Timoshenko Systems with Indefinite Damping*

Jaime E. Muñoz Rivera and Reinhard Racke

Abstract: We consider the Timoshenko system in a bounded domain $(0, L) \subset \mathbb{R}^1$. The system has an indefinite damping mechanism, i.e. with a damping function $a = a(x)$ possibly changing sign, present only in the equation for the rotation angle. We shall prove that the system is still exponentially stable under the same conditions as in the positive constant damping case, and provided $\overline{a} = \int_0^L a(x)\,dx > 0$ and $\|a - \overline{a}\|_{L^2} < \epsilon$, for ϵ small enough. The decay rate will be described explicitly.

In the arguments, we shall also give a new proof of exponential stability for the constant case $a \equiv \overline{a}$. Moreover, we give a precise description of the decay rate and demonstrate that the system has the spectrum determined growth (SDG) property, i.e. the type of the induced semigroup coincides with the spectral bound for its generator.

1 Introduction

Here we will consider the system

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0 \quad \text{in } (0, \infty) \times (0, L), \tag{1.1}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + a(x)\psi_t = 0 \quad \text{in } (0, \infty) \times (0, L), \tag{1.2}$$

with positive constants $\rho_1, k, \rho_2, b, \gamma, \rho_3, \kappa$ together with initial conditions

$$\varphi(0, \cdot) = \varphi_0, \quad \varphi_t(0, \cdot) = \varphi_1, \quad \psi(0, \cdot) = \psi_0, \quad \psi_t(0, \cdot) = \psi_1, \quad \theta(0, \cdot) = \theta_0 \quad \text{in } (0, L), \tag{1.3}$$

and boundary conditions

$$\varphi(t, 0) = \varphi(t, L) = \psi_x(t, 0) = \psi_x(t, L) = 0 \quad \text{in } (0, \infty). \tag{1.4}$$

It models the transverse displacement φ of a beam with reference configuration $(0, L) \subset \mathbb{R}^1$ and the rotation angle ψ of a filament. The well-posedness of (1.1)–(1.4) is standard, cp. [20].

[0] AMS subject classification: 35 B 40, 74 H 40, 47 D 06

[0] Keywords: Timoshenko system, exponential stability, spectrum determined growth property, indefinite damping

*Supported by a CNPq grant.

There is a damping mechanism present (only) in one equation, (1.2), given by $a(x)\psi_t$, where $a \in L^\infty((0,L))$ may change sign, but will satisfy

$$\overline{a} := \frac{1}{L}\int_0^L a(x)dx > 0. \tag{1.5}$$

For strictly positive a it was shown by Soufyane [20] that the system is exponentially stable if and only if

$$\frac{\rho_1}{k} = \frac{\rho_2}{b} \tag{1.6}$$

holds[1]. That is under this condition the damping in only one equation is strong enough for the exponential decay of the associated energy

$$\begin{aligned} E(t) &:= \frac{1}{2}\int_0^L (\rho_1\varphi_t^2 + \rho_2\psi_t^2 + b\psi_x^2 + k|\varphi_x + \psi|^2)(t,x)dx \\ &\equiv E(t,\varphi,\psi) \end{aligned}$$

(mostly dropping (t,x) in the sequel). In the author's paper [1] the damping $a\psi_t$ could be replaced by a memory term $\int_0^t g(t-s)\psi_{xx}ds$, and in [14] by a coupling to a heat equation, see also [15] for nonlinear systems.

Already for the wave equation

$$u_{tt} - u_{xx} + a(x)u_t = 0,$$

with Dirichlet boundary conditions, it is a subtle issue to see whether an indefinite damping with the function a just satisfying (1.5) still leads to exponential stability, and which additional conditions have to be added, respectively. The non-dissipative case with indefinite a seems to have been posed first by Chen, Fulling, Narcovich and Sun [3] where it was conjectured that the energy

$$E_0(t) = \int_0^L (u_t^2 + u_x^2)(t,x)dx$$

decays exponentially if

$$\exists\, \gamma > 0\ \forall\, n = 1,2,\ldots : \quad \int_0^L a(x)\sin^2(n\pi x/L)dx \geq \gamma$$

holds. Later Freitas [6] found that the latter condition on the moments is not sufficient to guarantee exponential stability when $\|a\|_{L^\infty}$ is large, but replacing a by εa, Freitas and

[1] In [12] it was pointed out that this conditions for real materials never holds, but the analysis gives inside for various problems.

Zuazua [8] proved that when a is of bounded variation and the condition on the moments holds, then there is $\varepsilon^* = \varepsilon^*(a)$ such that for all $\varepsilon \in (0, \varepsilon^*)$ the energy decays indeed exponentially. This result was extended to a differential equation of the type

$$u_{tt} - u_{xx} + \varepsilon a(x) u_t + b(x) u = 0 \tag{1.7}$$

by Benaddi and Rao [2]. K. Liu, Z. Liu and Rao [10] gave an abstract treatment of these results under certain conditions on the abstract damping operator. An extension to higher space dimensions was presented by Liu, Rao and Zhang [11], for unbounded domains see the recent work of Freitas and Krejčiřík [7].

In [16] the authors could show that is sufficient to require just

$$\|a - \overline{a}\|_{L^2} \quad \text{small enough.} \tag{1.8}$$

There it was also shown the there are certain pairs (a, L) with possibly negative moments $\int\limits_0^L a(x)\sin^2(n\pi x/L)dx$ but still leading to exponential decay. An extension to the type of equation (1.7) was given by Menz [13].

Fo the Timoshenko system under consideration, we shall demonstrate that the conditions (1.5)–(1.8) are sufficient to yield exponential stability.

Moreover, we shall precisely describe the *best rate of decay* d_0 for the energy in the estimate

$$\exists\, d_0 > 0\ \exists\, C_0 > 0\ \forall\, t \geq 0 : \quad E(t) \leq C_0 E(0) \mathrm{e}^{-2d_0 t}. \tag{1.9}$$

for the constant coefficient case $a = \overline{a}$. This is related the result to be proved here that the system has the so-called spectrum determined growth property (SDG property); that is, after having reformulated the system as a first-order system (see Section 2) $V_t = AV$. with a C_0-semigroup generator A in an appropriate Hilbert space, we shall prove that the type of the semigroup, the growth abscissa $\omega_0(A)$, equals the spectral bound $\omega_\sigma(A)$,

$$\omega_0(A) = \omega_\sigma(A). \tag{1.10}$$

Here, in general, the type of a C_0-semigroup generator G is defined as

$$\omega_0(G) = \lim_{t\to\infty} \frac{\ln \|\mathrm{e}^{Gt}\|}{t} = \inf_{t>0} \frac{\ln \|\mathrm{e}^{Gt}\|}{t}, \tag{1.11}$$

and the spectral bound is given as the least upper bound for the real parts of the values in the spectrum $\sigma(G)$ of G,

$$\omega_\sigma(G) = \sup\{\mathrm{Re}\lambda \mid \lambda \in \sigma(G)\}. \tag{1.12}$$

For a discussion of the SDG property see the work of Prüss [18], Huang [9] and Renardy [19]. We shall prove the SDG property for our system using a characterization given by Prüss [18] or Huang [9], saying

$$\omega_0(G) = \inf\{\omega \in \mathbb{R} \mid \exists\, M = M(\omega)\ \ \forall\, \lambda, \mathrm{Re}\lambda \geq \omega : \ \|(\lambda - G)^{-1}\| \leq M\}. \tag{1.13}$$

To prove our results for $\mathcal{A}$, we shall regard it as a perturbation of A, where A , as above, represents the same system but with $a(x)$ being replaced by a (*the*) constant $\overline{a} > 0$. This corresponds to a previously discussed constant damping in one equation, cp. [20, 15]. Here it will be demonstrated that it yields exponential stability — first another proof of this known result using (1.13), but second and additionally showing that

$$\omega_0(A) = \omega_\sigma(A) \tag{1.14}$$

holds, and in determining $\omega_\sigma(A)$ explicitly. An explicit representation of the inverse $(\lambda - A)^{-1}$ and a sophisticated analysis of $\|(\lambda - A)^{-1}\|$ will yield the result. Then a fixed point argument for $\mathcal{A}$, where (1.8) will describe the contraction precisely, will be used.

We remark that a stronger requirement of the smallness of $\|a\|_{L^\infty}$ as discussed for wave equations by other authors, see [8, 2, 11] could be treated in an even easier way for our system too.

Summarizing, our new contributions are, assuming (1.6),

- to show that in the positive constant damping case, the system has the SDG property, and to give for this situation a precise computation of the rate of decay (and the type of the semigroup),
- to show that also for the case of possibly indefinite damping, the system is still exponentially stable, for small $\|a - \overline{a}\|_{L^2}$.

The paper is organized as follows. In Section 2 we shall formulate the semigroup setting, in Section 3 we discuss the constant coefficient case yielding the SDG property there, and in Section 4 we finish the discussion of the original indefinite problem obtaining the result on exponential stability.

2 The semigroup setting

We rewrite the initial-boundary value problem (1.1)–(1.4) as a first-order system for $V := (\varphi, \varphi_t, \psi, \psi_t)'$, where the prime is used to denote the transpose. Then V satisfies

$$V_t = \mathcal{A}V, \qquad V(t = 0) = V_0, \tag{2.1}$$

where $V_0 := (\varphi_0, \varphi_1, \psi_0, \psi_1)'$ and $\mathcal{A}$ is the (formal) differential operator

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ k/\rho_1 \partial_x^2 & 0 & k/\rho_1 \partial_x & 0 \\ 0 & 0 & 0 & 1 \\ -k/\rho_2 \partial_x & 0 & b/\rho_2 \partial_x^2 - k/\rho_2 & -a/\rho_2 \end{pmatrix}. \tag{2.2}$$

Let

$$H := H_0^1((0, L)) \times L^2((0, L)) \times H^1((0, L)) \times L^2((0, L))$$

be the Hilbert space with norm given by

$$\begin{aligned}\|V\|_H^2 &= \|(\phi^1,\phi^2,\psi^1,\psi^2)'\|_H^2\\ &\equiv \rho_1\|\phi^2\|_{L^2}^2 + b\|\psi_x^1\|_{L^2}^2 + k\|\phi_x^1+\psi^1\|_{L^2}^2 + \rho_2\|\psi^2\|_{L^2}^2.\end{aligned}$$

Then $\mathcal{A}$, formally given in (2.2), with domain

$$\begin{aligned}D(\mathcal{A}) &:= \{V\in H \mid \phi^1\in H^2((0,L)),\ \ \phi^2\in H_0^1((0,L)),\ \ \psi^1\in H^2((0,L)),\\ &\quad \psi_x^1\in H_0^1((0,L)),\ \ \psi^2\in L^2((0,L))\},\end{aligned}$$

generates a semigroup $\{\mathrm{e}^{tA}\}_{t\geq 0}$. We observe that for a solution (φ,ψ) to (1.2)–(1.4), and the corresponding V, the norm $\|V(t)\|_H^2$ equals twice the energy $E(t)$ of (φ,ψ) defined by

$$E(t) := \frac{1}{2}\int_0^L (\rho_1|\varphi_t|^2 + \rho_2|\psi_t|^2 + b|\psi_x|^2 + k|\varphi_x+\psi|^2)(t,x)dx. \tag{2.3}$$

Replacing the function $a=a(x)$ in (1.2) by the constant $\overline{a}$, we write A for the arising constant coefficient operator instead of $\mathcal{A}$. We shall first give in the next section a precise description of the spectrum of A, and we show that the SDG property holds for A.

3 The constant coefficient case

Since it is not difficult to see that the inverse of the operator A is compact, we have to determine the eigenvalues of A in a way that allows us to determine $\omega_\sigma(A)$ and to estimate the resolvent operators uniformly. Therefore, let

$$(A-\lambda)W = 0$$

with $\lambda\in\mathbb{C}\setminus\{0\}$ and $W\in D(A)$. Then $W=(\varphi,\lambda\varphi,\psi,\lambda\psi)'$, and (φ,ψ) satisfy

$$\rho_1\lambda^2\varphi - k(\varphi_x+\psi)_x = 0 \quad \text{ in } (0,L), \tag{3.1}$$

$$\rho_2\lambda^2\psi - b\psi_{xx} + k(\varphi_x+\psi) + \overline{a}\lambda\psi = 0 \quad \text{ in } (0,L), \tag{3.2}$$

together with the boundary conditions

$$\varphi(0)=\varphi(L)=\psi_x(0)=\psi_x(L)=0. \tag{3.3}$$

Observing the boundary conditions, we can reduce the system for (φ,ψ) to a single one for φ by differentiating (3.1) and using (3.2), yielding

$$bk\varphi_{xxxx} - [(k\rho_2+b\rho_1)\lambda^2 + k\overline{a}\lambda]\varphi_{xx} + (\rho_2\lambda^2+\overline{a}\lambda+k)\rho_1\lambda^2\varphi = 0, \tag{3.4}$$

$$\varphi(0)=\varphi(L)=\varphi_{xx}(0)=\varphi_{xx}(L)=0. \tag{3.5}$$

For (3.4), (3.5), we have a complete orthonormal system of eigenfunctions: $\varphi_j(x) = \sqrt{\frac{2}{L}}\sin(\theta_j x)$ with $\theta_j := (j\pi)/L$. Then $\lambda = \lambda_j$ has to satisfy

$$P(\lambda,\theta_j) \equiv \rho_1\rho_2\lambda^4 + \overline{a}\rho_1\lambda^3 + [(k\rho_2 + b\rho_1)\theta_j^2 + k\rho_1]\lambda^2 + k\overline{a}\theta_j^2\lambda + bk\theta_j^4 = 0.$$

Dividing by $\rho_1\rho_2\lambda^2$ we find

$$\lambda^2 + \frac{\overline{a}}{\rho_2}\lambda + [(\frac{k}{\rho_1} + \frac{b}{\rho_2})\theta_j^2 + \frac{k}{\rho_2}] + \frac{k\overline{a}}{\rho_1\rho_2}\theta_j^2\frac{1}{\lambda} + \frac{bk}{\rho_1\rho_2}\theta_j^4\frac{1}{\lambda^2} = 0.$$

Rearranging terms we get

$$\lambda^2 + \frac{bk}{\rho_1\rho_2}\theta_j^4\frac{1}{\lambda^2} + [(\frac{k}{\rho_1} + \frac{b}{\rho_2})\theta_j^2 + \frac{k}{\rho_2}] + \frac{\overline{a}}{\rho_2}\left\{\frac{k\theta_j^2}{\rho_1}\frac{1}{\lambda} + \lambda\right\} = 0 \tag{3.6}$$

Now assuming and using the identity (1.6), and defining

$$y := \frac{k\theta_j^2}{\rho_1}\frac{1}{\lambda} + \lambda, \tag{3.7}$$

we obtain from (3.6)

$$y^2 + \frac{\overline{a}}{\rho_2}y + \frac{k}{\rho_2} = 0. \tag{3.8}$$

This implies

$$y \equiv y_{1,2} = -\frac{\overline{a}}{2\rho_2} \pm \sqrt{\frac{\overline{a}^2}{4\rho_2^2} - \frac{k}{\rho_2}}. \tag{3.9}$$

Multiplying (3.7) by λ we get

$$\lambda^2 - \lambda y + \frac{k\theta_j^2}{\rho_1} = 0, \tag{3.10}$$

implying

$$\lambda = \frac{y}{2} \pm \sqrt{\frac{y^2}{4} - \frac{k\theta_j^2}{\rho_1}}. \tag{3.11}$$

Therefore the following set

$$B := \left\{\frac{y}{2} \pm \sqrt{\frac{y^2}{4} - \frac{k\theta_j^2}{\rho_1}} \;\mid\; y = -\frac{\overline{a}}{2\rho_2} \pm \sqrt{\frac{\overline{a}^2}{4\rho_2^2} - \frac{k}{\rho_2}}, \quad \theta_j = \frac{j\pi}{L},\ j \in \mathbb{N}\right\} \tag{3.12}$$

is the candidate for the (point) spectrum of A. For $\lambda_j \in B$, a possible eigenfunction W has the form $W_j = const.(\varphi_j, \lambda_j\varphi_j, \psi_j, \lambda_j\psi_j)'$, where ψ_j is determined from (3.1), (3.2) as

$$\psi_j = \psi_j(x) = const.\frac{b\rho_1\theta_j\lambda_j^2 + bk\theta_j^3 - k\theta_j}{k(\rho_2\lambda_j^2 + k + \overline{a}\lambda_j)}\cos(\theta_j x),$$

provided

$$\rho_2\lambda_j^2 + k + \overline{a}\lambda_j \neq 0. \tag{3.13}$$

The validity of (3.13) can be seen as follows.

Observing (3.8), (3.13) and

$$\lambda_j = \lambda_j^r,\ r = 1, 2, 3, 4$$

with

$$\lambda_j^{1,2} = \frac{y_1}{2} \pm \sqrt{\frac{y_1^2}{4} - \frac{k\theta_j^2}{\rho_1}},\ \lambda_j^{3,4} = \frac{y_2}{2} \pm \sqrt{\frac{y_2^2}{4} - \frac{k\theta_j^2}{\rho_1}} \tag{3.14}$$

we have to show

$$\lambda_j^r \neq y_m \text{ for } r = 1, 2, 3, 4 \text{ and } m = 1, 2$$

Let w.l.o.g. $m = 1$. Then, by (3.8), (3.13) we immediately have $\lambda_j^1 \neq y_1$, $\lambda_j^2 \neq y_1$. The assumption $\lambda_j^3 = y_1$ is equivalent to

$$\frac{y_2}{2} + \sqrt{\frac{y_2^2}{4} - \frac{k\theta_j^2}{\rho_1}} = y_1 \tag{3.15}$$

We can exclude the case that $\overline{a}^2 = 4\rho_2 k$, because this would imply that $\lambda_j^1 = \lambda_j^3 = y_1$, a contradiction. For the case that $\overline{a}^2 > 4\rho_2 k$ we conclude

$$0 > y_1 > y_2 > \frac{y_2}{2} + \sqrt{\frac{y_2^4}{4} - \frac{k\theta_j^2}{\rho_1}} = y_1$$

again a contradiction ($\sqrt{\ldots}$ being imaginary is impossible too).
In the last case that $\overline{a}^2 < 4\rho_2 k$, we conclude

$$\sqrt{\frac{y_2^2}{4} - \frac{k\theta_j^2}{\rho_1}} = y_1 - \frac{y_2}{2} = \frac{-\overline{a} + i\sqrt{4\rho_2 k - \overline{a}^2}}{4\rho_2}$$

implying, after taking squares and comparing the imaginary part,

$$8\overline{a}\sqrt{4\rho_2 k - \overline{a}^2} = 0,$$

or $4\rho_2 k = \overline{a}^2$, again a contradiction.
Altogether we proved (3.13) and hence

Theorem 3.1 *Assume (1.6). Then*

$$\sigma(A) = \left\{ \frac{y}{2} \pm \sqrt{\frac{y^2}{4} - \frac{k\theta_j^2}{\rho_1}} \ \middle|\ y = -\frac{\overline{a}}{2\rho_2} \pm \sqrt{\frac{\overline{a}^2}{4\rho_2^2} - \frac{k}{\rho_2}}, \quad \theta_j = \frac{j\pi}{L},\ j \in \mathbb{N} \right\}$$

Next we determine $\omega_\sigma(A)$ explicitly.
<u>Case I:</u> $\overline{a}^2 \geq 4\rho_2 k$.
This implies

$$y_{1/2} \in \mathbb{R}, \quad 0 > y_1 \geq y_2$$

hence

$$\max_{j\in\mathbb{N}} \max_{r=1,2,3,4} \mathrm{Re}\lambda_j^r = \mathrm{Re}\left\{\frac{y_1}{2} + \sqrt{\frac{y_1^2}{4} - \frac{k\theta_1^2}{\rho_1}}\right\}$$

$$= \frac{-\overline{a} + \sqrt{\overline{a}^2 - 4\rho_2 k}}{4\rho_2} + \mathrm{Re}\sqrt{\left[\frac{-\overline{a} + \sqrt{\overline{a}^2 - 4\rho_2 k}}{8\rho_2}\right]^2 - \frac{k\pi^2}{\rho_1 L^2}} < 0 \qquad (3.16)$$

Case II: $\overline{a}^2 < 4\rho_2 k$.
Then

$$y_{1/2} = \frac{-\overline{a} \pm i\sqrt{4\rho_2 k - \overline{a}^2}}{2\rho_2} \equiv \psi_1 \pm i\psi_2,\ \psi_j \in \mathbb{R}. \qquad (3.17)$$

Let $\xi_j := \frac{k\theta_j^2}{\rho_1}$, then

$$\sqrt{\frac{y_1^2}{4} - \xi_j} \equiv \alpha + i\beta, \quad \alpha, \beta \in \mathbb{R}, \quad \alpha \geq 0$$

with

$$\alpha^2 - \beta^2 = \frac{\psi_1^2 - \psi_2^2}{4} - \xi_j, \quad 2\alpha\beta = \frac{\psi_1\psi_2}{2}$$

implying

$$\alpha^4 - (\frac{\psi_1^2 - \psi_2^2}{4} - \xi_j)\alpha^2 - \frac{\psi_1^2 - \psi_1^2}{16} = 0,$$

that is

$$\alpha = \sqrt{\frac{1}{2}\left(\frac{\psi_1^2 - \psi_2^2}{4} - \xi_j\right) + \sqrt{\frac{1}{4}\left[\frac{\psi_1^2 - \psi_2^2}{4} - \xi_j\right]^2 + \frac{\psi_1^2 - \psi_2^2}{16}}} \qquad (3.18)$$

observing

$$\psi_1^2 - 4_2^2 = \frac{\overline{a}^2 - (4\rho_2 k - \overline{a}^2)}{4\rho_2^2}, \quad \psi_1^2 4_2^2 = \frac{\overline{a}^2}{4\rho_2^2}\left(\frac{4\rho_2 k - \overline{a}^2}{4\rho_2^2}\right)$$

we conclude form (3.14), (3.17) and (3.18)

$$\max_{r=1,2,3,4} \mathrm{Re}\lambda_j^r = -\frac{\overline{a}}{2\rho_2} + \sqrt{\frac{d_1 - \xi_j}{2} + \sqrt{\left(\frac{d_1 - \xi_j}{2}\right)^2 + d_2}} \qquad (3.19)$$

where

$$d_1 := \frac{\psi_1^2 - \psi_2^2}{4}, \quad d_2 := \frac{\psi_1^2\psi_2^2}{16}.$$

Since $x \longmapsto f(x) := \sqrt{\frac{d_1 - x}{2} + \sqrt{\left(\frac{d_1 - x}{2}\right)^2 + d_2}}$ takes for $x \geq \xi_1 = \frac{k\pi^2}{\rho_1 L^2}$ its Maximum in ξ_1, we have from (3.19)

$$\max_{j\in\mathbb{N}} \max_{r=1,2,3,4} \mathrm{Re}\lambda_j^r = -\frac{\overline{a}}{2\rho_2}$$

$$+ \sqrt{\underbrace{\frac{1}{2}\left(\left(\frac{\overline{a}}{2\rho_2}\right)^2 - \frac{4\rho_2 k - \overline{a}^2}{4\rho_2^2} - \frac{k\pi^2}{\rho_1 L^2}\right)}_{=:z} + \sqrt{z^2 + \frac{\overline{a}}{4\rho_2^2}\frac{(4\rho_2 k - \overline{a}^2)}{4\rho_2^2}}} < 0. \qquad (3.20)$$

Summarizing the cases I, II we conclude from (3.16), (3.20)

Theorem 3.2 *Assume (1.6). Then*

$$0 > \omega_\sigma(A) = \max_{j\in\mathbb{N}} \max_{r=1,2,3,4} \mathrm{Re}\lambda_j^r$$

given in (3.16) if $\overline{a}^2 \geq 4\rho_2 k$, and given in (3.20) if $\overline{a}^2 < 4\rho_2 k$, respectively.

Finally, we shall investigate $\|(\lambda - A)^{-1}\|$ for $\mathrm{Re}\lambda > \omega_\sigma$ and demonstrate the SDG property. Let $\lambda \in \mathbb{C}$, $\mathrm{Re}\lambda \geq \omega_\sigma + \varepsilon$ for some $\varepsilon > 0$. The equation

$$(\lambda - A)W = F$$

implies $W = (\varphi, \lambda\varphi - F^1, \psi, \lambda\psi - F^3)'$ and $(\varphi, 4)$ solve

$$\rho_1\lambda^2\varphi - k(\varphi_x + \psi)_x = f_1 \tag{3.21}$$

$$\rho_2\lambda^2\varphi - b\psi_{xx} + k(\varphi_x + \psi) + \overline{a}\lambda\psi = f_2, \tag{3.22}$$

$$\varphi(0) = \varphi(L) = \psi_x(0) = \psi_x(L) = 0, \tag{3.23}$$

where

$$f_1 := \rho_1 F^2 + \rho_1\lambda F^1, \quad f_2 := \rho_2 F^4 + \rho_2\lambda F^3 + \overline{a}F^3. \tag{3.24}$$

The boundary conditions admit the expansions

$$\varphi_{(x)} = \sum_{j=1}^{\infty} g_j v_{j(x)}, \quad \psi_x = \sum_{j=1}^{\infty} h_j w_{j(x)} \tag{3.25}$$

where

$$v_j(x) := \sqrt{\frac{2}{L}}\sin(\theta_j x), \quad w_j(x) := \sqrt{\frac{2}{L}}\cos(\theta_j x), \quad \theta_j := \frac{j\pi}{L} \tag{3.26}$$

Then we obtain from (3.21), (3.22), (3.25) the relations

$$(\rho_1\lambda^2 + k\theta_j^2)g_j + k\theta_j h_j = f_{1,j}, \tag{3.27}$$

$$k\theta_j g_j + (\rho_2\lambda^2 + b\theta_j^2 + \overline{a}\lambda + k)h_j = f_{2,j}, \tag{3.28}$$

where $(f_{1,j})_j$ and $(f_{2,j})_j$ denote the Fourier coefficients of f_1 and f_2, respectively.
We compute

$$g_j = \frac{f_{1,j}(\rho_2\lambda^2 + \overline{a}\lambda + b\theta_j^2 + k) - f_{2,j}\ k\theta_j}{\rho_1\rho_2\lambda^2(y^2 + \frac{\overline{a}}{\rho_2}y + \frac{k}{\rho_2})}, \tag{3.29}$$

$$h_j = \frac{-k\theta_j f_{1,j} + f_{2,j}(\rho_1\lambda^2 + k\theta_j^2)}{\rho_1\rho_2\lambda^2(y^2 + \frac{\overline{a}}{\rho_2}y + \frac{k}{\rho_2})}, \tag{3.30}$$

where we used the transformation as in (3.5) – (3.7).
We have to estimate

$$\int_0^L |\varphi_x(y)|^2 dy, \quad \int_0^L |\lambda\varphi(y)|^2 dy, \quad \int_0^L |\varphi_x(y)|^2 dy, \quad \int_0^L |\lambda\varphi(y)|^2 dy, \tag{3.31}$$

in terms of $\|F\|_X^2$.
Rewriting

$$g_j = \frac{f_{1,j}(\rho_2\lambda^2 + b\theta_j^2)}{\rho_1\rho_2\lambda^2(y^2 + \frac{\overline{a}}{\rho_2}y + \frac{k}{\rho_2})} + \frac{f_{1,j}(\overline{a}\lambda + k) - f_{2,j}\ k\theta_j}{\rho_1\rho_2\lambda^2(y^2 + \frac{\overline{a}}{\rho_2}y + \frac{k}{\rho_2})} \tag{3.32}$$

we hence first prove a bound for

$$\mathrm{I} := \frac{\theta_{\mathrm{j}}^2\ |\rho_2\lambda^2 + \mathrm{b}\theta_{\mathrm{j}}^2|^2}{|\rho_1\rho_2\lambda^2(\mathrm{y}^2 + \frac{\overline{\mathrm{a}}}{\rho_2}\mathrm{y} + \frac{\mathrm{k}}{\rho_2})|^2}$$

which is uniform in j and λ, for $\mathrm{Re}\lambda \geq \omega_\sigma + \varepsilon$.
Observing (3.7) and the essential condition (1.6) again, we obtain

$$\rho_2\lambda^2 + b\theta_j^2 = \rho_2\lambda y \tag{3.33}$$

we have

$$\mathrm{I} = \left|\frac{\theta_{\mathrm{j}}^2\ \mathrm{y}^2}{\rho_1\lambda^2(\mathrm{y}^2 + \frac{\overline{\mathrm{a}}}{\rho_2}\mathrm{y} + \frac{\mathrm{k}}{\rho_2})^2}\right|$$

and since, by (3.7),

$$\frac{y}{\lambda} = \frac{k\theta_j^2}{\rho_1\lambda^2} + 1$$

implying

$$\frac{\theta_j^2}{\rho_1\lambda^2} = \frac{y}{k\lambda} - 1 \tag{3.34}$$

we have

$$\mathrm{I} \leq \left|\frac{\mathrm{y}^3}{\mathrm{k}\cdot\lambda(\mathrm{y}^2 + \frac{\overline{\mathrm{a}}}{\rho_2} + \frac{\mathrm{k}}{\rho_2})^2}\right| + \left|\frac{\mathrm{y}^2}{\mathrm{y}^2 + \frac{\overline{\mathrm{a}}}{\rho_2} + \frac{\mathrm{k}}{\rho_2}}\right| \equiv \mathrm{I}_1 + \mathrm{I}_2 \tag{3.35}$$

without loss of generality we assume $|\lambda| \geq 1$. There is $R_0 > 0$ such that for $|y| \geq R_0$ we know

$$\left|\frac{y^3}{(y^2 + \frac{\overline{a}}{\rho_2} + \frac{k}{\rho_2})^2}\right| \leq 1 \tag{3.36}$$

while in the compact set $\{y \mid (y) \leq R_0\}$ the quadratic polynomial in y in the nominator does not have a zero since $\mathrm{Re}\lambda \geq \omega_\sigma + \varepsilon$. Therefore there is a positive $c = c(\varepsilon)$ such that for $(y) \leq R_0$ we have

$$\left|\frac{y^3}{(y^2 + \frac{\overline{a}}{\rho_2} + \frac{k}{\rho_2})^2}\right| \leq c^2(\varepsilon). \tag{3.37}$$

Thus we can estimate I_1, and similarly I_2, uniformly to obtain

$$\mathrm{I} \leq \mathrm{c}^2(\varepsilon). \tag{3.38}$$

The remaining terms in the representation of g_j in (3.32) are now estimated as follows.

$$\begin{aligned}
\mathrm{II} \ &:= \ \frac{\theta_j^2|\overline{a}\lambda+k|^2}{|\rho_1\rho_2\lambda^2(y^2+\frac{\overline{a}}{\rho_2}+\frac{k}{\rho_2})|^2} \\
&\leq \ \left|\frac{\overline{a}}{\rho_1\rho_2}\right|^2 \ \left|\frac{\theta_j}{\lambda}\right|^2 \ \frac{1}{|y^2+\frac{\overline{a}}{\rho_2}+\frac{k}{\rho_2}|^2}+\frac{k}{|\rho_1\rho_2\lambda^2(y^2+\frac{\overline{a}}{\rho_2}+\frac{k}{\rho_2})|^2} \\
&\leq \ \left|\frac{\overline{a}}{\rho_1\rho_2}\right|^2 \ \frac{|y|}{k|\lambda|}+\frac{1}{|y^2+\frac{\overline{a}}{\rho_2}+\frac{k}{\rho_2}|^2}+\frac{1}{k}\ \frac{1}{|y^2+\frac{\overline{a}}{\rho_2}+\frac{k}{\rho_2}|^2}+\frac{k}{|\lambda|^2} \\
&\leq \ \text{const.}
\end{aligned} \tag{3.39}$$

where we used (3.34).

$$\mathrm{III} := \frac{\theta_{\mathrm{j}}^2}{|\rho_1\rho_2\lambda^2(\mathrm{y}^2+\frac{\bar{\mathrm{a}}}{\rho_2}\mathrm{y}+\frac{\mathrm{k}}{\rho_2})|^2} \leq \frac{\mathrm{c}}{|\lambda|} \tag{3.40}$$

c denoting as usual a positive constant that may vary from line to line, similarly $c(\varepsilon)$. The estimates (3.38) – (3.40) imply

$$|\theta_j g_j|^2 \leq c^2(\varepsilon)\ (|f_{1,j}|^2+|f_{2,j}|^2) \tag{3.41}$$

The terms f_1 and f_2 contain λF^1 and λF^3, respectively. Observing

$$\lambda F_j^1 = \frac{\lambda}{\theta_j}\ \theta_j F_j^1 = \frac{\lambda}{\theta_j}\ (\partial_x F^1)_j$$

and the fact that the factor $|\frac{\lambda}{\theta_j}|$ does not affect the reasoning to obtain (3.41), we have proved:

$$\int_0^L |\varphi_x(y)|^2 dy \leq c^2(\varepsilon)\ \|F\|_{\mathcal{H}}^2 \tag{3.42}$$

where $c(\varepsilon)$ depends at most on ε, not on λ for $\mathrm{Re}\lambda \geq \omega_\sigma+\varepsilon$.
Since $\lambda = \frac{\lambda}{\theta_j}\theta_j$ we obtain analogously

$$\int_0^L |\lambda\varphi(y)|^2 dy \ + \int_0^L |\varphi_x(y)|^2 dy \ + \int_0^L |\lambda\varphi(y)|^2 dy \ \leq c^2(\varepsilon)\ \|F\|_{\mathcal{H}}^2 \tag{3.43}$$

hence we proved

$$\exists c(\varepsilon)>0\ \forall\lambda, \mathrm{Re}\lambda \geq \omega_\sigma+\varepsilon\ \forall \mathrm{F}\in\mathcal{H} : \|(\lambda-\mathrm{A})^{-1}\mathrm{F}\|_{\mathcal{H}} \leq \mathrm{c}(\varepsilon)\|\mathrm{F}\|_{\mathcal{H}}$$

which implies by [9] or [18]:

Theorem 3.3 *Assume (1.6). Then the SDG property holds for $A, \omega_0(A)=\omega_\sigma(A)$.*

4 Exponential stability for indefinite damping

We return to the original system (1.1) – (1.4), or (2.1), with an indefinite damping $a = a(x)$. It will be shown that the system is exponential stable if $\|a - \overline{a}\|_{L^2}$ is small enough. Of course, we keep the basic assumptions (1.5) and (1.6), i.e. we assume

$$\overline{a} = \frac{1}{L}\int_0^L a(y)dy > 0 \tag{4.1}$$

and

$$\frac{\rho_1}{k} = \frac{\rho_2}{b}. \tag{4.2}$$

Theorem 4.1 *Assume (4.1) and (4.2). Then there is $\tau > 0$ such that if $\|a - \overline{a}\|_{L^2} < \tau$ the system (2.1) is exponentially stable, that is, the energy E, defined in (2.3), to the initial boundary value problem (1.1) – (1.4) satisfies*

$$\exists\, d > 0\; \exists\, C > 0 \;\; \forall t \geq 0: \;\; E(t) \leq Ce^{-2dt}E(0).$$

PROOF: Recalling [9, 18] again, it suffices to show that for sufficiently small $\tau > 0$ and for λ with Re $\lambda \geq \omega_\sigma + \varepsilon$, for some $\varepsilon > 0$ such that $\omega_\sigma + \varepsilon < 0$, $(\lambda - A)W = F$ is uniquely solvable for any $F \in \mathcal{H}$, and $\|W\|_{\mathcal{H}} \leq C\, \|F\|_{\mathcal{H}}$ with a constant $C > 0$ at most depending on τ and ε. A fixed point argument will be used. To solve

$$(\lambda - \mathcal{A})W = F$$

is equivalent to solving

$$\begin{aligned}(\lambda - A)\, W &= F + (\mathcal{A} - A)W \\ &= F - (a - \overline{a})BW\end{aligned}$$

with

$$B := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\rho_2 \end{pmatrix}.$$

Let Re$\lambda > \omega_\sigma$, then W should satisfy $W = (\varphi, \lambda\varphi - F^1, \psi, \lambda\psi - F^3)'$ with (φ, ψ) satisfying, cp. (3.21) – (3.24),

$$\rho_1\lambda^2\varphi - k(\varphi_x + \psi)_x = f_1, \tag{4.3}$$

$$\rho_2\lambda^2\psi - b\psi_{xx} + k(\varphi_x + \psi) + \overline{a}\lambda\psi = (\overline{a} - a)\lambda\psi + f_2, \tag{4.4}$$

$$\varphi(0) = \varphi(L) = \psi_x(0) = \psi_x(L) = 0, \tag{4.5}$$

where

$$f_1 := \rho_1 F^2 + \rho_1\lambda F^1, \quad f_2 := \rho_2 F^4 + \rho_2\lambda F^3 + aF^3. \tag{4.6}$$

(4.4) can be rewritten as

$$\psi_{xx} - (\underbrace{\frac{\rho_2\lambda^2 + \overline{a}\lambda + k}{b}}_{\equiv\alpha^2})\psi = \frac{k}{b}\varphi_x + \frac{a - \overline{a}}{b}\lambda\psi - \frac{1}{b}f_2. \tag{4.7}$$

Let $\mathcal{N}_\alpha(g)$ denote the solution v to the Neumann problem

$$v_{xx} - \alpha^2 v = g, \qquad v_x(0) = v_x(L) = 0.$$

This is well defined if $\alpha^2 \neq -\frac{j^2\pi^2}{L}^2$, for $j = 0, 1, 2, \ldots$, which is guaranteed if

$$\mathrm{Re}\lambda > -\mathrm{Re}\frac{-\overline{a} + \sqrt{\overline{a}^2 - 4\rho_1 k}}{2\rho_1} =: z_0. \tag{4.8}$$

The sufficiency of (4.8) can be seen from

$$\alpha^2 = -\frac{j^2\pi^2}{L^2} \Leftrightarrow \lambda = \frac{-\overline{a} \pm \sqrt{\overline{a}^2 - 4\rho_1(k + \frac{bj^2\pi^2}{L^2})}}{2\rho_1}.$$

Thus, (4.7) can be written as

$$\psi = \mathcal{N}_\alpha\left(\frac{k}{b}\varphi_x + \frac{a - \overline{a}}{b}\lambda\psi - \frac{1}{b}f_2\right), \tag{4.9}$$

hence (4.3), (4.4) turn into

$$\rho_1\lambda^2\varphi - k(\varphi_x + \psi)_x = f_1, \tag{4.10}$$

$$\begin{aligned}\rho_2\lambda^2\psi - b\psi_{xx} + k(\varphi_x + \psi) + \overline{a}\lambda\psi = \\ (\overline{a} - a)\lambda\left\{\frac{k}{b}\mathcal{N}_\alpha(\varphi_x) + \frac{\lambda}{b}\mathcal{N}_\alpha((a - \overline{a})\psi) - \frac{1}{b}\mathcal{N}_\alpha(f_2)\right\} + f_2.\end{aligned} \tag{4.11}$$

For (v, w) let

$$G(v, w) := \frac{k}{b}\mathcal{N}_\alpha(v_x) + \frac{\lambda}{b}\mathcal{N}_\alpha((a - \overline{a})w) - \frac{1}{b}\mathcal{N}_\alpha(f_2)$$

and consider the mapping

$$P:\ H_0^1((0, L)) \times H^1((0, L)) \longrightarrow H_0^1((0, L)) \times H^1((0, L)),$$

$$(v, w) \mapsto (\varphi, \psi),$$

defined as solution (φ, ψ) to

$$\rho_1\lambda^2\varphi - k(\varphi_x + \psi)_x = f_1, \tag{4.12}$$

$$\rho_2\lambda^2\psi - b\psi_{xx} + k(\varphi_x + \psi) + \overline{a}\lambda\psi = (\overline{a} - a)\lambda G(v, w) + f_2, \tag{4.13}$$

$$\varphi(0) = \varphi(L) = \psi_x(0) = \psi_x(L) = 0 \tag{4.14}$$

which is well defined since $\lambda \in \varrho(A)$. As a norm in the space of definition of P we define

$$\|(v,w)\|_\lambda^2 := \int_0^L \rho_1|\lambda v|^2 + \rho_2|\lambda w|^2 + b|w_x|^2 + k|v_x + w|^2 dx.$$

We shall prove that P has a fixed point (φ,ψ) provided $\|a-\overline{a}\|_{L^2}$ is small enough. This fixed point is also a solution to (4.3) – (4.5), which can be seen as follows: Let (φ,ψ) be this fixed point, and let

$$\hat{\psi} := G(\varphi,\psi) = \frac{k}{b}\mathcal{N}_\alpha(\varphi_x) + \frac{\lambda}{b}\mathcal{N}_\alpha((a-\overline{a})\psi) - \frac{1}{b}\mathcal{N}_\alpha(f_2),$$

hence

$$\hat{\psi}_{xx} - \alpha^2\hat{\psi} = \frac{k}{b}\varphi_x + \frac{\lambda}{b}(a-\overline{a})\psi - \frac{1}{b}f_2,$$

$$\hat{\psi}_x(0) = \hat{\psi}_x(L) = 0,$$

implying

$$\rho_2\lambda^2\hat{\psi} - b\hat{\psi}_{xx} + k(\varphi_x + \hat{\psi}) + \overline{a}\lambda\hat{\psi} = \lambda(\overline{a}-a)\psi + f_2. \tag{4.15}$$

Since (φ,ψ) is a fixed point of P, we also have

$$\rho_2\lambda^2\psi - b\psi_{xx} + k(\varphi_x + \psi) + \overline{a}\lambda\psi = \lambda(\overline{a}-a)\hat{\psi} + f_2. \tag{4.16}$$

We conclude for the difference $\Psi := \hat{\psi} - \psi$

$$\Psi_{xx} - \alpha^2\Psi = \frac{\lambda(a-\overline{a})}{b}\Psi.$$

or

$$\Psi = \mathcal{N}_\alpha(\frac{\lambda(a-\overline{a})}{b}\Psi).$$

With the estimates for $\mathcal{N}_\alpha$ to be proved below, we can conclude, with positive constants c_1, c_2,

$$|\Psi| \le c_1\|(a-\overline{a})\Psi\|_{L^1} \le c_1\|a-\overline{a}\|_{L^2}\|\Psi\|_{L^2},$$

hence

$$\|\Psi\|_{L^2} \le c_2\|a-\overline{a}\|_{L^2}\|\Psi\|_{L^2},$$

implying $\Psi = 0$ if $\|a-\overline{a}\|_{L^2} < 1/c_2$.

Now we shall prove that P is a contraction mapping provided $\|a-\overline{a}\|_{L^2}$ is small enough. For this purpose let

$$(\varphi^j,\psi^j) := P(v^j,w^j), \quad j = 1,2,$$

and

$$(\varphi,\psi) := (\varphi^1-\varphi^2, \psi^1-\psi^2), \quad (v,w) := (v^1-v^2, w^1-w^2).$$

Then $(\varphi,\psi),\ (v,w)$ satisfy (4.12) – (4.14) with $f_1 = f_2 = 0$. Multiplying (4.12) and (4.13) by $\overline{\lambda\varphi}$ and $\overline{\lambda\psi}$, respectively, and integrating we obtain

$$\rho_1\lambda\int_0^L|\lambda\varphi|^2dx + k\overline{\lambda}\int_0^L(\varphi_x+\psi)\overline{\varphi_x}dx = 0,$$

$$\rho_2\lambda\int_0^L|\lambda\psi|^2dx + b\overline{\lambda}\int_0^L|\psi_x|^2dx + k\overline{\lambda}\int_0^L(\varphi_x+\psi)\overline{\psi}dx + \overline{a}\int_0^L|\lambda\psi|^2dx = \lambda\int_0^L(\overline{a}-a)G\overline{\lambda\psi}dx.$$

Summing up we get

$$\mathrm{Re}\lambda\,\|(\varphi,\psi)\|_\lambda^2 + \overline{a}\int_0^L|\lambda\psi|^2dx = \mathrm{Re}\left\{\lambda\int_0^L(\overline{a}-a)G\overline{\lambda\psi}dx\right\}. \tag{4.17}$$

Multiplying (4.13) by $\overline{\varphi_x+\psi}$ and integrating we get

$$\rho_2\lambda^2\int_0^L\psi(\overline{\varphi_x+\psi})dx + b\int_0^L\psi_x(\overline{\varphi_x+\psi})_xdx + k\int_0^L|\varphi_x+\psi|^2dx + \overline{a}\lambda\int_0^L\psi(\overline{\varphi_x+\psi})dx =$$

$$\lambda\int_0^L(\overline{a}-a)G(\overline{\varphi_x+\psi})dx.$$

Using the assumption (4.2) on the coefficients we have

$$(\overline{\varphi_x+\psi})_x = \frac{\rho_1}{k}\overline{\lambda^2\varphi} = \frac{\rho_2}{b}\overline{\lambda^2}\overline{\varphi}$$

implying

$$k\int_0^L|\varphi_x+\psi|^2dx + \rho_2(\lambda^2-\overline{\lambda}^2)\int_0^L\psi(\overline{\varphi_x+\psi})dx + \rho_2\overline{\lambda^2}\int_0^L|\psi|^2dx + \overline{a}\lambda\int_0^L\psi(\overline{\varphi_x+\psi})dx =$$

$$\lambda\int_0^L(\overline{a}-a)G(\overline{\varphi_x+\psi})dx.$$

Then we conclude

$$\frac{k}{2}\int_0^L|\varphi_x+\psi|^2dx \le c\left(\int_0^L|\lambda\psi|^2dx + |\lambda\int_0^L(a-\overline{a})G(\overline{\varphi_x+\psi})dx| + |\lambda\int_0^L(a-\overline{a})G\overline{\lambda\psi}dx|\right) \tag{4.18}$$

where c will denote constants at most depending on the coefficients. We also used the fact that it is sufficient to prove everything for λ such that $\mathrm{Re}\lambda\in[d_0,d_1]$ with some negative d_0 and some sufficiently large, but *fixed* d_1.

Multiplying (4.13) by $\overline{\psi}$ and integrating we obtain

$$\rho_2\lambda^2\int\limits_0^L|\psi|^2dx+b\int\limits_0^L|\psi_x|^2dx+k\int\limits_0^L(\varphi_x+\psi)\overline{\psi}dx+\overline{a}\lambda\int\limits_0^L|\psi|^2dx=\lambda\int\limits_0^L(\overline{a}-a)G\overline{\psi}dx. \tag{4.19}$$

It is not difficult to show that $0\in\varrho(\mathcal{A})$, hence

$$\exists\, z_1>0\ \exists\, c_1>0\ \forall\,\lambda, |\lambda|\le z_1:\quad \lambda\in\varrho(\mathcal{A})\ \wedge\ \|(\lambda-\mathcal{A})^{-1}\|\le c_1. \tag{4.20}$$

That is, we assume in the sequel w.l.o.g. $|\lambda|\ge z_1$. Then

$$\int\limits_0^L|\psi|^2\le\frac{1}{z_1^2}\int\limits_0^L|\lambda\psi|^2. \tag{4.21}$$

Combining (4.19), (4.18) and (4.21) we get

$$b\int\limits_0^L|\psi_x|^2dx\le c\left(|\lambda|\int\limits_0^L|(\overline{a}-a)||G\psi|dx+|\lambda|\int\limits_0^L|(\overline{a}-a)||G(\varphi_x+\psi)|dx+\right.$$
$$\left.\int\limits_0^L|\lambda\psi|^2dx+|\lambda|^2\int\limits_0^L|(\overline{a}-a)||G\psi|dx\right). \tag{4.22}$$

Multiplying (4.12) by $\frac{\overline{\lambda}}{\lambda}\varphi$ and integrating we obtain

$$\rho_1\int\limits_0^L|\lambda\varphi|^2dx\le c\left(\int\limits_0^L|\varphi_x+\psi|^2dx+\int\limits_0^L|\psi|^2dx\right). \tag{4.23}$$

We conclude from (4.18) and (4.23)

$$\frac{k}{4}\int\limits_0^L|\varphi_x+\psi|^2dx+\frac{\rho_1k}{4c}\int\limits_0^L|\lambda\varphi|^2dx\le$$
$$\left(c\int\limits_0^L|\lambda\psi|^2dx+|\lambda\int\limits_0^L(a-\overline{a})G(\overline{\varphi_x+\psi})dx|+|\lambda\int\limits_0^L(a-\overline{a})G\overline{\lambda\psi}dx|\right) \tag{4.24}$$

Combining (4.22) and (4.24) we get

$$\frac{k}{4}\int\limits_0^L|\varphi_x+\psi|^2dx+\frac{\rho_1k}{4c}\int\limits_0^L|\lambda\varphi|^2dx+b\int\limits_0^L|\psi_x|^2dx\le$$
$$\left(c\int\limits_0^L|\lambda\psi|^2dx+|\lambda\int\limits_0^L(a-\overline{a})G(\overline{\varphi_x+\psi})dx|+|\lambda\int\limits_0^L(a-\overline{a})G\overline{\lambda\psi}dx|\right) \tag{4.25}$$

Multiplying (4.25) by $\frac{\overline{a}}{2c}$ and combining it with (4.17) yields that there exists $\gamma_0 > 0$ such that we have

$$(\mathrm{Re}\lambda + \gamma_0)\|(\varphi,\psi)\|_\lambda^2 \leq c\left((|\lambda| + |\lambda|^2)\int_0^L |(\overline{a}-a)||G\psi|dx + |\lambda|\int_0^L |(\overline{a}-a)||G(\varphi_x+\psi)|dx\right). \tag{4.26}$$

It will now be demonstrated that the right-hand side R of (4.26) can be estimated by

$$|R| \leq c\|a-\overline{a}\|_{L^2}\|(\varphi,\psi)\|_\lambda\|(v,w)\|_\lambda. \tag{4.27}$$

For this purpose we recall that

$$G(v,w) = \frac{k}{b}\mathcal{N}_\alpha(v_x) + \frac{\lambda}{b}\mathcal{N}_\alpha((a-\overline{a})w).$$

We have the representation

$$\mathcal{N}_\alpha(g) = -\frac{1}{\alpha}\frac{\cosh(\alpha x)}{\sinh(\alpha L)}\int_0^L \cosh(\alpha(L-s))g(s)ds + \frac{1}{\alpha}\int_0^L \sinh(\alpha(x-s))g(s)ds.$$

Decomposing $\alpha = a_1 + \mathrm{i}a_2$ and $\lambda = \gamma + \mathrm{i}\eta$ into its real and imaginary part, respectively, we have

$$\exists\,\beta > 0\;\forall\,\lambda, \mathrm{Re}\lambda \in [d_0,d_1]:\;\; |a_1| \geq \beta, \quad a_2 = \mathcal{O}(|\eta|), \quad (|\eta| \to \infty),$$

(cp. similar considerations in [16]). This allows us to conclude that

$$|\mathcal{N}_\alpha(v_x)(s)| \leq c\|v_x\|_{L^2}, \quad |\lambda|^2|\mathcal{N}_\alpha((a-\overline{a})w)(s)| \leq c\|a-\overline{a}\|_{L^2}\|\lambda w\|_{L^2}.$$

Thus

$$|\lambda G(v,w)(s)| \leq c\|(v,w)\|_\lambda$$

which implies (4.27). Combining (4.26), (4.27) we get for

$$\mathrm{Re}\lambda > \gamma_0 \tag{4.28}$$

the estimate

$$\|(\varphi,\psi)\|_\lambda \leq c\|a-\overline{a}\|_{L^2}\|(v,w)\|_\lambda \leq d\|(v,w)\|_\lambda \tag{4.29}$$

for some $d < 1$ provided $\|a-\overline{a}\|_{L^2}$ is small enough. The thus existing unique fixed point (φ,ψ) of P is the unique solution to (4.3)–(4.5), as explained above, and thus yields the unique solution W to $(\lambda - \mathcal{A})W = F$ through

$$W = (\varphi, \lambda\varphi, \psi, \lambda\varphi)' + (0, -F^1, 0, -F^3)'$$

which implies, using (4.29),

$$\|(\varphi,\psi)\|_\lambda \leq \|W\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}. \tag{4.30}$$

Let $\widetilde{W}$ be the solution to $(\lambda - A)\widetilde{W} = F$, i.e.

$$\widetilde{W} = (\widetilde{\varphi}, \lambda\widetilde{\varphi}, \widetilde{\psi}, \lambda\widetilde{\varphi})' + (0, -F^1, 0, -F^3)'$$

with

$$(\widetilde{\varphi}, \widetilde{\psi}) = P((0,0))$$

Then we obtain, using (4.29), (4.30),

$$\|W\|_{\mathcal{H}} - \|\widetilde{W}\|_{\mathcal{H}} \leq \|W - \widetilde{W}\|_{\mathcal{H}} = \|(\varphi, \psi) - (\widetilde{\varphi}, \widetilde{\psi})\|_{\lambda} =$$

$$\|P((\varphi, \psi)) - P((\widetilde{\varphi}, \widetilde{\psi}))\|_{\lambda} \leq d\|(\varphi, \psi) - (\widetilde{\varphi}, \widetilde{\psi})\|_{\lambda} \leq d\|W\|_{\mathcal{H}} + d\|F\|_{\mathcal{H}},$$

hence

$$\begin{aligned} \|W\|_{\mathcal{H}} &\leq \frac{1}{1-d}\|\widetilde{W}\|_{\mathcal{H}} + \frac{d}{1-d}\|F\|_{\mathcal{H}} \\ &\leq c\|F\|_{\mathcal{H}} \end{aligned}$$

where we used

$$\|\widetilde{W}\|_{\mathcal{H}} \leq c\|F\|_{\mathcal{H}}$$

which is justified since $\lambda \in \varrho(A)$. Thus we have proved that for

$$\mathrm{Re}\lambda > \max\{-\gamma_0, z_0\}$$

where z_0 is given in (4.8) and γ_0 is given through (4.26), we have $\lambda \in \varrho(\mathcal{A})$ and the norm of the inverse $(\lambda - \mathcal{A})^{-1}$ is uniformly bounded in λ. This completes the proof of Theorem 4.1.

Q.E.D.

We remark that $\overline{a} = \frac{1}{L}\int_0^L a(x)dx$ could be replaced by any positive, fixed $\hat{a}$ yielding a result for a situation near an exponentially stable situation ($a(x) \equiv \hat{a}$). But since $\overline{a}$ depends on a and tends to zero as $\|a\|_{L^1}$ tends to zero (in particular if $\|a\|_{L^\infty} \to 0$), our result is not just a perturbation result, because in the case $a = 0$ there is no energy decay.

References

[1] Ammar Khodja, F., Benabdallah, A., Muñoz Rivera, J.E., Racke R.: Energy decay for Timoshenko systems of memory type. *J. Differential Equations* **194** (2003), 82–115.

[2] Benaddi, A., Rao, B.: Energy decay rate of wave equations with indefinite damping. *J. Differential Equations* **161** (2000), 337–357.

[3] Chen, G., Fulling, S.A., Narcowich, F.J., Sun, S.: Exponential decay of energy of evolution equations with locally distributed damping. *SIAM J. Appl. Math.* **51** (1991), 266–301.

[4] Cox, S.J., Overton, M.L.: Perturbing the critically damped wave equation. *SIAM J. Appl. Math.* **56** (1994), 231–243.

[5] Cox, S., Zuazua, E.: The rate at which energy decays in a damped string. *Comm. PDE* **19** (1994), 213–243.

[6] Freitas, P.: On some eigenvalue problems related to the wave equation with indefinite damping. *J. Differential Equations* **127** (1996), 320–335.

[7] Freitas, P., Krejčiřík, D.: Instability results for the damped wave equation in unbounded domains. *J. Differential Equations,* to appear.

[8] Freitas, P., Zuazua, E.: Stability results for the wave equation with indefinite damping. *J. Differential Equations* **132** (1996), 338–353.

[9] Huang, F.: Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces. Ann. Diff. Eqs. 1 (1985), 43–56.

[10] Liu, K., Liu, Z., Rao, B.: Exponential stability of an abstract nondissipataive linear system. *SIAM J. Control Optim.* **40** (2001), 149–165.

[11] Liu, K., Rao, B., Zhang, X.: Stabilization of the wave equations with potential and indefinite damping. *J. Math. Anal. Appl.* **269** (2002), 747–769.

[12] Liu, Z., Rao, B.: Energy decay rate of the thermoelastic Bresse system. Preprint (2004).

[13] Menz, G.: Exponential stability of wave equations with potential and indefinite damping. *Konstanzer Schriften Math. Inf.* **224** (2007).

[14] Muñoz Rivera, J.E., Racke, R.: Mildly dissipative nonlinear Timoshenko systems — global existence and exponential stability. *J. Math. Anal. Appl.* **276** (2002), 248–278.

[15] Muñoz Rivera, J.E., Racke, R.: Global stability for damped Timoshenko systems. *Discr. Cont. Dyn. Sys. B* **9** (2003), 1625–1639.

[16] Muñoz Rivera, J.E., Racke, R.: Exponential stability for wave equations with non-dissipative damping. *Nonlinear Analysis, T.M.A.* (accepted).

[17] Neves, A.F., Ribeiro, H. de S., Lopes, O.: On the spectrum of evolution operators generated by hyperbolic systems. *J. Functional Anal.* **67** (1986), 320–344.

[18] Prüss, J.: On the spectrum of C_0-semigroups. Trans. AMS **284** (1984), 847–857.

[19] Renardy, M.: On the type of certain C_0-semigroups. *Comm. PDE* **18** (1993), 1299–1307.

[20] Soufyane, A.: Stabilisation de la poutre de Timoshenko. *C. R. Acad. Sci. Paris, Sér. I* **328** (1999), 731–734.

Jaime E. Muñoz Rivera, Department of Research and Development, National Laboratory for Scientific Computation, Rua Getulio Vargas 333, Quitandinha, CEP 25651-070 Petrópolis, RJ, and UFRJ, Rio de Janeiro, Brazil
rivera@lncc.br

Reinhard Racke, Department of Mathematics and Statistics, University of Konstanz, 78457 Konstanz, Germany
reinhard.racke@uni-konstanz.de