx < 2\pi$.

Proof: The assertion follows from Lemma 2.6 by well known characterizations of the exponential stability of semigroups, see the results by Prüss [17], cp. [12, Thm 1.3.1].

Q.E.D.

Now we consider Case (II), where finally $\|a\|_{L^\infty}$ will have to be sufficiently small, related to L, that is, (a, L) will satisfy certain relations. Contrasting these restrictions with respect to previous results for small a, e.g. by Freitas and Zuazua [7], we shall obtain examples where the moment $\int_0^L a(x)\sin^2(\pi x/L)\, dx$ is negative.
Similary as in Case (I) above, we wish to prove that for small $\varepsilon_1 > 0$ and any $\varepsilon_0 > \varepsilon_1$ we can choose (a, L) with a small enough such that for any $\varepsilon \in [\varepsilon_1, \varepsilon_0]$

$$\Gamma_\varepsilon^{II} := \{-\frac{\overline{a}}{2} + \varepsilon + \mathrm{i}\eta \;\mid\; \eta \in \mathbb{R}\} \subset \varrho(\mathcal{A}) \tag{2.22}$$

and that

$$\sup_{\varepsilon \in [\varepsilon_1, \varepsilon_0],\, \lambda \in \Gamma_\varepsilon^{II}} \|(\lambda - \mathcal{A})^{-1}\| < \infty. \tag{2.23}$$

We choose

$$\varepsilon_1 := \frac{\overline{a}}{4}, \quad \varepsilon_0 := \max\{2\varepsilon_1, 3|a|_\infty\}, \tag{2.24}$$

where $|a|_\infty := \|a\|_{L^\infty}$, and we observe that $\mathcal{A} - 3|a|_\infty$ is dissipative.
Now we shall regard $\mathcal{A}$ as a perturbation of the following operator $\mathcal{A}_0^{II}$, given by

$$\mathcal{A}_0^{II} U := -K U_x - \frac{a}{2} U$$

with domain

$$D(\mathcal{A}_0^{II}) := D(\mathcal{A}_0^{I}) = \left\{ \begin{pmatrix} p \\ q \end{pmatrix} \in H^1((0,L)) \times H^1((0,L)) \mid p(0) + q(0) = p(L) + q(L) = 0 \right\}$$

in the same Hilbert space

$$\tilde{\mathcal{H}} = L^2((0,L)) \times L^2((0,L)).$$

For $U \in D(\mathcal{A})$ we have

$$\mathcal{A}U = \mathcal{A}_0 U - \frac{a}{2} W U$$

with

$$W := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

${\mathcal{A}_0}^{-1}$ is compact which can be shown as for $\mathcal{A}$ above.

Lemma 2.8 $\sigma(\mathcal{A}_0) = \left\{ -\frac{\overline{a}}{L} + \frac{k\pi \mathrm{i}}{L} \;\mid\; k \in \mathbb{Z} \right\}$

PROOF: We investigate the the solvability of $(\lambda - \mathcal{A}_0)U = F$ for any $F = (f,g)' \in \tilde{\mathcal{H}}$. Using similar ideas as in the proof of Lemma 2.2 we obtain

$$\lambda \in \sigma(\mathcal{A}_0) \;\Leftrightarrow\; e^{-2\lambda L - \int_0^L a(y)dy} = 1 \Leftrightarrow \lambda = \lambda_k = -\frac{\overline{a}}{L} + \frac{k\pi \mathrm{i}}{L}, \quad k \in \mathbb{Z}.$$

Q.E.D.

For given, $0 < \gamma_0 < \gamma_1$ we define the set of admissible a and L as

$$\mathcal{K}(\gamma_0, \gamma_1) := \{(a, L) \mid \gamma_0 \leq \int_0^L a(x)dx \leq a_\infty L \leq \gamma_1\}.$$

We shall prove that, after fixing γ_0 and γ_1, we can determine the maximal possible L^∞-norm of a, and the admissible values of L, that imply exponential decay.

Lemma 2.9 *There are pairs $(a, L) \in \mathcal{K}(\gamma_0, \gamma_1)$ such that*

(i) $\forall \varepsilon \in [\varepsilon_1, \varepsilon_0] : \Gamma_\varepsilon = \{-\frac{\overline{a}}{2} + \varepsilon + \mathrm{i}\eta \;\mid\; \eta \in \mathbb{R}\} \subset \varrho(\mathcal{A})$.

(ii) $\sup\limits_{\varepsilon \in [\varepsilon_1, \varepsilon_0],\, \lambda \in \Gamma_\varepsilon} \|(\lambda - \mathcal{A})^{-1}\| < \infty.$

PROOF: We will show, for any admitted ε, that for $\lambda \in \Gamma_\varepsilon^{II}$ the equation $(\lambda - \mathcal{A})U = F$ is solvable for any $F \in \mathcal{H}$, and the postulated estimate on the inverse holds. We shall use again a fixed point argument to prove this. Now let $F = (f,g)' \in \mathcal{H}$ be given as well as $\lambda \in \Gamma_\varepsilon$. Let

$$\Phi : \tilde{\mathcal{H}} \longrightarrow \tilde{\mathcal{H}}, \quad V \to U = \Phi V$$

be defined as solution $U = (U_1, U_2)'$ to

$$(\lambda - \mathcal{A}_0)U = F - \frac{a}{2}WV$$

which is well defined since $\lambda \in \varrho(\mathcal{A}_0)$. Let us consider

$$U(x) = (\Phi V)(x) = E_0(x, 0, \lambda)U_0 + \int_0^x E_0(x, s, \lambda) \left(KF(s) - \frac{a(s)}{2} KWV(s) \right) ds \tag{2.25}$$

where $U_0 = (p_0, -p_0)$ with

$$p_0 = \frac{\int_0^L e_1(L, s, \lambda) \left(f(s) - \frac{a(s)}{2} V_2(s) \right) - e_2(L, s, \lambda) \left(g(s) - \frac{a(s)}{2} V_1(s) \right) ds}{e_2(L, 0, \lambda) - e_1(L, 0, \lambda)}. \tag{2.26}$$

Let

$$U^j := \Phi V^j, \quad j = 1, 2.$$

Then

$$U^1(x) - U^2(x) = E_0(x, 0, \lambda)(U_0^1 - U_0^2) + \int_0^x E_0(x, s, \lambda) \left(\frac{a(s)}{2} KW(V^2(s) - V^1(s)) \right) ds.$$

This implies

$$|U^1(x) - U^2(x)| \leq c_1 e^{(a_\infty+\varepsilon_0)L} |U_0^1 - U_0^2| + c_2 e^{(a_\infty+\varepsilon_0)L} a_\infty \int_0^x |V^1(s) - V^2(s)| ds \tag{2.27}$$

where c_2 denotes here and in the sequel a generic positive constant, in particular not depending on a or L. We conclude from (2.26)

$$|U_0^1 - U_0^2| \leq \frac{c_1 e^{(a_\infty+\varepsilon_0)L} a_\infty}{\sinh(\varepsilon_1 L)} \int_0^L |V^1(s) - V^2(s)| ds. \tag{2.28}$$

The last two inequalities yield

$$\|U^1 - U^2\| \leq c_2 \frac{a_\infty e^{a_\infty L} e^{\varepsilon_0 L}}{\sinh(\varepsilon_1 L)} \|V^1 - V^2\|.$$

Hence Φ is a contraction mapping if

$$c_2 \frac{a_\infty e^{a_\infty L} e^{\varepsilon_0 L}}{\sinh(\varepsilon_1 L)} < 1$$

Observing (2.24), this is satisfied for (a, L) in $\mathcal{K}$ if

$$\frac{c_2 e^{\gamma_1} e^{3\gamma_1} a_\infty}{\sinh(\gamma_0/8)} < 1$$

or, equivalently, if

$$a_\infty < \frac{\sinh(\gamma_0/2)}{c_2 e^{4\gamma_1}} =: c_0(\gamma_0, \gamma_1) \tag{2.29}$$

(i) Now fixing γ_0 and γ_1, the last inequality determines the maximal possible L^∞-norm of a.
(ii) Then the condition

$$\frac{\gamma_0}{a_\infty} < L \leq \frac{\gamma_1}{a_\infty} \tag{2.30}$$

determines the possible values of L.

(iii) Now fixing L, since $a_\infty L > \gamma_0$, we can choose a satisfying (i) as well as

$$\int_0^L a(x)dx \geq \gamma_0 \tag{2.31}$$

Altogether we have found in (i)–(iii) pairs $(a, L) \in \mathcal{K}$ for which Φ is a contraction.
Let $U \equiv (p, q)'$ be the unique fixed point. It satisfies

$$\lambda U + K U_x + \frac{a}{2} B U = F. \tag{2.32}$$

By definition we have $U \in D(\mathcal{A}_0)$. Since $F \in \mathcal{H}$ we obtain by integration of (2.32)

$$\begin{aligned} 0 = \int_0^L f(s) - g(s) ds &= \lambda \int_0^L p(s) - q(s) ds + \int_0^L (p+q)_x(s) ds \\ &= \lambda \int_0^L p(s) - q(s) ds. \end{aligned}$$

Without loss of generality we can assume that $\lambda \neq 0$. Then we conclude

$$U \in D(\mathcal{A}) \quad \text{ and } (\lambda - \mathcal{A})U = F.$$

Finally, we obtain the uniform boundedness of the inverses as in Case (I),namely let U be still the fixed point, and let $\tilde{U} := \Phi(0)$. Using the same arguments as in the proof of Lemma 2.6 we get

$$\|U\| \leq c_3 \|F\|$$

where c_3 depends at most on a, L, γ_0, γ_1. This proves assertion (ii).

Q.E.D.

As in Case (I) we conclude the exponential stability. Let

$$\mathcal{E}(t) := \|U(t,\cdot)\|^2 = \|e^{t\mathcal{A}}U_0\|^2$$

be the associated energy to problem (2.5). Then

$$\begin{aligned} \mathcal{E}(t) &= \left\| \begin{pmatrix} p \\ q \end{pmatrix} (t,\cdot) \right\|^2 = \left\| \begin{pmatrix} u_t - u_x \\ u_t + u_x \end{pmatrix} (t,\cdot) \right\|^2 \\ &= \int_0^L (u_t^2 + u_x^2)(t,x)dx. \end{aligned}$$

Theorem 2.10 *For sufficiently small* $|a|_\infty$ *and admissible* $(a, L) \in \mathcal{K}(\gamma_0, \gamma_1)$*, we have*

$$\exists\, c_0 \ > 0 \exists\, \alpha_0 > 0\ \forall t \geq 0 :\ \mathcal{E}(t) \leq c_0 e^{-2\alpha_0 t} \mathcal{E}(0).$$

Using Lemma 2.9 which gives information on the spectral radius $\omega_{\sigma(\mathcal{A})} := \sup_{\lambda \in \sigma(\mathcal{A})} \Re\lambda$, a result of Neves, Ribeiro and Lopes [16] saying that the essential type $\omega_e(\mathcal{A})$ is given by

$$\omega_e(\mathcal{A}) = -\frac{1}{2L}\int_0^L a(y)dy. \tag{2.33}$$

as well as using the general characterization (see [15])

$$\omega_0(\mathcal{A}) = \max\{\omega_e(\mathcal{A}), \omega_\sigma(\mathcal{A})\},$$

where $\omega_0(\mathcal{A})$ denotes the type of the semigroup,

$$\omega_0(\mathcal{A}) = \lim_{t\to\infty} \frac{\ln\|\mathrm{e}^{\mathcal{A}\mathrm{t}}\|}{t}.$$

Using this we can establish

Theorem 2.11 *Under the conditions of Theorem 2.10 we have*

$$\exists\, c_0 > 0\ \exists\, \alpha_0 = \alpha_0 \left(\frac{1}{L} \int_0^L a(x)dx \right) > 0\ \forall t \geq 0 :\ \ \mathcal{E}(t) \leq c_0 e^{-2\alpha_0 t} \mathcal{E}(0).$$

α_0 can be chosen as any number $-\overline{\alpha}$ with

$$0 > \overline{\alpha} > -\frac{1}{2L}\int_0^L a(x)dx + \varepsilon_1, \qquad e.g. \quad \alpha_0 := \frac{1}{4L}\int_0^L a(x)dx$$

We can now present an example of a function $a : [0,1] \longrightarrow \mathbb{R}$, for which exponential stability holds, but for which (1.10) is violated since we shall have

$$\int_0^1 a(x)\sin^2(\pi x)dx < 0. \tag{2.34}$$

This will not be a contradiction to a result of Freitas in [6] saying that if (1.10) is not valid, then the solution is not exponentially decaying if one replaces a by εa for sufficiently small $\varepsilon > 0$, because in our example replacing a by εa is not allowed in general because of the admissibility criteria to be observed in the construction of (a, L) in Case (II).
Let $0 < \delta_1 < \delta_2$, to be fixed later, and let

$$a(x) := \begin{cases} \delta_1 & \text{for } x \in [0,1/4) \cup (3/4,1] \\ -\delta_2 & \text{for } x \in [1/4,3/4] \end{cases}$$

Then $0 < \int_0^1 a(x)dx = (\delta_1 - \delta_2)/2$ provided $\delta_1 > \delta_2$. Moreover, we have

$$\int_0^1 a(x)\sin^2(\pi x)dx = \frac{1}{2\pi}((\pi-1)\delta_1 - (\pi+1)\delta_2) < 0$$

if and only if

$$\delta_1 < \frac{\pi+1}{\pi-1}\delta_2.$$

Hence choosing δ_1, δ_2 such that

$$\delta_2 < \delta_1 < \frac{\pi+1}{\pi-1}\delta_2 \tag{2.35}$$

we have a function a satisfying

$$0 < \int_0^1 a(x)dx \quad \text{and} \quad \int_0^1 a(x)\sin^2(\pi x)dx < 0.$$

Choose $\delta_2 := \alpha\delta_1$ such that (2.35) is satisfied, e.g. $\alpha := 1/1.92$ for the rest of the exposition. Then δ_1 is still free. Let us define the function $\hat{a}$ as

$$\hat{a}(x) := \frac{a(x/L)}{L} \quad \text{for } x \in (0,L)$$

where L will be chosen to satisfy (2.30). Since $a_\infty = \delta_1$ we have $\hat{a}_\infty = \delta_1/L$. We are free to choose γ_0 and γ_1. Fix γ_1. The condition (2.31) can be satisfied if

$$\gamma_0 \le (\delta_1 - \delta_2)/2 = \frac{23}{96}\delta_1$$

i.e. we choose $\delta_1 := \frac{96}{23}\gamma_0$, condition (2.29) can now be satisfied if

$$\frac{\delta_1}{L} < c_0(\gamma_0,\gamma_1) = c_{\gamma_1}\sinh(\gamma_0/2)$$

where

$$c_{\gamma_1} := \frac{1}{c_2\exp(\gamma_1)\exp(3\gamma_1)}$$

Since, for small γ_0,

$$\sinh(\gamma_0/2) \geq \frac{\gamma_0}{3}$$

it is hence sufficient to require

$$L > \frac{288}{23c_{\gamma_1}} \tag{2.36}$$

The last condition (2.30) can now be satisfied if $\frac{96}{23}\gamma_0 \leq \gamma_1$ e.g. take $\gamma_0 := \chi\frac{23}{96}\gamma_1$ with arbitrary, but then fixed $\chi \in (0,1]$. As conclusion from the discussion of Case (II) above we get that with such an L and the given $\hat{a}$ the solution to

$$u_{tt} - u_{xx} + \hat{a}(x)u_t = 0 \quad \text{in } (0,L)$$

plus initial and boundary conditions has exponentially decaying energy. In the way we have chosen $\hat{a}$ resp. a, we have now found u solving

$$u_{tt} - u_{xx} + a(x)u_t = 0 \quad \text{in } (0,1)$$

with exponentially decaying energy for a function a satisfying

$$0 < \int_0^1 a(x)dx \quad \text{and} \quad \int_0^1 a(x)\sin^2(\pi x)dx < 0.$$

We finish this section giving some higher norm estimates valid in both Cases (I) and (II), i.e. whenever exponential stability is given. Differentiating the differential equation (2.1) with respect to t,

$$(\partial_t^j u)_{tt} - (\partial_t^j u)_{xx} + a(x)(\partial_t^j u)_t = 0, \quad j \in \mathbb{N},$$

and using the fact that derivatives with respect to x can be computed from the differential equation successively, we get as a consequence of $a \in C^0([0,L],\mathbb{R})$, and $a \in C^{s-2}([0,L],\mathbb{R})$ if $s \geq 2$ the following theorem.

Theorem 2.12 *Under the conditions of the Theorems 2.7 and 2.10, respectively, we have*

$$\forall s \in \mathbb{N} \quad \exists C_s > 0 \quad \forall t \geq 0 : \quad \left\| \begin{pmatrix} u_t \\ u_x \end{pmatrix}(t,\cdot) \right\|_{H^s((0,L))} \leq C_s e^{-\alpha_0 t} \left\| \begin{pmatrix} u_1 \\ u_{0,x} \end{pmatrix} \right\|_{H^s(0,L)}$$

where α_0 is given in Theorem 2.11, and the data are assumed to be sufficiently smooth and to satisfy the usual compatibility conditions.

Remarks:

1. We only used $a \in L^\infty((0,L))$.

2. Without loss of generality we studied the equation

$$u_{tt} - d_0 u_{xx} + a(x)u_t = 0, \quad x \in (0,L),$$

for $d_0 = 1$, because if $d_0 > 0$ is arbitrary we may define

$$v(t,y) := u(t,\sqrt{d_0}y), \quad y \in \left(0, \frac{L}{\sqrt{d_0}}\right).$$

Then v satisfies

$$v_{tt} - v_{yy} + \tilde{a}(y)v_t = 0, \quad y \in \left(0, \frac{L}{\sqrt{d_0}}\right),$$

for which Theorem 2.11 can be applied directly replacing a by $\tilde{a}$ and L by $L/\sqrt{d_0}$. The decay rate $\tilde{\alpha}_0 = \tilde{\alpha}_0 \left(\frac{\sqrt{d_0}}{L} \int\limits_0^{L/\sqrt{d_0}} \tilde{a}(y)dy \right)$ turns into $\tilde{\alpha}_0 = \alpha_0 \left(\frac{1}{L} \int\limits_0^{L} a(y)dy \right)$ again since $\frac{\sqrt{d_0}}{L} \int\limits_0^{L/\sqrt{d_0}} \tilde{a}(y))dy = \frac{1}{L} \int\limits_0^{L} a(x)ds.$

3 Global existence for the nonlinear system

We now return to the nonlinear system (1.1)–(1.3) assuming again the positivity of the mean value (1.4) and also the condition (1.5) on the nonlinearity, which, for example, is satisfied in the classical model for a nonlinear string, where

$$\sigma(u_x) = \frac{u_x}{\sqrt{1+u_x^2}}.$$

After recalling the local well-posedness it is the aim to prove a global existence result for data (u_0, u_1) being sufficiently small in $H^4((0,L))$, and, quasi simultaneously, to obtain the exponential stability. The method used imitates that one which is well-known for nonlinear evolution equations and systems, see [18] for a presentation of the general approach for Cauchy problems ($x \in \mathbb{R}^n$, no boundary). Here we shall have to prove so-called high energy estimates and a weighted a priori estimate — describing the expected exponential decay — for a boundary value problem and a non-dissipative problem reflected in the possible negativity of the function a. To deal with the latter the condition (1.11) on the negative part of a, i.e.

$$\|a^-\|_{L^\infty} < \alpha_0,$$

will be used.
We assume that the conditions on a and on (a, L), respectively, are satisfied which assured the exponential stability of the linearized systems as given in Theorem 2.12.
Observing that the term $a(x)u_t$ is of lower order, we can recall the following local existence theorem, see for instance [4] or [8, p.97].

Theorem 3.1 *There is $T = T\ (\|(u_0, u_1)\|_{H^4 \times H^3}) > 0$ such that (1.1)–(1.3) has a unique local solution*

$$u \in \bigcap_{k=0}^{3} C^k([0,T], H^{4-k}((0,L)) \cap H_0^1((0,L)) \cap C^4([0,T], L^2((0,L)).$$

Remark: Of course u_0, u_1 have to satisfy the usual compatibility conditions.

Now we turn to the high energy estimates. For this purpose it is useful to rewrite (1.1)–(1.3) as a first-order system for

$$V := (u_t, u_x)'.$$

Then V satisfies

$$V_t + \underbrace{\begin{pmatrix} a & -d_0\partial_x \\ -\partial_x & 0 \end{pmatrix}}_{=:-A} V = \begin{pmatrix} b(u_x)\partial_x u_x \\ 0 \end{pmatrix} =: F(V, V_x),$$

$$V(t=0) = (u_1, \partial_x u_0)' =: V_0.$$

The first formally defined operator A generates a C_0-semigroup as usual, for $F = 0$ the solution V is given by

$$V(t) = e^{tA}V_0$$

and the (local) solution to (1.1)–(1.3) satisfies

$$V(t) = e^{tA}V_0 + \int_0^t e^{(t-r)A}F(V, V_x)(r)dr. \tag{3.1}$$

From Section 2 we conclude that

$$V := (u_t, u_x)$$

as solution of the linear system (1.1)–(1.3) written in first-order form satisfies

$$V(t) = e^{tA}V(t=0)$$

with a C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ satisfying

$$\|V(t)\|_{H^s} \leq c_s e^{-\alpha_0 t}\|V(t=0)\|_{H^s} \tag{3.2}$$

for $s = 0, 1, 2$ (cp. below). This follows from Theorem 2.1 for $s = 0$ and obtained for $s = 1, 2$ by differentiating the equation (1.8) with respect to t one and then twice, as well as using the differential equation to obtain information for derivatives in x. Let

$$a_\infty^- := \|a^-\|_{L^\infty} \tag{3.3}$$

in the sequel we assume without loss of generality that u_x is small enough a priori, i.e. such that $\sigma'(u_x)$ remains strictly positive (near $u_x = 0$, cp. (1.5)), or in other terms we can assume that there is $\eta > 0$ such that

$$d_0 - b(u_x) \geq \frac{d_0}{2} > 0 \qquad \text{if} \qquad |u_x| < \eta < 1. \tag{3.4}$$

Lemma 3.2 *There are constants $c_2, c_3 > 0$, not depending on V_0 or T, such that the local solution given in Theorem 3.1 satisfies for $t \in [0, T]$:*

$$\|V(t)\|_{H^3}^2 \leq c_2\|V_0\|_{H^3}^2 e^{a_\infty^- t} e^{c_3\int_0^t (\|V(r)\|_{H^2} + \|V(r)\|_{H^2}^2 + \|V(r)\|_{H^2}^3)dr}$$

Proof: Multiplying

$$u_{tt} - d_0 u_{xx} + a u_t = b(u_x)u_{xx} \tag{3.5}$$

by u_t in L^2 we obtain $\left(\int \equiv \int\limits_0^L\right)$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int u_t^2 + d_0 u_x^2 dx &= -\int a u_t^2 dx + \int b(u_x) u_{xx} u_t dx \\ &\leq \int a_\infty^- u_t^2 - \int (\partial_x b(u_x)) u_x u_t dx - \int b(u_x) u_x u_{xt} \\ &\equiv I.1 + I.2 + I.3, \end{aligned} \tag{3.6}$$

$$\begin{aligned} |I.2| &\leq \frac{1}{2}\|b'(u_x)u_{xx}\|_{L^\infty} \int u_x^2 + u_t^2 dx \\ &\leq c\|V\|_{H^2} \int u_x^2 + u_t^2 dx \end{aligned} \tag{3.7}$$

where c will denote a constant not depending on V^0 or on T.

$$\begin{aligned} I.3 &= -\frac{1}{2}\frac{d}{dt}\int b(u_x) u_x^2 dx + \frac{1}{2}\int (\partial_t b(u_x)) u_x^2 \\ &\equiv I.3.1 + I.3.2. \end{aligned} \tag{3.8}$$

The term I.3.2 can be estimated in the same way as I.2 in (3.7):

$$|I.3.2| \leq c\|V\|_{H^2} \int u_x^2. \tag{3.9}$$

The term I.3.1 can be incorporated into and be dominated by the left-hand side of inequality (3.6) after an integration with respect to t later on, since

$$\int\limits_0^t I.3.1(r) dr = -\frac{1}{2}\int b(u_x) u_x^2 dx + \frac{1}{2}\int b(u_x(t=0)) u_x^2(t=0) dx. \tag{3.10}$$

Summarizing (3.6)–(3.10) we have proved

$$\|V(t)\|_{L^2}^2 \leq c\|V_0\|_{L^2}^2 + \int\limits_0^t (a_\infty^- + c\|V(r)\|_{H^2})\|V(r)\|_{L^2}^2 dr. \tag{3.11}$$

In order to get estimates for the higher-order derivatives of V (resp. u) we differentiate equation (3.5) with respect to t to get

$$u_{ttt} - d_0 u_{txx} + a u_{tt} = b'(u_x) u_{xt} u_{xx} + b(u_x) u_{txx}. \tag{3.12}$$

Multiplying by u_{tt} in L^2 we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int u_{tt}^2 + d_0 u_{tx}^2 dx &\leq \int a_\infty^- u_{tt}^2 dx + \int b(u_x) u_{txx} u_{tt} dx + \int b'(u_x) u_{xt} u_{xx} u_{tt} dx \\ &\equiv I.4 + I.5 + I.6. \end{aligned} \tag{3.13}$$

The term I.5 can be treated like the term I.2 + I.3 from (3.6), see (3.7)–(3.11).

$$|I.6| \leq c\|V\|_{H^2} \int u_{tt}^2 + u_{tx}^2 dx. \tag{3.14}$$

Observe that the differential equation (3.5) yields the estimate

$$|u_{xx}|^2 \leq c(|u_{tt}|^2 + |u_t|^2). \tag{3.15}$$

Thus we obtain from (3.11), (3.13), (3.14)

$$\|V(t)\|^2_{H^1} \leq c\|V_0\|^2_{H^1} + \int_0^t (a^-_\infty + c\|V(r)\|_{H^2})\|V(r)\|^2_{H^1} dr. \tag{3.16}$$

Differentiating the differential equation (3.12) once more with respect to t we get

$$u_{tttt} - d_0 u_{ttxx} + a u_{ttt} = b''(u_x)u^2_{xt}u_{xx} + b'(u_x)u_{xtt}u_{xx} + 2b'(u_x)u_{xt}u_{xxt} + b(u_x)u_{ttxx}. \tag{3.17}$$

Multiplying by u_{ttt} in L^2 we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int u^2_{ttt} + d_0 u^2_{ttx} dx &\leq \int a^-_\infty u^2_{ttt} dx + \int b''(u_x)u^2_{xt}u_{xx}u_{ttt} dx \\ &\quad + \int b'(u_x)u_{xtt}u_{xx}u_{ttt} dx \\ &\quad + 2\int b'(u_x)u_{xt}u_{xxt}u_{ttt} dx + \int b(u_x)u_{ttxx}u_{ttt} dx \\ &\equiv I.7 + I.8 + I.9 + I.10 + I.11. \end{aligned} \tag{3.18}$$

The term I.11 is again dealt with like I.2 + I.3 in (3.7)– (3.11).

$$|I.8| + |I.9| + |I.10| \leq c(\|V\|^2_{H^2} + \|V\|_{H^2}) \int u^2_{xt} + u^2_{ttt} + u^2_{ttt} + u^2_{xtt} + u^2_{xxt} dx. \tag{3.19}$$

Hence we obtain from (3.16), (3.19) using (3.12) to estimate u_{txx},

$$\|V(t)\|^2_{H^2} \leq c\|V_0\|^2_{H^2} + \int_0^t a^-_\infty + c(\|V(r)\|_{H^2} + \|V(t)\|^2_{H^2})\|V(r)\|^2_{H^2} dr. \tag{3.20}$$

The final estimate is obtained after differentiating the differential equation a last time with respect to t yielding

$$\begin{aligned} u_{ttttt} - d_0 u_{tttxx} + a u_{tttt} &= b'''(u_x)u^3_{xt}u_{xx} + 3b''(u_x)u_{xt}u_{xtt}u_{xx} \\ &\quad + 3b''(u_x)u^2_{xt}u_{xxt} + b'(u_x)u_{xttt}u_{xx} + 3b'(u_x)u_{xtt}u_{xxt} \\ &\quad + 3b'(u_x)u_{xt}u_{xxtt} + b(u_x)u_{tttxx} \\ &\equiv \sum_{j=12}^{18} \theta_j. \end{aligned} \tag{3.21}$$

Remark: The derivatives of order five are formally not defined but the estimates aimed at will only involve derivatives of order four. A usual approximation argument with data $V_0 \in H^4((0,2))$ and the lower semicontinuity of the norms justifies our calculation finally for $V_0 \in H^3((0,2))$ only.

A multiplication of (3.21) by u_{tttt} in L^2 yields

$$\frac{1}{2}\frac{d}{dt}\int u^2_{tttt} + u^2_{tttx} dx \leq \int a^-_\infty u^2_{tttt} + \sum_{j=12}^{18} \int \theta_j u_{tttt} dx. \tag{3.22}$$

The term $\int \theta_{18} u_{tttt} dx$ can be dealt with like I.2. + I.3 in (3.7)–(3.11). The terms $\int \theta_j u_{tttx} dx$ for $j \neq 16$, 18 can be estimated easily as before by

$$\left| \sum_{\substack{j=12 \\ j\neq 16}}^{17} \int \theta_j u_{tttt} dx \right| \leq c(\|V\|_{H^2} + \|V\|_{H^2}^2 + \|V\|_{H^2}^3)\|V\|_{H^3}^2. \tag{3.23}$$

The only more difficult term is

$$\int \theta_{16} u_{tttt} = 3 \int b'(ux) u_{xtt} u_{xxt} u_{tttt} dx$$

involving three factors of order at least three. This term is estimated as follows using the Gagliardo-Nirenberg inequality.

$$\begin{aligned} \left| \int b'(ux) u_{ttx} u_{txx} u_{tttt} dx \right| &\leq c\|b'(u_x) u_{ttx}\|_{L^2} \|u_{tttt}\|_{L^2} \\ &\leq c\|b'(u_x) u_{ttx} u_{txx}\|_{L^2} \|V\|_{H^3}. \end{aligned} \tag{3.24}$$

$$\begin{aligned} \|b'(u_x) u_{ttx} u_{txx}\|_{L^2} &\leq \|\partial_x(b'(u_x) u_{tt}) \partial_x(u_{tx})\|_{L^2} + \|b''(u_x) u_{xx} u_{tt} u_{txx}\|_{L^2} \\ &\leq \|\partial_x(b'(u_x) u_{tt}) \partial_x(u_{tx})\|_{\ L^2} + c\|V\|_{H^2}^2 \|V\|_{H^3}. \end{aligned} \tag{3.25}$$

Using Young's inequality and the Gagliardo-Nirenberg inequality, and observing $L^\infty \hookrightarrow H^1$, we conclude

$$\begin{aligned} \|\partial_x(b'(u_x) u_{tt}) \partial_x(u_{tx})\|_{L^2} &\leq \|\partial_x(b'(u_x) u_{tt})\|_{L^4} \|\partial_x(u_{tx})\|_{L^4} \\ &\leq c\|b'(u_x) u_{tt}\|_{H^2}^{1/2} \|b'(u_x) u_{tt}\|_{L^\infty}^{1/2} \|u_{tx}\|_{H^2}^{1/2} \|u_{tx}\|_{L^\infty}^{1/2} \\ &= c\left(\|b'(u_x) u_{tt}\|_{L^\infty} \|u_{tx}\|_{H^2}\right)^{1/2} \left(\|b'(u_x) u_{tt}\|_{H^2} \|u_{tx}\|_\infty\right)^{1/2} \\ &\leq c\left(\|b'(u_x) u_{tt}\|_{L^\infty} \|u_{tx}\|_{H^2} + \|u_{tx}\|_{L^\infty} \|b'(u_x) u_{tt}\|_{H^2}\right) \\ &\leq c\|V\|_{H^2} \|V\|_{H^3}. \end{aligned} \tag{3.26}$$

From (3.20) and (3.22)–(3.26) we conclude

$$\|V(t)\|_{H^3}^2 \leq c\|V_0\|_{H^3}^2 + \int_0^t a_\infty^- + c(\|V\|_{H^2} + \|V\|_{H^2}^2 + \|V\|_{H^2}^3)(r)\|V(r)\|_{H^3}^2 dr$$

which yields the assertion of Lemma 3.2 using Gronwall's inequality.

Q.E.D.

Next we want to prove a weighted a priori estimate for $\|V(t)\|_{H^2}$.

Remark: Observe that we did not yet use the assumption (1.5) requiring $b'(0) = \sigma''(0) = 0$. Indeed, with this assumption it would be possible to remove the linear term $\|V(t)\|_{H^2}^1$ in the exponential in the estimate for $\|V(t)\|_{H^3}$ in Lemma 4.2, i.e. the estimate would read

$$\|V(t)\|_{H^3}^2 \leq C\|V_0\|_{H^3}^2 e^{a_\infty^- t} e^{c\int_0^t (\|V(r)\|_{H^2}^2 + \|V(r)\|_{H^2}^3) dr} \tag{3.27}$$

and without loss of generality the a priori boundedness — being proved a posteriori — of $\|V(t)\|_{H^2}$ would be used to achieve

$$\|V(t)\|^2_{H^3} \leq C\|V_0\|^2_{H^3} e^{a^-_\infty t} e^{c\int\limits_0^t (\|V(r)\|^2_{H^2} dr}. \tag{3.28}$$

Since we aim at exponential decay it will not matter if we use (3.28), (3.27) or the statement of Lemma 3.2.
Using the representation (3.1) and Theorem 2.12 — observing that the nonlinearity satisfies the compatibility conditions to estimate the H^2-norm — we can estimate

$$\begin{aligned} \|V(t)\|_{H^2} &\leq \|e^{At}V_0\|_{H^2} + \int\limits_0^t \|e^{(t-r)A}\mathcal{F}(V,V_x)(r)\|_{H^2} dr \\ &\leq c_1 e^{-\alpha_0 t}\|V_0\|_{H^2} + c_1 \int\limits_0^t e^{-\alpha_0(t-r)}\|F(V,V_x)\|_{H^2} dr \end{aligned} \tag{3.29}$$

and it will be in the following estimate for $\|F(V,V_x)\|_{H^2}$, where we really use assumption (1.5) to get an estimate we need later on in the weighted a priori estimate.

Lemma 3.3 $\exists\, c > 0\, \forall\, W \in H^3 \,:\, \|F(W,W_x)\|_{H^2} \leq c\|W\|^2_{H^2}\|W\|_{H^3}$.

Proof: (cp. [18] in $\mathbb{R}^n$) Let $u := W^1$.

Using $b(\tau) = \int\limits_0^1 b''(\mu\tau\nu) d\nu\mu d\mu\tau^2$ we obtain

$$\begin{aligned} \|b(u_x)u_{xx}\|_{H^2} &\leq c(\|b(u_x)\|_\infty \|u_{xx}\|_{H^2} + \|b(u_x)\|_{H^2}\|u_{xx}\|_{L^\infty}) \\ &\leq c(\|u_x\|^2_{L^\infty}\|u_x\|_{H^3} + \|u_x\|_{L^\infty}\|u_{xx}\|_{L^\infty}\|u_x\|_{H^3}) \\ &\leq c\|W\|^2_{H^2}\|W\|_{H^3} \end{aligned}$$

Q.E.D.

Using Lemma 3.3 we conclude from (3.29)

$$\|V(t)\|_{H^2} \leq ce^{-\alpha_0 t}\|V_0\|_{H^2} + c\int\limits_0^t e^{-\alpha_0(t-r)}\|V(r)\|^2_{H^2}\|V(r)\|_{H^3} dr \tag{3.30}$$

which is the starting point to prove the following weighted a priori estimate.

Lemma 3.4 *For* $0 \leq t \leq T$ *let*

$$M_2(t) := \sup_{0\leq r\leq t} (e^{\tau_0 r}\|V(r)\|_{H^2})$$

where $0 < \tau_0 \leq \alpha_0$. *Let (1.11) be satisfied, i.e.* $a^-_\infty < d_0$. *Then there are* $M_0 > 0$ *and* $\delta > 0$ *such that if* $\|V_0\|_{H^3} < \delta$ *we have for all* $0 \leq t \leq T$:

$$M_2(t) \leq M_0 < \infty$$

M_0 *is independent of* T *(and of* V_0*).*

PROOF: From (3.30) and the energy estimate in Lemma 3.2 we conclude

$$\begin{aligned}\|V(t)\|_{H^2} &\leq c\|V_0\|_{H^2}e^{-\alpha_0 t} + c\int_0^t e^{-\alpha_0(t-r)}\|V(r)\|^2_{H^2}\|V_0\|_{H^3}e^{\frac{a_{\overline{\infty}}}{2}r} \times \\ &\quad \times e^{c\int_0^r (\|V(\tau)\|_{H^2}+\|V(\tau)\|^2_{H^2}+\|V(\tau)\|^3_{H^2})d\tau} dr\end{aligned}$$

If $\|V_0\|_{H^3} \leq \delta$ (δ to be determined) we get

$$\begin{aligned}\|V(t)\|_{H^2} &\leq c\delta e^{-\alpha_0 t} + c\delta e^{c\int_0^t (\|V(\tau)\|_{H^2}+\|V(\tau)\|^2_{H^2}+\|V(\tau)\|^3_{H^2})d\tau}\int_0^t e^{-\alpha_0(t-r)}e^{\frac{a_{\overline{\infty}}r}{2}}\|V(r)\|^2_{H^2}dr \\ &\leq c\delta e^{-\alpha_0 t} + c\delta e^{c(M_2(t)+M_2^2(t)+M_2^3(t)\int_0^t e^{-\alpha_0 r}+e^{-2\alpha_0 r}+e^{-3\alpha_0 r}dr} \times \\ &\quad \times M_2^2(t)\int_0^t e^{-\alpha_0(t-r)}e^{\frac{a_{\overline{\infty}}r}{2}}e^{-2\alpha_0 r}dr\end{aligned}$$

which implies

$$\begin{aligned}M_2(t) &\leq c\delta + c\delta e^{c(M_2(t)+M_2^2(t)+M_2^3(t))} \times \\ &\quad \times M_2^2(t)\sup_{0\leq t<\infty} e^{\alpha_0 t}\int_0^t e^{-\alpha_0(t-r)}e^{\frac{a_{\overline{\infty}}}{2}r}e^{-2\alpha_0 r}dr.\end{aligned} \tag{3.31}$$

Since by assumption (1.11) it easily follows that

$$\sup_{0\leq t<\infty} e^{\alpha_0 t}\int_0^t e^{-\alpha_0(t-r)}e^{\frac{a_{\overline{\infty}}}{2}r}e^{-2\alpha_0 r}dr \leq c < \infty$$

we obtain from (3.31) for $0 \leq t \leq T$:

$$M_2(t) \leq c\delta + c\delta M_2^2(t)e^{c(M_2(t)+M_2^2(t)+M_2^3(t))}. \tag{3.32}$$

By standard arguments (cp. e.g. [18]), considering the function

$$f(x) := c\delta(1 + cx^2 e^{c(x+x^2+x^3)}) - x$$

it follows that $M_2(t)$ is uniformly bounded by the first zero M_0 of f provided δ and $M_2(0)$ are sufficiently small.
This proves Lemma 3.4.

Q.E.D.

Now we can formulate and prove the main theorem on global existence and exponential decay.

Theorem 3.5 *Let the assumptions (1.5) and (1.11) be satisfied. Then there exists $\delta > 0$ such that if $\|V_0\|_{H^3} < \delta$ there is a unique global solution u to (1.1)–(1.3) satisfying*

$$u \in \bigcap_{k=0}^{3} C^k\left([0,\infty),\, H^{4-k}((0,L)) \cap H_0^1((0,L))\right) \cap C^4\left([0,\infty), L^2((0,L))\right).$$

Moreover there are constants $c_0 = c_0(V_0) > 0$ *and* $c_1 > 0$ *such that*

$$\|V(t)\|_{H^2} \leq c_0 e^{-\alpha_0 t}$$

and

$$\|V(t)\|_{H^3} \leq c_1 \|V_0\|_{H^3} e^{a_\infty^- t}, \quad t \geq 0.$$

PROOF: From Lemma 3.2 and Lemma 3.4 we conclude for the local solution

$$\begin{aligned}
\|V(t)\|_{H^3} &\leq c\|V_0\|_{H^3} e^{a_\infty^- t} e^{c\int_0^t (\|V(r)\|_{H^2} + \|V(r)\|_{H^2}^2 + \|V(r)\|_{H^2}^3) dr} \\
&\leq c\|V_0\|_{H^3} e^{a_\infty^- t} e^{c(M_0 + M_0^2 + M_0^3)} \\
&\leq c\|V_0\|_{H^3} e^{a_\infty^- t},
\end{aligned}$$

c being independent of t or V_0, from where the global existence follows by the usual continuation argument. The claim on the exponential decay of $\|V(t)\|_{H^2}$ now is a consequence of Lemma 3.4.

Q.E.D.

The assumption (1.11), i.e.

$$a_\infty^- < 2\alpha_0$$

together with the explicit estimates for α_0 from Section 2 just requires that the possibly existing negative part of a is not too large in comparison to its positive part.

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