Elastic and Electro-Magnetic Waves in Infinite Waveguides

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Abstract: We consider initial boundary value problems for the equations of isotropic elasticity for several mixed boundary conditions in infinite wave guides, as well as Maxwell equations. With the help of decompositions of the displacement field into divergence- and curl-free parts, respectively, which are compatible with the boundary conditions, we obtain sharp decay rates for the solutions. The decomposed systems correspond to the second-order Maxwell equations for the electric and the magnetic field with electric and magnetic boundary conditions, respectively.

1 Introduction

We start considering the equations of elasticity

$$u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla\nabla^* u = f \tag{1.1}$$

for the displacement vector $u : [0,\infty)\times\Omega \to \mathbb{R}^n,\ n = 2,3$, arising for isotropic media (cf. [9]), in domains $\Omega \subset \mathbb{R}^n$ with infinite boundaries of waveguide type, that is,

$$\Omega = \mathbb{R}^l \times B, \qquad B \subset \mathbb{R}^{n-l} \text{ bounded,}$$

where $1 \le l \le n-1$. Following are the typical examples:

$n = 2,\ l = 1$: Infinite strip.

$n = 3,\ l = 2$: Domain between two planes.

$n = 3,\ l = 1$: Infinite cylinder with cross section $B \subset \mathbb{R}^2$ having a smooth boundary ∂B.

The differential equations (1.1) for u are completed by initial conditions

$$u(t=0) = u^0, \qquad u_t(t=0) = u^1 \tag{1.2}$$

and by the boundary conditions (1.7) or (1.9) below.

In our paper [15], in particular, the classical (nonlinear) wave equation for the scalar function v,

$$v_{tt} - \Delta v = g(v, v_t, \nabla v, \nabla v_t, \nabla^2 v), \tag{1.3}$$

was investigated together with initial conditions and the Dirichlet boundary condition

$$v(t,\cdot) = 0 \qquad \text{on } \partial\Omega.$$

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Sharp decay rates of solutions to the linearized problem ($g = 0$ or $g = g(t, x)$) were proved, yielding interesting new phenomenon, and then nonlinear well-posedness results could be proved under some conditions on the nonlinearity g.

Now formulating the corresponding Dirichlet boundary conditions for the displacement vector,

$$u(t, \cdot) = 0 \qquad \text{on } \partial\Omega,$$

does not allow for carrying over the methods from [15], this already out of the simple reason that a decomposition of the Laplacean Δ as $\Delta = \Delta' + \Delta''$ according to the decomposition of the space variable into $x = (x', x'') \in \mathbb{R}^l \times B$ is not possible for the operator $E := -\mu\Delta - (\mu + \lambda)\nabla\nabla^*$ because of the mixing of components in the $\nabla\nabla^*$-part.

On the other hand, if we think for a moment of the Cauchy problem $\Omega = \mathbb{R}^n$, the well-known decomposition of vector fields into its curl-free and divergence-free components, respectively,

$$\left((L^2(\mathbb{R}^n))\right)^n = \overline{\nabla H^1(\mathbb{R}^n)} \oplus D_0(\mathbb{R}^n) \tag{1.4}$$

where $D_0(\mathbb{R}^n)$ denotes the divergence-free fields, leads to a decomposition of the displacement into

$$u = u^{po} + u^s$$

which decomposes the differential equation (1.1) ($f = 0$),

$$u_{tt} - \mu\Delta u - (\mu + \lambda)\nabla\nabla^* u = 0$$

into two classical wave equations for the two projections:

$$u^{po}_{tt} - (2\mu + \lambda)\Delta u^{po} = 0, \qquad u^s_{tt} - \mu\Delta u^s = 0. \tag{1.5}$$

But for our problem, a decomposition into curl- and divergence-free parts is not compatible with the Dirichlet boundary condition.

Our observation is that we can study several interesting boundary conditions and find appropriate decompositions into curl-free and divergence-free components, respectively. Here are some examples, more are given in Section 3.

Consider a strip in $\mathbb{R}^2$: $\Omega = \mathbb{R} \times (0, 1)$ and the boundary condition

$$u_1(t, \cdot) = \partial_{\vec{n}} u_2(t, \cdot) = 0 \qquad \text{on } \partial\Omega \tag{1.6}$$

where u_1, u_2 denote the components of u, and $\partial_{\vec{n}}$ denotes the normal derivative, $\vec{n}$ denoting the exterior normal, which is given by $\vec{n} = (0, \pm 1)^*$ in this example. The boundary conditions (1.6) correspond to an elastic movement on the boundary where the movement into the normal direction is free, but the object does not move into the x_1-direction (no shear movement).

We recall the following formulae for vector functions H and scalar functions h, respectively,

$$\nabla \times H = \partial_1 H_2 - \partial_2 H_1, \qquad \nabla \times h = (\partial_2 h, -\partial_1 h)^*$$

Then the formula for the vector Laplacean,

$$\Delta = \nabla\nabla^* - \nabla \times \nabla \times$$

holds in both space dimensions $n = 2, 3$, and we have correspondingly

$$\vec{n} \times H = \vec{n}_1 H_2 - \vec{n}_2 H_1, \qquad \vec{n} \times h = (\vec{n}_2 h, -\vec{n}_1 h)^*.$$

Now we observe that we can reformulate the boundary conditions (1.6) equivalently as

$$\vec{n}(\cdot) \times u(t, \cdot) = 0, \qquad \nabla^* u(t, \cdot) = 0 \qquad \text{on } \partial\Omega. \tag{1.7}$$

As second example we consider in the strip the boundary conditions

$$u_2(t, \cdot) = \partial_{\vec{n}} u_1(t, \cdot) = 0 \qquad \text{on } \partial\Omega \tag{1.8}$$

which represents a mere shear movement at the boundary, i.e. a free movement into the x_1-direction, but no movement in the normal direction. This boundary condition is now equivalent to

$$\vec{n}(\cdot) u(t, \cdot) = 0, \qquad \vec{n}(\cdot) \times (\nabla \times u(t, \cdot)) = 0 \qquad \text{on } \partial\Omega. \tag{1.9}$$

As third and fourth examples we consider the domain between two plates in three space dimensions: $\Omega = \mathbb{R}^2 \times (0, 1)$ and the boundary conditions

$$u_1(t, \cdot) = u_2(t, \cdot) = \partial_{\vec{n}} u_3(t, \cdot) = 0 \qquad \text{on } \partial\Omega, \tag{1.10}$$

or

$$\partial_{\vec{n}} u_1(t, \cdot) = \partial_{\vec{n}} u_2(t, \cdot) = \ u_3(t, \cdot) = 0 \qquad \text{on } \partial\Omega, \tag{1.11}$$

respectively. The third example (1.10) represents a mere movement into the normal direction, no shear movement, while the fourth one (1.11) represents a mere shear movement at the boundary. It turns out that the boundary conditions (1.10) are equivalent to the ones formulated in (1.7), and the conditions (1.11) are equivalent to those in (1.9). Therefore, we will concentrate on these boundary conditions (1.7) and (1.9), respectively, also for infinite cylinders in $\mathbb{R}^3$. For a more detailed discussion of the boundary conditions see Section 3.

We prove that the boundary conditions are compatible with certain (different) decompositions of L^2 into spaces of curl-free and diverengence-free functions, respectively, see Section 2. If we denote the associated decomposition of u by $u = u^{po} + u^s$, then for $\beta \in \{po, s\}$

$$u^\beta_{tt} - \tau \Delta u^\beta = f^\beta \tag{1.12}$$

with $\tau \in \{2\mu + \lambda, \mu\}$, and u^β satisfies one of the boundary conditions (1.7), (1.9). This means that u^{po}, u^s are solutions of the second-order Maxwell equations together with the so-called electric and magnetic boundary condition, respectively, cf. [26, 27, 28]. It is important that for solutions of the maxwell equations these boundary conditions split up in a way such that appropriate boundary conditions at the boundary of the cross section B appear (which was trivial for Dirichlet boundary conditions, for example). Therefore we are able to carry over considerations for the pure wave equation under Dirichlet's boundary condition from [15] for each of the subsystems for u^{po} and u^s.

Combining the results for u^{po} and u^s we obtain rates of decay for the solution u. For those components u_k of u, where the boundary conditions do not cause null spaces and hence stationary solutions in the cross section, the decay in the strip or the domain between two planes is the same as for the Cauchy problem, that is in absence of the obstacles. If the domain is an infinite cylinder, we loose order one half; in terms of the decay of the L^∞-norm this means that

$$\|u_k(t,\cdot)\|_{L^\infty} \leq \text{const} \cdot t^{-l/2} \qquad \text{as } t \to \infty.$$

This decay rate is the same as was observed in [15] for the pure wave equation. For all other components u_j of u we obtain

$$\|u_j(t,\cdot)\|_{L^\infty} \leq \text{const} \cdot t^{-(l-1)/2} \qquad \text{as } t \to \infty.$$

If we suppose additionally that the curl-free and diverengence-free parts of the data u^0, u^1, f are orthogonal to the null spaces, then all components of u show the better decay.

We remark that the knowledge of the decay rates of the linear system is an important tool in solving the corresponding nonlinear systems, cp. [15].

Linear and nonlinear wave equations (1.3) (extending to Klein-Gordon type) in waveguides providing sharp decay rates and giving global well-posedness results were first studied in our paper [15], then improvements leading to more admissible nonlinearities were given by Metcalfe, Sogge and Stewart [17]. Conical sets with infinite boundaries instead of waveguides were the subject of Dreher [6] proving decay rates in the linear situation. A discussion of resonance behavior in waveguides was given for classical wave equations and for Maxwell's equations by Werner [29, 30, 31], for elasticity with Dirichlet boundary conditions see Lesky [14]. Anisotropic situations like cubic or rhombic media were studied by Stoth, see [23] and [25], see also Doll [5], leading to interesting effects in comparison to the special isotropic case. The local energy decay of solutions to the linearized problem in exterior domains and for Dirichlet type boundary conditions was investigated by Iwashita and Shibata [8] and Dan [4].

The special boundary conditions (1.7) and (1.9), respectively, were considered in different contexts for bounded domains or in exterior domains — not in waveguides — in our work [21, 22, 24], see Section 2.

Summarizing our new contributions, we present the first results on sharp decay rates for solutions to initial boundary value problems for systems in elasticity as well as for classical Maxwell systems in infinite waveguides. The essential ingredients are various decompositions of the vector fields in appropriate spaces that are compatible with the boundary conditions, and extensions of techniques from the scalar wave equation case [15].

The paper is organized as follows: In Section 2 we formulate the setting, in particular we introduce appropriate spaces and discuss the decompositions. In Section 3 the boundary conditions are characterized with examples. Section 4 collects the necessary background in elliptic theory for the Laplacean resp. Maxwell operator with respect to the boundary conditions under investigation. The main results on the decay rates for the Maxwell and the elastic systems are given in Section 5 ($f = 0$) and in Section 6 ($f = f(t, x)$), respectively.

2 Spaces and well-posedness

The starting point is the system of (homogeneous, isotropic) elasticity (2.1)–(2.3), or (2.1), (2.2), (2.4), respectively:

$$u_{tt} - \mu\Delta u - (\mu + \lambda)\nabla\nabla^* u = f, \tag{2.1}$$

where $u : \mathbb{R} \times \Omega \to \mathbb{R}^n,\ n = 2, 3$, with the Lamé constants λ, μ satisfying $\mu > 0$, $2\mu + n\lambda > 0$ (cf. [9]), and

$$\Omega = \mathbb{R}^l \times B, \qquad B \subset \mathbb{R}^{n-l} \text{ bounded},$$

with $1 \le l \le n - 1$. The superscript * denotes transposition, e.g. ∇^* is the divergence operator. $f : \mathbb{R} \times \Omega \to \mathbb{R}^n$ is assumed to be smooth; further assumptions on the topology of B will be given Section 4.

The differential equations (2.1) for u are completed by initial conditions

$$u(t = 0) = u^0, \qquad u_t(t = 0) = u^1 \tag{2.2}$$

and either the boundary conditions

$$\vec{n}(\cdot) \times u(t, \cdot) = 0, \qquad \nabla^* u(t, \cdot) = 0 \qquad \text{on } \partial\Omega \tag{2.3}$$

or the boundary conditions

$$\vec{n}(\cdot)u(t, \cdot) = 0, \qquad \vec{n}(\cdot) \times (\nabla \times u(t, \cdot)) = 0 \qquad \text{on } \partial\Omega. \tag{2.4}$$

The well-posedness of the system (2.1), (2.2) with the boundary condition (2.3) has been studied by Leis [11], cp. also [22] for exterior domains or [24]. For the well-posedness in case of boundary conditions (2.4) see [21, 24]. We remark that after the following decompositions into divergence- and curl-free components, respectively, the two arising subsystems are well-posed based on the knowledge for Maxwell's equations, thus giving another direct well-posedness argument.

First we consider the boundary condition (2.3), and we use the following decomposition of $L^2 = (L^2(\Omega))^n$:

$$L^2 = \overline{\nabla H_0^1(\Omega)} \oplus D_0(\Omega)$$

where $D_0(\Omega)$ denotes the fields with divergence zero.

The decomposition follows from the projection theorem and decomposes u into

$$u = u^{po} + u^s, \qquad u^{po} \in \overline{\nabla H_0^1(\Omega)},\ u^s \in D_0(\Omega). \tag{2.5}$$

The compatibility of the boundary conditions (2.3) with the decomposition (2.5) is reflected in the decoupling of the differential equation for u *and* the decoupling of the boundary conditions as follows. u^s satisfies

$$u^s_{tt} + \mu\nabla \times \nabla \times u^s = f^s, \qquad \nabla^* u^s = 0, \tag{2.6}$$

$$u^s(t=0) = u^{0,s}, \qquad u^s_t(t=0) = u^{1,s}, \tag{2.7}$$

$$\vec{n}(\cdot) \times u^s(t,\cdot) = 0 \qquad \text{on } \partial\Omega. \tag{2.8}$$

The boundary condition (2.8) will be satisfied in the weak sense,

$$u^s(t,\cdot) \in R^0(\Omega)$$

where $R^0(\Omega)$ generalizes the classical boundary condition,

$$R^0(\Omega) := \{v \in L^2 \,|\, \nabla \times v \in L^2, \text{ and } \forall\, F \in L^2,\, \nabla \times F \in L^2 : \langle v, \nabla \times F\rangle = \langle \nabla \times v, F\rangle\}$$

with $\langle\cdot,\cdot\rangle$ denoting the inner product in L^2 and corresponding norm $\|\cdot\|$. $R^0(\Omega)$ equals the completion of C_0^∞-fields with respect to the norm $\|\cdot\|_R := (\|\cdot\|^2 + \|\nabla\times\cdot\|^2)^{1/2}$, cf. [12, 21]. u^{po} satisfies

$$u^{po}_{tt} - (2\mu+\lambda)\nabla\nabla^* u^{po} = f^{po}, \qquad u^{po} \in \overline{\nabla H^1_0(\Omega)}, \tag{2.9}$$

$$u^{po}(t=0) = u^{0,po}, \qquad u^{po}_t(t=0) = u^{1,po}, \tag{2.10}$$

$$\vec{n}(\cdot) \times u^{po}(t,\cdot) = 0, \qquad \nabla^* u^{po}(t,\cdot) = 0 \qquad \text{on } \partial\Omega. \tag{2.11}$$

We remark that the boundary condition $\vec{n}(\cdot)\times u^{po}_{\partial\Omega} = 0$ is satisfied in the weak sense automatically since $u^{po} \in \overline{\nabla H^1_0(\Omega)} \subset R^0(\Omega)$, the boundary condition $\nabla^* u^{po}(t,\cdot)_{\partial\Omega} = 0$ is also defined in the usual weak sense: $\nabla^* u(t,\cdot) \in H^1_0(\Omega)$.

Thus, we obtain for $\beta \in \{po, s\}$ that

$$u^\beta_{tt} - \tau_\beta\, \Delta u^\beta = f^\beta, \tag{2.12}$$

$$u^\beta(t=0) = u^{0,\beta}, \qquad u^\beta_t(t=0) = u^{1,\beta}, \tag{2.13}$$

$$\vec{n}(\cdot) \times u^\beta(t,\cdot) = 0, \qquad \nabla^* u^\beta(t,\cdot) = 0 \qquad \text{on } \partial\Omega \tag{2.14}$$

with

$$\tau_\beta := \begin{cases} 2\mu+\lambda & \text{if } \beta = po, \\ \mu & \text{if } \beta = s. \end{cases} \tag{2.15}$$

Note that $\tau_\beta > 0$ by our assumptions $\mu > 0$, $2\mu + n\lambda > 0$.

The initial boundary value problem (2.12)–(2.14) is of Maxwell type corresponding to the second order equation for the electric field with so-called electric boundary conditions, see [26, 27, 28, 12]. The existence theory is well-known. It has also been studied in [11, 22, 24]. In [22] exterior domains were studied and polynomial decay rates were given, while in [24] bounded domains and the question of exponential stability under the presence of thermal damping were discussed.

If we now turn to the boundary condition (2.4), we use the following decomposition

$$L^2 = R_0(\Omega) \oplus \overline{\nabla \times R^0(\Omega)},$$

where $R_0(\Omega)$ denotes the fields with vanishing rotation. Note in the case $n = 2$ that $R^0(\Omega)$ has to be taken as a space of scalar-valued functions and equals $H_0^1(\Omega)$. u is now decomposed into

$$u = u^{po} + u^s, \qquad u^{po} \in R_0(\Omega),\, u^s \in \overline{\nabla \times R^0(\Omega)}. \tag{2.16}$$

The same argument as above yields that

$$u_{tt}^{\beta} - \tau_\beta \,\Delta u^\beta = f^\beta, \tag{2.17}$$

$$u^\beta(t=0) = u^{0,\beta}, \qquad u_t^\beta(t=0) = u^{1,\beta}, \tag{2.18}$$

$$\vec{n}(\cdot) \times (\nabla \times u^\beta(t,\cdot)) = 0, \qquad \vec{n}(\cdot)\, u^\beta(t,\cdot) = 0 \qquad \text{on } \partial\Omega \tag{2.19}$$

for $\beta \in \{po, s\}$.

The first part of the boundary condition is interpreted in the sense $\nabla \times u^s(t,\cdot) \in R^0$. The second part $\vec{n}v = 0$ on $\partial\Omega$ is formulated in L^2 by saying $v \in D^0(\Omega)$ with

$$D^0(\Omega) := \{v \in L^2 \,|\, \nabla^* v \in L^2, \text{ and } \forall\, f \in H^1(\Omega) : \langle v, \nabla f\rangle = -\langle \nabla^* v, f\rangle\},$$

$D^0(\Omega)$ equals the completion of C_0^∞-fields with respect to the norm $\|\cdot\|_D := (\|\cdot\|^2 + \|\nabla^* \cdot\|^2)^{1/2}$, cf. [12, 21].

The initial boundary value problem (2.17)–(2.19) is of Maxwell type corresponding to the second order equation for the magnetic field with so-called magnetic boundary conditions, see [26, 27, 28, 12]. The existence theory is well-known.

Summarizing we have found that for both boundary conditions (2.3) and (2.4), respectively, a decomposition of the displacement vector $u = u^{po} + u^s$ into a curl-free component u^{po} and a divergence-free component u^s is possible which is compatible with the boundary conditions leading to similar systems for u^{po} and u^s that correspond to Maxwell's equations for the electric field with electric boundary conditions in case of boundary condition (2.3), and to Maxwell's equations for the magnetic field with magnetic boundary conditions in case of boundary condition (2.4).

Consequently, in order to obtain decay rates for the displacement vector finally, we will look at Maxwell's equations under electric and magnetic boundary conditions, respectively, in Section 5.

To realize the connection between the equations of elasticity with the boundary conditions under investigation is a basic element of this paper. In the next section we examine examples for the elastic boundary conditions in the typical situations.

3 The boundary conditions

We noticed that the boundary conditions for the displacement that are considered here, (2.3) resp. (2.4), are just those well-known for Maxwell's equations, the electric boundary condition for the electric field resp. the magnetic boundary condition for the magnetic field. Here we examine the typical meaning of these boundary conditions in elasticity for the three types of waveguides that arise in two and three space dimensions.

First we consider the two-dimensional case where we have essentially one situation, Ω being a strip with cross section $(0,1)$ without loss of generality,

$$\Omega = \mathbb{R} \times (0,1) \subset \mathbb{R}^2.$$

The first boundary conditions are (2.3), i.e.

$$\vec{n} \times u = 0, \qquad \nabla^* u = 0 \qquad \text{on } \partial\Omega. \tag{3.1}$$

Observing in two resp. one space dimension(s) the rules for "$\times$" between vectors (and scalars), given in the introduction, we have from (3.1) equivalently on $\partial\Omega$

$$\begin{aligned} -\vec{n}_2 u_1 + \vec{n}_1 u_2 &= 0, \\ \partial_1 u_1 + \partial_2 u_2 &= 0, \end{aligned}$$

where $\partial_j = \partial/\partial x_j,\ j = 1, 2(, 3)$. Since $\vec{n} = (0, \pm 1)^*$ and $\partial/\partial_{\vec{n}} = \pm\partial_2$, this is equivalent to

$$u_1 = \partial_{\vec{n}} u_2 = 0 \qquad \text{on } \partial\Omega, \tag{3.2}$$

and hence represents a free movement in the normal direction and no shear movement. The second boundary conditions are (2.4), i.e.

$$\vec{n} u = 0, \qquad \vec{n} \times (\nabla \times u) = 0 \qquad \text{on } \partial\Omega. \tag{3.3}$$

Observing in two resp. one space dimension(s) the rules for the curl "$\nabla\times$" given in the introduction, we have from (3.3) equivalently on $\partial\Omega$

$$\begin{aligned} \vec{n}_1 u_1 + \vec{n}_2 u_2 &= 0, \\ \vec{n}_2(-\partial_2 u_1 + \partial_1 u_2) &= 0, \\ \vec{n}_1(\partial_2 u_1 - \partial_1 u_2) &= 0, \end{aligned}$$

or, equivalently,

$$\partial_{\vec{n}} u_1 = u_2 = 0 \qquad \text{on } \partial\Omega. \tag{3.4}$$

Hence (3.3) represents a free shear movement without movement in the normal direction.

Second, we consider the three-dimensional case $n = 3$ with $l = 2$ where we have essentially one situation, Ω being the region between two plates. Without loss of generality we consider

$$\Omega = \mathbb{R}^2 \times (0,1) \subset \mathbb{R}^3.$$

The first boundary conditions (3.1) are now equivalent on $\partial\Omega$ to

$$\left.\begin{aligned} \vec{n}_2 u_3 - \vec{n}_3 u_2 &= 0, \\ \vec{n}_3 u_1 - \vec{n}_1 u_3 &= 0, \\ \vec{n}_1 u_2 - \vec{n}_2 u_1 &= 0, \\ \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 &= 0. \end{aligned}\right\} \tag{3.5}$$

Observing $\vec{n} = (0, 0, \pm 1)^*$ and $\partial/\partial_{\vec{n}} = \pm\partial_3$, this is equivalent to

$$u_1 = u_2 = \partial_{\vec{n}} u_3 = 0 \qquad \text{on } \partial\Omega. \tag{3.6}$$

This is the analogon to the two-dimensional version (3.2) representing a mere movement in the normal direction, no shear movement.

The second boundary conditions (3.3) are equivalent on $\partial\Omega$ to

$$\left.\begin{array}{rcl} \vec{n}_1 u_1 + \vec{n}_2 u_2 + \vec{n}_3 u_3 & = & 0, \\ \vec{n}_2(\partial_1 u_2 - \partial_2 u_1) - \vec{n}_3(\partial_3 u_1 - \partial_1 u_3) & = & 0, \\ \vec{n}_3(\partial_2 u_3 - \partial_3 u_2) - \vec{n}_1(\partial_1 u_2 - \partial_2 u_1) & = & 0, \\ \vec{n}_1(\partial_3 u_1 - \partial_1 u_3) - \vec{n}_2(\partial_2 u_3 - \partial_3 u_2) & = & 0, \end{array}\right\} \tag{3.7}$$

or, equivalently,

$$\partial_{\vec{n}} u_1 = \partial_{\vec{n}} u_2 = u_3 = 0 \qquad \text{on } \partial\Omega. \tag{3.8}$$

This is the analogon to the two-dimensional version (3.4) and represents a free shear movement without movement in the normal direction.

Third, we have the three-dimensional infinite cylinder ($n = 3$, $l = 1$),

$$\Omega = \mathbb{R} \times B \subset \mathbb{R}^3,$$

where $B \subset \mathbb{R}^2$ is a bounded domain.

For the first boundary conditions (3.1) we obtain from (3.5), observing $\vec{n} = (0, \vec{n}_2, \vec{n}_3)^*$,

$$u_1 = 0, \quad \vec{n}_2 u_3 - \vec{n}_3 u_2 = 0, \quad \partial_2 u_2 + \partial_3 u_3 = 0 \qquad \text{on } \partial\Omega. \tag{3.9}$$

For the second boundary condition (3.3) we obtain from (3.7)

$$\vec{n}_2 u_2 + \vec{n}_3 u_3 = 0, \quad \vec{n}_2(\partial_1 u_2 - \partial_2 u_1) - \vec{n}_3(\partial_3 u_1 - \partial_1 u_3) = 0, \quad \partial_2 u_3 - \partial_3 u_2 = 0 \quad \text{on } \partial\Omega. \tag{3.10}$$

The boundary conditions (3.9) and (3.10), respectively, become more transparent for cylindrically symmetrical domains. This is the following situation where $\Omega = \mathbb{R} \times B$ is a classical cylinder, i.e. B is radially symmetrical which in turn means

$$x'' \in B \implies \forall\, R \in O(2): \quad Rx'' \in B$$

where $O(n)$ will denote for $n = 2, 3$ the set of orthogonal $n \times n$ real matrices. The typical examples are balls or annular domains. Let now Ω be a classical cylinder. We define

Definition 3.1 *A vector field $u : \Omega \to \mathbb{R}^3$ is called cylindrically symmetrical, if*

$$\forall\, x_1 \in \mathbb{R} \;\; \forall\, x'' = (x_2, x_3) \in B \;\; \forall\, R \in O(2):$$

$$u_1(x_1, Rx'') = u_1(x_1, x''), \quad (u_2, u_3)^*(x_1, Rx'') = R(u_2, u_3)^*(x_1, x'').$$

That is, u is cylindrically symmetrical if, for fixed x_1, the first component as a scalar, and the second and third component together as a vector field are radially symmetrical in B, cp. for example [9]. Therefore we have the following characterization (cp. [9, Lemma 4.5]),

Lemma 3.2 *$u : \Omega \to \mathbb{R}^3$ is cylindrically symmetrical $\iff$ There exist functions h, $\phi : \mathbb{R} \times \mathbb{R}_0^+$ such that for all $(x_1, x'') \in \Omega$*

$$u_1(x_1, x'') = h(x_1, r), \quad (u_2, u_3)^*(x_1, x'') = x''\phi(x_1, r),$$

where $r := |x''| = \sqrt{x_2^2 + x_3^2}$.

Our initial-boundary value problem with differential equation (2.1) and $f = 0$, initial conditions (2.2) and boundary conditions (2.3) resp. (2.4) turns out to be cylindrically invariant, we have

Lemma 3.3 *Let Ω be cylindrically symmetrical. If the data u^0, u^1 are cylindrically symmetrical, then the solution $u(t, \cdot)$ to (2.1), (2.2), (2.3) resp. (2.4), $f = 0$, is cylindrically symmetrical for all $t \geq 0$.*

PROOF: Let $R = (r_{ij})_{1 \leq i,j \leq 2} \in O(2)$, and let

$$\tilde{R} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & r_{11} & r_{12} \\ 0 & r_{21} & r_{22} \end{pmatrix}.$$

Then $\tilde{R} \in O(3)$. Set for $t \geq 0$, $x = (x_1, x'') \in \Omega$

$$v(t, x) := \tilde{R}^* u(t, \tilde{R}x).$$

Since

$$v_{tt}(t, x) = \tilde{R}^* u_{tt}(t, \tilde{R}x), \quad \Delta v(t, x) = \tilde{R}^*(\Delta u)(t, \tilde{R}x), \quad \nabla\nabla^* v(t, x) = \tilde{R}^*(\nabla\nabla^* u)(t, \tilde{R}x)$$

we conclude that v satisfies the same differential equation as u. Since the initial data are cylindrically symmetrical we have

$$v(0, \cdot) = u^0, \quad v_t(0, \cdot) = u^1,$$

and hence v has the same initial values as u. By the uniqueness of solutions it only remains to show that v satisfies the same boundary conditions as u, that is the invariance of the boundary conditions under cylindrical symmetry.

For the first boundary conditions (2.3) this can be seen as follows: First note that

$$\begin{pmatrix} \vec{n}_2(x_1, Rx'') \\ \vec{n}_3(x_1, Rx'') \end{pmatrix} = R \begin{pmatrix} \vec{n}_2(x_1, x'') \\ \vec{n}_3(x_1, x'') \end{pmatrix}$$

and by $\vec{n}_1 = 0$

$$\vec{n}(\tilde{R}x) = \tilde{R}\,\vec{n}(x). \tag{3.11}$$

This implies

$$\begin{aligned}\vec{n}(x)\times\tilde{R}^*u(t,\tilde{R}x) &= \left(\tilde{R}^*\tilde{R}\,\vec{n}(x)\right)\times\tilde{R}^*u(t,\tilde{R}x)\\ &= \det(R)\,\tilde{R}^*\left(\left(\tilde{R}\,\vec{n}(x)\right)\times u(t,\tilde{R}x)\right) = \det(R)\,\tilde{R}^*\left(\vec{n}(\tilde{R}x)\times u(t,\tilde{R}x)\right) = 0 \qquad \text{on } \partial\Omega\end{aligned}$$

by $\vec{n}\times u = 0$ on $\partial\Omega$. Hence we have proved

$$\vec{n}\times v(t,\cdot) = 0 \qquad \text{on } \partial\Omega. \tag{3.12}$$

A short calculation shows that $\nabla^*\left(\tilde{R}^*u(t,\tilde{R}x)\right) = (\nabla^*u)(t,\tilde{R}x)$. Therefore we have

$$\nabla^*v(t,x) = (\nabla^*u)(t,\tilde{R}x) = 0 \qquad \text{on } \partial\Omega \tag{3.13}$$

because of the boundary conditions given for u. This proves that v satisfies the same boundary condition (2.3) as u.

For the second boundary condition (2.4) it is easy to see that on $\partial\Omega$

$$\vec{n}(x)v(t,x) = \left(\tilde{R}\,\vec{n}(x)\right)u(t,\tilde{R}x) = \vec{n}(\tilde{R}x)u(t,\tilde{R}x) = 0 \tag{3.14}$$

holds. Using

$$\nabla\times\left(\tilde{R}^*u(t,\tilde{R}x)\right) = \det(R)\,\tilde{R}^*\,(\nabla\times u)(t,\tilde{R}x)$$

we obtain in the same way

$$\vec{n}(x)\times(\nabla\times v(t,x)) = \tilde{R}^*\left(\vec{n}(\tilde{R}x)\times(\nabla\times u)(t,\tilde{R}x)\right) = 0 \qquad \text{on } \partial\Omega \tag{3.15}$$

With (3.14) and (3.15) we have that v satisfies the same boundary condition (2.4) as u. This finishes the proof of Lemma 3.3.

Q.E.D.

For a cylindrically symmetrical solution

$$u(t,x_1,x'') = \begin{pmatrix} h(t,x_1,r) \\ x''\phi(t,x_1,r) \end{pmatrix}$$

we can rewrite the second boundary condition (2.4) as

$$h_r = 0, \quad \phi = 0, \tag{3.16}$$

or

$$\partial_{\vec{n}}u_1 = u_2 = u_3 = 0, \tag{3.17}$$

cp. (3.6). The first boundary boundary condition (2.3) can be rewritten for a cylindrically symmetrical solution as

$$h = 0, \quad 2\phi + r\phi_r = 0, \tag{3.18}$$

the latter following from $\partial_2u_2 + \partial_3u_3 = 0$. (3.18) represents a kind of Robin type boundary condition for (u_2,u_3).

We finally remark that the boundary conditions (2.3) for elasticity were already studied by Weyl[1] [33]; in particular, he investigated the asymptotic distribution of eigenvalues of the associated stationary problem, cp. the next section.

4 Elliptic estimates

We write

$$x = (x', x'') \in \Omega = \mathbb{R}^l \times B,$$

$$\Delta = \sum_{j=1}^{n} \partial_j^2, \quad \Delta' = \sum_{j=1}^{l} \partial_j^2, \quad \Delta'' = \sum_{j=l+1}^{n} \partial_j^2.$$

Let

$$A : D(A) \subset L^2(\Omega) \to L^2(\Omega),$$

$$D(A) := W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), \quad A\varphi := -\Delta\varphi,$$

where we use standard notations for Sobolev spaces (cp. [1]), similarly

$$A' : D(A') \subset L^2(\mathbb{R}^l) \to L^2(\mathbb{R}^l),$$

$$D(A') := W^{2,2}(\mathbb{R}^l), \quad A'\varphi := -\Delta'\varphi,$$

$$A'' : D(A'') \subset L^2(B) \to L^2(B),$$

$$D(A'') := W^{2,2}(B) \cap W_0^{1,2}(B), \quad A''\varphi := -\Delta''\varphi.$$

The operators A, A', A'' are selfadjoint, A'' is positive definite with compact inverse, having a complete orthogonal set $(w_j)_{j\in\mathbb{N}}$ of eigenfunctions corresponding to positive eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ being arranged such that

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \to \infty, \quad \text{as } j \to \infty.$$

The spectrum of A' resp. A is purely continuous and consists of

$$\sigma(A') = [0, \infty), \quad \sigma(A) = [\lambda_1, \infty),$$

cp. e.g. [13]. The following lemma is taken from Section 2 in our previous work [15].

Lemma 4.1 *(L^p-regularity for the Laplace operator) Let $j \in \mathbb{N}_0$, $\varphi \in D(A)$, $1 < p < \infty$, $\varphi \in L^p(\Omega)$, $A\varphi \in W^{j,p}(\Omega)$. Then it holds*

$$\|\varphi\|_{W^{2+j,p}(\Omega)} \le c\|A\varphi\|_{W^{j,p}(\Omega)},$$

where c is a positive constant at most depending on j and p.

[1] Weyl gave a motivation as follows: *"Sie* `[the boundary condition (2.3)]` *wird für uns dadurch wesentlich, dass sie nach dem Schema $\ll$ Elastischer Körper $\longrightarrow$ FRESNELs elastischer Aether $\longrightarrow$ elektromagnetischer Aether $\gg$ den Uebergang von der Elastizitätstheorie zur Potentialtheorie zu Wege bringt."*

In the case $n \leq 4$ the assumption $\varphi \in L^p(\Omega)$ can be omitted. This can be seen approximating φ by $\varphi_k(x) := \psi(\frac{x'}{k})\varphi(x)$, where ψ denotes a cut-off function with respect to the unbounded variable $x' \in \mathbb{R}^l$. Then $\varphi_k \in L^p(\Omega) \cap D(A)$ and $\|A\varphi_k - A\varphi\|_{L^p(\Omega)} \to 0$. This proves:

Corollary 4.2 *Let $n \leq 4$, $j \in \mathbb{N}_0$, $\varphi \in D(A)$, $1 < p < \infty$, $A\varphi \in W^{j,p}(\Omega)$. Then $\varphi \in W^{2+j,p}(\Omega)$ and*

$$\|\varphi\|_{W^{2+j,\,p}(\Omega)} \leq c\|A\varphi\|_{W^{j,\,p}(\Omega)},$$

where c is a positive constant at most depending on j and p.

We define the two Maxwell operators M_1, M_2 with

$$M_j : D(M_j) \ \subset L^2 \to L^2$$

by

$$\begin{aligned} D(M_1) &:= \{u \in L^2 \,|\, u \in R^0(\Omega),\, \nabla^* u \in H_0^1(\Omega),\, \Delta u \in L^2\} && (4.1)\\ D(M_2) &:= \{u \in L^2 \,|\, u \in D^0(\Omega),\, \nabla \times u \in R^0(\Omega),\, \Delta u \in L^2\}, && (4.2)\\ M_j u &:= -\tau\Delta u, && (4.3) \end{aligned}$$

where τ will be either of μ, $(2\mu + \lambda)$. We have from [30, 31] that M_j is a positive self-adjoint operator with purely continuous spectrum

$$\sigma(M_j) = [\alpha, \infty)$$

where α satisfies

$$\alpha \left\{ \begin{array}{c} > \\ = \end{array} \right\} 0 \quad \text{if} \quad \left\{ \begin{array}{l} \text{j=1 and } B \text{ is simply connected} \\ \text{j=2 or } B \text{ is multiply connected} \end{array} \right\}. \tag{4.4}$$

The following assertion is an extension from Kozono and Yanagisawa [10], where the case of a bounded domain is studied.

Lemma 4.3 *(L^p-regularity for the Maxwell operators) Let $m \in \mathbb{N}_0$, $u \in D(M_j)$, $j = 1, 2$, $1 < p < \infty$, $u \in L^p(\Omega)$, $M_j u \in W^{m,p}(\Omega)$. Then it holds $u \in W^{m+2,p}(\Omega)$ and*

$$\|u\|_{W^{m+2,p}(\Omega)} \leq c\,\|(M_j + 1)u\|_{W^{m,p}(\Omega)},$$

where c is a positive constant at most depending on m and p (and j).

As for the case of the classical wave equation ("scalar Δ-operator with Dirichlet boundary conditions") previously studied in [15], we will need knowledge on the eigenvalue distribution for the different operators acting on the bounded cross section B. It is important that the original boundary conditions in Ω split up into reasonable boundary conditions for the components in the cross section.

We have the following six cases: space dimensions $n = 2, 3$, $1 \leq l \leq n - 1$, boundary conditions (2.3), (2.4). We know from Section 2 that the decompositions $u = u^{po} + u^s$ according to (2.5) resp. (2.16) carry over the boundary conditions (2.3) resp. (2.4) to both u^{po} and u^s, while the elastic operator A_j is turned into the Laplace operator. That is we will have to check what kind of boundary conditions arise for the components in the cross section. For these boundary conditions we need the distributions of the eigenvalues $(\rho_m)_m$ of the Laplace operator, not necessarily taking into account the subspaces where the vector functions live in, but just in all of L^2, and in ascending order as usual. Let $N(\rho)$ denote the number of eigenvalues ρ_m satsifying $\rho_m \le \rho$.

Case **I**: $n = 2,\ l = 1,$ boundary conditions (2.3):
By (3.2) we need the behavior for the Dirichlet and Neumann boundary conditions in $B = (0, 1)$, which is well-known:

$$\rho_m = (m\pi)^2, \quad N(\rho) \sim \rho^{\frac{1}{2}},$$

with $m = 1, 2, \ldots$ for the Dirichlet condition, and $m = 0, 1, 2, \ldots$ for the Neumann condition on ∂B.

Case **II**: $n = 2,\ l = 1,$ boundary conditions (2.4):
By (3.4) we have the same situation as in case I.

Case **III**: $n = 3,\ l = 2,$ boundary conditions (2.3):
By (3.6) we have the same situation as in case I.

Case **IV**: $n = 3,\ l = 2,$ boundary conditions (2.4):
By (3.8) we have the same situation as in case I.

Case **V**: $n = 3,\ l = 1,$ boundary conditions (2.3):
By (3.9) we have for fixed $x' \in \mathbb{R}$ at ∂B the Dirichlet condition for $u_1(x', \cdot)$ and the boundary conditions (2.3) for the components $(u_2, u_3)(x', \cdot)$ and the normal vector $(\vec{n_2}, \vec{n_3})(x', \cdot)$. For the Dirichlet condition we have the well-known estimate, see [2], going back to Weyl [32, 33]:

$$N(\rho) \sim \rho^{\frac{2}{2}}$$

implying

$$\rho_m \ge c m^{\frac{2}{2}},$$

where $c > 0$ is independent of m.

For the boundary condition (2.3) we have the same result. This was proved first by Weyl [32, 33] for a three-dimensional domain and extended by Mehra [16] to the two-dimensional situation, the situation we encounter here.

Case **VI**: $n = 3,\ l = 1,$ boundary conditions (2.4), cylindrical symmetry:
The boundary conditions (3.10) give the boundary conditions (2.4) for fixed $x' \in \mathbb{R}$ at ∂B for the components $(u_2, u_3)(x', \cdot)$ and the normal vector $(\vec{n_2}, \vec{n_3})(x', \cdot)$. The condition for u_1 is not separated in general, but for the assumed cylindrical symmetry we get the Neumann boundary condition for $u_1(x', \cdot)$, and the Dirichlet boundary condition for u_2, u_3, see (3.17). Hence we obtain the same asymptotics for the eigenvalues as in case V.

Summarizing we have:

Lemma 4.4 *The eigenvalues* $(\rho_m)_m$ *for the Laplace-operator studied in the cross section* B *under the different boundary conditions arising in the cases I–VI satisfy*

$$\rho_m \geq c m^{\frac{2}{n-l}}$$

where $c > 0$ *is independent of* m*.*

We remark that the Laplace operator is studied in certain subspaces of L^2 according to the decomposition of u into $u^{po} + u^s$, and maybe under radial symmetry. Still the estimate on the *lower bound* for ρ_j in Lemma 4.4 remains valid for these subspaces, in some cases not being sharp, compare e.g. the distribution of the eigenvalues of the Dirichlet Laplace operator for the radially symmetrical case of the unit disk in $\mathbb{R}^2$ with $N(\rho) \sim \rho^{\frac{1}{2}}$, as in the one-dimensional case, while $N(\rho) \sim \rho^{\frac{2}{2}}$ for a general two-dimensional bounded domain, see [2].

5 The decay of solutions, $f = 0$

In the first part of this section we study Maxwell's equation for the electric and magnetic field, respectively, with right-hand side $f = 0$. In the second part we apply the results obtained to the original elastic system with $f = 0$. Results for non-vanishing right-hand side will be given in the next section.

Let $z : [0, \infty) \times \Omega \to \mathbb{R}^n$ be the solution of Maxwell's equation

$$z_{tt} - \tau \Delta z = 0, \tag{5.1}$$

where $\tau > 0$, together with initial conditions

$$z(t = 0) = z^0, \qquad z_t(t = 0) = z^1 \tag{5.2}$$

and either electric boundary conditions (2.3), i.e.

$$\vec{n}(\cdot) \times z(t, \cdot) = 0, \qquad \nabla^* z(t, \cdot) = 0 \qquad \text{on } \partial\Omega, \tag{5.3}$$

or magnetic boundary conditions (2.4), i.e.

$$\vec{n}(\cdot) z(t, \cdot) = 0, \qquad \vec{n}(\cdot) \times (\nabla \times z(t, \cdot)) = 0 \qquad \text{on } \partial\Omega. \tag{5.4}$$

We consider again the six cases $n = 2, 3$, $1 \leq l \leq n - 1$, boundary conditions (5.3), (5.4), having been denoted by I–VI in Section 4.

Cases **I, II, III, IV, VI**: $n = 2$ or $n = 3, l = 2$ or $n = 3, l = 1$ with boundary conditions (5.4) and cylindrical symmetry:
By (3.2), (3.4), (3.6), (3.8) and (3.17), respectively, we observe that all components satisfy either the Dirichlet or the Neumann boundary condition, and that at least one component satisfies the Dirichlet boundary condition and at least one the Neumann boundary condition. Thus we obtain from [15, Thm. 3.3 plus Remark] the decay rates for the $\|z_k(t, \cdot)\|_{L^q(\Omega)}$-norm $(2 \leq q \leq \infty)$ of

the components of the solution, see the following Theorem. In case of the Dirichlet condition we obtain the rate $t^{-l/2}$ for $\|z_k(t,\cdot)\|_{L^\infty(\Omega)}$, while for the Neumann condition this only holds on the space orthogonal to the null space of the Neumann operator (the constant in B), otherwise we have the rate $t^{-(l-1)/2}$.

Case **V**: $\quad n=3,\ l=1,$ boundary conditions (5.3):
By (3.9) we have for fixed $x'\in\mathbb{R}$ at ∂B the Dirichlet condition for $z_1(t,x',\cdot)$ and the boundary conditions (5.3) for the components $(z_2,z_3)(t,x',\cdot)$ and the normal vector $(\vec{n_2},\vec{n_3})(x',\cdot)$. Therefore z_1 can be treated as before, and for (z_2,z_3) the methods from [15] carry over, we sketch the steps:
The null space of M_1'' — denoting M_1 as operator in B instead of Ω now — equals $R_0(B)\cap R^0(B)\cap D_0(B)$, the dimension of which is 1, hence being generated by some $w_0=w_0(x'')\in\mathbb{R}^2$, see e.g. [18, 19, 20, 28]. Let $(w_m)_m$ denote the eigenfunctions of M_1 (in B) with eigenvalues $(\rho_m)_m$, $m=0,1,2,\ldots$, with $\rho_0=0$ and $0<\rho_1\le\rho_2\le\ldots$. Set $\tilde z:=(z_2,z_3)=(z_2,z_3)(t,x',x'')$ and

$$v_m(t,x'):=\langle\tilde z(t,x',\cdot),w_m\rangle_{L^2(B)},$$

for $m\in\mathbb{N}\cup\{0\}$. Then v_m satisfies

$$v_{m,tt}-\Delta' v_m+\rho_m v_m=0 \qquad \text{in } [0,\infty)\times\mathbb{R}^1, \tag{5.5}$$

$$v_m(0,x')=v_m^0(x'):=\langle\tilde z(0,x',\cdot),w_m\rangle_{L^2(B)}, \qquad \text{in } \mathbb{R}^1, \tag{5.6}$$

$$v_{m,t}(0,x')=v_m^1(x'):=\langle\tilde z_t(0,x',\cdot),w_m\rangle_{L^2(B)}, \qquad \text{in } \mathbb{R}^1. \tag{5.7}$$

Thus v_m satisfies a Klein-Gordon equation in $\mathbb{R}^1$ for $m\ge1$, and a pure wave equation ($\rho_0=0$) for $m=0$. Hence we have for $m\ge1$ the decay of v_m in L^∞ of order $t^{-1/2}$, and for v_0 no decay.

Expanding $\tilde z$ into a Fourier series with respect to the $(w_m)_m$ in $L^2(B)$ one can get the decay rate for $\tilde z$ from that of the v_m, here using Lemma 4.4 in a series argument, as in [15]. The L^2-L^2 energy estimate is also given, so we obtain the L^p-L^q-decay by interpolation. In this single case V, the interpolation result seems not yet to be given in the literature, but is expected to hold, cp. Guidetti [7] for the interpolation argument. The extension to the boundary conditions (5.3) we take as hypothesis here.

We summarize our results, using the estimates for wave equations from [15]. In order to have one unified result for all cases, we introduce a condition reflecting possible parts in the arising null spaces in case of the Neumann boundary condition and the electric boundary condition in case V.

Condition (N): *The initial data $z^0(x',\cdot)$, $z^1(x',\cdot)$ from (5.2) satisfy for every fixed $x'\in\mathbb{R}^l$ that their projections onto the null spaces of the operator's part in cross-direction (constant functions for the Neumann condition in cases I,II,III,IV,VI; in case V:* span $\{w_0\}$*) vanish.*

Theorem 5.1 *(Maxwell systems, $f=0$) Assume condition (N). Let z be the unique solution to (5.1)–(5.3) or (5.1), (5.2) and (5.4), and let $2\le q\le\infty$, $\quad 1/p+1/q=1$. Let*

$$K_2:=\left[\frac{n}{2}\right]+\left[\frac{n-l}{2}\right]+3,\quad K_3:=\left[\frac{l+3}{2}\right]+\left[\frac{n-l+1}{2}\right],$$

and

$$z_0 \in D(M_j^{K_2/2}) \cap W^{K_2+K_3+1,\,1}(\Omega),$$
$$z_1 \in D(M_j^{(K_2-1)/2}) \cap W^{K_2+K_3,\,1}(\Omega),$$

$j = 1, 2$ *corresponding to boundary condition (5.3) and (5.4), respectively. Then* z *satisfies*

$$\|(z(t), z_t(t), \nabla z(t))\|_{L^q(\Omega)} \le \frac{c}{(1+t)^{(1-\frac{2}{q})\frac{l}{2}}} \|(z_0, z_1, \nabla z_0)\|_{W^{\tilde{N}_p,\,p}(\Omega)},$$

where

$$\tilde{N}_p := \begin{cases} (1-\frac{2}{q})(K_2+K_3) & \text{if} \quad q \in \{2, \infty\}, \\ [(1-\frac{2}{q})(K_2+K_3)] + 1 & \text{if} \quad 2 < q < \infty, \end{cases}$$

and c *depends at most on* q *and* B.

In view of the applications to corresponding nonlinear systems, we remark that we have in Theorem 5.1 a real decay of the L^∞-norm like t^{-r}, with $r > 0$, in the cases III and IV *always* ($r = 1$ if condition (N) is satisfied, $r = 1/2$ otherwise), and in the other cases (only) if condition (N) is satisfied, then with $r = 1/2$.

Now let u be the solution to the original system (2.1), (2.2) with boundary condition (2.3) or (2.4), respectively. As in Section 2, we decompose $u = u^{po} + u^s$ according to

$$L^2(\Omega) = \mathcal{H}^{po} \oplus \mathcal{H}^s \tag{5.8}$$

where $\mathcal{H}^{po}$ and $\mathcal{H}^s$ are determined in the given decompositions in (2.5) resp. (2.16), which themselves depend on the choice of boundary conditions (2.3) resp. (2.4). Then in each case

$$z := u^\beta, \quad \beta \in \{po, s\}$$

satisfies (5.1), (5.2) and (5.3) or (5.4), respectively (cp. (2.7)–(2.10) and (2.17)–(2.19)). The initial data for z are given by the projections of u^0, u^1 onto the spaces $\mathcal{H}^{po}$ and $\mathcal{H}^s$, respectively:

$$z^0 = P^{po}u^0, \; z^1 = P^{po}u^1 \text{ if } \beta = po \quad \text{and} \quad z^0 = P^s u^0, \; z^1 = P^s u^1 \text{ if } \beta = s. \tag{5.9}$$

We apply the result of Theorem 5.1 and obtain for u (e.g.), and $q < \infty$,

$$\begin{aligned} \|u(t,\cdot)\|_{L^q(\Omega)} &\le \|u^{po}(t,\cdot)\|_{L^q(\Omega)} + \|u^s(t,\cdot)\|_{L^q(\Omega)} \\ &\le \frac{c}{(1+t)^{(1-\frac{2}{q})\frac{l}{2}}} \Big(\|(P^{po}u^0, P^{po}u^1, \nabla P^{po}u^0)\|_{W^{\tilde{N}_p,\,p}(\Omega)} + \\ &\qquad \|(P^s u^0, P^s u^1, \nabla P^s u^1)\|_{W^{\tilde{N}_p,\,p}(\Omega)} \Big). \end{aligned}$$

Thus we have proved, defining the elasitic operators E_1, E_2 as follows,

$$E_j : D(E_j) \subset L^2 \to L^2$$

by

$$\begin{aligned} D(E_1) &:= \{u \in L^2 \,|\, u \in R^0(\Omega),\, \nabla^* u \in H_0^1(\Omega),\, (\mu\Delta + (\mu+\lambda)\nabla\nabla^*)u \in L^2\} && (5.10) \\ D(E_2) &:= \{u \in L^2 \,|\, u \in D^0(\Omega),\, \nabla \times u \in R^0(\Omega),\, (\mu\Delta + (\mu+\lambda)\nabla\nabla^*)u \in L^2\}, && (5.11) \\ E_j u &:= (\mu\Delta + (\mu+\lambda)\nabla\nabla^*)u, && (5.12) \end{aligned}$$

Theorem 5.2 *(Elastic systems, $f = 0$) Assume that the initial data $z^0 = P^{po}u^0, z^1 = P^{po}u^1$ and $z^0 = P^s u^0, z^1 = P^s u^1$ satisfy condition (N). Let u be the unique solution to (2.1)–(2.3) resp. to (2.1), (2.2, (2.4), and let $2 \leq q < \infty, \quad 1/p + 1/q = 1$. Let*

$$K_2 := \left[\frac{n}{2}\right] + \left[\frac{n-l}{2}\right] + 3, \quad K_3 := \left[\frac{l+3}{2}\right] + \left[\frac{n-l+1}{2}\right],$$

and

$$u^0 \in D(E_j^{K_2/2}) \cap W^{K_2+K_3+1,\,1}(\Omega),$$
$$u^1 \in D(E_j^{(K_2-1)/2}) \cap W^{K_2+K_3,\,1}(\Omega),$$

$j = 1, 2$ corresponding to boundary condition (5.3) and (5.4), respectively. Then u satisfies

$$\|(u(t), u_t(t), \nabla u(t))\|_{L^q(\Omega)} \leq \frac{c}{(1+t)^{(1-\frac{2}{q})\frac{l}{2}}} \left(\|(P^{po}u^0, P^{po}u^1, \nabla P^{po}u^0)\|_{W^{\tilde{N}_p,\,p}(\Omega)} + \|(P^s u^0, P^s u^1, \nabla P^s u^0)\|_{W^{\tilde{N}_p,\,p}(\Omega)}, \right)$$

where

$$\tilde{N}_p := \begin{cases} (1-\frac{2}{q})(K_2+K_3) & \text{if} \quad q \in \{2, \infty\}, \\ [(1-\frac{2}{q})(K_2+K_3)] + 1 & \text{if} \quad 2 < q < \infty, \end{cases}$$

and c depends at most on q and B.

In order to remove the projections in the estimate of the last theorem we need to know the continuity of the projections P^β onto the space $\mathcal{H}^\beta$, $\beta \in \{po, s\}$, in the Sobolev space $W^{N,p}(\Omega)$,

$$\|P^\beta v\|_{W^{N,p}} \leq \text{const}\, \|v\|_{W^{N,p}}. \tag{5.13}$$

For bounded domains we could refer for $1 < p < \infty$ to Kozono and Yanagisawa [10], where the case $N = 0$ is discussed in detail. For our waveguides, we first present a proof for the decomposition (2.5), used for the boundary conditions (2.3).

Theorem 5.3 *Let $1 < p < \infty$, $m \in \mathbb{N}$. Let*

$$X := \left(L^2(\Omega) \cap W^{m,p}(\Omega)\right)^n, \quad Y := \overline{\nabla H_0^1(\Omega)} \cap (W^{m,p}(\Omega))^n, \quad Z := D_0(\Omega) \cap (W^{m,p}(\Omega))^n$$

with natural norm $\|\cdot\|_{W^{m,p}(\Omega)} + \|\cdot\|_{L^2(\Omega)}$. Then we have the direct sum decomposition

$$X = Y \oplus Z$$

and the projections P_Y onto Y and P_Z onto Z, respectively, are continuous.

Proof: First, using the Poincaré inequality, we observe that

$$\overline{\nabla H_0^1(\Omega)} = \nabla H_0^1(\Omega),$$

and that X is a Banach space, and Y and Z are subspaces. Let $u \in X$, then the decomposition (2.5) in L^2 yields

$$u = u_1 + u_2, \quad u_1 = \nabla g \in \nabla H_0^1(\Omega), \quad u_2 \in D_0.$$

g can be determined by solving

$$\Delta g = \nabla^* u, \quad g \in H_0^1(\Omega),$$

the solution g of which has (elliptic) regularity $g \in W^{m+1,p}(\Omega)$ (cp. Corollary 4.2). Hence

$$u_1 \in W^{m,p}(\Omega).$$

Then $u_2 = u - u_1 \in W^{m,p}(\Omega)$ too. This proves the direct decomposition $X = Y + Z$. To prove the continuity of the projections it is now sufficient to show the closedness of Y *and* Z (cp. [3, p.189]). But this easily follows for Y using the Poincaré estimate again, and for Z using the definition of D_0.

Q.E.D.

In the application to the estimate of u in Theorem 5.2 we note that $1 < p \leq 2$ and $m = \tilde{N}_p \geq \frac{n}{2}$. Then we obtain from Theorem 5.3 and Sobolev's inequality

$$\|P^\beta u^j\|_{W^{m,p}} \leq c \left(\|u^j\|_{W^{m,p}} + \|u^j\|_{L^2}\right) \leq c\,\|u^j\|_{W^{m,p}}. \tag{5.14}$$

Now we give a corresponding result in the case $n = 2$ for the decomposition (2.16), used for the boundary condition (2.4). We remark that the case $n = 3$ remains open.

Theorem 5.4 *Suppose that $n = 2$ and that $1 < p < \infty$, $m \in \mathbb{N}$,*

$$u \in W^{m,p}(\Omega) \cap \{u \mid \nabla \times u \in L^2(\Omega)\}.$$

Then $P^\beta u \in W^{m,p}(\Omega)$, and there is a constant $c > 0$ independent of u such that

$$\|P^\beta u\|_{W^{m,p}} \leq c\,\|u\|_{W^{m,p}} \qquad (\beta \in \{po, s\}).$$

Proof: Let g be the (scalar-valued) solution to

$$-\Delta g = \nabla \times u, \quad g \in H_0^1(\Omega) = R^0(\Omega).$$

Then

$$\nabla \times (u - \nabla \times g) = \nabla \times u - \nabla \times \begin{pmatrix} \partial_2 g \\ -\partial_1 g \end{pmatrix} = \nabla \times u - \Delta g = 0$$

and hence $u - \nabla \times g \in R_0(\Omega)$. This implies

$$u = (u - \nabla \times g) + \nabla \times g, \quad u - \nabla \times g \in R_0(\Omega), \quad \nabla \times g \in \nabla \times R^0(\Omega)$$

which means that

$$P^{po} u = \nabla \times g \in W^{m,p}(\Omega), \qquad P^s u = u - \nabla \times g \in W^{m,p}(\Omega),$$

using elliptic regularity. Now we obtain from Corollary 4.2 that

$$\|P^{po} u\|_{W^{m,p}} \leq \|g\|_{W^{m+1,p}} \leq c\|\nabla \times u\|_{W^{m-1},p} \leq c\|u\|_{W^{m,p}}.$$

The boundedness of P^s follows from $P^s u = u - P^{po}u$.

Q.E.D.

Using Theorems 5.3 and 5.4, we immediately conclude from Theorem 5.2:

Theorem 5.5 *(Elastic systems, $f = 0$) Assume that the initial data $z^0 = P^{po}u^0, z^1 = P^{po}u^1$ and $z^0 = P^s u^0, z^1 = P^s u^1$ satisfy condition (N). Let u be the unique solution to (2.1)–(2.3) resp. to (2.1), (2.2), (2.4), in the latter case assuming $n = 2$, and let $2 \le q < \infty$, $\quad 1/p+1/q = 1$. Let*

$$K_2 := \left[\frac{n}{2}\right] + \left[\frac{n-l}{2}\right] + 3, \quad K_3 := \left[\frac{l+3}{2}\right] + \left[\frac{n-l+1}{2}\right],$$

and

$$u^0 \in D(E_j^{K_2/2}) \cap W^{K_2+K_3+1,1}(\Omega),$$
$$u^1 \in D(E_j^{(K_2-1)/2}) \cap W^{K_2+K_3,1}(\Omega),$$

$j = 1, 2$ corresponding to boundary condition (5.3) and (5.4), respectively. Then u satisfies

$$\|(u(t), u_t(t), \nabla u(t))\|_{L^q(\Omega)} \le \frac{c}{(1+t)^{(1-\frac{2}{q})\frac{l}{2}}} \|(u^0, u^1, \nabla u^0)\|_{W^{\tilde{N}_p, p}(\Omega)},$$

where

$$\tilde{N}_p := \begin{cases} (1-\frac{2}{q})(K_2+K_3) & \text{if } \ q \in \{2, \infty\}, \\ [(1-\frac{2}{q})(K_2+K_3)] + 1 & \text{if } \ 2 < q < \infty, \end{cases}$$

and c depends at most on q and B.

6 The decay of solutions, general $f = f(t, x)$

As in [15] we have to overcome the difficulty that one cannot directly apply Duhamel's principle of the variation of constants because of the boundary conditions to be observed. Here, additionally, one has operators that may have zero in the spectrum, cp (4.4). Therefore we have to extend the approach in [15] in the following way which works analogously for any operator being bounded from below.

Let A be any of the operators M_j defined in Section 4, $j = 1, 2$. We first look for estimates of the solution $u : [0, \infty) \to \mathbb{R}^n$ to

$$u_{tt} + Au = f, \tag{6.1}$$

$$u(t) \in D(A), \tag{6.2}$$

$$u(0, \cdot) = u^0, \quad u_t(0, \cdot) = u^1, \tag{6.3}$$

$$u \in \bigcap_{j=0}^{2K+2} C^j\big([0, \infty), W^{2K+2-j,2}(\Omega)\big), \tag{6.4}$$

where $K \in \mathbb{N}$. If the solution exists, the following necessary conditions have to be satisfied:

$$f \in \bigcap_{j=0}^{2K} C^j\big([0, \infty), W^{2K-j,2}(\Omega)\big) \tag{6.5}$$

and

$$u^k \in D(A) \cap W^{2K+2-j,2}(\Omega) \text{ for } k = 0, 1, \ldots, 2K, \quad u^{2K+1} \in D(A^{1/2}), \quad u^{2K+2} \in L^2(\Omega); \tag{6.6}$$

here u^0, u^1 denote the given data and inductively

$$u^j := f^{(j-2)}(0) - Au^{j-2} \quad \text{for } j = 2, \ldots, 2K+2, \tag{6.7}$$

which means that $u^j = \partial_t^j u(0, \cdot)$.

Now suppose that (6.5) and (6.6) hold and that u is solution of (6.1) – (6.4). We define the invertible Operator $B := A + 1$ on $D(B) := D(A)$ and rewrite (6.1) as

$$(\partial_t^2 - 1)u + Bu = f. \tag{6.8}$$

Then we can do the same trick used in section 4 of [15], we only have to replace the operator ∂_t^2 by $(\partial_t^2 - 1)$. In particular we set

$$v := (-B)^{-K}(\partial_t^2 - 1)^K u. \tag{6.9}$$

By induction with respect to k we conclude from (6.8) that

$$(-B)^{-k}(\partial_t^2 - 1)^k u = \sum_{j=0}^{k-1} (-B)^{-(j+1)}(\partial_t^2 - 1)^j f + u$$

and

$$v = \sum_{j=0}^{K-1} (-B)^{-(j+1)}(\partial_t^2 - 1)^j f + u. \tag{6.10}$$

Hence v is the solution of

$$v_{tt} + Av = (-B)^{-K}(\partial_t^2 - 1)^K f, \tag{6.11}$$

$$v(t) \in D(A), \tag{6.12}$$

$$v(0, \cdot) = v^0 := \sum_{j=0}^{K-1} (-B)^{-(j+1)}(\partial_t^2 - 1)^j f(0) + u^0, \tag{6.13}$$

$$v_t(0, \cdot) = v^1 := \sum_{j=0}^{K-1} (-B)^{-(j+1)}(\partial_t^2 - 1)^j \partial_t f(0) + u^1, \tag{6.14}$$

$$v \in \bigcap_{j=0}^{2} C^j\big([0, \infty), W^{2K+2-j,2}(\Omega)\big), \tag{6.15}$$

where (6.15) follows from (6.4), (6.9) and elliptic regularity theory, cp. Lemma 4.3.

On the other hand, if v is the solution of (6.11) – (6.15), then set

$$\tilde{u} := v - \sum_{j=0}^{K-1} (-B)^{-(j+1)}(\partial_t^2 - 1)^j f \tag{6.16}$$

As in the proof of Theorem 4.1 in [15] we can show that $\tilde{u}$ solves (6.1) – (6.3), and (6.4) follows from (6.15) and (6.5) by $\tilde{u}_{tt} = f - A\tilde{u}$. Then $\tilde{u} = u$ by uniqueness of the solution.

Note that the right-hand side of (6.11) satisfies boundary conditions as element of $D(B^K) = D(A^K)$. Analogously, $v^0 = B^{-K}(\partial_t^2 - 1)^K u(0,.) \in D(A^{K+1})$ and $v^1 = B^{-K}(\partial_t^2 - 1)^K \partial_t u(0,.) \in D(A^{K+1/2})$ by (6.6). Therefore we can apply Duhamel's principle and the results of the previous section to obtain decay-estimates for v and hence for u. In the following we give only the results, the remaining proofs can be carried over from section 4 of [15], here using the L^p-regularity for the Maxwell operators M_j given in Lemma 4.3.

We say that (f, u^0, u^1) satisfies the *compatibility condition of order K for the operator A*, if u^0, u^2 and u^j defined by (6.7) satisfy

$$u^k \in D(A) \text{ for } k = 0, 1, \ldots, 2K, \quad u^{2K+1} \in D(A^{1/2}). \tag{6.17}$$

Moreover, we naturally extend condition (N) to non-zero f:

Condition (N): *$z^0(x', \cdot)$, $z^1(x', \cdot)$ and $f(t, x', .)$ satisfy for every fixed $t \geq 0$ and $x' \in \mathbb{R}^l$ that their projections onto the null spaces of the operator's part in cross-direction (constant functions for the Neumann condition in cases I,II,III,IV, VI; in case V:* span $\{w_0\}$*) vanish.*

Theorem 6.1 *(Maxwell systems, L^p-L^q-decay)) Assume condition (N). Let K_2, K_3, d be defined as in Theorem 5.1. Suppose*

$$K \geq \frac{K_2 + K_3 + 1}{2} = \frac{1}{2}\left(\left[\frac{n}{2}\right] + \left[\frac{l+1}{2}\right] + n - l + 1\right),$$

$$f \in \bigcap_{j=0}^{2K} C^j\left([0, \infty, W^{2K-j,2}(\Omega) \cap W^{2K-j,\,1}(\Omega)\right),$$

$$z^0 \in W^{2K+2,2}(\Omega) \cap W^{2K+2,1}(\Omega), \quad z^1 \in W^{2K+1,2}(\Omega) \cap W^{2K+1,1}(\Omega),$$

that (z^0, z^1, f) satisfies the compatibility condition of order $2K$ for the operator M_1 or M_2, respectively, and that z is the unique solution of (5.1) – (5.3) or (5.1), (5.2), (5.4). Then for every $q \in [2, \infty)$ and $\frac{1}{q} + \frac{1}{p} = 1$ the solution z satisfies

$$\begin{aligned} &\|(z(t,\cdot), z_t(t,\cdot), \nabla z(t,\cdot))\|_{L^q(\Omega)} \\ &\leq \frac{c}{(1+t)^{(1-\frac{2}{q})\frac{l}{2}}} \left(\|(z^0, z^1, \nabla z^0)\|_{W^{2K,\,p}(\Omega)} + \sum_{j=0}^{2K-1} \|f^{(j)}(0)\|_{W^{2K-1-j,\,p}(\Omega)} \right) \\ &\quad + c\int_0^t \frac{1}{(1+t-\tau)^{(1-\frac{2}{q})\frac{l}{2}}} \sum_{j=0}^{2K} \|f^{(j)}(\tau)\|_{L^p(\Omega)} d\tau + c\sum_{j=0}^{2K-1} \|f^{(j)}(t)\|_{W^{2K-1-j,\,p}(\Omega)}, \end{aligned}$$

where the constant $c > 0$ does not depend on z^0, z^1, f and t.

Finally, we obtain the general decay results for the elastic system (2.1)–(2.3)/(2.4), decomposing u and f into

$$u = u^{po} + u^s, \quad f = f^{po} + f^s.$$

Theorem 6.2 *(Elastic systems, L^p-L^q-decay) Assume that the projections $P^{po}u^0, P^{po}u^1, P^{po}f$ and $P^s u^0, P^s u^1, P^s f$ satisfy condition (N). Let K_2, K_3, d be defined as in Theorem 5.1. Suppose*

$$K \geq \frac{K_2+K_3+1}{2} = \frac{1}{2}\left(\left[\frac{n}{2}\right] + \left[\frac{l+1}{2}\right] + n - l + 1\right),$$

$$f \in \bigcap_{j=0}^{2K} C^j\left([0,\infty), W^{2K-j,2}(\Omega) \cap W^{2K-j,1}(\Omega)\right),$$

$$u^0 \in W^{2K+2,2}(\Omega) \cap W^{2K+2,1}(\Omega), \quad u^1 \in W^{2K+1,2}(\Omega) \cap W^{2K+1,1}(\Omega),$$

that (u^0, u^1, f) satisfies the compatibility condition of order $2K$, now for the operator E_1 or E_2, respectively, and that u is the unique solution of (2.1)–(2.3) or (2.4), respectively. Then for every $q \in [2,\infty)$ and $\frac{1}{q} + \frac{1}{p} = 1$ the solution u satisfies

$$\begin{aligned}
&\|(u(t,\cdot), u_t(t,\cdot), \nabla u(t,\cdot))\|_{L^q(\Omega)} \\
\leq\ & \frac{c}{(1+t)^{(1-\frac{2}{q})\frac{l}{2}}} \Big(\|(P^{po}u^0, P^{po}u^1, \nabla P^{po}u^0, P^s u^0, P^s u^1, \nabla P^s u^0)\|_{W^{2K,p}(\Omega)} \\
&\left. + \sum_{j=0}^{2K-1} \|P^{po}f^{(j)}(0), P^s f^{(j)}(0)\|_{W^{2K-1-j,p}(\Omega)}\right) + \\
& c\int_0^t \frac{1}{(1+t-\tau)^{(1-\frac{2}{q})\frac{l}{2}}} \sum_{j=0}^{2K} \|P^{po}f^{(j)}(\tau), P^s f^{(j)}(\tau)\|_{L^p(\Omega)} d\tau + \\
& c\sum_{j=0}^{2K-1} \|P^{po}f^{(j)}(t), P^s f^{(j)}(t)\|_{W^{2K-1-j,p}(\Omega)},
\end{aligned}$$

where the constant $c > 0$ does not depend on u^0, u^1, f.

Using Theorems 5.3 and 5.4 we can remove the projection operators appearing in the last theorem.

Theorem 6.3 *(Elastic systems, L^p-L^q-decay) Assume condition (N). Let K_2, K_3, d be defined as in Theorem 5.1. Suppose*

$$K \geq \frac{K_2+K_3+1}{2} = \frac{1}{2}\left(\left[\frac{n}{2}\right] + \left[\frac{l+1}{2}\right] + n - l + 1\right),$$

$$f \in \bigcap_{j=0}^{2K} C^j([0,\infty), W^{2K-j,2}(\Omega) \cap W^{2K-j,1}(\Omega)),$$

$$u^0 \in W^{2K+2,2}(\Omega) \cap W^{2K+2,1}(\Omega), \quad u^1 \in W^{2K+1,2}(\Omega) \cap W^{2K+1,1}(\Omega),$$

that (u^0, u^1, f) satisfies the compatibility condition of order $2K$, now for the operator E_1 or E_2, respectively, and that u is the unique solution of (2.1)–(2.3) resp. to (2.1), (2.2), (2.4), in the *latter case assuming $n=2$. Then for every $q\in[2,\infty)$ and $\frac{1}{q}+\frac{1}{p}=1$ the solution u satisfies*

$$\begin{aligned}
&\|(u(t,\cdot),u_t(t,\cdot),\nabla u(t,\cdot))\|_{L^q(\Omega)}\\
&\leq \frac{c}{(1+t)^{(1-\frac{2}{q})\frac{l}{2}}}\left(\|(u^0,u^1,\nabla u^0)\|_{W^{2K,p}(\Omega)}+\sum_{j=0}^{2K-1}\|f^{(j)}(0)\|_{W^{2K-1-j,p}(\Omega)}\right)\\
&\quad +c\int_0^t \frac{1}{(1+t-\tau)^{(1-\frac{2}{q})\frac{l}{2}}}\sum_{j=0}^{2K}\|f^{(j)}(\tau)\|_{L^p(\Omega)}d\tau+c\sum_{j=0}^{2K-1}\|f^{(j)}(t)\|_{W^{2K-1-j,p}(\Omega)}\,,
\end{aligned}$$

where the constant $c>0$ does not depend on u^0,u^1,f.

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References

[1] Adams, R.A. *Sobolev spaces*, Academic Press: New York (1975).

[2] Agmon, S. *Lectures on elliptic boundary value problems.* Van Nostrand: New York (1965).

[3] Alt, H.W. *Lineare Funktionalanalysis.* Springer-Verlag: Berlin (1985).

[4] Dan, W.: On the low-frequency asymptotic expansion for some second-order elliptic systems in a two-dimensional exterior domain. *Math. Meth. Appl. Sci.* **19** (1996), 1073–1090.

[5] Doll, M.S.: L^p-L^q-estimate of the linear equations of thermoelasticity for rhombic media in $\mathbb{R}^2$. *Konstanzer Schriften Math. Inf.* **210** (2005).

[6] Dreher, M.: Asymptotic behavior of solutions to wave equations in conical sets. Manuscript (2005).

[7] Guidetti, D.: On interpolation with boundary conditions. *Math. Zeitschrift* **207** (1991), 439–460.

[8] Iwashita, H., Shibata, Y.: On the analyticity of spectral functions for some exterior boundary value problems. *Glasnik Mat.* **23** (1988), 291–313.

[9] Jiang, S., Racke, R.: *Evolution equations in thermoelasticity,* π Monographs Surveys Pure Appl. Math. **112.** Chapman & Hall/CRC, Boca Raton (2000).

[10] Kozono, H., Yanagisawa, T.: L^r-variational inequality for vector fields and the Helmholtz-Weyl decomposition in bounded domains. Preprint (submitted) (2006).

[11] Leis, R.: Exterior initial boundary value problems in thermoelasticity. In: G.F. Roache, ed., *Univ. Strathclyde Sem. Appl. Math. Anal.: Vibration theory.* Shiva, Nantwich, Ches (1982).

[12] Leis, R.: *Initial boundary value problems in mathematical physics.* B.G. Teubner-Verlag, Stuttgart; John Wiley & Sons, Chichester (1986).

[13] Lesky, P. Resonance phenomena for a class of partial differential equations of higher order in cylindrical waveguides. Math. Meth. Appl. Sci. **1989**, *11*, 697–723.

[14] Lesky, P.: Long-time asymptotics for the wave equation of linear elasticity in cylindrical waveguides. *Preprint Math. Inst. Univ. Stuttgart* **2000-14** (2000).

[15] Lesky, P., Racke, R.: Nonlinear wave equations in infinite waveguides. *Comm. PDE* **28** (2003), 1265–1301.

[16] Mehra, M.L.: Zur asymptotischen Verteilung der Eigenwerte des Maxwellschen Randwertproblems. *Bonner Math. Schriften* **109** (1978).

[17] Metcalfe, J., Sogge, C.D., Stewart, A.: Nonlinear hyperbolic equations in infinite homogeneous waveguides. *Comm. PDE* **30** (2005), 643–661.

[18] Picard, R.: On the boundary value problems of electro- and magnetostatics. *Proc. Roy. Soc. Edinburgh* **92A** (1982), 165–174.

[19] Picard, R.: Ein Hodge-Satz für Mannigfaltigkeiten mit nicht-glattem Rand. *Math. Meth. Appl. Sci.* **5** (1983), 151–163.

[20] Picard, R.: On the low-frequency asymptotics in electromagnetic theory. *J. Reine Angew. math.* **354** (1984), 50–73.

[21] Racke, R.: Ein spezielles Anfangsrandwertproblem zu den Gleichungen der Thermoelastizitätstheorie — Existenztheorie und asymptotische Untersuchungen. Diploma Thesis, University of Bonn (1982).

[22] Racke, R.: $L^p - L^q$-estimates for solutions to the equations of linear thermoelasticity in exterior domains. *Asymptotic Analysis* **3** (1990), 105–132.

[23] Racke, R.: *Lectures on nonlinear evolution equations. Initial value problems.* Aspects of Mathematics **E19**. Friedr. Vieweg & Sohn, Braunschweig/Wiesbaden (1992).

[24] Racke, R.: Exponential decay for a class of initial boundary value problems in thermoelasticity. *Comp. Appl. Math.* **12** (1993), 67–80.

[25] Stoth, M.: Decay estimates for solutions of linear elasticity for anisotropic media. *Math. Meth. Appl. Sci.* **19** (1996), 15–31.

[26] Werner, P.: Selfadjoint extensions of the Laplace operator with respect to electric and magnetic boundary conditions. *J. Math. Anal. Appl.* **70** (1979), 131–160.

[27] Werner, P.: Regularity properties of the Laplace operator with respect to electric and magnetic boundary conditions. *J. Math. Anal. Appl.* **87** (1982), 560–602.

[28] Werner, P.: Spectral properties of the Laplace operator with respect to electric and magnetic boundary conditions. *J. Math. Anal. Appl.* **92** (1983), 1–65.

[29] Werner, P.: Ein Resonanzphänomen in der Theorie akustischer und elektromagnetischer Wellen. *Math. Meth. Appl. Sci.* **6** (1984), 104–128.

[30] Werner, P.: Resonance phenomena in cylindrical waveguides. *J. Math. Anal. Appl.* **121** (1987), 173–214.

[31] Werner, P.: Aperiodic electromagnetic waves in cylindrical waveguides. *J. Math. Anal. Appl.* **121** (1987), 215–272.

[32] Weyl, H.: Das asymptotische Verteilungsgesetz der Eigenwerte linearer patieller Differentialgleichungen. *Math. Ann.* **71** (1912), 441–479.

[33] Weyl, H.: Das asymptotische Verteilungsgesetz der Eigenschwingungen eines beliebig gestalteten elastischen Körpers. *Rend. Circolo Mat. Palermo* **39** (1915), 1–50.

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