L^p-resolvent estimates and time decay for generalized thermoelastic plate equations

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Abstract: We consider the Cauchy problem for a coupled system generalizing the thermoelastic plate equations. First we prove resolvent estimates for the stationary operator and conclude the analyticity of the associated semigroup in L^p-spaces, $1 < p < \infty$, for certain values of the parameters of the system; here the Newton polygon method is used. Then we prove decay rates of the $L^q(\mathbb{R}^n)$-norms of solutions, $2 \leq q \leq \infty$, as time tends to infinity.

1 Introduction

We consider the Cauchy problem

$$u_{tt} + aSu - bS^\beta\theta = 0, \tag{1.1}$$

$$d\theta_t + gS^\alpha\theta + bS^\beta u_t = 0, \tag{1.2}$$

$$u(0,\cdot) = u_0, u_t(0,\cdot) = u_1, \theta(0,\cdot) = \theta_0 \tag{1.3}$$

for the functions $u, \theta : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$, where $S := (-\Delta)^\eta, \eta > 0$, and $\alpha, \beta \in [0,1]$ are parameters of the "α-β-system" (1.1) (1.2). The constants a, b, d, g are positive and assumed to be equal to one in the sequel w.l.o.g. For $\eta = 2$ and $\alpha = \beta = 1/2$ we have the thermoelastic plate equations ([8])

$$u_{tt} + a\Delta^2 u + b\Delta\theta = 0, \tag{1.4}$$

$$d\theta_t - g\Delta\theta - b\Delta u_t = 0$$

which has been widely discussed in particular for bounded reference configurations $\Omega \ni x$, see the work of Kim [6], Muñoz Rivera & Racke [18], Liu & Zheng [16], Avalos & Lasiecka [2], Lasiecka & Triggiani [9, 10, 11, 12] for the question of exponential stability of the associated semigroup (for various boundary conditions), and Russell [21], Liu & Renardy [13], Liu & Liu [14], Liu & Yong [15] for proving its analyticity, see also the book of

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Liu & Zheng [17] for a survey. In our paper [19] we introduced the more general α-β-system (1.1), (1.2), in a general Hilbert space $\mathcal{H}$, S self-adjoint, and also proved for $\beta = 1/2$ polynomial decay rates of L^∞-norms $\|u(t,\cdot),\ (-\Delta)^{\gamma/2}u(t,\cdot),\ \theta(t,\cdot)\|_{L^\infty(\Omega)}$ of the solutions for $\Omega = \mathbb{R}^n$ or Ω being an exterior domain, $\mathcal{H} = L^2(\Omega)$ essentially. It was demonstrated that the α-β-system may also describe viscoelastic equations of memory type with even non convolution type kernels for $(\beta = 1/2, \alpha = 0)$, and that it captures features of second-order thermoelasticity for $(\beta = 1/2, \alpha = 1/2)$.
In [19] the region D of parameters where the system has a smoothing property,

$$D = \{(\beta,\alpha)|1-2\beta < \alpha < 2\beta, \alpha < 2\beta - 1\} \tag{1.5}$$

see Figure 1, was described.
The α-β-system was independently introduced by Ammar Khodja & Benabdallah [1].

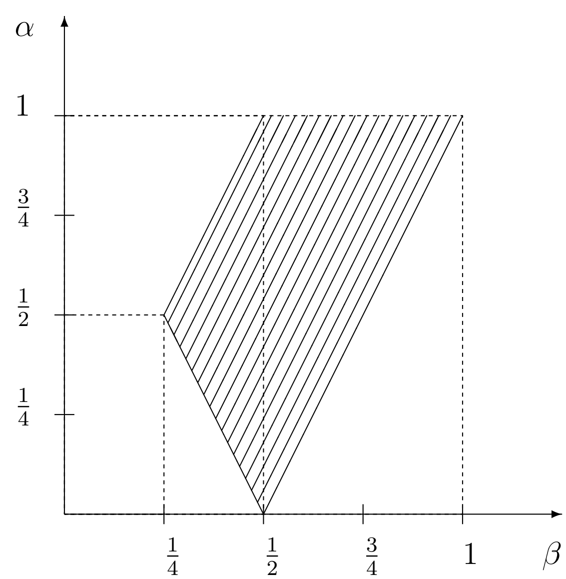

Figure 1: Area of smoothing

In particular they proved the analyticity of the associated semigroup for $\alpha = 1$ and, if and only if, $3/4 \le \beta \le 1$. Also Liu & Liu [14] and Liu & Yong [15] studied general α-β-systems in the Hilbert space case ("*bounded domains* Ω"), in particular in [15] they obtained analyticity in the region

$$\widetilde{\mathfrak{A}} := \{(\beta,\alpha)|\alpha > \beta, \alpha \le 2\beta - 1/2\}. \tag{1.6}$$

Shibata [22] obtained the analyticity in L^p-spaces, $1 < p < \infty$, for the classical thermoelastic plate, i.e. for $(\beta,\alpha) = (1/2, 1/2)$. All but the last one of the above mentioned papers work in Hilbert spaces, none can replace $L^2(\Omega)$ by $L^p(\Omega), 1 < p < \infty$ (if $(\beta,\alpha) \neq (1/2, 1/2)$), and none gives (polynomial) decay rates — if β is different from 1/2. So our goals and new contributions are

- to discuss the α-β-system in $L^p(\mathbb{R}^n)$-spaces,$1 < p < \infty$, and to describe the region $\mathfrak{A}$ of parameters (β, α) of analyticity of the semigroup, and
- to obtain sharp polynomial decay rates for $\|u(t,\cdot),\ S^{1/2}u(t,\cdot),\ \theta(t,\cdot)\|_{L^q(\Omega)}$ for $2 \leq q \leq \infty$, and (β, α) in the analyticity region $\mathfrak{A}$, but also for $1/4 \leq \beta \leq 3/4$ while $\alpha = 1/2$ (exemplarily).

We shall obtain the following region of analyticity

$$\mathfrak{A} = \{(\beta, \alpha) |\ \alpha \leq \beta,\ \alpha \geq 2\beta - 1/2\}, \tag{1.7}$$

see Figure 2 (cp.(1.6)) in proving resolvent estimates in L^p-spaces using the theory of parameter-elliptic mixed-order systems by Denk, Mennicken & Volevich [4].
The polynomial decay estimates will be obtained in applying the Fourier transform and

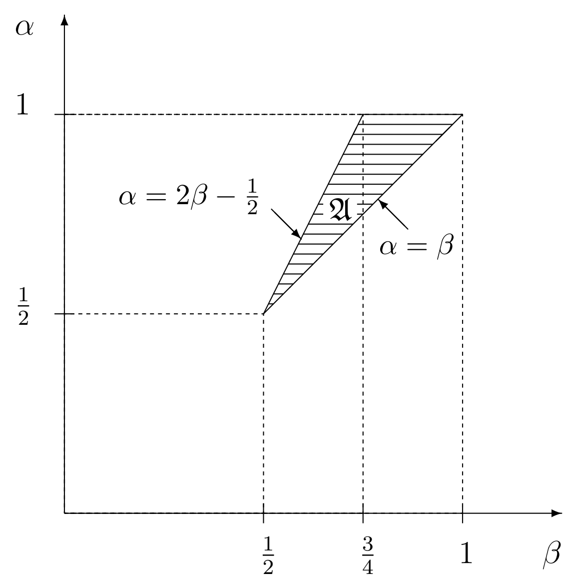

Figure 2: Area of analyticity

analysing the arising characteristic polynomial

$$P(\xi, \lambda) := \lambda^3 + \rho^\alpha \lambda^2 + (\rho^{2\beta} + \rho)\lambda + \rho^{1+\alpha} \tag{1.8}$$

carefully, where $\rho := |\xi|^{2\eta}$. In particular we describe the asymptotic expansion as $\lambda \to 0$ (and $\lambda \to \infty$). The results here will also be basic for further investigations of boundary value problems in exterior domains.

The paper is organized as follows: In Section 2 we review the relevant parts of the theory of parameter-elliptic mixed-order systems. The application to the α-β-system is given in Section 3. In Section 4 we prove the decay estimates for solutions as time tends to infinity.

2 Remarks on mixed order systems

The theory of mixed order systems usually deals with matrices of partial differential operators. As the generalized thermoelastic plate equation leads to a matrix with pseudo-differential operators with constant symbols, we will formulate the definitions and results for such matrices. It is also possible to consider general pseudo-differential operators (see, for instance, the book of Grubb [5] in this context). However, for the present case such general framework is not necessary, and we will deal only with Fourier multipliers.

In the following, the letter $\mathcal{F}$ stands for the Fourier transform in $\mathbb{R}^n$, acting in the Schwartz space of tempered distributions $\mathcal{F}\colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$. For a symbol $a(\xi)$ (belonging to some symbol class), the pseudo-differential operator $a(D)$ is defined by $a(D) := \mathcal{F}^{-1}a(\xi)\mathcal{F}$. If $a(\xi)$ is homogeneous with respect to ξ of non-negative degree μ, then the pseudo-differential operator $a(D)$ has order μ. In an obvious way, for $r > 0$ the order of symbols like $|\xi|^r$ and $1 + |\xi|^r$ are equal to r.

In the following we will consider operator matrices of the form $A(D) = (A_{ij}(D))_{i,j=1,\dots,n}$ where every entry $A_{ij}(D)$ is a Fourier multiplier of the form

$$A_{ij}(D) = |D|^{\alpha_{ij}} = \mathcal{F}^{-1}|\xi|^{\alpha_{ij}}\mathcal{F}.$$

In this case, $\alpha_{ij} = \mathrm{ord} A_{ij}(D)$. For a permutation $\pi : \{1, \dots, n\} \to \{1, \dots, n\}$ we define

$$R(\pi) := \alpha_{1\pi(1)} + \dots + \alpha_{n\pi(n)}.$$

We set $R := \max_\pi R(\pi)$. Then there exist real numbers $s_1, \dots, s_n$ and $r_1, \dots, r_n$ such that

$$\alpha_{ij} \le s_i + t_j,$$
$$\sum_{i=1}^{n}(s_i + t_i) = R.$$

For differential operators, this was shown by Volevich in [23]. The case of non-integer orders follows in exactly the same way.

Definition 2.1 *The matrix $A(D)$ and the corresponding symbol $A(\xi)$ are called elliptic in the sense of Douglis-Nirenberg (or elliptic mixed order system) if*
(i) $A(\xi)$ is non-degenerate, i.e. $R = \deg\det A(\xi)$.
(ii) $\det A(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$.

For an elliptic matrix $A(\xi)$ the principal part is defined as

$$A^0_{ij}(\xi) := \begin{cases} A_{ij}(\xi) & \text{if } \mathrm{ord} A_{ij} = s_i + t_j, \\ 0 & \text{else.} \end{cases}$$

Note that the numbers s_i and t_j are defined up to translations of the form

$$(s_1, \dots, s_n, t_1, \dots, t_n) \mapsto (s_1 - \kappa, \dots, s_n - \kappa, t_1 + \kappa, \dots, t_n + \kappa). \tag{2.1}$$

On the diagonal we have the orders

$$r_i := s_i + t_i \quad (i = 1, \dots, n). \tag{2.2}$$

Now let $A(D)$ be an elliptic mixed order system of the form indicated above. To solve the Cauchy problem $\frac{d}{dt}U = A(D)U$, one can consider the parameter-dependent symbol $\lambda - A(\xi)$ where the complex parameter λ belongs to some sector in the complex plane. In the homogeneous case, this is the standard approach to parabolic equations, see, e.g., [3]. Here the homogeneity of the determinant $\det(\lambda - A(\xi))$ as a function of λ and ξ is essential for resolvent estimates.

For mixed order systems of the form $\lambda - A(D)$, however, the definition of parabolicity and parameter-ellipticity is not obvious. In [4] a definition of this notion and several equivalent descriptions can be found. We will recall and slightly generalize some definitions and main results of [4].

We start with the definition of the Newton polygon associated to $A(\xi)$. In the case considered in the present paper, the determinant $P(\xi, \lambda) := \det(\lambda - A(\xi))$ is a polynomial in $\lambda \in \mathbb{C}$ and $|\xi|$ for $\xi \in \mathbb{R}^n$ which can be written in the form

$$P(\xi, \lambda) = \sum_{\gamma,k} a_{\gamma k} |\xi|^\gamma \lambda^k. \tag{2.3}$$

Here the exponents γ of $|\xi|$ are, in general, non-integer. The Newton polygon $N(P)$ of $P(\xi, \lambda)$ is defined as the convex hull of all points (γ, k) for which the coefficient $a_{\gamma k}$ in (2.3) does not vanish, and the projections of these points onto the coordinate axes. For instance, consider the symbol $\lambda^3 + |\xi|^3\lambda^2 + |\xi|^{9/2}\lambda$. The associated Newton polygon is the convex hull of the points $(0, 3), (3, 2), (\frac{9}{2}, 1), (\frac{9}{2}, 0)$ and $(0, 0)$. A Newton polygon is called regular if it has no edges parallel to one of the axes but not belonging to this axis.

In the following, let $\mathscr{L}$ be a closed sector in the complex plane with vertex at the origin. The constant C stands for an unspecified constant which may vary from line to line but which is independent of the free variables. The following definition is a slight modification of the definition in [4].

Definition 2.2 *a) Let $N(P)$ be the Newton polygon of the symbol* (2.3)*. Then this symbol is called parameter-elliptic in $\mathscr{L}$ if there exists a $\lambda_0 > 0$ such that the inequality*

$$|P(\xi, \lambda)| \geq C W_P(\xi, \lambda) \quad (\lambda \in \mathscr{L},\ |\lambda| \geq \lambda_0,\ \xi \in \mathbb{R}^n) \tag{2.4}$$

holds where W_P denotes the weight function associated to P:

$$W_P(\xi, \lambda) := \sum_{\gamma,k} |\xi|^\gamma |\lambda|^k. \tag{2.5}$$

The last sum runs over all indices (γ, k) which are vertices of the Newton polygon $N(P)$.

b) The mixed order system $\lambda - A(D)$ is called parameter-elliptic in $\mathscr{L}$ if the Newton polygon of $P(\xi, \lambda) := \det(\lambda - A(D))$ is regular and if P is parameter-elliptic in $\mathscr{L}$.

There are several equivalent descriptions of parameter-ellipticity for mixed order systems (see [4]). The Newton polygon approach is a geometric description of the various homogeneities contained in the determinant $P(\xi, \lambda) = \det(\lambda - A(\xi))$. For $r > 0$ and a polynomial of the form (2.3) define the r-order of P by

$$d_r(P) := \max\{\gamma + rk : a_{\gamma k} \neq 0\}.$$

The r-principal part $P_r(\xi, \lambda)$ is given by

$$P_r(\xi, \lambda) := \sum_{\gamma + rk = d_r(P)} a_{\gamma k} |\xi|^\gamma \lambda^k.$$

The following result is a straightforward generalization of Theorem 2.2 in [4] where polynomial entries were considered.

Theorem 2.3 *Let $A(D)$ be a mixed order system and $P(\xi, \lambda) = \det(\lambda - A(\xi))$. Then the following statements are equivalent.*

(a) *The operator matrix $\lambda - A(D)$ is parameter-elliptic in $\mathscr{L}$.*

(b) *There exist constants $C > 0$, $\lambda_0 > 0$ such that*

$$\big|P(\xi, \lambda)\big| \geq C \prod_{i=1}^{n} (|\xi|^{r_i} + |\lambda|) \quad (\lambda \in \mathscr{L}, |\lambda| \geq \lambda_0, \xi \in \mathbb{R}^n). \tag{2.6}$$

Here the numbers r_i are defined in (2.2).

(c) *For every $r > 0$,*

$$P_r(\xi, \lambda) \neq 0 \quad (\lambda \in \mathscr{L} \setminus \{0\},\ \xi \in \mathbb{R}^n \setminus \{0\}). \tag{2.7}$$

The condition of parameter-ellipticity is equivalent to a uniform estimate of the entries of the inverse matrix, see [4], Proposition 3.10. Applying Plancherel's theorem, we immediately obtain L_2-estimates for the solution. When we deal with L_p-spaces, we want to apply Michlin's theorem. For this we need another estimate which is contained in the following theorem.

Theorem 2.4 *Let $P(\xi, \lambda)$ be parameter-elliptic in the sector $\mathscr{L}$ and assume, for simplicity, that $N(P)$ is regular. Let $(\sigma, \kappa) \in \mathbb{R}^2$ be a point belonging to the Newton polygon $N(P)$. Then there exists a $\lambda_0 > 0$ and for every $\alpha \in \mathbb{N}_0^n$ a constant $C_\alpha = C_\alpha$ such that*

$$\Big|\partial_\xi^\alpha \Big(\frac{|\xi|^\gamma |\lambda|^\kappa}{P(\xi, \lambda)}\Big)\Big| \leq C_\alpha |\xi|^{-\alpha} \quad (\xi \in \mathbb{R}^n \setminus \{0\},\ \lambda \in \mathscr{L},\ |\lambda| \geq \lambda_0). \tag{2.8}$$

If $P(\xi, \lambda) \neq 0$ for all $\xi \in \mathbb{R}^n$ and $\lambda \in \mathscr{L}$ with $|\lambda| \geq \varepsilon$ for some $\varepsilon > 0$, then inequality (2.8) *holds for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and all $\lambda \in \mathscr{L}$ with $|\lambda| \geq \varepsilon$.*

Proof. For a point $(\sigma,\kappa)\in N(P)$ we have, by convexity and Jensen's inequality,

$$|\xi|^\sigma|\lambda|^\kappa \le W_P(\xi,\lambda).$$

Let $\alpha=0$. By definition of parameter-ellipticity, there exists a $\lambda_0>0$ such that

$$\Big|\,|\xi|^\gamma\lambda^\kappa\Big| \le C|P(\xi,\lambda)| \quad (\lambda\in\mathscr{L},\ |\lambda|\ge\lambda_0,\ \xi\in\mathbb{R}^n).$$

Thus, the case $\alpha=0$ follows directly from the definition of parameter-ellipticity.

Now let $|\alpha|=1$ and assume, without loss of generality, that $\partial_\xi^\alpha=\partial_{\xi_1}$. We have

$$\begin{aligned}\Big|\partial_{\xi_1}\Big(|\xi|^\gamma\lambda^\kappa\Big)\Big| &= \Big|\lambda^\kappa\Big(\sum_{i=1}^n\xi_i^2\Big)^{\frac{\gamma}{2}-1}\cdot\frac{\gamma}{2}\cdot 2\xi_1\Big|\\ &= \Big|\gamma\cdot\lambda^\kappa\xi_1|\xi|^{\gamma-2}\Big|\\ &\le C|\lambda|^k|\xi|^{\gamma-1}. \end{aligned}\tag{2.9}$$

In the same way we can estimate

$$\Big|\partial_{\xi_1}P(\xi,\lambda)\Big| \le W_P(\xi,\lambda)\cdot|\xi|^{-1}. \tag{2.10}$$

We write

$$\Big|\partial_{\xi_1}\Big(\frac{|\xi|^\gamma\lambda^\kappa}{P(\xi,\lambda)}\Big)\Big| = \Big|\frac{P(\xi,\lambda)\partial_{\xi_1}(|\xi|^\gamma\lambda^\kappa)-|\xi|^\gamma\lambda^\kappa\partial_{\xi_1}P(\xi,\lambda)}{P(\xi,\lambda)^2}\Big|$$

and obtain the first statement of the theorem by (2.9), (2.10) and the definition of parameter-ellipticity (2.4).

The case of higher derivatives (general α) follows by iteration.

Now assume that $P(\xi,\lambda)\neq0$ for all $\xi\in\mathbb{R}^n$ and $\lambda\in\mathscr{L}$, $|\lambda|\ge\varepsilon$. By the regularity of $N(P)$, we can write

$$P(\xi,\lambda)=a_{\gamma_0,0}|\xi|^{\gamma_0}+\sum_{\substack{\gamma,k\\ k>0}}a_{\gamma k}|\xi|^\gamma\lambda^k.$$

In the last sum, only exponents of $|\xi|$ appear with $\gamma<\gamma_0$. We obtain

$$\lim_{|\xi|\to\infty}\frac{P(\xi,\lambda)}{|\xi|^{\gamma_0}}=a_{\gamma_0,0}\neq0 \quad (\lambda\in\mathscr{L},\ \varepsilon\le|\lambda|\le\lambda_0).$$

In the same way,

$$W_P(\xi,\lambda)=|\xi|^{\gamma_0}+\sum_{\substack{\gamma,k\\ k>0}}|\xi|^\gamma\lambda^k,$$

and

$$\lim_{|\xi|\to\infty}\frac{W_P(\xi,\lambda)}{|\xi|^{\gamma_0}}=1 \quad (\lambda\in\mathscr{L},\ \varepsilon\le|\lambda|\le\lambda_0).$$

Now we use $P(\xi,\lambda)\neq 0$ and a compactness argument to see that

$$\begin{aligned} |P(\xi,\lambda)| &\geq C|\xi|^{\gamma_0} && (\xi\in\mathbb{R}^n, \lambda\in\mathscr{L},\, \varepsilon\leq|\lambda|\leq\lambda_0),\\ |P(\xi,\lambda)| &\geq CW_P(\xi,\lambda) && (\xi\in\mathbb{R}^n, \lambda\in\mathscr{L},\, \varepsilon\leq|\lambda|\leq\lambda_0). \end{aligned}$$

From these inequalities we obtain (2.8) for all $\lambda\in\mathscr{L}$ with $|\lambda|\geq\varepsilon$ in the same way as in the first part of the proof.

Remark 2.5 *As we can see from the proof of the preceding theorem, we can also estimate*

$$\Big|\partial_\xi^\alpha\Big(\frac{\xi^\beta\lambda^\kappa}{P(\xi,\lambda)}\Big)\Big| \leq C_\alpha|\xi|^{-\alpha} \quad (\xi\in\mathbb{R}^n\setminus\{0\},\ \lambda\in\mathscr{L},\ |\lambda|\geq\lambda_0)$$

where now β is a multi-index such that $(|\beta|,\kappa)$ belongs to the Newton polygon.

3 Resolvent estimates for the generalized thermoelastic plate equation

To apply the results mentioned above to the generalized linear thermoelastic plate equation (1.1), (1.2) we rewrite this equation as a first-order system, setting $U := (S^{1/2}u,\ u_t,\ \theta)^t$. We get

$$U_t = A(D)U := \begin{pmatrix} 0 & S^{1/2} & 0\\ -S^{1/2} & 0 & S^\beta\\ 0 & -S^\beta & -S^\alpha \end{pmatrix} U.$$

The symbol of this system is given by

$$A(\xi) = \begin{pmatrix} 0 & \rho^{1/2} & 0\\ -\rho^{1/2} & 0 & \rho^\beta\\ 0 & -\rho^\beta & -\rho^\alpha \end{pmatrix}$$

with $\rho := |\xi|^{2\eta}$. Thus we have $\mathrm{ord}A_{ij}(\xi)\leq s_i+t_j$ with

$$s := 2\eta\cdot\begin{pmatrix} \frac12\\ 2\beta-\alpha\\ \beta \end{pmatrix},\quad t := 2\eta\cdot\begin{pmatrix} \frac12+\alpha-2\beta\\ 0\\ \alpha-\beta \end{pmatrix}. \tag{3.1}$$

Consequently, the weight vector is given by

$$\begin{pmatrix} r_1\\ r_2\\ r_3 \end{pmatrix} = 2\eta\begin{pmatrix} 1+\alpha-2\beta\\ 2\beta-\alpha\\ \alpha \end{pmatrix}. \tag{3.2}$$

With the order vectors s and t defined as above, the matrix $A(\xi)$ coincides with its principal part. The determinant of this system equals

$$P(\xi,\lambda) := \det(\lambda - A(\xi)) = \lambda^3+\lambda^2\rho^\alpha+\lambda(\rho^{2\beta}+\rho)+\rho^{1+\alpha}. \tag{3.3}$$

From (3.3) we can see that the Newton polygon $N(P)$ is the convex hull of the points

$$(0,3),\ (2\eta\alpha,2),\ (4\eta\beta,1),\ (2\eta+2\eta\alpha,0),\ (0,0)$$

(see Figure 3).

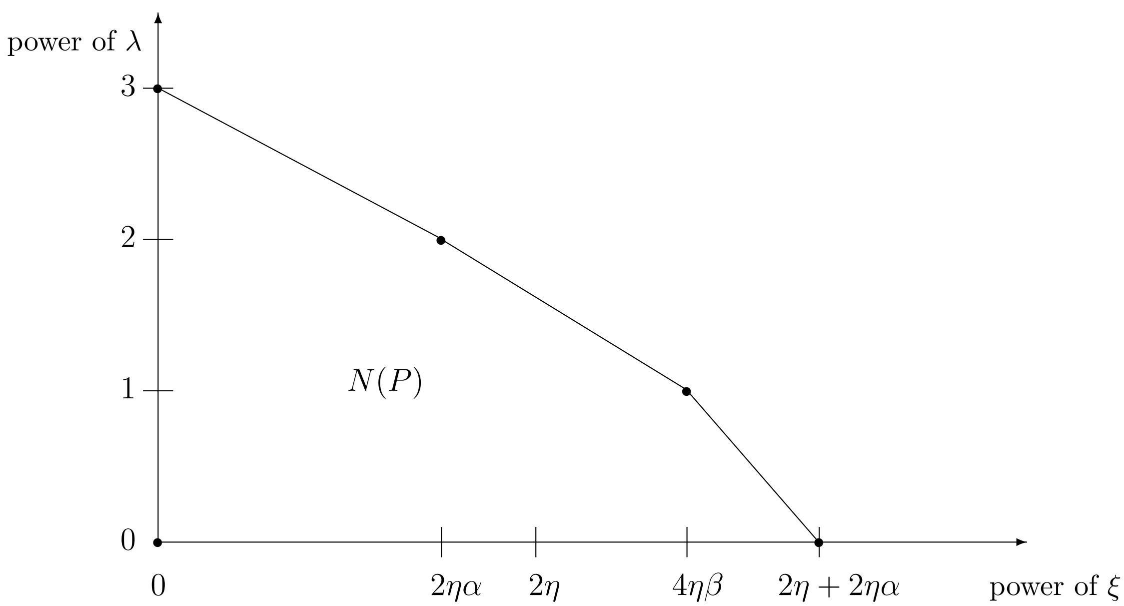

Figure 3: The Newton polygon of the mixed order system $\lambda - A(\xi)$.

Lemma 3.1 *Assume that* $(\beta,\alpha)\in\mathfrak{A}$, *i.e. that*

$$\alpha\ge\beta \quad \textit{and} \quad 2\beta-\alpha\ge\frac{1}{2}. \tag{3.4}$$

Then the matrix $\lambda - A(D)$ *is parameter-elliptic in* $\mathbb{C}_+ := \{\lambda\in\mathbb{C} : \operatorname{Re}\lambda\ge 0\}$.

Proof. We will check the conditions of Theorem 2.3 (c). Let us first assume $\alpha>\beta$ and $2\beta-\alpha>\frac{1}{2}$. Then we have $r_1<r_2<r_3$ in (3.2), and the r-principal part of $P(\xi,\lambda)$ is given by

$$\begin{aligned}
&P_r(\xi,\lambda)=\lambda^3, && r>2\eta\alpha,\\
&P_r(\xi,\lambda)=\lambda^3+\lambda^2\rho^\alpha, && r=2\eta\alpha,\\
&P_r(\xi,\lambda)=\lambda^2\rho^\alpha, && 4\eta\beta-2\eta\alpha<r<2\eta\alpha,\\
&P_r(\xi,\lambda)=\lambda^2\rho^\alpha+\lambda\rho^{2\beta}, && r=4\eta\beta-2\eta\alpha,\\
&P_r(\xi,\lambda)=\lambda\rho^{2\beta}, && 2\eta+2\eta\alpha-4\eta\beta<r<4\eta\beta-2\eta\alpha,\\
&P_r(\xi,\lambda)=\lambda\rho^{2\beta}+\rho^{1+\alpha}, && r=2\eta+2\eta\alpha-4\eta\beta,\\
&P_r(\xi,\lambda)=\rho^{1+\alpha}, && 0<r<2\eta+2\eta\alpha-4\eta\beta.
\end{aligned}$$

We immediately see that $P_r(\xi,\lambda)\neq 0$ for all $\xi\in\mathbb{R}^n\setminus\{0\}$ and $\lambda\in\mathbb{C}\setminus(-\infty,0]$.

In the case $2\beta-\alpha>\frac{1}{2}$ and $\alpha=\beta$ we have $r_1<r_2=r_3$. For $r=2\eta\alpha=2\eta\beta$ the r-principal part of $P(\xi,\lambda)$ is now given as

$$P_r(\xi,\lambda)=\lambda^3+\lambda^2\rho^\alpha+\lambda\rho^{2\alpha}.$$

The zeros of $P_r(\xi,\lambda)$ are $\lambda=0$ and $\lambda=\frac{1}{2}(-1\pm\sqrt{3}i)$, so we have $P_r(\xi,\lambda)\neq 0$ for $\xi\neq 0$ and $\lambda\in\mathbb{C}_+\setminus\{0\}$.

In a similar way the other boundary cases can be handled. We see that for every $(\beta,\alpha)\in\mathfrak{A}$ the system is parameter-elliptic in $\mathbb{C}_+$.
Q.e.d.

Remark 3.2 *a) As we can see from the proof of Lemma 3.1, the system is parameter-elliptic in every closed sector of the complex plane which does not contain the negative real axis, provided that (β,α) lies in the interior of $\mathfrak{A}$.*

b) The conditions on (β,α) are essential for parameter-ellipticity. For instance, consider the case $\frac{1}{2}<\alpha<\beta<1$. Then for $r=2\eta\beta$ the r-principal part of $P(\xi,\lambda)$ is given by

$$P_r(\xi,\lambda)=\lambda^3+\lambda\rho^{2\beta}.$$

As this polynomial has purely imaginary roots, it is not parameter-elliptic in $\mathbb{C}_+$. Note that this holds also for (β,α) which belong to the area $\widetilde{\mathfrak{A}}$ of smoothing.

In the next step we will prove uniform resolvent estimates (a priori estimates) for the mixed order system $A(D)$. We will show resolvent estimates in the standard L_p-Bessel potential spaces $W_p^r(\mathbb{R}^n)$ with norm

$$\|u\|_{W_p^r(\mathbb{R}^n)}:=\|\mathcal{F}^{-1}(1+|\xi|^2)^{r/2}\mathcal{F}u\|_{L_p(\mathbb{R}^n)}.$$

We still assume $\alpha\geq\beta$ and $2\beta-\alpha\geq\frac{1}{2}$. We choose as a basic space

$$X:=W_p^{\eta(2\beta-1)}(\mathbb{R}^n)\times W_p^{2\eta(\alpha-\beta)}(\mathbb{R}^n)\times L_p(\mathbb{R}^n). \tag{3.5}$$

The domain of the operator $A(D)$ will be defined as

$$Y:=W_p^{\eta(1+2\alpha-2\beta)}(\mathbb{R}^n)\times W_p^{2\eta\beta}(\mathbb{R}^n)\times W_p^{2\eta\alpha}(\mathbb{R}^n). \tag{3.6}$$

Lemma 3.3 *The operator $A(D):Y\to X$ is well-defined and continuous.*

Proof. This follows immediately from

$$A(D)=\begin{pmatrix} 0 & (-\Delta)^\eta & 0\\ -(-\Delta)^\eta & 0 & (-\Delta)^{\eta\beta}\\ 0 & -(-\Delta)^{\eta\beta} & -(-\Delta)^{\eta\alpha}\end{pmatrix}$$

and the fact that powers of the negative Laplacian are continuous in the corresponding scale of Bessel potential spaces.
Q.e.d.

Theorem 3.4 *Let $(\beta,\alpha)\in\mathfrak{A}$, and let $1<p<\infty$. Then there exists a $\lambda_0>0$ such that for all $\lambda\in\mathbb{C}_+$, $|\lambda|\geq\lambda_0$, the equation $(\lambda-A(D))U=F$ has a unique solution $U=(v,w,\theta)^t\in Y$ for every $F=(f,g,h)^t\in X$. Moreover, the estimate*

$$|\lambda|\cdot\|U\|_X+\|U\|_{D(A)}\leq C\|F\|_X$$

holds for all $\lambda\in\mathbb{C}_+$ with $|\lambda|\geq\lambda_0$.

Proof. We already know from Lemma 3.1 that $\lambda-A(D)$ is parameter-elliptic in Σ_ε. In particular, for $\xi\in\mathbb{R}^n$ and large $\lambda\in\Sigma_\varepsilon$, the determinant $P(\xi,\lambda)=\det(\lambda-A(\xi))$ does not vanish.

(i) First we show that there exists a $\lambda_0>0$ and for every multi-index $\gamma\in\mathbb{N}_0^n$ a constant C_γ such that

$$\Big|\partial_\xi^\gamma\big[\lambda M_X(\xi)(\lambda-A(\xi))^{-1}\cdot M_X(\xi)^{-1}\big]\Big|\leq C_\gamma|\xi|^{-|\gamma|}\quad(\xi\in\mathbb{R}^n\setminus\{0\},\lambda\in\Sigma_\varepsilon,\ |\lambda|\geq\lambda_0).\tag{3.7}$$

Here the diagonal matrix $M_X(\xi)$ is defined as

$$M_X(\xi):=\begin{pmatrix}(1+\rho)^{\beta-\frac12}&0&0\\0&(1+\rho)^{\alpha-\beta}&0\\0&0&1\end{pmatrix}\tag{3.8}$$

where we again have set $\rho=|\xi|^{2\eta}$. To prove the inequality above, we compute the inverse matrix explicitly. We have

$$(\lambda-A(\xi))^{-1}=\frac{1}{P(\xi,\lambda)}\begin{pmatrix}\lambda(\lambda+\rho^\alpha)+\rho^{2\beta}&\rho^{\frac12}(\lambda+\rho^\alpha)&\rho^{\beta+\frac12}\\-\rho^{\frac12}(\lambda+\rho^\alpha)&\lambda(\lambda+\rho^\alpha)&\lambda\rho^\beta\\\rho^{\beta+\frac12}&-\lambda\rho^\beta&\lambda^2+\rho\end{pmatrix}.$$

Therefore, the the matrix $\lambda M_X(\xi)(\lambda-A(\xi))^{-1}\cdot M_X(\xi)^{-1}$ is given by $P(\xi,\lambda)^{-1}$ times

$$\begin{pmatrix}\lambda^2(\lambda+\rho^\alpha)+\lambda\rho^{2\beta}&\lambda\rho^{\frac12}(1+\rho)^{2\beta-\alpha-\frac12}(\lambda+\rho^\alpha)&\lambda\rho^{\beta+\frac12}(1+\rho)^{\beta-\frac12}\\-\lambda\rho^{\frac12}(1+\rho)^{\frac12+\alpha-2\beta}(\lambda+\rho^\alpha)&\lambda^2(\lambda+\rho^\alpha)&\lambda^2\rho^\beta(1+\rho)^{\alpha-\beta}\\\lambda\rho^{\beta+\frac12}(1+\rho)^{-\beta+\frac12}&-\lambda^2\rho^\beta(1+\rho)^{\beta-\alpha}&\lambda(\lambda^2+\rho)\end{pmatrix}.\tag{3.9}$$

We want to apply Theorem 2.4 to every component of this matrix. As an example, let us consider the left lower corner

$$\lambda\rho^{\beta+\frac12}(1+\rho)^{-\beta+\frac12}.\tag{3.10}$$

It is easily seen that $(\frac{\rho}{1+\rho})^\beta$ is a Fourier multiplier in the sense that for every multi-index $\gamma\in\mathbb{N}_0^n$

$$\Big|\partial_\xi^\gamma\Big(\frac{\rho}{1+\rho}\Big)^\beta\Big|\leq C_\gamma|\xi|^{-|\gamma|}\quad(\xi\in\mathbb{R}^n\setminus\{0\}).$$

Therefore, it suffices to consider $\lambda\rho^{\frac{1}{2}}(1+\rho)^{\frac{1}{2}}$ instead of 3.10. In the same way, $\frac{1+\rho^{1/2}}{(1+\rho)^{1/2}}$ is a Fourier multiplier, so we have to estimate $\lambda\rho^{\frac{1}{2}} + \lambda\rho$. These two terms correspond to the pairs $(2,1)$ and $(4,1)$ of exponents. Both points belong to the Newton polygon $N(P)$, and by Theorem 2.4 the desired result follows.

The same proof works for all components of the matrix (3.9). More precisely, we get the following exponents (in the order of appearance in the matrix (3.9)):

$$\begin{array}{lll} (0,3),(2\eta\alpha,2),(4\eta\beta,1); & (4\eta\beta-2\eta\alpha,2),(4\eta\beta,1); & (4\eta\beta,1); \\ (2\eta+2\eta\alpha-4\eta\beta,2),(2\eta+4\eta\alpha-4\eta\beta,1); & (0,3),(2\eta\alpha,2); & (2\eta\alpha,2); \\ (2\eta,1); & (4\eta\beta-2\eta\alpha,2); & (0,3),(2\eta,1). \end{array}$$

Due to our conditions on the parameters α and β, all points appearing in this list are in the interior or on the boundary of the Newton polygon $N(P)$. In Figure 4 the points are marked with "■".

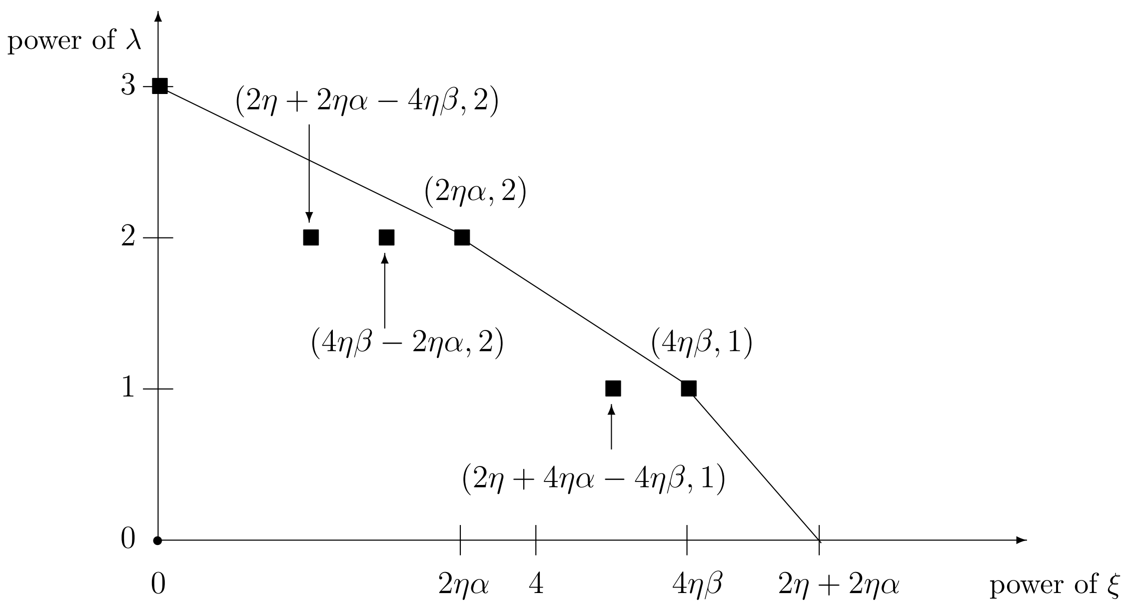

Figure 4: Exponents appearing in the matrix (3.9).

(ii) Now we apply Michlin's theorem which tells us that for $U := (\lambda - A(D))^{-1}F$ (being defined for $\lambda \in \mathbb{C}_+$ with $|\lambda| \geq \lambda_0$) the inequality

$$|\lambda|\,\|M_X(D)U\|_{L_p} \leq \|M_X(D)F\|_{L_p}$$

holds. Due to the definition of the space X, this can be reformulated as

$$|\lambda| \cdot \|U\|_X \leq C\|F\|_X.$$

(iii) Now we show that for any $F \in X$ the formula

$$U := (\lambda - A(\xi))^{-1} F$$

defines a solution in the space Y. This can be shown in exactly the same way as the resolvent estimate but now we have to apply Michlin's theorem to the coefficients of the matrix

$$M_Y(\xi)(\lambda - A(\xi))^{-1} M_X(\xi)^{-1}.$$

Here we set

$$M_Y(\xi) := \begin{pmatrix} (1+\rho)^{\frac{1}{2}+\alpha-\beta} & 0 & 0 \\ 0 & (1+\rho)^{\beta} & 0 \\ 0 & 0 & (1+\rho)^{\alpha} \end{pmatrix}.$$

The matrix $M_Y(\xi)(\lambda - A(\xi))^{-1} M_X(\xi)^{-1}$ is given by $P(\xi, \lambda)^{-1}$ times

$$\begin{pmatrix} (1+\rho)^{1+\alpha-2\beta}[\lambda(\lambda+\rho^{\alpha}) + \rho^{2\beta}] & \rho^{\frac{1}{2}}(1+\rho)^{\frac{1}{2}}(\lambda+\rho^{\alpha}) & \rho^{\beta+\frac{1}{2}}(1+\rho)^{\frac{1}{2}+\alpha-\beta} \\ -\rho^{\frac{1}{2}}(1+\rho)^{\frac{1}{2}}(\lambda+\rho^{\alpha}) & \lambda(1+\rho)^{2\beta-\alpha}(\lambda+\rho^{\alpha}) & \lambda\rho^{\beta}(1+\rho)^{\beta} \\ \rho^{\beta+\frac{1}{2}}(1+\rho)^{\alpha-\beta+\frac{1}{2}} & -\lambda\rho^{\beta}(1+\rho)^{\beta} & (1+\rho)^{\alpha}(\lambda^2+\rho) \end{pmatrix}.$$

As before, we can see that all exponents appearing in this matrix belong to $N(P)$, and applying Theorem 2.2 and Michlin's theorem, we obtain the desired inequality

$$\|U\|_Y \le C\|F\|_X.$$

Q.e.d.

As a consequence we obtain

Theorem 3.5 *The semigroup associated to A is analytic in the region $\mathfrak{A}$.*

We note that the choice of the spaces X and Y above is in some sense unique. More precisely, consider the spaces

$$X = W_p^{2\eta t_1}(\mathbb{R}^n) \times W_p^{2\eta t_2}(\mathbb{R}^n) \times W_p^{2\eta t_3}(\mathbb{R}^n)$$

and

$$Y = W_p^{2\eta s_1}(\mathbb{R}^n) \times W_p^{2\eta s_2}(\mathbb{R}^n) \times W_p^{2\eta s_3}(\mathbb{R}^n)$$

with $t_i, s_i \in \mathbb{R}$. We are looking for indices s_i, t_i such that the following conditions are satisfied:

(i) $A(D)\colon X \to Y$ is well defined and continuous.

(ii) $\lambda(\lambda - A(D))^{-1}\colon X \to X$ is continuous for $\lambda \in \mathbb{C}_+$, $|\lambda| \ge \lambda_0$ with norm bounded by a constant independent of λ.

(iii) $(\lambda - A(D))^{-1}\colon X \to Y$ is continuous for $\lambda \in \mathbb{C}_+$, $|\lambda| \ge \lambda_0$ with norm bounded by a constant independent of λ.

By Michlin's theorem, this implies that the corresponding symbols are bounded for $|\xi| \to \infty$. Setting

$$M_X(\xi) := \begin{pmatrix} (1+\rho)^{t_1} & 0 & 0 \\ 0 & (1+\rho)^{t-2} & 0 \\ 0 & 0 & (1+\rho)^{t_3} \end{pmatrix}$$

and

$$M_Y(\xi) := \begin{pmatrix} (1+\rho)^{t_1} & 0 & 0 \\ 0 & (1+\rho)^{t-2} & 0 \\ 0 & 0 & (1+\rho)^{t_3} \end{pmatrix},$$

we see that that (i)-(iii) implies that the following matrices are bounded for $|\xi| \to \infty$ and $\lambda \in \Sigma_\varepsilon$, $|\lambda| \geq \lambda_0$:

$$\begin{aligned} N_1(\xi) &:= M_X(\xi)A(\xi)M_Y(\xi)^{-1}, \\ N_2(\xi) &:= \lambda M_X(\xi)(\lambda - A(\xi))^{-1}M_X(\xi), \\ N_3(\xi) &:= M_Y(\xi)(\lambda - A(\xi))^{-1}M_X(\xi). \end{aligned}$$

The boundedness of N_1 for large $|\xi|$ implies that every exponent of ξ appearing in this matrix is less or equal to 0. For instance, in the first row and second column of $N_1(\xi)$ we have the coefficient $(1+\rho)^{t_1}\rho^{1/2}(1+\rho)^{-s_2}$. The boundedness implies

$$t_1 - s_2 + \frac{1}{2} \leq 0.$$

In the same way we get an inequality for every non-zero coefficient of N_1.

Both N_2 and N_3 have the form $\frac{1}{P(\xi,\lambda)}\widetilde{N}_i(\xi,\lambda)$ $(i = 2,3)$ where the coefficients of the matrix $\widetilde{N}_i$ are sums of terms of the form $\rho^\sigma\lambda^k$. From the properties of the Newton polygon, it can easily be seen that an expression of the form

$$\frac{\rho^\sigma\lambda^k}{P(\xi,\lambda)}$$

can only be bounded for $|\xi| \to \infty$ if $(2\eta\sigma, k)$ belongs to the Newton polygon. For instance, in the first row and second column of $\widetilde{N}_2$ we have the exponents $(4t_1 - 4t_2 + 2, 2)$ and $(4t_1 - 4t_2 + 4\alpha + 2, 1)$. These points belong to $N(P)$ if

$$t_1 - t_2 + \frac{1}{2} \leq \alpha \quad \text{and} \quad t_1 - t_2 + \alpha + \frac{1}{2} \leq 2\beta.$$

Altogether we obtain a set of inequalities for s_i and t_i, $i = 1,2,3$. Note that for a solution s_i, t_i also $s_i + \tau, t_i + \tau$ with arbitrary $\tau \in \mathbb{R}$ is a solution. If we assume $t_3 = 0$, a simple but lengthy calculation shows that

$$\begin{aligned} s_1 &= 2 + 4\alpha - 4\beta, \qquad & s_2 &= 4\beta, \qquad & s_3 &= 4\alpha, \\ t_1 &= 4\beta - 2, & t_2 &= 4\alpha - 4\beta, & t_3 &= 0 \end{aligned}$$

is the only solution of all inequalities.

4 Decay rates

Denoting by

$$\big(v(t,\xi) := (\mathcal{F}u(t,\cdot))(\xi), \psi(t,\xi) := (\mathcal{F}\theta(t,\cdot))(\xi)\big)$$

the Fourier transform of the solution (u,θ) to the α-β-system (1.1), (1.2) with initial conditions (1.3), it is easy to see that both v and ψ satisfy the following third-order equation (w. o. l. g. $a = b = d = g = 1$ again)

$$w_{ttt} + \rho^{\alpha} w_{tt} + (\rho^{2\beta} + \rho) w_t + \rho^{1+\alpha} w = 0, \tag{4.1}$$

where $\rho := |\xi|^{4\gamma}$, and with initial conditions

$$w(0,\cdot) = w_0 := \mathcal{F}u_0, \quad w_t(0,\cdot) = \mathcal{F}u_1, \quad w_{tt}(0,\cdot) = \mathcal{F}u_{tt}(0,\cdot) \tag{4.2}$$

(cp. [19]). Then w is given by

$$w(t,\xi) = \sum_{j=1}^{3} b_j(\rho) e^{\lambda j(\rho) t}, \tag{4.3}$$

where $\lambda j(\rho), j = 1,2,3$, are the roots of the characteristic equation

$$P(\lambda,\rho) = \lambda^3 + \rho^{\alpha}\lambda^2 + (\rho^{2\beta} + \rho)\lambda + \rho^{1+\alpha} = 0 \tag{4.4}$$

and

$$b_j(\rho) := \sum_{k=0}^{2} b_j^k(\rho) w_k(\rho)$$

with

$$b_j^0 := \frac{\prod\limits_{l\neq j} \lambda_e}{\prod\limits_{l\neq j}(\lambda_j - \lambda_e)}, \quad b_j^1 := \frac{\sum\limits_{l\neq j} \lambda_e}{\prod\limits_{l\neq j}(\lambda_j - \lambda_e)}, \quad b_j^2 := \frac{1}{\prod\limits_{l\neq j}(\lambda_j - \lambda_e)}.$$

The study of the asymptotic behavior of $\lambda_j(\rho)$ when $\rho \to \infty$ gives the information on the smoothing effect of the solutions, that is: if for any $j = 1,2,3$ the real part of λ_j tends to infinity as $\rho \to \infty$, then the smoothing property holds; if there exists a subscript j for which the real part does not tend to infinity, then the solution cannot be more regular than the initial data, cp. [19].

On the other hand, the behavior of the real part of λ_j as $\rho \to 0$ gives the information on the asymptotic behavior of the solution as time goes to infinity, such as uniform polynomial decay rates. In the following theorem the equalities are to be read up to terms of lower/higher order in ρ as $\rho \to \infty/0$.

Theorem 4.1
(1) We have in $\mathfrak{A}$ as $\rho \to \infty$:

$$\begin{array}{rrcl}
\textit{In the interior of } \mathfrak{A} \textit{ and for } \alpha = 1: & \lambda_1 & = & k_1 \rho^\alpha, \\
& \lambda_{2,3} & = & k_{2a,2b} \rho^{2\beta-\alpha} \\
\textit{For } \alpha = \beta > \dfrac{1}{2}: & \lambda_1 & = & k_3 \rho^{1-\alpha}, \\
& \lambda_{2,3} & = & k_4 \rho^\alpha \pm \mathrm{i} k_5 \rho^\alpha \\
\textit{For } \alpha = \beta = \dfrac{1}{2}: & \lambda_1 & = & k_6 \rho^{1/2}, \\
& \lambda_{2,3} & = & k_7 \rho^{1/2} \pm \mathrm{i} k_8 \rho^{1/2} \\
\textit{For } \dfrac{1}{2} < \alpha = 2\beta - \dfrac{1}{2}: & \lambda_1 & = & k_9 \rho^\alpha, \\
& \lambda_{2,3} & = & k_{10} \rho^{1/2} \pm \mathrm{i} k_{11} \rho^{1/2}
\end{array}$$

The constant coefficients $k_m > 0, m = 1, ..., 11$, can be given explicitly, they are not the same in all of $\mathfrak{A}$, but can jump coming from the interior to the boundary; the same holds for the positive constants r_m below.

(2) We have in $\mathfrak{A}$ as $\rho \to 0$:

$$\begin{array}{rrcl}
\textit{In the interior of } \mathfrak{A} \textit{ and for } \alpha = 1: & \lambda_1 & = & r_1 \rho^\alpha, \\
& \lambda_{2,3} & = & r_2 \rho^{\alpha+2\beta-1} \pm \mathrm{i} r_3 \lambda^{1/2} \\
\textit{For } \alpha = \beta > \dfrac{1}{2}: & \lambda_1 & = & r_4 \rho^\alpha, \\
& \lambda_{2,3} & = & r_5 \rho^{3\alpha-1} \pm \mathrm{i} r_6 \rho^{1/2} \\
\textit{For } \alpha = \beta = \dfrac{1}{2}: & \lambda_1 & = & r_7 \rho^\alpha, \\
& \mathrm{Re}\lambda_{2,3} & = & r_8 \rho^{1/2} \\
\textit{For } \dfrac{1}{2} < \alpha = 2\beta - \dfrac{1}{2}: & \lambda_1 & = & r_9 \rho^\alpha, \\
& \lambda_{2,3} & = & r_{10} \rho^{2\alpha-1/2} \pm \mathrm{i} r_{11} \rho^{1/2}
\end{array}$$

(3) For $\beta = 1/2,\ 0 \le \alpha \le 1$ we have as $\rho \to 0$

$$\mathrm{Re}\lambda_j = \begin{cases} c_j \rho^\alpha, & \frac{1}{2} \le \alpha \le 1 \\ c_j \rho^{1-\alpha}, & 0 \le \alpha \le 1/2 \end{cases}$$

where c_j denotes (different) positive constants.

(4) For $\alpha = 1/2,\ \frac{1}{4} \le \beta \le 1/2$: $\quad \mathrm{Re}\lambda_j = c_j \rho^{1/2}$

(5) For $\alpha = 1/2,\ \frac{1}{2} \le \beta \le 3/4$: $\quad \mathrm{Re}\lambda_j = c_j \rho^{2\beta-1/2}$

Proof. The roots λ_j of the cubic polynomial $P(\lambda, \rho)$ in (4.4) are given by

$$\lambda_1 = \frac{1}{3}(\rho^\alpha + B - D^{1/3}), \tag{4.5}$$

$$\lambda_{2,3} = \frac{1}{3}\rho^\alpha + \frac{1}{6}(D^{1/3} - B) \mp \frac{\sqrt{3}}{6}(D^{1/3} + B), \tag{4.6}$$

where

$$B := \frac{3_\rho - \rho^{2\alpha} + 3\rho^{2\beta}}{D^{1/3}} \tag{4.7}$$

and

$$D \;:=\; \frac{1}{2}\Big(-2\rho^{3\alpha} - 18\rho^{1+\alpha} + 9\rho^{\alpha+2\beta} + \tag{4.8}$$

$$\sqrt{4(3\rho - \rho^{2\alpha} + 3\rho^{2\beta})^3 + (-2\rho^{3\alpha} - 18\rho^{1+\alpha} + 9\rho^{\alpha+2\beta})^2}\Big) \tag{4.9}$$

By a straightforward but lengthy analysis of the terms in (4.4)–(4.7) as $\rho \to \infty$ and $\rho \to 0$, respectively, studying the different cases $(1), (2), (4), (5)$, we arrive at the expansion claimed in Theorem 4.1. Claim (3) was already given in [19]. Combining the representation of the solution with the asymptotic properties given in Theorem 4.1 for $\rho \to 0$, we can conclude

Theorem 4.2 *For the solution (ψ, θ) to the α-β-system (1.1), (1.2), (1.3) we have for $2 \le q \le \infty, \frac{1}{q} + \frac{1}{q'} = 1$, and $t > 0$:*

$$\|(u_t(t,\cdot), (-\Delta)^\gamma u(t,\cdot), \theta(t,\cdot))\|_{L^q(\mathbb{R}^n)} \le c_{q,n} t^{-\frac{n}{2\gamma d}(1-\frac{2}{q})} \|(u_1, (-\Delta)^\gamma u_0, \theta_0)\|_{L^{q'}(\mathbb{R}^n)}$$

where $c_{q,n}$ is a positive constant at most depending on q and the space dimension n, and where d is given as follows:

(1) For $(\beta, \alpha) \in \mathfrak{A}$: $\quad d = \alpha$.

(2) For $\beta = \frac{1}{2}, 0 \le \alpha \le 1$: $\quad d = \begin{cases} \alpha, & \frac{1}{2} \le \alpha \le 1 \\ 1-\alpha, & 0 \le \alpha \le 1/2 \end{cases}$.

(3) For $\alpha = \frac{1}{2}, \frac{1}{4} \le \beta \le 1/2$: $\quad d = \frac{1}{2}$.

(4) For $\alpha = 1/2, \frac{1}{2} \le \beta \le \frac{3}{4}$: $\quad d = 2\beta - \frac{1}{2}$.

Proof. Claim (2) was already proved in [19]. For the remaining cases the L^∞-decay $(g = \infty)$ follows in a standard way from the asymptotic expansions given in Theorem 4.1, see e.g. [18] or [20]. The L^2-"decay" $(q = 2 = q\prime)$ is given by the dissipation of the system, so the claims follow by interpolation.
Q.e.d.

As already remarked in [19] we again see the very special rôle of $(\beta, \alpha) = (\frac{1}{2}, \frac{1}{2})$, i.e. of the classical thermoelastic plate system. It is also interesting to notice the (non-)dependence of the decay rate on the parameter β.

We studied case (3) and (4), respectively case (4) and (5) in Theorem 4.1, just exemplarily. Further cases for (β, α) can be treated similarly.

The analysis of the roots as given in 4.4, 4.5 gives three real roots $\lambda_1, \lambda_2, \lambda_3$ for $(\beta, \alpha) \in \mathfrak{A}$, fitting to the analyticity there. On the other hand, another asymptotic analysis for $\alpha > \beta \geq \frac{1}{2}$, $\alpha > 2\beta - \frac{1}{2}$, i.e. for (β, α) *outside* the region $\mathfrak{A}$, yields

$$\left|\frac{\mathrm{Im}\lambda_{2/3}}{\mathrm{Re}\lambda_{2/3}}\right| = \rho^{\alpha-2\beta+1/2} + \text{l.o.t.} \to \infty \quad \text{as } \rho \to \infty \tag{4.10}$$

excluding analyticity because of a non-sectorial operator appearing. $\mathfrak{A}$ is expected to be *the* analyticity region.

Finally we remark that the next step is to consider domains $\Omega \subsetneqq \mathbb{R}^n$ with boundaries. For this the domains of the operators, the admissible or meaningful boundary conditions and the appropriate Sobolev spaces have to be determined, at least, pointing out possible future research.

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