Qualitative aspects in dual-phase-lag thermoelasticity

Ramón Quintanilla[1]

Department of Applied Mathematics II
UPC Terrassa, Colom 11, 08222 Terrassa, Spain
ramon.quintanilla@upc.edu

Reinhard Racke[2]

Department of Mathematics and Statistics
University of Konstanz
78457 Konstanz, Germany
reinhard.racke@uni-konstanz.de

Abstract: We consider the system of dual-phase-lag thermoelasticity proposed by Chandrasekharaiah and Tzou. First, we prove that the solutions of the problem are generated by a semigroup of quasi-contractions. Thus, the problem of third-order in time is well-posed. Then the expontial stability is investigated. Finally the spatial behavior of solutions is analyzed in a semi-infinite cylinder and a resuult on the domain of influence is obtained.

1 Introduction

It is well known that the usual theory of heat conduction based on Fourier's law predicts infinite heat propagation speed. Heat transmission at low temperature has been observed to propagate by means of waves. These aspects have caused intense activity in the field of heat propagation. Extensive reviews on the so-called second sound theories (hyperbolic heat conduction) are given in Chandrasekharaiah [3] and in the books of Müller and Ruggeri [20] and Jou *et al.* [18]. A theory of heat conduction in which the evolution equation contains a third order derivative with respect to time was proposed in [8]. Several instability results have been obtained for the theory, see e.g. [7, 23], as well as proof of the nonexistence of global solutions in the nonlinear theory [29].

In 1995, Tzou [34] proposed a theory of heat conduction in which the Fourier law is

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replaced by an approximation of the equation

$$\mathbf{q}(\mathbf{x}, t+\tau_q) = -k\nabla\theta(\mathbf{x}, t+\tau_\theta), \quad \tau_q > 0, \ \tau_\theta > 0, \tag{1.1}$$

where τ_q is the phase-lag of the heat flux and τ_θ is the phase-lag of the gradient of temperature. The relation (1.1) states that the gradient of temperature at a point in the material at time $t+\tau_\theta$ corresponds to the heat flux vector at the same point at time $t+\tau_q$. The delay time τ_θ is caused by microstructural interactions such as phonon scattering or phonon-electron interactions. The delay τ_q is interpreted as the relaxation time due to fast-transient effects of thermal inertia. The thermoelastic model was proposed in [3]

$$\mu u_{i,jj} + (\lambda+\mu)u_{j,ji} - m\theta_{,i} = \rho\ddot{u}_i \tag{1.2}$$

$$-q_{i,i} - m\theta_0\dot{u}_{,i} = c\dot{\theta} \tag{1.3}$$

$$q_i(\cdot, t+\tau_q) = -k\theta_{,i}(\cdot, t+\tau_\theta) \tag{1.4}$$

where ρ, θ_0, c are positive constants. $\mu > 0$ and λ are the Lamé moduli satisfying $\mu^* > 0$, where μ^* is defined in (2.16) depending on the space dimension.

Here and in the sequel we use the Einstein summation convention with indices in the range $1 \ldots n$, where $n = 1, 2, 3$ denotes the space dimension. Instead of Fourier's law, being equivalent to assuming

$$\tau_q = \tau_\theta = 0 \tag{1.5}$$

and leading to the classical hyperbolic-parabolic system of thermoelasticity together with the physical paradoxon of infinite propagation speed through the heat conduction part, we consider the model proposed by Chandrasekharaiah and Tzou [3], where

$$\tau_q > 0, \quad \tau_\theta > 0$$

are positive relaxation times, and where a second-order approximation for $\mathbf{q}$ and a first-order approximation for θ is used, turning (1.4) into

$$q_i + \tau_q\dot{q}_i + \frac{\tau_q^2}{2}\ddot{q}_i = -k\theta_{,i} - k\tau_\theta\dot{\theta}_{,i} \tag{1.6}$$

Thus, in this paper, we consider the theory developed by taking a Taylor series expansion on both sides of (1.1) and retaining terms up to the second-order in τ_q, but only to the first-order in τ_θ. The model that we consider here involves a system of two coupled partial differential equations. It is of hyperbolic type (cp. Section 3). One of them is the usual second order in time equation of motion in the major part of thermoelastic systems and the other has a third-order derivative with respect to time. This system of equations have not received much attention in the literature (until now), but Hetnarski and Ignaczak consider it within the nonclassical approach of thermoelasticity in their review [10]. However, we can recall several references in the case that we do not consider mechanical deformations, cp. [15, 25]. It is known that, when $\tau_\theta = 0$, solutions of heat conduction are not determined by means of a semigroup [9] (see p.125). However in [25], it was established that whenever $\tau_\theta > 0$, one can obtain solutions by means of a semigroup. Thus, the term $\tau_\theta \triangle \dot{\theta}$ plays a role in the stabilization for the equation. In this paper we extend some of the results on existence and stability obtained for the heat conduction to the thermoelastic problem.

The case $\tau_\theta > 0$ but $\tau_q = 0$, also leading to a hyperbolic system, the system of Lord and Shulman, has been studied before, and, for example, the exponential stability has been obtained for bounded reference configurations as well as the nonlinear stability near the equilibrium, see [31, 32].

A natural question is the determination of the time parameters τ_q and τ_θ (see [10]) and our work is motivated by this question. One might expect that mathematical analysis of existence, uniqueness and stability issues, for example, would furnish certain restrictions on the parameters. One condition to be satisfied by solutions of a heat equation should be exponential stability (or at least stability). In [25], exponential stability (for the heat conduction) was established whenever

$$\tau_\theta > \tau_q/2. \tag{1.7}$$

We also recall that in [10] Hetnarski and Ignaczak asked (p.474) for a *general domain of influence theorem as well as a principle of Saint-Venant's type* for this theory. We note that results of this kind were obtained in [15] for the heat conduction. In this paper we also extend some of the results concerning the time asymptotic and the spatial behavior obtained for the heat conduction in order to include mechanical deformations.

Thus, under condition (1.7), one has a heat theory with a third-order derivative in time in the equation that predicts stability. This is of interest in the light of the results obtained in the theory proposed in [8]. By means of several exact solutions instability of solutions was also established in [25] whenever the condition (1.7) is violated. Thus, one may assume that the condition (1.7) must be satisfied in order to use this model to describe heat transmission. In fact one of the objects of this paper is to extend stability results to the thermoelastic problem. In [26] it was demonstrated for a bounded interval $(0, L) \subset \mathbb{R}$ that for the boundary conditions

$$u = \theta_x = 0 \tag{1.8}$$

which allows a nice series expansion of the solutions into $\sin(nx), \cos(nx)$ terms, exponential stability is to be expected since the relevant spectrum of the associated stationary

operator lies strictly in the right-half complex plane.
We shall investigate here the more complicated boundary conditions

$$u = 0, \quad \theta = 0 \tag{1.9}$$

and prove the exponential stability of the associated semigroup.

So in this paper we study three kinds of questions. One is to determine the suitable frame where the third-order problem of thermoelasticity of Chandrasekharaiah and Tzou type is well posed and where the solutions are stable. Second is to prove the exponential stability for bounded reference configurations, and third to determine the spatial behavior of the solutions of the thermoelasticity in a semi-infinite cylinder in $\mathbb{R}^3$.

This paper is organized as follows: in Section 2 we set down the field equations and the boundary and initial conditions of the problem we consider in this paper. A uniqueness and existence result is proved in Section 3. In Section 4 we prove the exponential stability for bounded reference configurations. In Section 5, we obtain some results of Saint-Venant's type concerning the spatial behavior of solutions in a semi-infinte cylinder and some consequences of them as obtained in Section 6. the last section is devoted to the study of the spatial behavior of solutions of a non-standard problem.

When we study the spatial behavior of solutions of some problems concerning the dual-phase-lag thermoelastic system we shall denote the three-dimensional semi-infinite cylinder R with cross-section D. The finite end face of the cylinder is in the plane $x_3 = 0$. The boundary ∂D is supposed regular enough to allow the use of the divergence theorem. We denote by $R(z)$ the set of points of the cylinder R such that x_3 is greater than z and by $D(z)$ the cross-section of the points such that $x_3 = z$. The spatial evolution with distance from the end for solutions of elliptic equations is relevant to the study of Saint-Venant's principle in continuum mechanics (see, e.g. [6, 11, 12, 13] for reviews of this work). Such results for parabolic equations have also been obtained (see [6, 11, 12, 13, 14]) and more recently for hyperbolic equations (see [2] and the references cited therein).

2 Preliminaries

We consider the homogeneous isotropic case. In this paper we study solutions $(\mathbf{u}, \theta) = (\mathbf{u}(\mathbf{x}, t), \theta(\mathbf{x}, t))$ of the thermoelastic system for the C-T theory. The equations are

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ji} - m\theta_{,i} = \rho \ddot{u}_i \tag{2.1}$$

$$k\hat{\theta}_{,ii} - m\theta_0 \dot{\tilde{u}}_{,i} = c\dot{\tilde{\theta}} \tag{2.2}$$

We have used the notation

$$\hat{f} = f + \tau_\theta \dot{f}, \quad \tilde{f} = f + \tau_q \dot{f} + \frac{\tau_q^2}{2} \ddot{f}, \tag{2.3}$$

where $\tau_\theta > 0$, $\tau_q > 0$ are the dimensionless time lag parameters.

We study the qualitative behavior of classical solutions subject to the initial conditions

$$u_i(\mathbf{x},0) = u_i^0(\mathbf{x}),\ \dot{u}_i(\mathbf{x},0) = v_i^0(\mathbf{x}), \theta(\mathbf{x},0) = \theta^0(\mathbf{x}),\ \dot{\theta}(\mathbf{x},0) = \vartheta^0(\mathbf{x}),\ \ddot{\theta}(\mathbf{x},0) = \phi^0(\mathbf{x}), \tag{2.4}$$

and the boundary conditions

$$u_i(\mathbf{x},t) = \theta(\mathbf{x},t) \ = \ 0,\ \ \mathbf{x}\in\partial D \ \times \ [0,\infty), \tag{2.5}$$

$$u_i(x_1,x_2\ ,0,\ t) \ = f_i(x_\alpha\ ,t)\ \ \theta(x_1,x_2\ ,0,\ t) \ = \ g(x_\alpha\ ,t) \quad \text{on } D(0)\ \times\ [0,\infty), \tag{2.6}$$

where the prescribed boundary data f_i, g on the end $x_3 = 0$ is such that $f_i(x_\alpha, 0) = u_i^0(x_\alpha)$, $\theta^0(x_\alpha)$ and f_i, g are assumed to vanish on $\partial D(0)\times\ [0,\infty)$. We will not make any *a priori* assumption regarding the behavior of solutions as $x_3 \longrightarrow \infty$.

Observe that in the limit as τ_θ and $\tau_q \to 0$, we recover from (2.1) or (2.2) the usual thermoelastic system where, in this limiting case, only the three first of (2.4) are assumed to hold. In this limit the existence, stability and the spatial evolution of solutions has been studied in a variety of contexts (see, e.g., [17, 24, 4] and the references cited therein). When τ_q and τ_θ are positive the results to be described in the sequel will be seen to be similar to those obtained previously for such equations (see, e.g., [2] and the references cited therein).

In the course of our calculations, we will use the fact that the eigenvalues of the real symmetric positive definite matrix

$$\begin{pmatrix} a & b \\ b & l \end{pmatrix} \tag{2.7}$$

are

$$\lambda^{\pm} = \frac{1}{2}\Big(a + l \pm \sqrt{(a-l)^2 + 4b^2}\Big), \tag{2.8}$$

so that the smallest eigenvalue is:

$$\lambda^{-} = \frac{1}{2}\Big(a + l - \sqrt{(a-l)^2 + 4b^2}\Big). \tag{2.9}$$

We will use (2.7) in two particular cases. When

$$a = \tau_q + \tau_\theta,\ b = \frac{\tau_q^2}{2},\ l = \frac{\tau_q^2\tau_\theta}{2}, \tag{2.10}$$

it can be easily verified on using (1.2) that the matrix (2.7) is indeed positive definite and so its smallest positive eigenvalue, denoted by λ_0, is given by

$$\lambda_0 = \frac{1}{2}\Big(\tau_q + \tau_\theta + \frac{1}{2}\tau_q^2\tau_\theta - \sqrt{\tau_q^4 + \tau_q^2 + \tau_\theta^2 + \frac{1}{4}\tau_q^4\tau_\theta^2 + 2\tau_q\tau_\theta - \tau_\theta^2\tau_q^2 - \tau_q^3\tau_\theta}\Big). \tag{2.11}$$

When

$$a=\frac{2}{\gamma}+(\tau_q+\tau_\theta),\quad b=\frac{1}{2}\tau_q^2\ ,\ \ l=\frac{1}{2}\tau_q^2\tau_\theta+\frac{2}{\gamma}(\tau_\theta\tau_q-\frac{1}{2}\tau_q^2),\quad \gamma\ >\ 0, \tag{2.12}$$

the matrix (2.9) is again positive definite with the smallest eigenvalue, denoted by μ_γ, given by

$$\begin{aligned}\mu_\gamma \ &= \ \frac{1}{2\gamma}\Big(2+\gamma(\tau_q+\tau_\theta)+\frac{\gamma}{2}\tau_q^2\tau_\theta+2(\tau_\theta\tau_q-\frac{1}{2}\tau_q^2)\\ &\quad -\sqrt{[2+\gamma(\tau_q+\tau_\theta)-\frac{\gamma}{2}\tau_q^2\tau_\theta-2(\tau_\theta\tau_q-\frac{1}{2}\tau_q^2)]^2+\gamma^2\tau_q^4}\Big).\end{aligned} \tag{2.13}$$

To be used later, it will be worth using the following notation

$$T_{ij}=\mu u_{i,j}+(\lambda+\mu)\delta_{ij}u_{r,r}-m\delta_{ij}\theta. \tag{2.14}$$

We have that the estimate

$$T_{ji}T_{ji}\le(1+\epsilon)\mu^*[\mu u_{i,j}u_{i,j}+(\lambda+\mu)u_{r,r}u_{s,s}]+3m^2(1+\epsilon^{-1})\theta^2, \tag{2.15}$$

is satisfied, for every positive ϵ, where

$$\mu^*=\begin{cases}2\mu+\lambda, & n=1\\ \max\{\mu,2\lambda+3\mu\}, & n=2\\ \max\{\mu,3\lambda+4\mu\}, & n=3\end{cases} \tag{2.16}$$

μ^* is the maximal positive eigenvalue of the quadratic form

$$Q(\zeta):=\mu\zeta_{ij}\zeta_{ij}+(\lambda+\mu)\zeta_{rr}\zeta_{ss}$$

cp. [4].

When we study the qualitative aspects concerning existence, uniqueness and exponential stability, and without loss of generality, we assume $\rho=c=1$. However, when we study the spatial behavior of solutions we relax this condition to assume that mass density and thermal capacity are positive because we wish to demonstrate the dependence of the decay parameters on ρ, c explicitly.

3 Well-posedness

We shall formulate the problem for the semi-infinite cylinder R in three space dimensions, but the well-posedness holds for general domains, see the remarks following Theorem 3.3.

The well-posedness result for the third-order in time system can be achieved by an appropriately sophisticated choice of variables and spaces which reflect the special structure of the system.

We first transform the system (2.1)–(2.6) to zero boundary conditions on all of ∂R by defining

$$v_i(x_\alpha, 0, t) := u_i(x_\alpha, 0.t) - f_i(x_\alpha, t), \qquad v_i(x_\alpha, x_3, t) := u_i(x_\alpha, x_3.t) \quad \text{for } x_3 > 0, \tag{3.1}$$

$$\psi(x_\alpha, 0, t) := \theta(x_\alpha, 0, t) - g(x_\alpha, t), \qquad \psi(x_\alpha, x_3, t) := \theta(x_\alpha, x_3, t) \quad \text{for } x_3 > 0, \tag{3.2}$$

and using (u_i, θ) instead of (v_i, ψ) again, we obtain the initial boundary value problem

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ji} - m\theta_{,i} = \ddot{u}_i - h_i, \tag{3.3}$$

$$k\hat{\theta}_{,ii} - m\theta_0 \dot{\tilde{u}}_{i,i} = \dot{\tilde{\theta}} - p, \tag{3.4}$$

$$u_i(\mathbf{x}, 0) = u_i^0(\mathbf{x}),\ \dot{u}_i(\mathbf{x}, 0) = v_i^0(\mathbf{x}), \theta(\mathbf{x}, 0) = \theta^0(\mathbf{x}),\ \dot{\theta}(\mathbf{x}, 0) = \vartheta^0(\mathbf{x}),\ \ddot{\theta}(\mathbf{x}, 0) = \phi^0(\mathbf{x}), \tag{3.5}$$

$$u_i(\mathbf{x}, t) = \theta(\mathbf{x}, t) = 0,\ \ \mathbf{x} \in \partial R\ \times\ [0, \infty) \tag{3.6}$$

where the given external force $\mathbf{h}$ and heat supply p arise from the transformation (3.1), (3.2) in terms of the boundary data $\mathbf{f}$ and g, respectively.
For the transformation to a first-order system that finally will be characterized by a semigroup, we apply the differential operator ˜ from (2.3) to the differential equation (3.3) and obtain

$$\mu \tilde{u}_{i,jj} + (\lambda + \mu) \tilde{u}_{j,ji} - m\tilde{\theta}_{,i} = \ddot{\tilde{u}}_i + \tilde{h}_i. \tag{3.7}$$

We remark that finding a solution $(\tilde{\mathbf{u}}, \theta)$ allows to determine the desired solutions $(\mathbf{u}, \theta)$ of the original system.
Defining

$$\mathbf{V} := (\tilde{\mathbf{u}}, \tilde{\mathbf{u}}_t, \theta, \theta_t.\theta_{tt})'$$

we obtain

$$\mathbf{V}_t = A\mathbf{V} + \mathbf{F}, \qquad V(0) = V^0 \tag{3.8}$$

with the (yet formal) differential operator A given by the symbol

$$A_f := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \mu\Delta + (\lambda+\mu)\nabla\nabla' & 0 & -m\nabla & -\tau_q m \nabla & -\frac{\tau_q^2 m}{2}\nabla \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -\frac{2m\theta_0}{\tau_q^2}\nabla' & \frac{2k}{\tau_q^2}\Delta & \frac{2}{\tau_q^2}(k\tau_\theta\Delta - 1) & -\frac{2}{\tau_q} \end{pmatrix}$$

the right-hand side $\mathbf{F}$ given by

$$\mathbf{F} := (0, \mathbf{h}, 0, 0, 0, p)'$$

and the initial value

$$\mathbf{V}^0(\mathbf{x}) := (\tilde{\mathbf{u}}, \tilde{\mathbf{u}}_t, \theta, \theta_t.\theta_{tt})'(\mathbf{x}, 0)$$

with its components being given in terms of the originally prescribed initial data in (3.5) by using the differential equations.
As underlying Hilbert space we choose

$$\mathcal{H} := (H_0^1(R))^n \times (L^2(R))^n \times H_0^1(R) \times H_0^1(R) \times L^2(R)$$

with inner product

$$\langle V, W, \rangle_{\mathcal{H}} := \frac{4}{\tau_q^4}\left(\langle \theta_0 V^2, W^2\rangle + \langle \theta_0 \mu \nabla V^1, \nabla W^1\rangle + \langle \theta_0(\lambda+\mu)\nabla' V^1, \nabla' W^1\rangle\right)$$
$$+\langle \frac{2}{\tau_q^2} V^4, W^4\rangle + \langle \frac{2\tau_\theta k}{\tau_q^2}\nabla V^4, \nabla W^4\rangle + \langle V^5, W^5\rangle + \langle \frac{2k}{\tau_q^2}\nabla V^3, \nabla W^4\rangle$$
$$+\langle \frac{2k}{\tau_q^2}\nabla V^4, \nabla W^3\rangle + b_0\langle \nabla V^3, \nabla W^3\rangle$$

where $\langle\cdot,\cdot\rangle$ denotes the usual $L^2(R)$-inner product, and where b_0 is chosen appropriately large in dependence of the coefficients to assure that the bilinear form $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ is positive definite. The operator A is now given as

$$A : D(A) \subset \mathcal{H} \mapsto \mathcal{H}, \qquad AV := A_f V$$

with

$$D(A) := \{V \in \mathcal{H} \mid V^2 \in H_0^1(R)^n,\ V^5 \in H_0^1(R),\ A_f V \in \mathcal{H}\}.$$

The choice of the inner product is special, of course, and extends similar considerations from [25] for the pure heat conduction problem.

Lemma 3.1 *There exists a constant $c_1 > 0$ such that for all $V \in D(A)$*

$$|\langle AV, V\rangle_{\mathcal{H}}| \le c_1 \|V\|_{\mathcal{H}}^2$$

holds.

PROOF: We have

$$\langle AV, V\rangle_{\mathcal{H}} = -\frac{4m\theta_0}{\tau_q^2}\langle \nabla V^4, V^2\rangle - \frac{4m\theta_0}{\tau_q^2}\langle \nabla V^4, V^2\rangle - \frac{2}{\tau_q}\langle V^5, V^5\rangle$$
$$+\frac{2k}{\tau_q^2}\langle \nabla V^4, \nabla V^4\rangle + b_0\langle \nabla V^3, \nabla V^3\rangle$$

which implies the assertion.
QED
As a consequence we see that for $d > c_1$ the operator $A - d$ is dissipative and invertible.

Lemma 3.2 *For all $d > c_1$ we have that the range of $A - d$ is all of $\mathcal{H}$.*

PROOF: The solvability of $(A-d)V = F$ is equivalent to solving

$$\begin{aligned} V^2 - dV^1 &= F^1, & (3.9)\\ \mu\Delta V^1 + (\lambda+\mu)\nabla\nabla' V^1 - m\nabla V^3 - \tau_q m \nabla V^4 - \frac{\tau_q^2 m}{2}\nabla V^5 - dV^2 &= F^2, & (3.10)\\ V^4 - dV^3 &= F^3, & (3.11)\\ V^5 - dV^4 &= F^4, & (3.12)\\ -\frac{2m\theta_0}{\tau_q^2}\nabla' V^2 + \frac{2k}{\tau_q^2}\Delta V^3 + \frac{2}{\tau_q^2}(k\tau_\theta\Delta - 1)V^4 - \frac{2}{\tau_q}V^5 - dV^5 &= F^5. & (3.13) \end{aligned}$$

Eliminating V^2, V^4, and V^5, and using

$$E := -\mu\Delta - (\lambda+\mu)\nabla\nabla'$$

we have to solve

$$-EV^1 + d^2V^1 + \underbrace{(m + \tau_q md + \frac{\tau_q^2 md^2}{2})}_{=:\alpha_1}\nabla V^3 = F^2 + dF^1 + (\tau_q m + \frac{\tau_q^2 md}{2})\nabla F^3 + \frac{\tau_q^2 m}{2}\nabla F^4, \quad (3.14)$$

$$-\underbrace{(\frac{2k}{\tau_q^2} + \frac{2k\tau_\theta d}{\tau_q^2})}_{=:\gamma_1}\Delta V^3 + \underbrace{(\frac{2d}{\tau_q^2} + \frac{2d^2}{\tau_q} + d^3)}_{=:\delta_1} V^3 + \underbrace{\frac{2m\theta_0 d}{\tau_q^2}}_{=:\beta_1}\nabla' V^1 =$$

$$-F^5 - \frac{2m\theta_0}{\tau_q^2}\nabla' F^1 - \frac{2}{\tau_q^2}F^3 - (\frac{2}{\tau_q} + d)(dF^3 + F^4) + \frac{2k\tau_\theta}{\tau_q^2}\Delta F^3. \quad (3.15)$$

Hence we consider for $G^1 \in L^2(R)^3$ and $G^2 \in H^{-1}$, the dual space to $H_0^1(R)$, the system

$$\begin{aligned} -EV^1 + d^2V^1 + \alpha_1\nabla V^3 &= G^1, & (3.16)\\ -\gamma\Delta V^3 + \delta V^3 + \beta\nabla' V^1 &= G^2 & (3.17) \end{aligned}$$

where $\alpha_1, \beta_1, \gamma_1, \delta_1$ are positive. If $(V^1, V^3) \in (H_0^1(R))^n \times H_0^1(R)$ solve (3.16),(3.17), then V^2, V^4 and V^5 can be determined from the equations (3.9), (3.11) and (3.12), respectively, and $V \in D(A)$ will solve $(A-d)V = F$.
$-E + d^2$ can be regarded as a positive self-adjoint operator the inverse of which maps $L^2(R)^3 \mapsto (H^2(R) \cap H_0^1(R))^n$, hence V^1 should satisfy

$$V^1 = (-E + d^2)^{-1}(G^1 - \alpha_1\nabla V^3).$$

Plugging this into (3.17) it remains to determine V^3 as a solution in $H_0^1(R)$ of

$$-\gamma_1 \Delta V^3 + \delta_1 V^3 - \alpha_1\beta_1 \nabla'(-E+d^2)^{-1}\nabla V^3 = G^2 - \beta_1\nabla'(-E+d^2)^{-1}G^1. \qquad (3.18)$$

But (3.18) can be solved easily because the bilinear form

$$B(g,h) := \gamma_1\langle\nabla g, \nabla h\rangle + \delta_1\langle g, h\rangle + \alpha_1\beta_1\langle(-E+d^2)^{-1/2}\nabla g, (-E+d^2)^{-1/2}\nabla h\rangle$$

is positive on $H_0^1(R)$, and hence the Lax & Milgram Lemma yields the solvability of (3.18) for any right-hand side in H^{-1}. This proves the assertion of the Lemma.
QED
Now we conclude from the last two Lemmata that A generates a C_0-semigroup, and hence the initial (boundary) value problem (3.8) is uniquely solvable:

Theorem 3.3 *For any $F \in C^0([0,\infty), D(A))$ or $F \in C^1([0,\infty),\mathcal{H})$ and any $V^0 \in D(A)$ there is a unique solution V to (3.8) with $V \in C^1([0,\infty),\mathcal{H}) \cap C^0([0,\infty), D(A))$.*

The well-posedness consideration in this section extend naturally to other domains $\Omega \subset \mathbb{R}^n$, $n = 1,2,3$, instead of the three-dimensional cylinder R, e.g. literally to smoothly bounded domains and to convex domains (where elliptic H^2-regularity up to the boundary holds).

The system under consideration is of hyperbolic type, as we shall demonstrate in the one-dimensional case. Here the differential equations (3.7), (3.4) turn into

$$\tilde{u}_{tt} = \alpha_*\tilde{u}_{xx} - \frac{\tau_q^2 m}{2}\theta_{ttx} - \tau_q m\theta_{tx} - m\theta_x, \qquad (3.19)$$

$$\theta_{ttt} = -\frac{2}{\tau_q}\theta_{tt} - \frac{2}{\tau_q^2}\theta_t - \frac{2m\theta_0}{\tau_q^2}\tilde{u}_{tx} + \frac{2\tau_\theta k}{\tau_q^2}\theta_{txx} + \frac{2k}{\tau_q^2}\theta_{xx} \qquad (3.20)$$

where $\alpha_* := 2\mu + \lambda$ and the right-hand sides are assumed to be zero.
Defining

$$\mathbf{W} := (\tilde{u}_x, \tilde{u}_t, \theta_x, \theta_t, \theta_{tx}, \theta_{tt})'$$

we obtain

$$\mathbf{W}_t = B\mathbf{W}_x + D\mathbf{W}$$

where

$$B := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \alpha_* & 0 & 0 & 0 & 0 & -\frac{\tau_q^2 m}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\frac{2m\theta_0}{\tau_q^2} & \frac{2k}{\tau_q^2} & 0 & \frac{2\tau_\theta k}{\tau_q^2} & 0 \end{pmatrix}$$

and

$$D := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -m & 0 & -\tau_q m & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2}{\tau_q^2} & 0 & -\frac{2}{\tau_q} \end{pmatrix}.$$

The eigenvalues of B are

$$\lambda_1 = \lambda_2 = 0, \quad \lambda_{3|4|5|6} = \pm \frac{1}{\sqrt{2}} \sqrt{(m^2\theta_0 + \frac{2\tau_\theta k}{\tau_q^2} + \alpha_*) \pm \sqrt{(m^2\theta_0 + \frac{2\tau_\theta k}{\tau_q^2} + \alpha_*)^2 - \frac{8\alpha_* \tau_\theta k}{\tau_q^2}}}.$$

which are all real, thus characterizing a hyperbolic system. The hyperbolicity also becomes apparent in the results on the domains of dependence in the following sections.

4 Exponential stability

We recall that in classical thermoelasticity as well as in several other thermoelastic models like the Lord-Shulman theory or the model of type III, the exponential stability of the system could be proved for bounded domains in one space dimension as well as for radially symmetric situations in higher dimensions, see e.g. [17, 31, 32, 28]. We shall demonstrate the exponential stability in one space dimension. Let (u, θ) satisfy (cp. (3.7), (3.4) or (3.19), (3.20))

$$\tilde{u}_{tt} - \alpha_* \tilde{u}_{xx} + m\tilde{\theta}_x = 0 \tag{4.1}$$

$$\tilde{\theta}_t + m\theta_0 \tilde{u}_{tx} - k\hat{\theta}_{xx} = 0 \tag{4.2}$$

with boundary conditions

$$u = \theta = 0 \qquad \text{for } x = 0, L \tag{4.3}$$

and initial conditions given in terms of the original initial conditions $u(\cdot, 0),\ u_t(\cdot, 0),\ \theta(\cdot, 0)$. As in the previous section we define

$$\mathbf{V} := (\tilde{\mathbf{u}}, \tilde{\mathbf{u}}_t, \theta, \theta_t.\theta_{tt})'$$

and we have

$$\begin{aligned} \|V(t)\|_{\mathcal{H}} &= \int_0^L \left\{ \frac{4\theta_0}{\tau_q^4} \tilde{u}_t^2 + \frac{4\theta_0 \alpha_*}{\tau_q^4} \tilde{u}_x^2 + \frac{2}{\tau_q^2} \theta_t^2 + \frac{2\tau_\theta k}{\tau_q^2} \theta_{tx}^2 + \theta_{tt}^2 + \frac{4k}{\tau_q^2} \theta_x \theta_{tx} + b_0 \theta_x^2 \right\} dx \\ &\equiv 2E_{\mathcal{H}}(t) \end{aligned} \tag{4.4}$$

defining the first "energy" term $E_{\mathcal{H}}(t)$. Another energy term is defined by

$$E(t) := \frac{1}{2}\int_0^L \left\{\theta_0\tilde{u}_t^2 + \theta_0\alpha_*\tilde{u}_x^2 + \tilde{\theta}^2 + \frac{\tau_q^2\tau_\theta k}{2}\theta_{tx}^2 + k(\tau_q+\tau_\theta)\theta_x^2 + \frac{k\tau_q^2}{2}\theta_x\theta_{tx}\right\} dx. \quad (4.5)$$

The aim will be to find a suitable Lyapunov functional for the energy terms that proves the exponential stability.

Multiplying the differential equation (4.1) by $\theta_0\tilde{u}_t$ and (4.2) by $\tilde{\theta}$, integrating and performing partial integrations we obtain

$$\begin{aligned}\frac{d}{dt}E(t) &= -k\int_0^L \theta_x^2 dx - \tau_q(\tau_\theta - \frac{\tau_q}{2})k\int_0^L \theta_{tx}^2 dx \\ &\leq -c_1\int_0^L \left\{\theta_x^2 + \theta_{tx}^2\right\} dx \end{aligned} \quad (4.6)$$

for some positive constant c_1, if the condition (1.7) holds. Then (4.6) reflects the dissipative character of the system. We shall assume (1.7) in the sequel, cp. [26], where the sufficiency and necessity of (1.7) was investigated for the boundary conditions (1.8).

Multiplying the differential equation (4.2) by θ_{tt} and integrating we get

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_0^L \left\{\theta_{tt}^2 + \frac{2}{\tau_q^2}\theta_t^2 + \frac{4k}{\tau_q^2}\theta_x\theta_{tx} + \frac{2k\tau_\theta}{\tau_q^2}\theta_{tx}^2\right\} dx &= -\frac{2}{\tau_q}\int_0^L \theta_{tt}^2 dx + \frac{2k}{\tau_q^2}\int_0^L \theta_{tx}^2 dx \\ &\quad -\frac{2m\theta_0}{\tau_q^2}\int_0^L \tilde{u}_{tx}\theta_{tt} dx. \end{aligned} \quad (4.7)$$

Moreover,

$$\frac{d}{dt}\frac{1}{2}\int_0^L b_0\theta_x^2 dx = \int_0^L b_0\theta_x\theta_{tx} dx \leq \frac{b_0}{2}\int_0^L \theta_x^2 dx + \frac{b_0}{2}\int_0^L \theta_{tx}^2 dx. \quad (4.8)$$

Multiplying (4.1) by $\frac{4\theta_0}{\tau_q^4}\tilde{u}_t$ and integrating we obtain

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_0^L \{\frac{4\theta_0}{\tau_q^4}\tilde{u}_t^2 + \frac{4\theta_0\alpha_*}{\tau_q^4}\tilde{u}_x^2\} dx &= \frac{2m\theta_0}{\tau_q^2}\int_0^L \theta_{tt}\tilde{u}_{tx} dx - \frac{4m\theta_0}{\tau_q^3}\int_0^L \theta_{tx}\tilde{u}_t dx \\ &\quad -\frac{4m\theta_0}{\tau_q^4}\int_0^L \theta_x\tilde{u}_t dx. \end{aligned} \quad (4.9)$$

We conclude from (4.7)–(4.9)

$$\begin{aligned}\frac{d}{dt}E_{\mathcal{H}}(t) &\leq -\frac{2}{\tau_q}\int_0^L \theta_{tt}^2 dx + \frac{2k}{\tau_q^2}\int_0^L \theta_{tx}^2 dx + \frac{b_0}{2}\int_0^L \theta_x^2 dx + \frac{b_0}{2}\int_0^L \theta_{tx}^2 dx \\ &\quad -\frac{4m\theta_0}{\tau_q^3}\int_0^L \theta_{tx}\tilde{u}_t dx - \frac{4m\theta_0}{\tau_q^4}\int_0^L \theta_x\tilde{u}_t dx \\ &\leq -\frac{2}{\tau_q}\int_0^L \theta_{tt}^2 dx + (\frac{b_0}{2} + \frac{4m\theta_0}{\tau_q^4\epsilon_1})\int_0^L \theta_x^2 dx + (\frac{2k}{\tau_q^2} + \frac{b_0}{2} + \frac{4m\theta_0}{\tau_q^3\epsilon_1})\int_0^L \theta_{tx}^2 dx \\ &\quad +2\epsilon_1\int_0^L \tilde{u}_t^2 dx \end{aligned} \quad (4.10)$$

where $\epsilon_1 > 0$ will be chosen later appropriately small. Combining (4.6) and (4.10) we get

$$\begin{aligned}\frac{d}{dt}(E_{\mathcal{H}}(t)+KE(t)) \;\leq\; & -\frac{2}{\tau_q}\int_0^L \theta_{tt}^2 dx - [Kk - (\frac{4m\theta_0}{\tau_q^4\epsilon_1}+\frac{b_0}{2})]\int_0^L \theta_x^2 dx \\ & -[Kk\tau_q(\tau_\theta - \frac{\tau_q}{2}) - (\frac{4m\theta_0}{\tau_q^3\epsilon_1}+\frac{b_0}{2}+\frac{2k}{\tau_q^2})]\int_0^L \theta_{tx}^2 dx \\ & +2\epsilon_1\int_0^L \tilde{u}_t^2 dx \end{aligned} \quad (4.11)$$

where $K > 0$ will be chosen below appropriately large. Once ϵ_1 will be fixed, we shall fix K such that the coefficients in $[\cdot]$-brackets in front of the two integrals of the right-hand side in (4.11) will be strictly positive.

Now we follow ansätze described in [17] for classical thermoelasticity but we have to add essential modifications in order to deal with the higher-order system and the different structure under investigation.

If we multiply the differential equation (4.1) by $\frac{1}{\alpha_*}\tilde{u}_{xx}$ and integrate we obtain after partial integrations

$$\frac{1}{\alpha_*}\frac{d}{dt}\int_0^L \tilde{u}_{tx}\tilde{u}_x dx \leq -\frac{2}{3}\int_0^L \tilde{u}_{xx}^2 dx + \frac{1}{\alpha_*}\int_0^L \tilde{u}_{tx}^2 dx + C\int_0^L \tilde{\theta}_x^2 dx \quad (4.12)$$

where capital C will denote a positive constant that may change from line to line in the sequel. Multiplying the differential equation (4.2) by $\frac{3}{\alpha_* m\theta_0}\tilde{u}_{tx}$ and integrating, using (4.1), yields

$$\begin{aligned}\frac{3}{\alpha_*}\int_0^L \tilde{u}_{tx}^2 dx \;=\; & -\frac{3}{\alpha_* m\theta_0}\int_0^L \tilde{\theta}_t\tilde{u}_{tx}dx - \frac{3k}{\alpha_* m\theta_0}\frac{d}{dt}\int_0^L \hat{\theta}_x(\frac{1}{\alpha_*}\tilde{u}_{tt}+\frac{m}{\alpha_*}\tilde{\theta}_x)dx \\ & +\frac{3k}{\alpha_* m\theta_0}\int_0^L \hat{\theta}_{tx}\tilde{u}_{xx} + \frac{3k}{\alpha_* m\theta_0}[\hat{\theta}_x\tilde{u}_{tx}]_{x=0}^{x=L},\end{aligned}$$

hence

$$\begin{aligned}\frac{3}{\alpha_*^2 m\theta_0}\frac{d}{dt}\int_0^L \{\hat{\theta}_x\tilde{u}_{tt}+m\hat{\theta}_x\tilde{\theta}_x\}dx \;\leq\; & -\frac{2}{\alpha_*}\int_0^L \tilde{u}_{tx}^2 dx + \frac{1}{6}\int_0^L \tilde{u}_{xx}^2 dx + C\int_0^L \{\hat{\theta}_{tx}^2+\tilde{\theta}_t^2\}dx \\ & +C\|\hat{\theta}_x\|_{L^\infty(\{0,L\})}\|\tilde{u}_{tx}\|_{L^\infty(\{0,L\})}\end{aligned} \quad (4.13)$$

where $\|f\|_{L^\infty(\{0,L\})} := \max\{|f(0)|,|f(L)|\}$ denotes the sup-norm on the boundary. Combining (4.12) and (4.13) we obtain

$$\begin{aligned}\frac{d}{dt}\int_0^L \{\frac{1}{\alpha_*}\tilde{u}_{tx}\tilde{u}_x + \frac{3k}{\alpha_*^2 m\theta_0}\hat{\theta}_x\tilde{u}_{tt} + \frac{3k}{\alpha_*^2\theta_0}\hat{\theta}_x\tilde{\theta}_x\}dx \leq & -\frac{1}{\alpha_*}\int_0^L \tilde{u}_{tx}^2 dx - \frac{1}{2}\int_0^L \tilde{u}_{xx}^2 dx \\ +C\int_0^L \{\tilde{\theta}_x^2+\hat{\theta}_{tx}^2+\tilde{\theta}_t^2\}dx + \frac{C}{\epsilon_2}\|\hat{\theta}_x\|_{L^\infty(\{0,L\})}^2 + \epsilon_2\|\tilde{u}_{tx}\|_{L^\infty(\{0,L\})}^2 &\end{aligned} \quad (4.14)$$

where $\epsilon_2 > 0$ will be chosen appropriately small later. The differential equation (4.2) yields

$$\int_0^L \hat{\theta}_{xx}^2 dx \leq C \int_0^L \{\tilde{\theta}_t^2 + \tilde{u}_{tx}^2\} dx.$$

Using this and the Sobolev imbedding $W^{1,1}((0,L)) \hookrightarrow L^\infty((0,L))$ we arrive at

$$\|\hat{\theta}_x\|^2_{L^\infty(\{0,L\})} \leq \frac{C}{\epsilon_2^2} \int_0^L \{\hat{\theta}_x^2 + \tilde{\theta}_t^2\} dx + C\epsilon_2^2 \int_0^L \tilde{u}_{tx}^2 dx.$$

Inserting this into (4.14) we conclude for sufficiently small ϵ_2

$$\frac{d}{dt}\int_0^L \{\frac{1}{\alpha_*}\tilde{u}_{tx}\tilde{u}_x + \frac{3k}{\alpha_*^2 m\theta_0}\hat{\theta}_x\tilde{u}_{tt} + \frac{3k}{\alpha_*^2\theta_0}\hat{\theta}_x\tilde{\theta}_x\} dx \leq -\frac{1}{2\alpha_*}\int_0^L \tilde{u}_{tx}^2 dx - \frac{1}{2}\int_0^L \tilde{u}_{xx}^2 dx$$
$$+\frac{C}{\epsilon_2^3}\int_0^L \{\tilde{\theta}_x^2 + \hat{\theta}_{tx}^2 + \tilde{\theta}_t^2\} dx + C\epsilon_2 \|\tilde{u}_{tx}\|^2_{L^\infty(\{0,L\})}. \tag{4.15}$$

In order to estimate the boundary term, we use a well-known technique exploiting in the multipliers a smooth extension of the normal at the boundary which means in one dimension to use the following function Φ with

$$\Phi(x) := \frac{1}{2} - \frac{x}{L}. \tag{4.16}$$

Differentiation of (4.1) with respect to t, multiplying with $\Phi\tilde{u}_{tx}$ and partially integrating yields

$$\begin{aligned} 0 \;=\; & \frac{d}{dt}\int_0^L \tilde{u}_{tt}\Phi\tilde{u}_{tx} dx + \frac{1}{2}\int_0^L \Phi_x(\tilde{u}_{tt}^2 + \tilde{u}_{tx}^2) dx + \frac{\alpha_*}{2}(\tilde{u}_{tx}^2(0) + \tilde{u}_{tx}^2(L)) \\ & + m\int_0^L (\theta_{tx} + \tau_q\theta_{ttx} + \frac{\tau_q^2}{2}\theta_{tttx})\Phi\tilde{u}_{tx} dx \end{aligned}$$

whence

$$\begin{aligned} \frac{d}{dt}\int_0^L \tilde{u}_{tt}\Phi\tilde{u}_{tx} dx \;\leq\; & -\frac{\alpha_*}{4}(\tilde{u}_{tx}^2(0) + \tilde{u}_{tx}^2(L)) + C\int_0^L \{\tilde{u}_{tt}^2 + \tilde{u}_{tx}^2 + \theta_{tx}^2 + \theta_{ttx}^2\} dx \\ & -\frac{m\tau_q^2}{2}\int_0^L \theta_{tttx}\Phi\tilde{u}_{tx} dx \end{aligned} \tag{4.17}$$

follows. Using the differential equation (4.2) again, we obtain

$$
\begin{aligned}
m\theta_0 \int_0^L \theta_{tttx}\Phi\tilde{u}_{tx}dx &= k\frac{d}{dt}\int_0^L \theta_{ttx}\Phi\hat{\theta}_{xx}dx - k\int_0^L \theta_{ttx}\Phi\hat{\theta}_{txx}dx + \int_0^L \theta_{ttt}\Phi\theta_{tx}dx \\
&\quad + \int_0^L \theta_{ttt}\Phi_x\theta_t dx + \tau_q\int_0^L \theta_{ttt}\Phi\theta_{ttx}dx + \tau_q\int_0^L \theta_{ttt}\Phi_x\theta_{tt}dx \\
&\quad + \frac{\tau_q^2}{4}\int_0^L \Phi_x\theta_{ttt}^2 dx \\
&\le k\frac{d}{dt}\int_0^L \theta_{ttx}\Phi\hat{\theta}_{xx}dx - \frac{k}{\tau_\theta}\int_0^L \hat{\theta}_{tx}\Phi\hat{\theta}_{txx}dx + \frac{k}{\tau_\theta}\int_0^L \theta_{tx}\Phi\hat{\theta}_{txx}dx \\
&\quad + C\int_0^L \{\theta_t^2 + \theta_{tt}^2 + \theta_{tx}^2 + \theta_{ttt}^2 + \theta_{ttx}^2\}dx \\
&= k\frac{d}{dt}\int_0^L \theta_{ttx}\Phi\hat{\theta}_{xx}dx + \frac{k}{4\tau_\theta}[\hat{\theta}_{tx}^2(0) + \hat{\theta}_{tx}^2(L)] \\
&\quad + \frac{d}{dt}\frac{k}{\tau_\theta}\int_0^L \theta_{tx}\Phi\hat{\theta}_{xx}dx - \frac{k}{\tau_\theta}\int_0^L \theta_{ttx}\Phi\hat{\theta}_{xx}dx \\
&\quad + C\int_0^L \{\theta_t^2 + \theta_{tt}^2 + \theta_{tx}^2 + \theta_{ttt}^2 + \theta_{ttx}^2\}dx. \qquad (4.18)
\end{aligned}
$$

Inserting (4.18) into (4.17) and using (4.1) again we get

$$
\begin{aligned}
\frac{d}{dt}\int_0^L \{\tilde{u}_{tt}\Phi\tilde{u}_{tx} + \frac{k\tau_q^2}{2\theta_0}\theta_{ttx}\Phi\hat{\theta}_{xx} + \frac{k\tau_q^2}{2\theta_0\tau_\theta}\theta_{tx}\Phi\hat{\theta}_{xx}\}dx &\le -\frac{\alpha_*}{4}(\tilde{u}_{tx}^2(0) + \tilde{u}_{tx}^2(L)) \\
+C\int_0^L \{\tilde{u}_{tx}^2 + \tilde{u}_{xx}^2 + \theta_t^2 + \theta_x^2 + \theta_{tt}^2 + \theta_{tx}^2 + \theta_{ttt}^2 + \theta_{ttx}^2 + \hat{\theta}_{xx}^2\}dx. \qquad (4.19)
\end{aligned}
$$

We still have to produce a term $-\int_0^L \hat{\theta}_{xx}^2 dx$-term on the right-hand side. This is obtained as follows. We have from (4.2)

$$
\hat{\theta}_{xx} = \frac{1}{k}\tilde{\theta}_t + \frac{m\theta_0}{k}\tilde{u}_{tx}
$$

which, inserted into (4.19), yields

$$
\begin{aligned}
\frac{d}{dt}\int_0^L \{\tilde{u}_{tt}\Phi\tilde{u}_{tx} + \frac{k\tau_q^2}{2\theta_0}\theta_{ttx}\Phi\hat{\theta}_{xx} + \frac{k\tau_q^2}{2\theta_0\tau_\theta}\theta_{tx}\Phi\hat{\theta}_{xx}\}dx &\le -\frac{\alpha_*}{4}(\tilde{u}_{tx}^2(0) + \tilde{u}_{tx}^2(L)) \\
+C\int_0^L \{\tilde{u}_{tx}^2 + \tilde{u}_{xx}^2 + \theta_t^2 + \theta_x^2 + \theta_{tt}^2 + \theta_{tx}^2 + \theta_{ttt}^2 + \theta_{ttx}^2\}dx. \qquad (4.20)
\end{aligned}
$$

A multiplication of (4.20) by $\epsilon_3 > 0$, and then a combination with (4.15) yields

$$
\frac{d}{dt}\int_0^L \{\frac{1}{\alpha_*}\tilde{u}_{tx}\tilde{u}_x + \frac{3k}{\alpha_*^2 m\theta_0}\hat{\theta}_x\tilde{u}_{tt} + \frac{3k}{\alpha_*^2\theta_0}\hat{\theta}_x\tilde{\theta}_x + \epsilon_3\tilde{u}_{tt}\Phi\tilde{u}_{tx} + \epsilon_3\frac{k\tau_q^2}{2\theta_0}\theta_{ttx}\Phi\hat{\theta}_{xx} + \epsilon_3\frac{k\tau_q^2}{2\theta_0\tau_\theta}\theta_{tx}\Phi\hat{\theta}_{xx}\}dx
$$

$$\le -\frac{1}{2\alpha_*}\int_0^L \tilde{u}_{tx}^2 dx - \frac{1}{2}\int_0^L \tilde{u}_{xx}^2 dx + \frac{C}{\epsilon_2^3}\int_0^L \{\hat{\theta}_t^2 + \hat{\theta}_{tx}^2 + \tilde{\theta_x}^2\} dx$$
$$-[\frac{\alpha_*\epsilon_3}{4} - C_1\epsilon_2](\tilde{u}_{tx}^2(0) + \tilde{u}_{tx}^2(L))$$
$$+C_2\epsilon_3\int_0^L \{\tilde{u}_{tx}^2 + \tilde{u}_{xx}^2\} dx + C_2\varepsilon_3\int_0^L \{\theta_t^2 + \theta_{tt}^2 + \theta_{tx}^2 + \theta_{ttt}^2 + \theta_{ttx}^2 + \tilde{\theta}_x^2\} dx$$

where C_1, C_2 are positive constants. Now choosing

$$\epsilon_3 := \min\{\frac{1}{4\alpha_* C_2}, \frac{1}{2C_2}\}$$

and then

$$\epsilon_2 := \frac{\alpha_*\epsilon_3}{4C_1}$$

we obtain

$$\frac{d}{dt}\int_0^L \{\frac{1}{\alpha_*}\tilde{u}_{tx}\tilde{u}_x + \frac{3k}{\alpha_*^2 m\theta_0}\hat{\theta}_x\tilde{u}_{tt} + \frac{3k}{\alpha_*^2\theta_0}\hat{\theta}_x\tilde{\theta}_x + \epsilon_3\tilde{u}_{tt}\Phi\tilde{u}_{tx} + \epsilon_3\frac{k\tau_q^2}{2\theta_0}\theta_{ttx}\Phi\hat{\theta}_{xx} + \epsilon_3\frac{k\tau_q^2}{2\theta_0\tau_\theta}\theta_{tx}\Phi\hat{\theta}_{xx}\} dx$$

$$\begin{aligned} \le\ & -\frac{1}{4\alpha_*}\int_0^L \tilde{u}_{tx}^2 dx - \frac{1}{4}\int_0^L \tilde{u}_{xx}^2 dx \\ & + C\int_0^L \{\theta_t^2 + \theta_{tt}^2 + \theta_{tx}^2 + \theta_{ttt}^2 + \theta_{ttx}^2 + \tilde{\theta}_t^2 + \tilde{\theta}_x^2\} dx. \end{aligned} \tag{4.21}$$

We observe that by Poincaré's estimates and (4.1) we have

$$\int_0^L \{\tilde{u}_t^2 + \tilde{u}_x^2\} dx \le C\int_0^L \{\tilde{u}_{tx}^2 + \tilde{u}_{xx}^2\} dx, \quad \int_0^L \tilde{u}_{tt}^2 dx \le C\int_0^L \{\tilde{u}_{xx}^2 + \tilde{\theta}_x^2\} dx. \tag{4.22}$$

Now let $E(t)$ and $E_{\mathcal{H}}(t)$ be given as defined in (4.4) and (4.5), respectively, and define for $K > 0$ (yet to be determined)

$$W_1(t) \equiv E_1(u,\theta;t) := E_{\mathcal{H}}(t) + KE(t), \qquad W_2(t) := W_1(u_t,\theta_t;t)$$

and the final energy term

$$\mathcal{W}(t) := W_1(t) + W_2(t)$$

where we now choose ϵ_1 small enough such that the terms $2\epsilon_1\int_0^L\{\tilde{u}_t^2 + \tilde{u}_{tt}^2\}dx$ are absorbed by arising corresponding negative terms (cp. (4.21), (4.22)). Then we choose K large enough to make sure that the coefficients in $[\cdot]$-brackets in (4.11) are positive. Defining for $\epsilon > 0$ the Lyapunov functional L by

$$\begin{aligned} L(t) \ :=\ & \frac{1}{\epsilon}\mathcal{W}(t) + \int_0^L \{\frac{1}{\alpha_*}\tilde{u}_{tx}\tilde{u}_x + \frac{3k}{\alpha_*^2 m\theta_0}\hat{\theta}_x\tilde{u}_{tt} + \frac{3k}{\alpha_*^2\theta_0}\hat{\theta}_x\tilde{\theta}_x + \epsilon_3\tilde{u}_{tt}\Phi\tilde{u}_{tx} \\ & + \epsilon_3\frac{k\tau_q^2}{2\theta_0}\theta_{ttx}\Phi\hat{\theta}_{xx} + \epsilon_3\frac{k\tau_q^2}{2\theta_0\tau_\theta}\theta_{tx}\Phi\hat{\theta}_{xx}\} dx \end{aligned}$$

we now find from (4.11), (4.21), (4.22), observing

$$\frac{k}{8m\theta_0\alpha_*}\int_0^L \hat{\theta}_{xx}^2 dx \leq \frac{1}{8\alpha_*}\int_0^L \tilde{u}_{tx}^2 dx + \frac{1}{8m\theta_0\alpha_*}\int_0^L \tilde{\theta}_t^2 dx$$

and choosing ϵ small enough that

$$\frac{d}{dt}L(t) \leq -C_3\mathcal{W}(t) \tag{4.23}$$

for some constant $C_3 > 0$. Moreover, we have for ϵ small enough

$$\exists\, K_1, K_2 > 0\ \forall\, t \geq 0: \quad K_1\mathcal{W}(t) \leq L(t) \leq K_2\mathcal{W}(t). \tag{4.24}$$

Combining (4.23) and (4.24) we have thus proved the exponential stability

Theorem 4.1 *The system (4.1)–(4.3) is exponentially stable,*

$$\exists\, d_1, d_2 > 0 \forall\, t \geq 0: \qquad \mathcal{W}(t) \leq d_1 \mathrm{e}^{-d_2 t}\mathcal{W}(0)$$

The Dirichlet-Neumann type boundary conditions

$$u_x = \theta = 0 \qquad \text{for } x = 0, L$$

or

$$u = \theta_x = 0 \qquad \text{for } x = 0, L$$

could be treated similarly. It is even likely that one can work just with the first energy $W_1(t)$ (instead of $W_1(t) + W_2(t)$). Moreover, the radially symmetric case in two or three space dimensions should be accessible.

The exponential stability result is first a result for θ and $\tilde{u}$. But we obtain an exponential decay result also for u itself observing that for functions $w, h : [0,\infty) \times (0, L) \to \mathbb{R}$ satisfying

$$\ddot{w} + \frac{2}{\tau_q}\dot{w} + \frac{2}{\tau_q^2}w = h \quad (:= \tilde{u}(t,x))$$

and

$$\exists\, d_1, d_2 > 0\ \ \forall\, t \geq 0: \quad \int_0^L |h(x,t)|^2 dx \leq d_1 \mathrm{e}^{-2d_2 t} C_0^2$$

where C_0 depends on the initial data according to Theorem 4.1, we conclude for $z := (w, \dot{w})'$,

$$\exists\, d_3, d_4 > 0\ \ \forall\, t \geq 0: \quad \int_0^L |z(x,t)|^2 dx \leq d_3 \mathrm{e}^{-2d_4 t}(|z(0)|^2 + C_0^2).$$

Here d_4 can be any positive number smaller than $\min\{1/\tau_q, d_2\}$ which becomes apparent observing that the characteristic values for the ODE for w are

$$\beta_{1,2} = -\frac{1}{\tau_q} \pm \mathrm{i}\frac{1}{\tau_q}.$$

5 First spatial estimates

In this Section we establish results on the spatial evolution of solutions of (2.1)-(2.6), provided that the initial data of (2.4) is assumed to be bounded in a certain energy norm.

We begin by considering

$$F(z,t) = -\int_0^t \int_{D(z)} (\tilde{T}_{i3}\dot{\tilde{u}}_i + \frac{1}{\theta_0}k\hat{\theta}_{,3}\tilde{\theta})dAds. \tag{5.1}$$

From (5.1) we find that

$$\frac{\partial F(z,t)}{\partial t} = -\int_{D(z)} (\tilde{T}_{i3}\dot{\tilde{u}}_i + \frac{1}{\theta_0}k\hat{\theta}_{,3}\tilde{\theta})dA, \tag{5.2}$$

and, on using (2.1), the divergence theorem on $D(z)$ and (2.4), (2.5), we obtain

$$\frac{\partial F(z,t)}{\partial z} = -\frac{1}{2}\int_{D(z)} \Big(\rho\dot{\tilde{u}}_i\dot{\tilde{u}}_i + \mu\tilde{u}_{i,j}\tilde{u}_{i,j} + (\lambda+\mu)\tilde{u}_{r,r}\tilde{u}_{s,s} + \frac{c}{\theta_0}(\tilde{\theta})^2 \tag{5.3}$$

$$+\frac{k}{\theta_0}\Big((\tau_q+\tau_\theta)|\nabla\theta|^2 + \frac{1}{2}\tau_q^2\tau_\theta|\nabla\dot{\theta}|^2 + \tau_q^2\nabla\theta\nabla\dot{\theta}\Big)\Big)dA$$

$$-\int_0^t\int_{D(z)}\frac{k}{\theta_0}\Big(|\nabla\theta|^2 + (\tau_\theta\tau_q - \frac{1}{2}\tau_q^2)|\nabla\dot{\theta}|^2\Big)dAds + E_1(z)$$

where

$$E_1(z) = \frac{1}{2}\int_{D(z)} \Big(\rho\tilde{v}_i^0\tilde{v}_i^0 + \mu\tilde{u}_{i,j}^0\tilde{u}_{i,j}^0 + (\lambda+\mu)\tilde{u}_{r,r}^0\tilde{u}_{s,s}^0\Big)dA+ \tag{5.4}$$

$$+\frac{1}{2\theta_0}\int_{D(z)}\Big(c(\tilde{\theta}^0)^2 + k\Big((\tau_q+\tau_\theta)|\nabla\theta^0|^2 + \frac{1}{2}\tau_q^2\tau_\theta|\nabla\vartheta^0|^2 + \tau_q^2\nabla\theta^0\nabla\vartheta^0\Big)\Big)dA.$$

Note that $E_1(z)$ depends only on the initial data (2.4) and we note that several time derivative at the time zero can be obtained assuming the continuity of the solutions at time $t = 0$. Re-writing (5.3) with z replaced by the variable η, and integrating with respect to η from 0 to z, we get

$$F(z,t) - F(0,t) = -\frac{1}{2}\int_0^z\int_{D(\eta)} \Big(\rho\dot{\tilde{u}}_i\dot{\tilde{u}}_i + \mu\tilde{u}_{i,j}\tilde{u}_{i,j} + (\lambda+\mu)\tilde{u}_{r,r}\tilde{u}_{s,s}\Big)dV \tag{5.5}$$

$$-\frac{1}{2\theta_0}\int_0^z\int_{D(\eta)}\Big(c(\tilde{\theta})^2 + k\Big((\tau_q+\tau_\theta)|\nabla\theta|^2 + \frac{k}{2}\tau_q^2\tau_\theta|\nabla\dot{\theta}|^2 + k\tau_q^2\nabla\theta\nabla\dot{\theta}\Big)\Big)dV$$

$$-\frac{k}{\theta_0}\int_0^t\int_0^z\int_{D(\eta)}\Big(|\nabla\theta|^2 + (\tau_\theta\tau_q - \frac{1}{2}\tau_q^2)|\nabla\dot{\theta}|^2\Big)dVds$$

$$+\frac{1}{2}\int_0^z\int_{D(\eta)}\Big(\rho\tilde{v}_i^0\tilde{v}_i^0 + \mu\tilde{u}_{i,j}^0\tilde{u}_{i,j}^0 + (\lambda+\mu)\tilde{u}_{r,r}^0\tilde{u}_{s,s}^0\Big)dV+$$

$$+\frac{1}{2\theta_0}\int_0^z\int_{D(\eta)}\Big(c(\tilde{\theta}^0)^2+k\big((\tau_q+\tau_\theta)|\nabla\theta^0|^2+\frac{1}{2}\tau_q^2\tau_\theta|\nabla\vartheta^0|^2+\tau_q^2\nabla\theta^0\nabla\vartheta^0\big)\Big)dV.$$

Our next step is to establish an inequality between the time and spatial derivatives of $F(z,t)$. By virtue of (1.2), the second integral on the right in (5.3) is non-negative. The last three terms in the integrand in the first integral on the right in (5.3) are a quadratic form and may be bounded below on using the smallest positive eigenvalue λ_0 of (2.7), (2.10) given in (2.11). Thus we find that

$$\begin{aligned}\frac{\partial F(z,t)}{\partial z} \quad &\leq \quad -\frac{1}{2}\int_{D(z)}\Big(\rho\dot{\tilde{u}}_i\dot{\tilde{u}}_i+\mu\tilde{u}_{i,j}\tilde{u}_{i,j}+(\lambda+\mu)\tilde{u}_{r,r}\tilde{u}_{s,s}\\ &+\frac{c}{\theta_0}(\tilde{\theta})^2+\frac{k\lambda_0}{\theta_0}(|\nabla\theta|^2+|\nabla\dot{\theta}|^2)\Big)dA+E_1(z). \qquad (5.6)\end{aligned}$$

Applying Schwarz's inequality in (5.2) and using (2.3), we get

$$|\frac{\partial F}{\partial t}|\leq\frac{1}{2}\int_{D(z)}[\frac{\epsilon_1}{\rho}\tilde{T}_{ij}\tilde{T}_{ij}+\frac{\rho}{\epsilon_1}\dot{\tilde{u}}_i\dot{\tilde{u}}_i+\frac{c}{\epsilon_2\theta_0}(\tilde{\theta})^2+\frac{\epsilon_2k^2}{c\theta_0}\hat{\theta}_{,3}\hat{\theta}_{,3}]dA \qquad (5.7)$$

$$\leq\frac{1}{2}\int_{D(z)}[\frac{\epsilon_1}{\rho}(1+\epsilon)\mu^*[\mu\tilde{u}_{i,j}\tilde{u}_{i,j}+(\lambda+\mu)\tilde{u}_{r,r}\tilde{u}_{s,s}]+\frac{1}{\epsilon_1}(\rho\dot{\tilde{u}}_i\dot{\tilde{u}}_i)$$

$$+(\frac{1}{\epsilon_2}+\frac{3m^2\epsilon_1\theta_0}{\rho c}(1+\epsilon^{-1}))\frac{c}{\theta_0}(\tilde{\theta})^2+\frac{\epsilon_2k(1+\tau_\theta^2)}{c\lambda_0}[\frac{k\lambda_0}{\theta_0}(|\nabla\theta|^2+|\nabla\dot{\theta}|^2)]dA$$

where the weighted arithmetic-geometric mean inequality has been employed and where ϵ_i are arbitrary positive constants.

Now, we equate the coefficients of the energetic terms in the last integral of the (5.7). We get

$$\frac{1}{\epsilon_1}=\frac{\epsilon_1}{\rho}(1+\epsilon)\mu^*=\frac{1}{\epsilon_2}+\frac{3m^2\epsilon_1\theta_0}{\rho c}(1+\epsilon^{-1})=\frac{\epsilon_2k(1+\tau_\theta^2)}{c\lambda_0}. \qquad (5.8)$$

That is

$$\epsilon_1=\beta^{-1},\epsilon_2=\frac{c\lambda_0\beta}{k(1+\tau_\theta^2)},\ \beta=\sqrt{\frac{(1+\epsilon_0)\mu^*}{\rho}}, \qquad (5.9)$$

where ϵ_0 is the positive root of the second order equation

$$x^2+\left(1-\frac{\rho k(1+\tau_0^2)}{\mu^*\lambda_0c}-\frac{3m^2\theta_0}{\mu^*c}\right)-\frac{3m^2\theta_0}{\mu^*c}=0. \qquad (5.10)$$

In view of (5.6) we can write (5.7) as

$$|\frac{\partial F}{\partial t}|+\beta\frac{\partial F}{\partial z}\leq\beta E_1(z), \qquad (5.11)$$

where β is defined at (5.9).

The inequality (5.11) implies that

$$\frac{\partial F}{\partial t}+\beta\frac{\partial F}{\partial z}\leq\beta E_1(z) \tag{5.12}$$

and

$$\frac{\partial F}{\partial t}-\beta\frac{\partial F}{\partial z}\geq-\beta E_1(z). \tag{5.13}$$

Integrating (5.12) and recalling the definition of $E_1(z)$ in (5.4) we obtain

$$F(z,\beta^{-1}(z-z^*))\leq\frac{1}{2}\int_{z^*}^{z}\int_{D(\eta)}\Big(\rho\tilde{v}_i^0\tilde{v}_i^0+\mu\tilde{u}_{i,j}^0\tilde{u}_{i,j}^0+(\lambda+\mu)\tilde{u}_{r,r}^0\tilde{u}_{s,s}^0\Big)dV, \tag{5.14}$$

$$+\frac{1}{2\theta_0}\int_{z^*}^{z}\int_{D(\eta)}\Big(c(\tilde{\theta}^0)^2+k\big((\tau_q+\tau_\theta)|\nabla\theta^0|^2+\frac{1}{2}\tau_q^2\tau_\theta|\nabla\vartheta^0|^2+\tau_q^2\nabla\theta^0\nabla\vartheta^0\big)\Big)dV$$

where $z\geq z^*$. Similarly on integrating (5.1) we obtain

$$F(z,\beta^{-1}(z^{**}-z))\geq-\frac{1}{2}\int_{z}^{z^{**}}\int_{D(\eta)}\Big(\rho\tilde{v}_i^0\tilde{v}_i^0+\mu\tilde{u}_{i,j}^0\tilde{u}_{i,j}^0+(\lambda+\mu)\tilde{u}_{r,r}^0\tilde{u}_{s,s}^0\Big)dV, \tag{5.15}$$

$$-\frac{1}{2\theta_0}\int_{z}^{z^{**}}\int_{D(\eta)}\Big(c(\tilde{\theta}^0)^2+k\big((\tau_q+\tau_\theta)|\nabla\theta^0|^2+\frac{1}{2}\tau_q^2\tau_\theta|\nabla\vartheta^0|^2+\tau_q^2\nabla\theta^0\nabla\vartheta^0\big)\Big)dV$$

where $z^{**}\geq z$. Let

$$\mathcal{E}(z,t):=\frac{1}{2}\int_{R(z)}\Big(\rho\dot{\tilde{u}}_i\dot{\tilde{u}}_i+\mu\tilde{u}_{i,j}\tilde{u}_{i,j}+(\lambda+\mu)\tilde{u}_{r,r}\tilde{u}_{s,s}\Big)dV \tag{5.16}$$

$$+\frac{1}{2\theta_0}\int_{R(z)}\Big(c(\tilde{\theta})^2+k\Big((\tau_q+\tau_\theta)|\nabla\theta|^2+\frac{k}{2}\tau_q^2\tau_\theta|\nabla\dot{\theta}|^2+k\tau_q^2\nabla\theta\nabla\dot{\theta}\Big)\Big)dV$$

$$+\frac{k}{\theta_0}\int_0^t\int_{R(z)}\Big(|\nabla\theta|^2+(\tau_\theta\tau_q-\frac{1}{2}\tau_q^2)|\nabla\dot{\theta}|^2\Big)dV\,ds.$$

If we now assume that the initial data (2.4) is such that

$$\mathcal{E}(0,0)=\frac{1}{2}\int_{R}\Big(\rho\tilde{v}_i^0\tilde{v}_i^0+\mu\tilde{u}_{i,j}^0\tilde{u}_{i,j}^0+(\lambda+\mu)\tilde{u}_{r,r}^0\tilde{u}_{s,s}^0\Big)dV \tag{5.17}$$

$$+\frac{1}{2\theta_0}\int_{R}\Big(c(\tilde{\theta}^0)^2+k\Big((\tau_q+\tau_\theta)|\nabla\theta^0|^2+\frac{1}{2}\tau_q^2\tau_\theta|\nabla\vartheta^0|^2+\tau_q^2\nabla\theta^0\nabla\vartheta^0\Big)\Big)dV<\infty.$$

Then the inequalities (5.15), (5.16) imply that, for each finite time t,

$$\lim_{z\to\infty}F(z,t)=0. \tag{5.18}$$

Thus, we may rewrite

$$F(z,t)=\frac{1}{2}\int_{R(z)}\Big(\rho\dot{\tilde{u}}_i\dot{\tilde{u}}_i+\mu\tilde{u}_{i,j}\tilde{u}_{i,j}+(\lambda+\mu)\tilde{u}_{r,r}\tilde{u}_{s,s}\Big)dV \tag{5.19}$$

$$+\frac{1}{2\theta_0}\int_{R(z)}\Big(c(\tilde{\theta})^2+k\Big((\tau_q+\tau_\theta)|\nabla\theta|^2+\frac{k}{2}\tau_q^2\tau_\theta|\nabla\dot{\theta}|^2+k\tau_q^2\nabla\theta\nabla\dot{\theta}\Big)\Big)dV$$

$$+\frac{k}{\theta_0}\int_0^t\int_{R(z)}\Big(|\nabla\theta|^2+(\tau_\theta\tau_q-\frac{1}{2}\tau_q^2)|\nabla\dot{\theta}|^2\Big)dVds$$

$$-\frac{1}{2}\int_{R(z)}\Big(\rho\tilde{v}_i^0\tilde{v}_i^0+\mu\tilde{u}_{i,j}^0\tilde{u}_{i,j}^0+(\lambda+\mu)\tilde{u}_{r,r}^0\tilde{u}_{s,s}^0\Big)dV.$$

$$-\frac{1}{2\theta_0}\int_{R(z)}\Big(c(\tilde{\theta}^0)^2+k\Big((\tau_q+\tau_\theta)|\nabla\theta^0|^2+\frac{1}{2}\tau_q^2\tau_\theta|\nabla\vartheta^0|^2+\tau_q^2\nabla\theta^0\nabla\vartheta^0\Big)\Big)dV.$$

Now the inequality (5.12) implies that

$$\mathcal{E}(z,t)\le\mathcal{E}(z^*,0) \tag{5.20}$$

where z, z^* and t are related by $t=\beta^{-1}(z-z^*)$. In a similar way, we get

$$\mathcal{E}(z,t)\ge\mathcal{E}(z^{**},0) \tag{5.21}$$

for $t=\beta^{-1}(z^{**}-z)$. From the inequalities (5.19) and (5.21), we conclude that

$$\mathcal{E}(z,t)\le\mathcal{E}(z^*,t^*) \tag{5.22}$$

for $|t-t^*|\le\beta^{-1}(z-z^*)$. Thus, we have proved:

Theorem 5.1 *Let* $(\mathbf{u},\theta)$ *be a solution of the initial-boundary-value problem (2.1)-(2.6). Then the energy function* $\mathcal{E}(z,t)$ *defined in (5.20) satisfies the inequality (5.22) whenever* $|t-t^*|\le\beta^{-1}(z-z^*)$*, provided that the initial data satisfy (5.16).*

We note that this result gives an answer to the question proposed by Hetnarski and Ignaczak ([10], p.474) a principle of Saint-Venant's type in this theory.

If one defines the measure

$$\mathcal{E}^*(z,t)=\int_0^t\mathcal{E}(z,s)ds, \tag{5.23}$$

the following inequalities can be obtained as in [2]:

$$\mathcal{E}^*(z,t)\le\beta^{-1}\int_{z-\beta t}^{z}\mathcal{E}(\eta,0)d\eta,\ \ \beta t\le z, \tag{5.24}$$

$$\mathcal{E}^*(z,t)\le\beta^{-1}\int_0^z\mathcal{E}(\eta,0)d\eta+\Big(1-\frac{z}{\beta t}\Big)\mathcal{E}^*(0,t),\ \ \beta t\ge z. \tag{5.25}$$

6 Consequences of the estimates (5.22), (5.24), (5.25)

In this section we show some consequences of the estimates (5.22), (5.24) and (5.25).

First, we assume that the initial conditions (2.4) are homogeneous. In this case we see that $\mathcal{E}(0,0) = 0$. Estimate (5.22) implies that $\mathcal{E}(z,t) = 0$ whenever $\beta t \leq z$. In view of the definition (5.20) we obtain that

$$\tilde{u}_i = 0, \;\; \theta = 0 \tag{6.1}$$

whenever $\beta t \leq z$. Then for every $\mathbf{x} = (x_1, x_2, z)$ such that $\beta t \leq z$, the functions $u_i(\mathbf{x}, t)$ satisfy the ordinary differential equation $\tilde{u}_i = 0$ with null initial conditions. Thus, we also conclude that

$$u_i = 0, \tag{6.2}$$

when $\beta t \leq z$. This is a result of the kind of the domain of dependence of the solutions. We have proved:

Theorem 6.1 *Let* $(\mathbf{u}, \theta)$ *be a solution of the initial-boundary-value problem (2.1)-(2.6) when the initial conditions are null. Then* $(\mathbf{u}, \theta) = (\mathbf{0}, 0)$ *whenever* $\beta t \leq z$.

We note that this result gives an answer to the question proposed by Hetnarski and Ignaczak ([10], p.474) concerning a general domain of influence theorem in this theory.

In this situation it is natural to look for estimates for

$$\mathcal{H}(z,t) := \int_z^\infty \mathcal{E}(\xi, t) d\xi, \tag{6.3}$$

where $z \leq \beta t$. We have that

$$\mathcal{H}(z,t) = \int_z^{\beta t} \mathcal{E}(\xi, t) d\xi. \tag{6.4}$$

But

$$\mathcal{E}(z,t) \leq \mathcal{E}(0, z^*), \tag{6.5}$$

when $z^* \geq t - \beta^{-1} z \geq 0$. Thus, it follows that

$$\mathcal{H}(z,t) \leq \frac{\beta t - z}{t} \int_0^t \mathcal{E}(0,s) ds = \frac{\beta t - z}{t} \mathcal{E}^*(0,t). \tag{6.6}$$

The second natural question we are interested with is to obtain spatial estimates for some norm of the solutions. We had obtained the estimates (5.22), (5.24) and (5.25), but they are expressed in a combination of the solution and its time derivatives. Now, we give explicit spatial estimates. From (5.20), (5.22), (5.24) and (5.25) we have

$$\mathcal{J}(z,t) = \frac{1}{2\theta_0} \int_{R(z)} \Big(k \Big((\tau_q + \tau_\theta) |\nabla \theta|^2 + \frac{1}{2} \tau_q^2 \tau_\theta |\nabla \dot{\theta}|^2 + \tau_q^2 \nabla \theta \nabla \dot{\theta} \Big) \Big) dV \tag{6.7}$$

$$+\frac{k}{\theta_0}\int_0^t\int_{R(z)}\Big(|\nabla\theta|^2+(\tau_\theta\tau_q-\frac{1}{2}\tau_q^2)|\nabla\dot\theta|^2\Big)dV\,ds\le\mathcal{E}(z,t),$$

for $|t-t^*|\le\beta^{-1}(z-z^*)$. Also, we obtain that

$$\mathcal{J}^*(z,t)\le\beta^{-1}\int_{z-\beta t}^{z}\mathcal{E}(\eta,0)d\eta,\;\;\beta t\le z,\tag{6.8}$$

and

$$\mathcal{J}^*(z,t)\le\beta^{-1}\int_0^{z}\mathcal{E}(\eta,0)d\eta+(1-\frac{z}{\beta t})\mathcal{E}^*(0,t),\;\;\beta t\ge z\tag{6.9}$$

where

$$\mathcal{J}^*(z,t)=\int_0^t\mathcal{J}(z,s)ds.\tag{6.10}$$

Now, we will obtain estimates for the mechanical part. To this end, we note that

$$\begin{aligned}\int_0^t(\tilde f)^2ds &= \int_0^t(f^2+\frac{\tau_q^4}{4}(\ddot f)^2)ds+(\tau_q(f^2(t)+\tau_q f(t)\dot f(t)+\frac{\tau_q^2}{2}\dot f^2(t))\\ &\quad -(\tau_q(f^2(0)+\tau_q f(0)\dot f(0)+\frac{\tau_q^2}{2}\dot f^2(0)).\end{aligned}\tag{6.11}$$

It is worth noting that the expression

$$\tau_q(f^2(t)+\tau_q f(t)\dot f(t)+\frac{\tau_q^2}{2}\dot f^2(t))$$

is positive in the sense that it is equivalent to the measure defined by $f^2(t)+\dot f^2(t)$. Thus, if we define

$$\begin{aligned}\mathcal{M}^*(z,t)=\frac{1}{2}\int_0^t\int_{R(z)}&\Big(\rho(\dot u_i\dot u_i+\frac{\tau_q^4}{4}\dddot u_i\dddot u_i)+\mu(u_{i,j}u_{i,j}+\frac{\tau_q^4}{4}\ddot u_{i,j}\ddot u_{i,j})+(\lambda+\mu)(u_{r,r}u_{s,s}\\ &+(\lambda+\mu)(u_{r,r}u_{s,s}+\frac{\tau_q^4}{4}\ddot u_{r,r}\ddot u_{s,s})\Big)dV\,ds\\ +\frac{\tau_q}{2}\int_{R(z)}&\Big(\rho(\dot u_i\dot u_i+\tau_q\dot u_i\ddot u_i+\frac{\tau_q^2}{2}\ddot u_i\ddot u_i)+\mu(u_{i,j}u_{i,j}+\tau_q u_{i,j}\dot u_{i,j}+\frac{\tau_q^2}{2}\dot u_{i,j}\dot u_{i,j})\\ &+(\lambda+\mu)(u_{r,r}u_{s,s}+\tau_q u_{r,r}\dot u_{s,s}+\frac{\tau_q^2}{2}\dot u_{r,r}\dot u_{s,s})\Big)dV,\end{aligned}\tag{6.12}$$

we obtain the estimates

$$\mathcal{M}^*(z,t)\le\beta^{-1}\int_{z-\beta t}^{z}\mathcal{E}(\eta,0)d\eta+\mathcal{P}(z),\;\;\beta t\le z\tag{6.13}$$

and

$$\mathcal{M}^*(z,t)\le\beta^{-1}\int_0^{z}\mathcal{E}(\eta,0)d\eta+(1-\frac{z}{\beta t})\mathcal{E}^*(0,t)+\mathcal{P}(z),\;\;\beta t\ge z,\tag{6.14}$$

where

$$\mathcal{P}(z) = \frac{\tau_q}{2}\int_{R(z)} \Big(\rho(v_i^0v_i^0 + \tau_q v_i^0 z_i^0 + \frac{\tau_q^2}{2} z_i^0 z_i^0) + \mu(u_{i,j}^0 u_{i,j}^0 + \tau_q u_{i,j}^0 v_{i,j}^0 + \frac{\tau_q^2}{2} v_{i,j}^0 v_{i,j}^0) \quad (6.15)$$

$$+(\lambda+\mu)(u_{r,r}^0 u_{s,s}^0 + \tau_q u_{r,r}^0 v_{s,s}^0 + \frac{\tau_q^2}{2} v_{r,r}^0 v_{s,s}^0)\Big)dV$$

and

$$z_i^0 = (\mu u_{i,j}^0 + (\lambda+\mu)u_{r,r}^0\delta_{ij} + m\delta_{ij}\theta)_{,j}. \quad (6.16)$$

It is worth noting that it is also possible to obtain estimates in the L^2-norm of the temperature and its two first times derivatives in a similar way of the estimates (6.13), (6.14)

7 A non-standard problem for (2.1), (2.2)

In this Section, we briefly discuss the behavior of solutions of (2.1), (2.2) subject to the boundary condition (2.5), (2.6) and the non-standard conditions

$$u_i(\mathbf{x},T) = \alpha u_i(\mathbf{x},0),\ \dot{u}_i(\mathbf{x},T) = \alpha \dot{u}_i(\mathbf{x},0),$$

$$\theta(\mathbf{x},T) = \alpha\theta(\mathbf{x},0),\ \dot{\theta}(\mathbf{x},T) = \alpha\dot{\theta}(\mathbf{x},0),\ \ddot{\theta}(\mathbf{x},T) = \alpha\ddot{\theta}(\mathbf{x},0), \quad (7.1)$$

where $\alpha > 1$. Such non-standard conditions have been the subject of much recent attention (see, e.g. [1, 16, 33] in the context of the heat equation, [21] for generalized heat conduction and [22] for viscous flows, [19] for the isothermal elasticity and [30], [27] for some thermoelastic theories).

The boundary data in (2.6) is assumed compatible with (6.1),(6.2).

The analysis begins by considering the function

$$F_\gamma(z) = -\int_0^T\int_{D(z)} \exp(-\gamma s)(\tilde{T}_{i3}\dot{\tilde{u}}_i + \frac{1}{\theta_0}k\hat{\theta}_{,3}\tilde{\theta})dAds, \quad (7.2)$$

where, guided by results established in [1, 16, 33], the positive constant γ is given by

$$\gamma = \frac{2}{T}\ln\alpha. \quad (7.3)$$

We have

$$F_\gamma(z) = F_\gamma(0) + \frac{\gamma}{2}\int_0^T\int_0^z\int_{D(\eta)} \exp(-\gamma s)\Big(\rho\dot{\tilde{u}}_i\dot{\tilde{u}}_i + \mu\tilde{u}_{i,j}\tilde{u}_{i,j} + (\lambda+\mu)\tilde{u}_{r,r}\tilde{u}_{s,s} \quad (7.4)$$

$$+\frac{c}{\theta_0}(\tilde{\theta})^2 + \frac{k}{\theta_0}\Big((\tau_q+\tau_\theta)|\nabla\theta|^2 + \frac{1}{2}\tau_q^2\tau_\theta|\nabla\dot{\theta}|^2 + \tau_q^2\nabla\theta\nabla\dot{\theta}\Big)\Big)dA$$

$$+\gamma\frac{k}{\theta_0}\int_0^T\int_0^z\int_{D(\eta)}\exp(-\gamma s)\Big(\frac{1}{\gamma}|\nabla\theta|^2+\frac{1}{\gamma}(\tau_\theta\tau_q-\frac{1}{2}\tau_q^2)|\nabla\dot{\theta}|^2\Big)dV\,ds.$$

An argument similar to the one used in the case of the standard initial conditions leads to the estimate

$$|F_\gamma|\leq\gamma\beta_\gamma\frac{\partial F_\gamma}{\partial z},\tag{7.5}$$

where β_γ is defined in the same form of β defined in (5.9), but changing the parameter λ_0 by μ_γ defined at (2.13).

This inequality is well-known in the study of spatial decay estimates. It implies that

$$F_\gamma\leq\gamma\beta_\gamma\frac{\partial F_\gamma}{\partial z},\ \text{and}\ -F_\gamma\leq\gamma\beta_\gamma\frac{\partial F_\gamma}{\partial z}.\tag{7.6}$$

From (7.6), we can obtain an alternative of Phragmen-Lindelöf type which states (see [5]) that the solutions either grow exponentially for z sufficiently large or solutions decay exponentially in the form

$$\mathcal{E}_\gamma(z)\leq\mathcal{E}_\gamma(0)\exp\Big(-\gamma^{-1}\beta_\gamma^{-1}z\Big)\tag{7.7}$$

for all $z\geq 0$, where

$$\mathcal{E}_\gamma(z)=\frac{\gamma}{2}\int_0^T\int_0^z\int_{D(\eta)}\exp(-\gamma s)\Big(\rho\dot{\tilde{u}}_i\dot{\tilde{u}}_i+\mu\tilde{u}_{i,j}\tilde{u}_{i,j}+(\lambda+\mu)\tilde{u}_{r,r}\tilde{u}_{s,s}\tag{7.8}$$
$$+\frac{c}{\theta_0}(\tilde{\theta})^2+\frac{k}{\theta_0}\Big((\tau_q+\tau_\theta)|\nabla\theta|^2+\frac{1}{2}\tau_q^2\tau_\theta|\nabla\dot{\theta}|^2+\tau_q^2\nabla\theta\nabla\dot{\theta}\Big)\Big)dV\,ds$$
$$+\frac{k}{\theta_0}\int_0^T\int_0^z\int_{D(\eta)}\exp(-\gamma s)\Big(|\nabla\theta|^2+(\tau_\theta\tau_q-\frac{1}{2}\tau_q^2)|\nabla\dot{\theta}|^2\Big)dV\,ds.$$

The decay rate in (7.7) depends explicitly on γ given in (7.3). Thus, we have proved

Theorem 7.1 *Let $(\mathbf{u},\theta)$ be a solution of the initial-boundary-value problem (2.1), (2.2) (2.5), (2.6) and (7.1). Then either the solutions grow exponentially or the estimate (7.7) is satisfied, where $\mathcal{E}_\gamma$ is defined at (7.8).*

Estimate (7.7) implies that

$$\mathcal{J}_\gamma(z)\leq\mathcal{E}_\gamma(0)\exp\Big(-\gamma^{-1}\beta_\gamma^{-1}z\Big)\tag{7.9}$$

where

$$\mathcal{J}_\gamma(z)=\frac{\gamma}{2\theta_0}\int_0^T\int_{R(z)}\exp(-\gamma s)\Big(k\Big((\tau_q+\tau_\theta)|\nabla\theta|^2+\frac{1}{2}\tau_q^2\tau_\theta|\nabla\dot{\theta}|^2+\tau_q^2\nabla\theta\nabla\dot{\theta}\Big)\Big)dV\,ds\tag{7.10}$$
$$+\frac{k}{\theta_0}\int_0^T\int_{R(z)}\exp(-\gamma s)\Big(|\nabla\theta|^2+(\tau_\theta\tau_q-\frac{1}{2}\tau_q^2)|\nabla\dot{\theta}|^2\Big)dV\,ds.$$

To obtain an estimate on the mechanical part, we first note that

$$\int_0^T \exp(-\gamma s)(\tilde{f})^2 ds = \int_0^T \exp(-\gamma s)((f^2 + \frac{\tau_q^4}{4}(\ddot{f})^2) + \gamma(\tau_q(f^2 + \tau_q f\dot{f} + \frac{\tau_q^2}{2}\dot{f}^2)\Big)ds \quad (7.11)$$

$$+\exp(-\gamma T)(\tau_q(f^2(t) + \tau_q f(T)\dot{f}(T) + \frac{\tau_q^2}{2}\dot{f}^2(T)) - (\tau_q(f^2(0) + \tau_q f(0)\dot{f}(0) + \frac{\tau_q^2}{2}\dot{f}^2(0)).$$

If we define

$$\mathcal{M}_\gamma(z) = \frac{1}{2}\int_0^T \int_{R(z)} \exp(-\gamma s)\Big(\rho(\dot{u}_i\dot{u}_i + \frac{\tau_q^4}{4}\dddot{u}_i\dddot{u}_i) + \mu(u_{i,j}u_{i,j} + \frac{\tau_q^4}{4}\ddot{u}_{i,j}\ddot{u}_{i,j})$$

$$+(\lambda+\mu)(u_{r,r}u_{s,s} + \frac{\tau_q^4}{4}\ddot{u}_{r,r}\ddot{u}_{s,s})\Big)dV ds$$

$$+\frac{\gamma\tau_q}{2}\int_0^T \int_{R(z)} \exp(-\gamma s)\Big(\rho(\dot{u}_i\dot{u}_i + \tau_q\dot{u}_i\ddot{u}_i + \frac{\tau_q^2}{2}\ddot{u}_i\ddot{u}_i) + \mu(u_{i,j}u_{i,j} + \tau_q u_{i,j}\dot{u}_{i,j} + \frac{\tau_q^2}{2}\dot{u}_{i,j}\dot{u}_{i,j})$$

$$+(\lambda+\mu)(u_{r,r}u_{s,s} + \tau_q u_{r,r}\dot{u}_{s,s} + \frac{\tau_q^2}{2}\dot{u}_{r,r}\dot{u}_{s,s})\Big)dV ds, \quad (7.12)$$

we obtain the estimate

$$\mathcal{M}_\gamma(z) \le \mathcal{E}_\gamma(0)\exp\Big(-\gamma^{-1}\beta_\gamma^{-1}z\Big), \quad (7.13)$$

which is a spatial decay estimate.

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