L_p theory for the linear thermoelastic plate equations in bounded and exterior domains

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Abstract

The paper is concerned with linear thermoelastic plate equations in a domain Ω:

$$u_{tt} + \Delta^2 u + \Delta\theta = 0 \text{ and } \theta_t - \Delta\theta - \Delta u_t = 0 \quad \text{in } \Omega \times (0, \infty),$$

subject to Dirichlet boundary condition: $u|_\Gamma = D_\nu u|_\Gamma = \theta|_\Gamma = 0$ and initial condition: $(u, u_t, \theta)|_{t=0} = (u_0, v_0, \theta_0) \in W^2_{p,D}(\Omega) \times L_p \times L_p$. Here, Ω is a bounded or exterior domain in $\mathbb{R}^n$ ($n \geq 2$). We assume that the boundary Γ of Ω is a C^4 hypersurface and we define $W^2_{p,D}$ by the formula: $W^2_{p,D} = \{u \in W^2_p \mid u|_\Gamma = D_\nu u|_\Gamma = 0\}$. We show that for any $p \in (1, \infty)$, the associated semigroup $\{T(t)\}_{t\geq 0}$ is analytic. Moreover, if Ω is bounded, then $\{T(t)\}_{t\geq 0}$ is exponentially stable.

1 Introduction

Let Ω be a bounded domain or an exterior domain (domain with bounded complement) in $\mathbb{R}^n$ ($n \geq 2$), the boundary Γ of which is a C^4 hypersurface. In this paper, we consider initial boundary value problem of linear thermoelastic plate equations:

$$u_{tt} + \Delta^2 u + \Delta\theta = 0 \text{ and } \theta_t - \Delta\theta - \Delta u_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+ \tag{1.1}$$

subject to the initial condition:

$$u(x, 0) = u_0(x), \;\; u_t(x, 0) = v_0(x), \;\; \theta(x, 0) = \theta_0(x) \tag{1.2}$$

and Dirichlet boundary condition:

$$u|_\Gamma = D_\nu u|_\Gamma = \theta|_\Gamma = 0, \tag{1.3}$$

where $D_\nu = \sum_{j=1}^n \nu_j D_j$ ($D_j = \partial/\partial x_j$) and $\nu = (\nu_1, \ldots, \nu_n)$ denotes the unit outer normal to Γ. In (1.1), u stands for a mechanical variable denoting the vertical displacement of the plate, while θ stands for a thermal variable describing the temperature relative to a constant reference temperature $\bar{\theta}$. Since the equations (1.1) represent the transfer of the mechanical energy to the thermal energy through coupling, we expect that total energy of the system decays, because of the thermal damping. In fact, when Ω is a bounded reference configuration, the exponential stability of the associated semigroup under several different kind of boundary conditions have been proved by Kim [4], Munõz Rivera and Racke [14], Liu and Zheng [12], Avalos and Lasiecka

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[1], Lasiecka and Triggiani [5, 6, 7, 8] and Shibata [19]. But, a more significant aspect that the equations (1.1) have is that the associated semigroup is analytic. Namely, although the first equations in (1.1) is a simple dispersive equation (the product of two Schrödinger equations), the effect from the heat equation through coupling is strong enough to have analyticity of the total system. This fact was first proved by Liu and Renardy [9] and then it has been studied by Russell [17], Liu and Liu [10], Liu and Yong [11], Munõz Rivera and Racke [15] in the L_2 or Hilbert space setting (see also the book of Liu and Zheng [13] for a survey).

The original equations derived by Lagnese [3] describing the motion and transfer of the energy of thermo-elastic plate is non-linear and it is widely accepted that the L_p approach is more relevant to handle with the non-linear problem under less regularity assumption and compatibility condition on initial data in the L_p setting. Concerning the generation of L_p analytic semigroup and its decay property for linear thermoelastic plate equations, Denk and Racke [2] studied the Cauchy problem for (1.1) in the whole space $\mathbb{R}^n$ and Naito and Shibata [16] studied the initial boundary value problem for (1.1) with Dirichlet boundary condition in the half-space $\mathbb{R}^n_+$.

The purpose of this paper is to study the generation of an L_p analytic semigroup and its decay property when the reference configuration Ω is a bounded domain or an exterior domain in $\mathbb{R}^n$ $(n \geq 2)$. To formulate the problem (1.1) – (1.3) in the semigroup setting, introducing the unknown function $v = u_t$, we rewrite it in the matrix form:

$$U_t = AU \quad \text{in } \Omega \times \mathbb{R}_+, \quad U|_{t=0} = U_0, \tag{1.4}$$

where we have set

$$U = \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}, \ U_0 = \begin{pmatrix} u_0 \\ v_0 \\ \theta_0 \end{pmatrix}, \ A = \begin{pmatrix} 0 & 1 & 0 \\ -\Delta^2 & 0 & -\Delta \\ 0 & \Delta & \Delta \end{pmatrix}. \tag{1.5}$$

To solve initial boundary value problem (1.4) with (1.3), we consider the corresponding resolvent problem:

$$(\lambda I - A)U = F \quad \text{in } \Omega \tag{1.6}$$

subject to the boundary condition (1.3), where I denotes the $n \times n$ unit matrix. To state our main result precisely, we introduce several spaces and some symbols at this point. For a general domain $\mathcal{O}$, $L_p(\mathcal{O})$ and $W^m_p(\mathcal{O})$ stand for the usual Lebesgue space and Sobolev space, respectively, $m \in \mathbb{N}_0$, $1 < p < \infty$.. Let $\| \ \|_{L_p(\mathcal{O})}$ and $\| \ \|_{W^m_p(\mathcal{O})}$ denote their norms. For a general domain $\mathcal{O}$ with C^1 boundary $\partial\mathcal{O}$, we introduce the spaces $W^2_{p,0}(\mathcal{O})$ and $W^m_{p,D}(\mathcal{O})$ $(m = 2, 4)$ as follows:

$$\begin{aligned} W^2_{p,0}(\mathcal{O}) &= \{u \in W^2_p(\mathcal{O}) \mid u|_{\partial\mathcal{O}} = 0\}, \\ W^m_{p,D}(\mathcal{O}) &= \{u \in W^m_p(D) \mid u|_{\partial\mathcal{O}} = D_\nu u|_{\partial\mathcal{O}} = 0\} \quad (m = 2, 4), \end{aligned} \tag{1.7}$$

where $\nu = (\nu_1, \ldots, \nu_n)$ denotes the unit outer normal to $\partial\mathcal{O}$ and $D_\nu = \sum_{j=1}^n \nu_j D_j$. Let $\mathcal{H}_p(\mathcal{O})$ and $\mathcal{D}_p(\mathcal{O})$ be the spaces defined by the following formulas:

$$\begin{aligned} \mathcal{H}_p(\mathcal{O}) &= \{F = {}^T(f, g, h) \mid f \in W^2_{p,D}(\mathcal{O}), \ g \in L_p(\mathcal{O}), \ h \in L_p(\mathcal{O})\}, \\ \mathcal{D}_p(\mathcal{O}) &= \{U = {}^T(u, v, \theta) \mid u \in W^4_{p,D}(\mathcal{O}), \ v \in W^2_{p,D}(\mathcal{O}), \ \theta \in W^2_{p,0}(\mathcal{O})\}. \end{aligned} \tag{1.8}$$

Here and hereafter, ${}^TM$ denotes the transposed of M. We define the norms $\| \cdot \|_{\mathcal{H}_p(\mathcal{O})}$ and $\| \cdot \|_{\mathcal{D}_p(\mathcal{O})}$ by the following formulas:

$$\begin{aligned} \|F\|_{\mathcal{H}_p(\mathcal{O})} &= \|f\|_{W^2_p(\mathcal{O})} + \|(g, h)\|_{L_p(\mathcal{O})} \quad (F = {}^T(f, g, h) \in \mathcal{H}_p(\mathcal{O})), \\ \|U\|_{\mathcal{D}_p(\mathcal{O})} &= \|u\|_{W^4_p(\mathcal{O})} + \|(v, \theta)\|_{W^2_p(\mathcal{O})} \quad (U = {}^T(u, v, \theta) \in \mathcal{D}_p(\mathcal{O})). \end{aligned} \tag{1.9}$$

Let $\mathcal{A}_p$ be an operator whose domain is $\mathcal{D}_p(\Omega)$ and its operation is defined by the formula:

$$\mathcal{A}_p U = AU \quad \text{for } U \in \mathcal{D}_p(\Omega). \tag{1.10}$$

Then, we have the following theorem.

Theorem 1.1. *Let $1 < p < \infty$. Then, $\mathcal{A}_p$ generates an analytic semigroup $\{T_p(t)\}_{t\geq 0}$ on $\mathcal{H}_p(\Omega)$.*

Moreover, if Ω is a bounded domain, then $\{T_p(t)\}_{t\geq 0}$ is exponentially stable, that is there exists a positive constant σ such that

$$\|T_p(t)F\|_{\mathcal{H}_p(\Omega)} \leq Ce^{-\sigma t}\|F\|_{\mathcal{H}_p(\Omega)}$$

for any $t > 0$ and $F \in \mathcal{H}_p(\Omega)$ with some constant C independent of t and F.

Theorem 1.1 immediately follows from the following theorem concerning the anaylysis of the resolvent of $\mathcal{A}_p$.

Theorem 1.2. *Let $1 < p < \infty$. Let $\rho(\mathcal{A}_p)$ be the resolvent set of $\mathcal{A}_p$. Let*

$$\mathbb{C}_+ = \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq 0\}$$

where $\mathbb{C}$ denotes the set of all complex numbers. Then, we have the following two assertions.

(1) *Assume that Ω is a bounded domain. Then, $\rho(\mathcal{A}_p) \supset \mathbb{C}_+$.*

Moreover, there exists a constant C depending on p and Ω such that for any $\lambda \in \mathbb{C}_+$ and $F \in \mathcal{H}_p(\Omega)$ there holds the estimate:

$$|\lambda|\|(\lambda I - \mathcal{A}_p)^{-1}F\|_{\mathcal{H}_p(\Omega)} + \|(\lambda I - \mathcal{A}_p)^{-1}F\|_{\mathcal{D}_p(\Omega)} \leq C\|F\|_{\mathcal{H}_p(\Omega)}.$$

(2) *Assume that Ω is an exterior domain. Then, $\rho(\mathcal{A}_p) \supset \mathbb{C}_+ \setminus \{0\}$.*

Moreover, for any $\lambda_0 > 0$ there exists a constant C depending on λ_0, p and Ω such that for any $\lambda \in \mathbb{C}_+$ with $|\lambda| \geq \lambda_0$ and $F \in \mathcal{H}_p(\Omega)$ there holds the estimate:

$$|\lambda|\|(\lambda I - \mathcal{A}_p)^{-1}F\|_{\mathcal{H}_p(\Omega)} + \|(\lambda I - \mathcal{A}_p)^{-1}F\|_{\mathcal{D}_p(\Omega)} \leq C\|F\|_{\mathcal{H}_p(\Omega)}.$$

Since Theorem 1.1 follows from Theorem 1.2 immediately (cf. Vrabie [21, Proof of Theorem 7.1.1], we shall only prove Theorem 1.2 in what follows.

We remark that replacing the Dirichlet boundary conditions (1.3) by the boundary conditions

$$u|_\Gamma = \Delta u|_\Gamma = \theta|_\Gamma = 0 \tag{1.11}$$

usually simplifies the situation, cf. [16, Sec.6] or [9], and hence allows to obtain similar theorems as presented above.

The paper is organized as follows: In section 2, we quote results due to Naito and Shibata [16] concerning the resolvent problem in $\mathbb{R}^n$ and $\mathbb{R}^n_+$. In section 3, we treat the resolvent problem in the bent half space. In section 4, we prove the *a priori* estimate for the resolvent problem in a general bounded or exterior domain whose boundary is assumed to be a $C^{3,1}$, compact hypersurface. In section 5, we shall show Theorem 1.2 when Ω is a bounded domain. In section 6, we shall show Theorem 1.2 when Ω is an exterior domain.

2 On a resolvent estimate in $\mathbb{R}^n$ and $\mathbb{R}^n_+$.

In this section, we shall quote results obtained by Naito and Shibata [16] concerning the resolvent problem in the whole space $\mathbb{R}^n$ and its half space $\mathbb{R}^n_+$ which is defined by the formula: $\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$, whose boundary is the set $\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$. Note that

$$
\begin{aligned}
&W^2_{p,0}(\mathbb{R}^n) = W^2_p(\mathbb{R}^n), \quad W^m_{p,D}(\mathbb{R}^n) = W^m_p(\mathbb{R}^n),\\
&W^2_{p,0}(\mathbb{R}^n_+) = \{u \in W^2_p(\mathbb{R}^n_+) \mid u|_{x_n=0} = 0\},\\
&W^m_{p,D}(\mathbb{R}^n_+) = \{u \in W^m_p(\mathbb{R}^n_+) \mid u|_{x_n=0} = D_n u|_{x_n=0} = 0\} \ \ (m = 2, 4).
\end{aligned}
$$

Theorem 2.1. *Let $1 < p < \infty$. Set*

$$\Sigma_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| < \pi - \epsilon\}. \tag{2.1}$$

Then, there exists an ϵ $(0 < \epsilon < \pi/2)$ such that for any $F = {}^T(f, g, h) \in \mathcal{H}_p(\mathbb{R}^n)$ and $\lambda \in \Sigma_\epsilon$ there exists a unique $U = {}^T(u, v, \theta) \in \mathcal{D}_p(\mathbb{R}^n)$ which solves the resolvent problem:

$$(\lambda I - A)U = F \quad \text{in } \mathbb{R}^n \tag{2.2}$$

uniquely and satisfies the estimates :

$$
\begin{aligned}
\sum_{j=0}^{2} |\lambda|^{\frac{2-j}{2}} \|(\nabla^{j+2}u, \nabla^j v, \nabla^j \theta)\|_{L_p(\mathbb{R}^n)} &\le C\|(\nabla^2 f, g, h)\|_{L_p(\mathbb{R}^n)},\\
\sum_{j=0}^{1} |\lambda|^{\frac{4-j}{2}} \|\nabla^j u\|_{L_p(\mathbb{R}^n)} &\le C\|(|\lambda| f, g, h)\|_{L_p(\mathbb{R}^n)}.
\end{aligned}
\tag{2.3}
$$

Here and hereafter,

$$
\begin{aligned}
&\nabla^1 u = \nabla u = (D_1 u, \ldots, D_n u), \ \ \nabla^j u = (D^\alpha_x u \mid |\alpha| = j) \ (j \ge 2)\\
&D_j u = \partial u / \partial x_j, \ \ D^\alpha_x u = D_1^{\alpha_1} \cdots D_n^{\alpha_n} u \ \ (\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n_0),
\end{aligned}
$$

$\mathbb{N}$ denotes the set of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Theorem 2.2. *Let $1 < p < \infty$ and let $\mathbb{C}_+$ be the same as in Theorem 1.2. Then for any $\lambda \in \mathbb{C}_+ \setminus \{0\}$ and $F = {}^T(f, g, h) \in \mathcal{H}_p(\mathbb{R}^n_+)$, there exists a $U = {}^T(u, v, \theta) \in \mathcal{D}_p(\mathbb{R}^n_+)$ which solves the resolvent problem:*

$$(\lambda I - A)U = F \quad \text{in } \mathbb{R}^n_+ \tag{2.4}$$

uniquely and satisfies the estimate:

$$
\begin{aligned}
\sum_{j=0}^{2} |\lambda|^{\frac{2-j}{2}} \|(\nabla^{j+2}u, \nabla^j v, \nabla^j \theta)\|_{L_p(\mathbb{R}^n_+)} &\le C\|(\nabla^2 f, g, h)\|_{L_p(\mathbb{R}^n_+)},\\
\sum_{j=0}^{1} |\lambda|^{\frac{4-j}{2}} \|\nabla^j u\|_{L_p(\mathbb{R}^n_+)} &\le C\|(|\lambda| f, g, h)\|_{L_p(\mathbb{R}^n_+)}.
\end{aligned}
$$

3 On a resolvent problem in a bent half space

Let $\omega : \mathbb{R}^{n-1} \to \mathbb{R}$ be a bounded function in $C^{3,1}$ class whose derivatives up to order 4 are all essentially bounded in $\mathbb{R}^{n-1}$. Let H_ω be a bent half space defined by the formula:

$$H_\omega = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > \omega(x') \ \ (x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1})\}.$$

∂H_ω denotes the boundary of H_ω, which is given by the formula:

$$\partial H_\omega = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n = \omega(x') \ \ (x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1})\}.$$

$\nu(x)$ denotes the unit outer normal to ∂H_ω, which is defined by the formula:

$$\nu(x) = (\nabla'\omega, -1)/\sqrt{1 + |\nabla'\omega|^2}, \quad \nabla'\omega = (D_1\omega, \dots, D_{n-1}\omega).$$

We shall prove the following theorem in this section.

Theorem 3.1. *Let $1 < p < \infty$. Then, there exist numbers δ and λ_0 with $0 < \delta \le 1$ and $\lambda_0 \ge 1$ such that if $\|\nabla'\omega\|_{L_\infty(\mathbb{R}^{n-1})} \le \delta$, then for any $\lambda \in \mathbb{C}_+$ with $|\lambda| \ge \lambda_0$, $F = {}^T(f, g, h) \in \mathcal{H}_p(H_\omega)$ there exists a unique $U = {}^T(u, v, \theta) \in \mathcal{D}_p(H_\omega)$ which solves the equation:*

$$(\lambda I - A)U = F \ \text{ in } H_\omega \tag{3.1}$$

and satisfies the estimate:

$$\sum_{j=0}^{2} |\lambda|^{\frac{2-j}{2}} (\|u\|_{W_p^{2+j}(H_\omega)} + \|(v, \theta)\|_{W_p^j(H_\omega)}) \le C\{\|f\|_{W_p^2(H_\omega)} + \|(g, h)\|_{L_p(H_\omega)}. \tag{3.2}$$

To prove Theorem 3.1, we reduce problem (3.1) to the half space problem by using the map $\Phi : H_\omega \to \mathbb{R}^n_+$ defind by the formula: $y = \Phi(x) = (x', x_n - \omega(x'))$. Given function $w(x)$ defined on H_Ω, we set $\underline{w}(y) = w(x) = w(y', y_n + \omega(y'))$. We have

$$\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j} - \omega_j \frac{\partial}{\partial y_n} \ \ (j = 1, \dots, n-1), \ \ \frac{\partial}{\partial x_n} = \frac{\partial}{\partial y_n}. \tag{3.3}$$

Here and hereafter, we set $\omega_j = D_j\omega$. Since the operator A contains Δ and Δ^2, we have to represent these operators in the new coordinate. In fact, by (3.3) we have

$$\begin{aligned} \Delta w =& \Delta\underline{w} - \sum_{j=1}^{n-1} 2\omega_j D_j D_n \underline{w} + |\nabla'\omega|^2 D_n^2 \underline{w} - (\Delta'\omega) D_n \underline{w}, \\ \Delta^2 w =& \Delta^2 \underline{w} + a_4(\nabla'\omega, \nabla^4 \underline{w}) \\ &+ a_3(\bar{D}^1_{x'}\nabla'\omega, \nabla^3\underline{w}) + a_2(\bar{D}^2_{x'}\nabla'\omega, \nabla^2\underline{w}) + a_1(\bar{D}^3_{x'}\nabla'\omega, \nabla\underline{w}), \end{aligned} \tag{3.4}$$

where we have set

$$\Delta'\omega = \sum_{j=1}^{n-1} D_j^2 \omega, \quad \bar{D}^k_{x'}\nabla'\omega = (D^{\alpha'}_{x'}\omega_j \mid j = 1, \dots, n-1, \ \ |\alpha'| \le k),$$

$$a_4(\nabla'\omega, \nabla^4\underline{w}) = -4\sum_{j=1}^{n-1} \omega_j \Delta D_j D_n \underline{w} + 2|\nabla'\omega|^2 \Delta D_n^2 \underline{w} + 4\sum_{j,k=1}^{n-1} \omega_j \omega_k D_j D_k D_n^2 \underline{w}$$

$$- 4\sum_{j=1}^{n-1}\omega_j|\nabla'\omega|^2 D_j D_n^3\underline{w} + |\nabla'\omega|^4 D_n^4\underline{w}\,,$$

$$a_3(\bar{D}^1_{x'}\nabla'\omega, \nabla^3\underline{w}) = -4\sum_{j=1}^{n-1}(\nabla'\omega_j)\cdot(\nabla' D_j D_n\underline{w}) + 2(\nabla'|\nabla'\omega|^2)\cdot(\nabla' D_n^2\underline{w})$$

$$- 2(\Delta'\omega)\Delta D_n\underline{w} + 4\sum_{j,k=1}^{n-1}\omega_j(D_jD_k\omega)D_k^2D_n^2\underline{w} - 2\sum_{j=1}^{n-1}\omega_j(D_j|\nabla'\omega|^2)D_n^3\underline{w}$$

$$+ 4\sum_{j=1}^{n-1}\omega_j(\Delta'\omega)D_jD_n^2\underline{w} - (\Delta'\omega)|\nabla'\omega|^2 D_n^3\underline{w}\,,$$

$$a_2(\bar{D}^2_{x'}\nabla'\omega, \nabla^2\underline{w}) = (\Delta'\omega)^2 D_n^2\underline{w} - 4\sum_{j=1}^{n-1}(\Delta'\omega_j)D_jD_n\underline{w} + 2\sum_{j=1}^{n-1}\omega_j D_j(\Delta'\omega)D_n^2\underline{w}$$

$$+ (\Delta'|\nabla'\omega|^2)D_n^2\underline{w}\,,$$

$$a_1(\bar{D}^3_{x'}\nabla'\omega, \nabla\underline{w}) = -((\Delta')^2\omega)D_n\underline{w}\,.$$

Let $U = {}^T(u, v, \theta)$ satisfy (3.1) with $F = {}^T(f, g, h)$. If we write (3.1) componentwise, then we have

$$\left.\begin{aligned} \lambda u - v &= f \\ \lambda v + \Delta^2 u + \Delta\theta &= g \\ \lambda\theta - \Delta\theta - \Delta v &= h \end{aligned}\right\} \quad \text{in } H_\omega, \tag{3.5}$$
$$u|_{\partial H_\omega} = D_\nu u|_{\partial H_\omega} = \theta|_{\partial H_\omega} = 0,$$

where $D_\nu = \nu\cdot\nabla = (1+|\nabla'\omega|^2)^{-1/2}\{\sum_{j=1}^{n-1}(D_j\omega)D_j - D_n\}$. Therefore, by (3.4) $(\underline{u}, \underline{v}, \underline{\theta})$ satisfies the equations:

$$\left.\begin{aligned} \lambda\underline{u} - \underline{v} &= \underline{f} \\ \lambda\underline{v} + \Delta^2\underline{u} + \Delta\underline{\theta} &= \underline{g} + G(\omega, \underline{u}, \underline{\theta}) \\ \lambda\underline{\theta} - \Delta\underline{\theta} - \Delta\underline{v} &= \underline{h} + H(\omega, \underline{v}, \underline{\theta}) \end{aligned}\right\} \quad \text{in } \mathbb{R}^n_+, \tag{3.6}$$
$$\underline{u}|_{x_n=0} = D_n\underline{u}|_{x_n=0} = \underline{\theta}|_{x_n=0} = 0,$$

where we have set

$$G(\omega, \underline{u}, \underline{\theta}) = - a_4(\nabla'\omega, \nabla^4\underline{u}) - a_3(\bar{D}^1_{x'}\nabla'\omega, \nabla^3\underline{u}) - a_2(\bar{D}^2_{x'}\nabla'\omega, \nabla^2\underline{u})$$
$$- a_1(\bar{D}^3_{x'}\nabla'\omega, \nabla\underline{u}) + 2\sum_{j=1}^{n-1}\omega_jD_jD_n\underline{\theta} - (\sum_{j=1}^{n-1}\omega_j^2)D_n^2\underline{\theta} - (\Delta'\omega)D_n\underline{\theta},$$

$$H(\omega, \underline{v}, \underline{\theta}) = - 2\sum_{j=1}^{n-1}\omega_jD_jD_n(\underline{\theta}+\underline{v}) + (\sum_{j=1}^{n-1}\omega_j^2)D_n^2(\underline{\theta}+\underline{v}) + (\Delta'\omega)D_n(\underline{\theta}+\underline{v}).$$

Obviously, if $(\underline{u}, \underline{v}, \underline{\theta})$ satisfies (3.6), then (u, v, θ) satisfies (3.5), and therefore we shall solve (3.6) in what follows. Since (3.6) is linear, we solve the following two systems of equations:

$$\left.\begin{aligned} \lambda\underline{u} - \underline{v} &= 0 \\ \lambda\underline{v} + \Delta^2\underline{u} + \Delta\underline{\theta} &= \underline{g} + G(\omega, \underline{u}, \underline{\theta}) \\ \lambda\underline{\theta} - \Delta\underline{\theta} - \Delta\underline{v} &= \underline{h} + H(\omega, \underline{v}, \underline{\theta}) \end{aligned}\right\} \quad \text{in } \mathbb{R}^n_+, \tag{3.7}$$

$$\underline{u}|_{x_n=0} = D_n\underline{u}|_{x_n=0} = \underline{\theta}|_{x_n=0} = 0,$$

and

$$\left.\begin{aligned} \lambda\underline{u} - \underline{v} &= \underline{f} \\ \lambda\underline{v} + \Delta^2\underline{u} + \Delta\underline{\theta} &= 0 \\ \lambda\underline{\theta} - \Delta\underline{\theta} - \Delta\underline{v} &= 0 \end{aligned}\right\} \quad \text{in } \mathbb{R}^n_+, \tag{3.8}$$
$$\underline{u}|_{x_n=0} = D_n\underline{u}|_{x_n=0} = \underline{\theta}|_{x_n=0} = 0.$$

First, we shall show the existence of a unique solution of equation (3.7) by contraction mapping principle. For the notational simplicity we introduce the following symbols:

$$\mathcal{I}_\lambda(\underline{u},\underline{v},\underline{\theta}) = \sum_{j=0}^{4} |\lambda|^{\frac{4-j}{2}} \|\nabla^j\underline{u}\|_{L_p(\mathbb{R}^n_+)} + \sum_{j=0}^{2} |\lambda|^{\frac{2-j}{2}} \|\nabla^j(\underline{v},\underline{\theta})\|_{L_p(\mathbb{R}^n_+)},$$
$$K_1 = \|\nabla'\omega\|_{L_\infty(\mathbb{R}^{n-1})}, \quad K_j = \|\bar{D}^{j-1}_{x'}\nabla'\omega\|_{L_\infty(\mathbb{R}^{n-1})} \ (j=2,3,4).$$

Since we shall choose K_1 small enough and $|\lambda|$ large enough later, we may assume that $0 < K_1 \le 1$ and $|\lambda| \ge 1$ *a priori*. By Theorem 2.2 we know that given $(\underline{u}^1,\underline{v}^1,\underline{\theta}^1) \in \mathcal{D}_p(\mathbb{R}^n_+)$, there exists a $(\underline{u}^2,\underline{v}^2,\underline{\theta}^2) \in \mathcal{D}_p(\mathbb{R}^n_+)$ which uniquely solves the equations:

$$\left.\begin{aligned} \lambda\underline{u}^2 - \underline{v}^2 &= 0 \\ \lambda\underline{v}^2 + \Delta^2\underline{u}^2 + \Delta\underline{\theta}^2 &= \underline{g} + G(\omega,\underline{u}^1,\underline{\theta}^1) \\ \lambda\underline{\theta}^2 - \Delta\underline{\theta}^2 - \Delta\underline{v}^2 &= \underline{h} + H(\omega,\underline{v}^1,\underline{\theta}^1) \end{aligned}\right\} \quad \text{in } \mathbb{R}^n_+, \tag{3.9}$$
$$\underline{u}^2|_{x_n=0} = D_n\underline{u}^2|_{x_n=0} = \underline{\theta}^2|_{x_n=0} = 0$$

and satisfies the estimate:

$$\mathcal{I}_\lambda(\underline{u}^2,\underline{v}^2,\underline{\theta}^2) \le C(\|(\underline{g},\underline{h})\|_{L_p(\mathbb{R}^n_+)} + \|(G(\omega,\underline{u}^1,\underline{\theta}^1), H(\omega,\underline{v}^1,\underline{\theta}^1))\|_{L_p(\mathbb{R}^n_+)}). \tag{3.10}$$

From the definition of $G(\omega,\underline{u},\underline{\theta})$ and $H(\omega,\underline{v},\underline{\theta})$ we have

$$\begin{aligned} &\|(G(\omega,\underline{u},\underline{\theta}), H(\omega,\underline{v},\underline{\theta}))\|_{L_p(\mathbb{R}^n_+)} \\ &\quad \le C(K_1 + K_2|\lambda|^{-\frac{1}{2}} + (K_2^2 + K_3)|\lambda|^{-1} + K_4|\lambda|^{-\frac{3}{2}})\mathcal{I}_\lambda(\underline{u},\underline{v},\underline{\theta}). \end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11) implies that

$$\begin{aligned} &\mathcal{I}_\lambda(\underline{u}^2.\underline{v}^2,\underline{\theta}^2) \\ &\quad \le C\{(K_1 + K_2|\lambda|^{-\frac{1}{2}} + (K_2^2 + K_3)|\lambda|^{-1} + K_4|\lambda|^{-\frac{3}{2}})\mathcal{I}_\lambda(\underline{u}^1,\underline{v}^1,\underline{\theta}^1) + \|(\underline{g},\underline{h})\|_{L_p(\mathbb{R}^n_+)}\}. \end{aligned} \tag{3.12}$$

If we choose K_1 and $\lambda_0 \ge 1$ in such a way that

$$CK_1 \le \frac{1}{4}, \quad C(K_2\lambda_0^{-\frac{1}{2}} + (K_2^2 + K_3)\lambda_0^{-1} + K_4\lambda_0^{-\frac{3}{2}}) \le \frac{1}{4}, \tag{3.13}$$

then by (3.12) and the linearity of the equation (3.7) the map $(\underline{u}^1,\underline{v}^1,\underline{\theta}^1) \mapsto (\underline{u}^2,\underline{v}^2,\underline{\theta}^2)$ is a contraction on $\mathcal{D}_p(\mathbb{R}^n_+)$. Therefore, (3.7) admits a unique solution $(\underline{u}',\underline{v}',\underline{\theta}') \in \mathcal{D}_p(\mathbb{R}^n_+)$, which satisfies (3.12) with $\underline{u}^1 = \underline{u}^2 = \underline{u}'$, $\underline{v}^1 = \underline{v}^2 = \underline{v}'$ and $\underline{\theta}^1 = \underline{\theta}^2 = \underline{\theta}'$. By (3.13) we have

$$\mathcal{I}_\lambda(\underline{u}',\underline{v}',\underline{\theta}') \le 2C\|(\underline{g},\underline{h})\|_{L_p(\mathbb{R}^n_+)}. \tag{3.14}$$

On the other hand, by Theorem 2.2 we know that (3.8) admits a unique solution $(\underline{u}'', \underline{v}'', \underline{\theta}'') \in \mathcal{D}_p(\mathbb{R}^n_+)$ which satisfies the estimate:

$$\begin{aligned} &\sum_{j=0}^{2} |\lambda|^{\frac{2-j}{2}} \|(\nabla^{j+2}\underline{u}'', \nabla^j\underline{v}'', \nabla^j\underline{\theta}'')\|_{L_p(\mathbb{R}^n_+)} \le C\|\nabla^2\underline{f}\|_{L_p(\mathbb{R}^n_+)}, \\ &\sum_{j=0}^{1} |\lambda|^{\frac{2-j}{2}} \|\nabla^j\underline{u}''\|_{L_p(\mathbb{R}^n_+)} \le C\|\underline{f}\|_{L_p(\mathbb{R}^n_+)}. \end{aligned} \tag{3.15}$$

Using the interpolation inequality: $\|\nabla\underline{u}''\|_{L_p(\mathbb{R}^n_+)} \le C\|\nabla^2\underline{u}''\|^{\frac{1}{2}}_{L_p(\mathbb{R}^n_+)}\|\underline{u}''\|^{\frac{1}{2}}_{L_p(\mathbb{R}^n_+)}$, by (3.15) we have $|\lambda|\|\underline{u}''\|_{W^2_p(\mathbb{R}^n_+)} \le C\|\underline{f}\|_{W^2_p(\mathbb{R}^n_+)}$. Finally, noting that $|\lambda| \ge 1$, by (3.15) we have

$$\sum_{j=0}^{2} |\lambda|^{\frac{2-j}{2}} (\|\underline{u}''\|_{W^{j+2}_p(\mathbb{R}^n_+)} + \|(\underline{v}'', \underline{\theta}'')\|_{W^j_p(\mathbb{R}^n_+)}) \le C\|\underline{f}\|_{W^2_p(\mathbb{R}^n_+)}$$

Therefore, $\underline{u} = \underline{u}' + \underline{u}''$, $\underline{v} = \underline{v}' + \underline{v}''$ and $\underline{\theta} = \underline{\theta}' + \underline{\theta}''$ solve equation (3.6) and satisfy the estimate:

$$\sum_{j=0}^{2} |\lambda|^{\frac{2-j}{2}} (\|\underline{u}\|_{W^{2+j}_p(\mathbb{R}^n_+)} + \|(\underline{v}, \underline{\theta})\|_{W^j_p(\mathbb{R}^n_+)}) \le C\{\|\underline{f}\|_{W^2_p(\mathbb{R}^n_+)} + \|(\underline{g}, \underline{h})\|_{L_p(\mathbb{R}^n_+)}\}.$$

If we define $u(x)$, $v(x)$ and $\theta(x)$ by $u(x) = \underline{u}(y)$, $v(x) = \underline{v}(y)$ and $\theta(x) = \underline{\theta}(y)$ with $y = \Phi(x) = (x', x_n - \omega(x'))$, then (u, v, θ) is a required solution to (3.1). We have also the uniqueness for equation (3.1), because both equations (3.8) and (3.9) are uniquely solvable. This completes the proof of Theorem 3.1.

4 A priori estimate

In this section, we shall show an *a priori* estimate of problem:

$$\left.\begin{aligned} \lambda u - v &= f \\ \lambda v + \Delta^2 u + \Delta\theta &= g \\ \lambda\theta - \Delta\theta - \Delta v &= h \end{aligned}\right\} \quad \text{in } \Omega, \qquad u|_\Gamma = D_\nu u|_\Gamma = \theta|_\Gamma = 0. \tag{4.1}$$

More precisely, we shall show the following theorem.

Theorem 4.1. *Let $1 < p < \infty$. Then, there exist constants $\lambda_1 \ge 1$ and $C > 0$ such that for any $\lambda \in \mathbb{C}_+$ with $|\lambda| \ge \lambda_1$ and $U = {}^t(u, v, \theta) \in \mathcal{D}_p(\Omega)$ there holds the estimate:*

$$\sum_{j=0}^{2} |\lambda|^{\frac{2-j}{2}} (\|u\|_{W^{2+j}_p(\Omega)} + \|(v, \theta)\|_{W^j_p(\Omega)}) \le C\{\|f\|_{W^2_p(\Omega)} + \|(g, h)\|_{L_p(\Omega)}\}$$

where $W^0_p(\Omega) = L_p(\Omega)$ and ${}^T(f, g, h) = (\lambda I - A)^T(u, v, \theta)$.

To prove the theorem, we localize problem (4.1). Let φ be a cut-off function in $C^\infty(\mathbb{R}^n)$. Then, $(\varphi u, \varphi v, \varphi\theta)$ enjoys the equations:

$$\left.\begin{aligned} \lambda(\varphi u) - (\varphi v) &= \varphi f \\ \lambda(\varphi v) + \Delta^2(\varphi u) + \Delta(\varphi\theta) &= \varphi g + G_\varphi(u, \theta) \\ \lambda(\varphi\theta) - \Delta(\varphi\theta) - \Delta(\varphi v) &= \varphi h + H_\varphi(v, \theta) \end{aligned}\right\} \quad \text{in } \Omega, \tag{4.2}$$

$$(\varphi u)|_\Gamma = D_\nu(\varphi u)|_\Gamma = (\varphi\theta)|_\Gamma = 0.$$

Here and hereafter, $G_\varphi(u,\theta)$ and $H_\varphi(v,\theta)$ denote the symbols defined by the following formulas:

$$\begin{aligned} G_\varphi(u,\theta) =& -\Delta(\nabla\cdot((\nabla\varphi)u)) - \Delta((\nabla\varphi)\cdot(\nabla u)) - \nabla\cdot((\nabla\varphi)\Delta u) \\ & - (\nabla\varphi)\cdot(\nabla\Delta u) - \nabla\cdot((\nabla\varphi)\theta) - (\nabla\varphi)\cdot(\nabla\theta), \\ H_\varphi(v,\theta) =& \nabla\cdot((\nabla\varphi)(\theta+v)) + (\nabla\varphi)\cdot(\nabla(\theta+v)). \end{aligned} \tag{4.3}$$

For the notational simplicity, we shall omit u, v and θ in the representation of G_φ and H_φ until the end of the proof of Theorem 4.1 unless any confusion occurs. Let $b>0$ be a large number such that $B_b \supset \Omega$ when Ω is bounded and $B_b \supset \mathbb{R}^n\setminus\Omega$ when Ω is exterior, where $B_b = \{x\in\mathbb{R}^n \mid |x|<b\}$. Pick up $x_0\in\partial\Omega$. Consider a small neighborhood $B_\sigma(x_0) = \{x\in\mathbb{R}^n \mid |x-x_0|<\sigma\}$ of x_0 with some $\sigma>0$. Let us choose $\varphi\in C^\infty(\mathbb{R}^n)$ in such a way that $\varphi(x)=1$ on $B_{\sigma/2}(x_0)$ and $\varphi(x)=0$ for $x\notin B_\sigma(x_0)$. Since we choose $\sigma>0$ small enough, we may assume that $\operatorname{supp}\varphi\subset B_b$. We shall reduce (4.2) to the bent half space problem studied in section 3. Let $-\nu(x_0)=\zeta_n={}^T(\zeta_{1n},\dots,\zeta_{nn})$ and $\zeta_j={}^T(\zeta_{1j},\dots,\zeta_{nj})$ $(j=1,\dots,n-1)$ be vectors such that $\zeta_j\cdot\zeta_k=\delta_{jk}$, and set

$$\mathcal{O} = (\zeta_1,\dots,\zeta_n) = \begin{pmatrix} \zeta_{11} & \zeta_{12} & \cdots & \zeta_{1n} \\ \zeta_{21} & \zeta_{22} & \cdots & \zeta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{n1} & \zeta_{n2} & \cdots & \zeta_{nn} \end{pmatrix}.$$

Since $\mathcal{O}$ is an orthogonal matrix, under the change of variables: $y={}^T\mathcal{O}(x-x_0)$ the operators Δ and Δ^2 do not change. Therefore, defining the functions $U(y)$, $V(y)$ and $\Theta(y)$ and a domain $\tilde\Omega$ by the formulas: $(\varphi u)(x)=U(y)$, $(\varphi v)(x)=V(y)$, $(\varphi\theta)(x)=\Theta(y)$ and $\tilde\Omega={}^T\mathcal{O}(\Omega-\{x_0\})$, we see that (U,V,Θ) enjoys the equations:

$$\left.\begin{aligned} \lambda U - V &= \widetilde{\varphi f} \\ \lambda V + \Delta^2 U + \Delta\Theta &= \widetilde{\varphi g} + \widetilde{G_\varphi} \\ \lambda\Theta - \Delta\Theta - \Delta V &= \widetilde{\varphi h} + \widetilde{H_\varphi} \end{aligned}\right\} \quad \text{in } \tilde\Omega, \tag{4.4}$$
$$U|_{\partial\tilde\Omega} = D_{\tilde\nu}U|_{\partial\tilde\Omega} = \Theta|_{\partial\tilde\Omega} = 0,$$

where $\partial\tilde\Omega$ denotes the boundary of $\tilde\Omega$, and $\tilde\nu$ denotes the unit outer normal to $\partial\tilde\Omega$. Here and hereafter, given function $w(x)$ defined on Ω, $\tilde w(y)$ denotes the function defined by the relationship: $\tilde w(y)=w(x)=w(x_0+\mathcal{O}y)$. In particular, we have

$$\tilde\nu(0) = {}^T(0,\dots,0,-1), \tag{4.5}$$

because $\tilde\nu(y)={}^T\mathcal{O}\nu(x)$, where $\nu(x)$ denotes the unit outer normal to Γ. Let δ_0 be a small positive number chosen later and choose $\sigma>0$ in such a way that $\operatorname{supp}U$, $\operatorname{supp}V$, $\operatorname{supp}\Theta\subset B_{\delta_0}=\{y\in\mathbb{R}^n \mid |y|<\delta_0\}$. Let $\delta_1>0$ be a small number such that

$$\begin{aligned} B_{\delta_0}\cap\tilde\Omega &\subset \{y=(y_1,\dots,y_n)\in\mathbb{R}^n \mid y_n>\psi(y'),\ \ y'\in B'_{\delta_1}(0)\}, \\ B_{\delta_0}\cap\partial\tilde\Omega &\subset \{y=(y_1,\dots,y_n)\in\mathbb{R}^n \mid y_n=\psi(y'),\ \ y'\in B'_{\delta_1}(0)\} \end{aligned} \tag{4.6}$$

for some $\psi\in C^{3,1}(B'_{\delta_1}(0))$, where $y'=(y_1,\dots,y_{n-1})$ and $B'_{\delta_1}(0)=\{y'\in\mathbb{R}^{n-1} \mid |y'|<\delta_1\}$. From (4.5) it follows that

$$\psi(0)=0,\ \ \nabla'\psi(0)=0,\ \ \tilde\nu=(\nabla'\psi,-1)/\sqrt{1+|\nabla'\psi|^2}, \tag{4.7}$$

where $\nabla'\psi = (D_1\psi, \ldots, D_{n-1}\psi)$. Let $\rho(y')$ be a function in $C_0^\infty(\mathbb{R}^{n-1})$ such that $\rho(y') = 1$ for $|y'| \le 1$ and $\rho(y') = 0$ for $|y'| \ge 2$. Setting $\omega(y') = \rho(y'/\delta_0)\psi(y')$ and using $\omega(y')$ we define a bent half space H_ω, its boundary and unit outer normal by the following formulas:

$$\begin{aligned}
H_\omega &= \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_n > \omega(y'),\ \ y' \in \mathbb{R}^{n-1}\},\\
\partial H_\omega &= \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_n = \omega(y'),\ \ y' \in \mathbb{R}^{n-1}\},\\
\nu_\omega &= (\nabla'\omega, -1)/\sqrt{1 + |\nabla'\omega|^2},\ \ \nabla'\omega = (D_1\omega,\ \ldots, D_{n-1}\omega).
\end{aligned}$$

If we choose $\delta_0 > 0$ so small that $0 < 2\delta_0 < \delta_1$, then it follows from (4.6) that (U, V, Θ) satisfies problem in H_ω:

$$\left.\begin{aligned}
\lambda U - V &= \widetilde{\varphi f}\\
\lambda V + \Delta^2 U + \Delta\Theta &= \widetilde{\varphi g} + \widetilde{G_\varphi}\\
\lambda\Theta - \Delta\Theta - \Delta V &= \widetilde{\varphi h} + \widetilde{H_\varphi}
\end{aligned}\right\}\quad \text{in } H_\omega, \qquad (4.8)$$
$$U|_{\partial H_\omega} = D_{\nu_\omega} U|_{\partial H_\omega} = \Theta|_{\partial H_\omega} = 0.$$

By (4.7) we see easily that

$$\begin{aligned}
|\nabla'_{y'}\omega(y')| \le \Big\{ &\sup_{|y'/\delta_0| \le 2} (\,|\rho(y'/\delta_0)||y'/\delta_0|\,) \sup_{|y'| \le \delta_1} |\nabla^2_{y'}\psi(y')|\\
&+ \sup_{|y'/\delta_0| \le 2} (\,|(\nabla_{y'}\rho)(y'/\delta_0)||y'/\delta_0|^2\,) \sup_{|y'| \le \delta_1} |\nabla^2_{y'}\psi(y')|\Big\}\delta_0\\
\le\ & C(\delta_1)\delta_0
\end{aligned}$$

with some constant $C(\delta_1)$ independent of δ_0. Therefore, we have

$$\|\nabla_{y'}\omega\|_{L_\infty(\mathbb{R}^{n-1})} \le C(\delta_1)\delta_0. \qquad (4.9)$$

Let $\delta > 0$ be the number given in Theorem 3.1 and choose δ_0 so small that $C(\delta_1)\delta_0 \le \delta$. Then, by Theorem 3.1 there exists a $\lambda_0 = \lambda_0(x_0) \ge 1$ such that

$$\begin{aligned}
&\sum_{j=0}^{2} |\lambda|^{\frac{2-j}{2}} \|U\|_{W_p^{2+j}(H_\omega)} + \sum_{j=0}^{2} |\lambda|^{\frac{2-j}{2}} \|(V, \Theta)\|_{W_p^j(H_\omega)}\\
&\qquad \le C(x_0)(\|\widetilde{\varphi f}\|_{W_p^2(H_\omega)} + \|(\widetilde{\varphi g}, \widetilde{\varphi h})\|_{L_p(H_\omega)} + \|(\widetilde{G_\varphi}, \widetilde{H_\varphi})\|_{L_p(H_\omega)})
\end{aligned} \qquad (4.10)$$

for any $\lambda \in \mathbb{C}_+$ with $|\lambda| \ge \lambda_0$. Here and hereafter, $C(x_0)$ denotes a generic constant depending on x_0 and φ. Noting that $\operatorname{supp}\varphi \subset B_b$, we have

$$\|(\widetilde{G_\varphi}, \widetilde{H_\varphi})\|_{L_p(H_\omega)} \le C(\|u\|_{W_p^3(\Omega_b)} + \|(v, \theta)\|_{W_p^1(\Omega_b))}), \qquad (4.11)$$

where $\Omega_b = \Omega \cap B_b = \{x \in \Omega \mid |x| < b\}$. Combining (4.10) and (4.11) we have

$$\begin{aligned}
&\sum_{j=0}^{2} |\lambda|^{\frac{2-j}{2}} (\|U\|_{W_p^{j+2}(H_\omega)} + \|(V, \Theta)\|_{W_p^j(H_\omega)})\\
&\qquad \le C(x_0)(\ \|f\|_{W_p^2(\Omega)} + \|(g, h)\|_{L_p(\Omega)} + \|u\|_{W_p^3(\Omega_b)} + \|(v, \theta)\|_{W_p^1(\Omega_b)}).
\end{aligned} \qquad (4.12)$$

Since $\varphi = 1$ on $B_{\sigma/2}(x_0)$, from (4.12) finally we have that there exist $\sigma = \sigma(x_0) > 0$, $\lambda(x_0) \geq 1$ and $C(x_0) > 0$ such that for any $\lambda \in \mathbb{C}_+$ with $|\lambda| \geq \lambda(x_0)$ there holds the estimate:

$$\begin{aligned}\sum_{j=0}^{2} |\lambda|^{\frac{2-j}{2}} (\|u\|_{W_p^{2+j}(B_{\sigma/2}(x_0))} + \|(v,\theta)\|_{W_p^j(B_{\sigma/2}(x_0))}) \\ \leq C(x_0)\{\|f\|_{W_p^2(\Omega)} + \|(g,h)\|_{L_p(\Omega)} + \|u\|_{W_p^3(\Omega_b)} + \|(v,\theta)\|_{W_p^1(\Omega_b)}\}.\end{aligned} \tag{4.13}$$

Since Γ is compact, there exists a finite number of points $x_j \in \Gamma$ $(j = 1, \ldots, N)$ such that

$$\Gamma \subset \bigcup_{j=1}^{N} B_{\sigma(x_j)/2}(x_j). \tag{4.14}$$

If we set $C = \sum_{j=1}^{N} C(x_j)$, $\lambda_0 = \max_{j=1,\ldots,N} \lambda(x_j)$, and $E = \Omega \cap (\bigcup_{j=1}^{N} B_{\sigma(x_j)/2}(x_j))$, then it follows from (4.13) and (4.14) that for any $\lambda \in \mathbb{C}_+$ with $|\lambda| \geq \lambda_0$ there holds the estimate:

$$\begin{aligned}\sum_{j=0}^{2} |\lambda|^{\frac{2-j}{2}} (\|u\|_{W_p^{2+j}(E)} + \|(v,\theta)\|_{W_p^j(E)}) \\ \leq C(\|f\|_{W_p^2(\Omega)} + \|(g,h)\|_{L_p(\Omega)} + \|u\|_{W_p^3(\Omega_b)} + \|(v,\theta)\|_{W_p^1(\Omega_b)}).\end{aligned} \tag{4.15}$$

Let $\kappa > 0$ be a small number such that $E \supset \{x \in \Omega \mid \mathrm{dist}\,(x, \Gamma) < 3\kappa\}$. Let φ be a function in $C^\infty(\mathbb{R}^n)$ such that

$$\varphi(x) = \begin{cases} 1 & \text{for } x \in \Omega \cap \{x \in \mathbb{R}^n \mid \mathrm{dist}\,(x,\Gamma) > 2\kappa\}, \\ 0 & \text{for } x \in \Omega \cap \{x \in \mathbb{R}^n \mid \mathrm{dist}\,(x,\Gamma) < \kappa\} \\ 0 & \text{for } x \notin \Omega. \end{cases}$$

Since $\mathrm{supp}\,\varphi \subset \mathrm{int}(\Omega)$, we have (4.2), replacing Ω by $\mathbb{R}^n$. And therefore, applying Theorem 2.1, for $\lambda \in \mathbb{C}_+$ with $|\lambda| \geq 1$ we have

$$\begin{aligned}\sum_{j=0}^{2} |\lambda|^{\frac{2-j}{2}} \|(\nabla^{j+2}(\varphi u), \nabla^j(\varphi v), \nabla^j(\varphi\theta))\|_{L_p(\mathbb{R}^n)} \\ \leq C(\|f\|_{W_p^2(\Omega)} + \|(g,h)\|_{L_p(\Omega)} + \|u\|_{W_p^3(\Omega_b)} + \|(v,\theta)\|_{W_p^1(\Omega_b)}),\end{aligned} \tag{4.16}$$

$$|\lambda|\|\varphi u\|_{L_p(\mathbb{R}^n)} \leq C\{\|f\|_{W_p^2(\Omega)} + |\lambda|^{-1}\|(g,h)\|_{L_p(\Omega)} + |\lambda|^{-1}(\|u\|_{W_p^3(\Omega_b)} + \|(v,\theta)\|_{W_p^1(\Omega_b)})\}, \tag{4.17}$$

where we have used the fact that $\mathrm{supp}\, D^\alpha\varphi \subset E \subset \Omega_b$ for any multi-index α with $|\alpha| \geq 1$. Using the interpolation inequality: $\|\nabla(\varphi u)\|_{L_p(\mathbb{R}^n)} \leq C\|\nabla^2(\varphi u)\|_{L_p(\mathbb{R}^n)}^{1/2}\|\varphi u\|_{L_p(\mathbb{R}^n)}^{1/2}$ and also the fact that $\varphi(x) = 1$ for $x \in \Omega \setminus E$, from (4.16) and (4.17) it follows that for any $\lambda \in \mathbb{C}_+$ with $|\lambda| \geq 1$ there holds the estimate:

$$\begin{aligned}\sum_{j=0}^{2} |\lambda|^{\frac{2-j}{2}} (\|u\|_{W_p^{2+j}(\mathbb{R}^n\setminus E)} + \|(v,\theta)\|_{W_p^j(\mathbb{R}^n\setminus E)}) \\ \leq C(\|f\|_{W_p^2(\Omega)} + \|(g,h)\|_{L_p(\Omega)} + \|u\|_{W_p^3(\Omega_b)} + \|(v,\theta)\|_{W_p^1(\Omega_b)}),\end{aligned}$$

which combined with (4.15) implies that

$$\begin{aligned}\sum_{j=0}^{2}|\lambda|^{\frac{2-j}{2}}(\|u\|_{W_p^{2+j}(\Omega)}+\|(v,\theta)\|_{W_p^j(\Omega)})& \\ \le C(\|f\|_{W_p^2(\Omega)}+\|(g,h)\|_{L_p(\Omega)}+\|u\|_{W_p^3(\Omega_b)}+\|(v,\theta)\|_{W_p^1(\Omega_b)})&\end{aligned}\tag{4.18}$$

for any $\lambda\in\mathbb{C}_+$ with $|\lambda|\ge\lambda_0$. Choosing $\lambda_1\ge\lambda_0$ so large that $\lambda_1^{\frac{1}{2}}\ge 2C$ in (4.18) implies that

$$\sum_{j=0}^{2}|\lambda|^{\frac{2-j}{2}}(\|u\|_{W_p^{2+j}(\Omega)}+\|(v,\theta)\|_{W_p^j(\Omega)})\le 2C(\|f\|_{W_p^2(\Omega)}+\|(g,h)\|_{L_p(\Omega)})$$

for $\lambda\in\mathbb{C}_+$ with $|\lambda|\ge\lambda_1$, which completes the proof of Theorem 4.1.

5 A proof of the unique existence theorem in the bounded domain case

In what follows, $\mathcal{H}_p(\Omega)$ and $\mathcal{D}_p(\Omega)$ denote the spaces defined in (1.8) with $\mathcal{O}=\Omega$ and $W^2_{p,0}(\Omega)$, $W^m_{p,D}(\Omega)$ $(m=2,4)$ denote the spaces defined in (1.7) with $\mathcal{O}=\Omega$. Recall that

$$\mathcal{H}_p(\Omega)=W^2_{p,D}(\Omega)\times L_p(\Omega)\times L_p(\Omega),\ \ \mathcal{D}_p(\Omega)=W^4_{p,D}(\Omega)\times W^2_{p,D}(\Omega)\times W^2_{p,0}(\Omega).\tag{5.1}$$

We shall show Theorem 1.2 by a compact perturbation method from the $\lambda=0$ case. To study the $\lambda=0$ case, we shall use the following well-known results.

Lemma 5.1. *Let $1<p<\infty$ and let Ω be a bounded domain in $\mathbb{R}^n$ $(n\ge 2)$. Then, we have the following two assertions:*
(1) *Assume that the boundary Γ of Ω is a C^2 hypersurface. Then, for any $f\in L_p(\Omega)$ there exists a unique solution $u\in W^2_{p,0}(\Omega)$ to the Laplace equation: $\Delta u=f$ in Ω.*
(2) *Assume that Γ is a C^4 hypersurface. Then, for any $f\in L_p(\Omega)$ there exists a unique solution $u\in W^4_{p,D}(\Omega)$ to the biharmonic equation: $\Delta^2 u=f$ in Ω.*

Proof. Both assertions are well-known (cf. Simader [20, Theorem 10.10]) [1]. □

Lemma 5.2. *Let $1<p<\infty$ and let Ω be a bounded domain. Assume that the boundary Γ of Ω is a C^4 hypersurface. Let $\mathcal{A}_p$ be the operator defined in (1.10) and $\rho(\mathcal{A}_p)$ its resolvent set. Then, $0\in\rho(\mathcal{A}_p)$.*

Proof. Since $\mathcal{D}_p(\Omega)$ is a closed subspace of $W^4_p(\Omega)\times W^2_p(\Omega)\times W^2_p(\Omega)$, to prove that $0\in\rho(\mathcal{A}_p)$ in view of the closed graph theorem of S. Banach it suffices to prove that the operator $\mathcal{A}_p:\mathcal{D}_p(\Omega)\to\mathcal{H}_p(\Omega)$ is bijective. The surjectivity follows from the existence of solution $U={}^T(u,v,\theta)\in\mathcal{D}_p(\Omega)$ to the equation $AU=F$ in Ω for given $F={}^T(f,g,h)\in\mathcal{H}_p(\Omega)$. To solve this equation, using the formula (1.5) we rewrite it componentwise as follows:

$$v=f,\ \ -\Delta^2 u-\Delta\theta=g\ \text{ and }\ \Delta v+\Delta\theta=h\ \text{ in }\Omega.\tag{5.2}$$

[1] Here, we quote a result due to Simader [20] concerning the unique existence of solutions to the Dirichlet problem for the biharmonic operator, and he assumed that Γ is a C^4 hypersurface. Therefore, in this paper we assume that Γ is a C^4 hypersurface, but we think that it is enough to assume that Γ is a $C^{3,1}$ hypersurface.

Since $F \in \mathcal{H}_p(\Omega)$, from (5.1) it follows that $v = f \in W^2_{p,D}(\Omega)$. Inserting this fact into the last equation in (5.2) and using Lemma 5.1 (1), we have the existence of a solution $\theta \in W^2_{p,0}(\Omega)$ to the equation: $\Delta\theta = h - \Delta v = h - \Delta f$ in Ω. Finally, inserting the first and last relations of (5.2) into the second equation of (5.2) and using Lemma 5.1 (2), we have the existence of a solution $u \in W^4_{p,D}(\Omega)$ to the equation: $\Delta^2 u = -g - \Delta\theta = -g - h + \Delta f$ in Ω. This completes the proof of the surjectivity of the map $\mathcal{A}_p : \mathcal{D}_p(\Omega) \to \mathcal{H}_p(\Omega)$. To show the injectivity of this map, let $U = {}^T(u, v, \theta) \in \mathcal{D}_p(\Omega)$ satisfy the relation: $\mathcal{A}_p U = 0$, which is rewritten componentwise as follows:

$$v = 0, \quad -\Delta^2 u - \Delta\theta = 0 \text{ and } \Delta v + \Delta\theta = 0 \text{ in } \Omega. \tag{5.3}$$

Since $\theta \in W^2_{p,0}(\Omega)$ as follows from (5.1) and since $\Delta\theta = 0$ in Ω as follows from (5.3), by Lemma 5.1 (1) we have $\theta = 0$. Therefore, by (5.1) and the second equation of (5.3) we have $u \in W^4_{p,D}(\Omega)$ and $\Delta^4 u = 0$ in Ω, which combined with Lemma 5.1 (2) implies that $u = 0$. Summing up, we have proved that $U = 0$, which implies the the injectivity of the map $\mathcal{A}_p : \mathcal{D}_p(\Omega) \to \mathcal{H}_p(\Omega)$. This completes the proof of the lemma. □

Now, we shall discuss the uniqueness.

Lemma 5.3. *Let $1 < p < \infty$ and let Ω be a bounded domain. Assume that the boundary Γ of Ω is a C^4 hypersurface. Let $\mathbb{C}_+$ be the same set as in Theorem 1.2 and let $\lambda \in \mathbb{C}_+$. If $U \in \mathcal{D}_p(\Omega)$ satisfies the homogeneous equation:*

$$(\lambda I - A)U = 0 \quad \text{in } \Omega, \tag{5.4}$$

then $U = 0$.

Proof. When $\lambda = 0$, we have already seen the lemma from the proof of Lemma 5.2, and therefore we assume that $\lambda \neq 0$ in what follows. In view of (1.5), $U = {}^T(u, v, \theta) \in \mathcal{D}_p(\Omega)$ satisfies the equations:

$$\lambda u = v, \quad \lambda v + \Delta^2 u + \Delta\theta = 0 \text{ and } \lambda\theta - \Delta\theta - \Delta v = 0 \text{ in } \Omega. \tag{5.5}$$

Inserting the relation: $\lambda u = v$ into other two equations in (5.5) we have

$$\lambda^2 u + \Delta^2 u + \Delta\theta = 0 \text{ and } \lambda\theta - \Delta\theta - \lambda\Delta u = 0 \text{ in } \Omega. \tag{5.6}$$

If $p = 2$, then multiplying the first and second equations by $\bar\lambda\bar u$ and $\bar\theta$, respectively, integrating the resultant formulas over Ω and using the fact that $(u, \theta) \in W^4_{p,D}(\Omega) \times W^2_{p,0}(\Omega)$, by the divergence theorem of Gauss and the Green formula we have

$$0 = |\lambda|^2\lambda\|u\|^2_{L_2(\Omega)} + \bar\lambda\|\Delta u\|^2_{L_2(\Omega)} + \bar\lambda(\Delta\theta, u)_\Omega + \lambda\|\theta\|^2_{L_2(\Omega)} + \|\nabla\theta\|^2_{L_2(\Omega)} - \lambda(u, \Delta\theta)_\Omega$$

where we have set $(a, b)_\Omega = \int_\Omega a(x)\overline{b(x)}\,dx$. Taking real part and using the assumption that $\mathrm{Re}\,\lambda \geq 0$, we have $\|\nabla\theta\|^2_{L_2(\Omega)} = 0$, which combined with the fact that $\theta|_\Gamma = 0$ implies that $\theta = 0$. Since $\lambda \neq 0$, the second equation of (5.6) implies that $\Delta u = 0$ in Ω, which combined with the fact that $u \in W^4_{p,D}(\Omega) \subset W^2_{p,0}(\Omega)$ implies that $u = 0$. Therefore, if $p = 2$, then we have $U = 0$. Since Ω is bounded, if $p > 2$, then $\mathcal{D}_p(\Omega) \subset \mathcal{D}_2(\Omega)$, and therefore $U = 0$ when $2 < p < \infty$ as well.

When $1 < p < 2$, we shall show that $U \in \mathcal{D}_p(\Omega)$ and (5.4) imply that $U \in \mathcal{D}_2(\Omega)$. Since $u \in W^4_p(\Omega)$ and $\theta \in W^2_p(\Omega)$, by Sobolev's imbedding theorem we see that $u \in W^2_q(\Omega)$ and $\theta \in L_q(\Omega)$ with exponent $q > p$ such that $n(1/p - 1/q) = 2$. By Lemma 5.1 there exists a solution $\tau \in W^2_{q,0}(\Omega)$ to the equation: $\Delta\tau = \lambda\theta - \lambda\Delta u$ in Ω. Since Ω is bounded, τ also belongs to $W^2_{p,0}(\Omega)$, and therefore the uniqueness of solutions in Lemma 5.1 implies that $\theta = \tau$, which means that $\theta \in W^2_{q,0}(\Omega)$. Therefore, $\lambda^2 u + \Delta\theta \in L_q(\Omega)$. By Lemma 5.1 there exists a solution $w \in W^4_{q,D}(\Omega)$ to the equation: $\Delta^4 w = -(\lambda u + \Delta\theta)$ in Ω. Since $w \in W^4_{q,D}(\Omega) \subset W^4_{p,D}(\Omega)$, the uniqueness of solutions implies that $u = w$, which means that $u \in W^4_{q,D}(\Omega)$. Therefore, we have seen that $U \in \mathcal{D}_q(\Omega)$. If $q \geq 2$, then we have $U = 0$. If q is still less than 2, then repeating the same argument finitely many times, finally we arrive at the stage that $U \in \mathcal{D}_2(\Omega)$, and therefore we have $U = 0$. This completes the proof of the lemma. □

Now, we shall give a

Proof of Theorem 1.2 (1). Let $\mathcal{A}_p^{-1}$ denote the inverse operator of $\mathcal{A}_p$, the existence of which was proved in Lemma 5.2. $\mathcal{A}_p^{-1}$ is a bounded linear operator from $\mathcal{H}_p(\Omega)$ onto $\mathcal{D}_p(\Omega)$. If we write $\lambda I - \mathcal{A}_p = -(I - \lambda\mathcal{A}_p^{-1})\mathcal{A}_p$, then the existence of the inverse operator $(\lambda I - \mathcal{A}_p)^{-1}$ is equivalent to that of the inverse operator $(I - \lambda\mathcal{A}_p^{-1})^{-1}$. In fact, if $(I - \lambda\mathcal{A}_p^{-1})^{-1}$ exists as a bounded linear operator on $\mathcal{H}_p(\Omega)$, then

$$(\lambda I - \mathcal{A}_p)^{-1} = -\mathcal{A}_p^{-1}(I - \lambda\mathcal{A}_{p,K}^{-1})^{-1} \tag{5.7}$$

exists as a bounded linear operator from $\mathcal{H}_p(\Omega)$ onto $\mathcal{D}_p(\Omega)$. By the Rellich compactness theorem, $\mathcal{D}_p(\Omega)$ is compactly imbedded into $\mathcal{H}_p(\Omega)$, and therefore $\lambda\mathcal{A}_p^{-1}$ is an entire function of λ with its value in $\mathcal{L}_C(\mathcal{H}_p(\Omega))$, where $\mathcal{L}_C(\mathcal{H}_p(\Omega))$ denotes the set of all compact linear operators on $\mathcal{H}_p(\Omega)$. Therefore, by the Harazov-Seeley theorem (cf. Seeley [18]) we see that $(I - \lambda\mathcal{A}_p^{-1})^{-1}$ is defined for all $\lambda \in \mathbb{C}$ as a finitely meromorphic function with its value in $\mathcal{L}(\mathcal{H}_p(\Omega))$, where $\mathcal{L}(\mathcal{H}_p(\Omega))$ denotes the set of all bounded linear operators on $\mathcal{H}_p(\Omega)$. Let Λ denote the set of all poles of $(I - \lambda\mathcal{A}_p^{-1})^{-1}$, and then Λ is a discrete set in $\mathbb{C}$. We shall show the following lemma.

Lemma 5.4. *$\Lambda \cap \mathbb{C}_+ = \emptyset$.*

Postponing the proof of Lemma 5.4, we continue the proof of Theorem 1.2 (1). By Lemma 5.4 we see that $(I - \lambda\mathcal{A}_p^{-1})^{-1}$ exists for all $\lambda \in \mathbb{C}_+$ and depends continuously on $\lambda \in \mathbb{C}_+$. Especially, for any $\lambda_1 > 0$ there exists a $K > 0$ such that

$$\|(I - \lambda\mathcal{A}_p^{-1})^{-1}\|_{\mathcal{L}(\mathcal{H}_p(\Omega))} \leq K \quad \text{whenever } \lambda \in \mathbb{C}_+ \text{ and } |\lambda| \leq \lambda_1. \tag{5.8}$$

where $\|\cdot\|_{\mathcal{L}(\mathcal{H}_p(\Omega))}$ denotes the operator norm of $\mathcal{L}(\mathcal{H}_p(\Omega))$. Since $\mathcal{A}_p^{-1}$ is a bounded linear operator from $\mathcal{H}_p(\Omega)$ onto $\mathcal{D}_p(\Omega)$, by (5.7) $(\lambda I - \mathcal{A}_p)^{-1}$ is also a bounded linear operator from $\mathcal{H}_p(\Omega)$ onto $\mathcal{D}_p(\Omega)$ and by (5.8) we see that for any $F \in \mathcal{H}_p(\Omega)$ and $\lambda \in \mathbb{C}_+$ with $|\lambda| \leq \lambda_1$ there holds the estimate:

$$\|(\lambda I - \mathcal{A}_p)^{-1}F\|_{\mathcal{D}_p(\Omega)} \leq K\|\mathcal{A}_p^{-1}\|_{\mathcal{L}(\mathcal{H}_p(\Omega),\mathcal{D}_p(\Omega))}\|F\|_{\mathcal{H}_p(\Omega)} \tag{5.9}$$

where $\|\cdot\|_{\mathcal{L}(\mathcal{H}_p(\Omega),\mathcal{D}_p(\Omega))}$ denotes the operator norm for the bounded linear operator from $\mathcal{H}_p(\Omega)$ into $\mathcal{D}_p(\Omega)$. On the other hand, recalling the definition of the norms for the spaces $\mathcal{H}_p(\Omega)$ and $\mathcal{D}_p(\Omega)$ given in (1.9), by Theorem 4.1 we see that there exists a $\lambda_1 \geq 1$ such that for any $\lambda \in \mathbb{C}_+$ with $|\lambda| \geq \lambda_1$ and $F \in \mathcal{H}_p(\Omega)$ there holds the estimate:

$$|\lambda|\|(\lambda I - \mathcal{A}_p)^{-1}F\|_{\mathcal{H}_p(\Omega)} + \|(\lambda I - \mathcal{A}_p)^{-1}F\|_{\mathcal{D}_p(\Omega)} \leq C\|F\|_{\mathcal{H}_p(\Omega)}. \tag{5.10}$$

Combining (5.9) and (5.10) completes the proof of Theorem 1.2 (1) if we finish the proof of Lemma 5.4. Finally, we give a

Proof of Lemma 5.4. Assume that $\Lambda \cap \mathbb{C}_+ \neq \emptyset$. Let $\lambda_0 \in \Lambda \cap \mathbb{C}_+$. By the definition of Λ, there exists a $\sigma_0 > 0$ and $N \in \mathbb{N}$ such that

$$(I - \lambda \mathcal{A}_p^{-1})^{-1} = \sum_{j=1}^{N} A_j(\lambda - \lambda_0)^{-j} + B_\lambda \quad (\lambda \in U_{\sigma_0} \setminus \{\lambda_0\}) \tag{5.11}$$

where $A_j \in \mathcal{L}(\mathcal{H}_p(\Omega))$, B_λ is a holomorphic function defined on U_{σ_0} with its value in $\mathcal{L}(\mathcal{H}_p(\Omega))$ and $U_{\sigma_0} = \{z \in \mathbb{C} \mid |z - \lambda_0| < \sigma_0\}$. We may assume that $A_N \not\equiv 0$, so that there exists at least one $F \in \mathcal{H}_p(\Omega)$ such that $A_N F \neq 0$. In view of (5.7), for $\lambda \in U_{\sigma_0} \setminus \{\lambda_0\}$ we set $V_\lambda = -\mathcal{A}_p^{-1}(I - \lambda \mathcal{A}_p^{-1})^{-1} F$. Then, $V_\lambda \in \mathcal{D}_p(\Omega)$ and V_λ satisfies the equation: $(\lambda I - \mathcal{A}_p)V_\lambda = F$ in Ω. In view of (5.11), we have

$$V_\lambda = -\sum_{j=1}^{N} (\lambda - \lambda_0)^{-j} \mathcal{A}_p^{-1} A_j F - \mathcal{A}_p^{-1} B_\lambda F$$

and therefore

$$(\lambda - \lambda_0)^N F = (\lambda - \lambda_0)^N (\lambda I - \mathcal{A}_p) V_\lambda = (\lambda I - \mathcal{A}_p)(-\mathcal{A}_p^{-1} A_N F) + O(|\lambda - \lambda_0|).$$

Letting $\lambda \to \lambda_0$, we have

$$(\lambda_0 I - \mathcal{A}_p)(-\mathcal{A}_p^{-1} A_N F) = 0 \quad \text{in } \Omega.$$

Since $-\mathcal{A}_p^{-1} A_N F \in \mathcal{D}_p(\Omega)$, by Lemma 5.3 $\mathcal{A}_p^{-1} A_N F = 0$. Applying $\mathcal{A}_p$, we have $A_N F = 0$, which contradicts the assumption that $A_N F \neq 0$. Therefore, $\Lambda \cap \mathbb{C}_+ = \emptyset$, which completes the proof of Lemma 5.4.

6 A proof of the unique existence theorem in the exterior domain case

We shall show Theorem 1.2 (2) in this section. Our main step is to prove the following theorem.

Theorem 6.1. *Let $1 < p < \infty$. Let Ω be an exterior domain in $\mathbb{R}^n$ $(n \geq 2)$ and assume that the boundary Γ of Ω is a C^4 hypersurface. Let b be a large number such that $\mathbb{R}^n \setminus \Omega \subset B_b = \{x \in \mathbb{R}^n \mid |x| < b\}$. Set*

$$\begin{aligned} L_{p,b+3}(\Omega) &= \{g \in L_p(\Omega) \mid g(x) = 0 \quad \textit{for } |x| \geq b+3\}, \\ L_{p,b+3}(\Omega)^2 &= \{(g,h) \mid g, h \in L_{p,b+3}(\Omega)\} \\ \mathcal{J}_p(\Omega) &= \{(u,\theta) \mid u \in W^4_{p,D}(\Omega), \ \ \theta \in W^2_{p,0}(\Omega)\}, \end{aligned}$$

where $W^4_{p,D}(\Omega)$ and $W^2_{p,0}(\Omega)$ are the spaces defined in (1.7) with $\mathcal{O} = \Omega$. Let λ_0 and λ_1 be any numbers such that $0 < \lambda_0 < \lambda_1 < \infty$. Then, for any $(g,h) \in L_{p,b+3}(\Omega)^2$ and $\lambda \in \mathbb{C}_+$ with $\lambda_0 \leq |\lambda| \leq \lambda_1$ there exists a unique solution $(u,\theta) \in \mathcal{J}_p(\Omega)$ to the equations:

$$\lambda^2 u + \Delta^2 u + \Delta\theta = g \ \textit{ and } \ \lambda\theta - \Delta\theta - \lambda\Delta u = h \ \textit{ in } \Omega, \tag{6.1}$$

which satisfies the estimate:

$$\|u\|_{W^4_p(\Omega)} + \|\theta\|_{W^2_p(\Omega)} \leq C\|(g,h)\|_{L_p(\Omega)} \tag{6.2}$$

for some constant C depending only on p, Ω, λ_0 and λ_1.

Proof. Before starting our proof of Theorem 6.1, we should remark the following fact: If $U = (u, v, \theta)$ satisfies the equation:

$$(\lambda I - A)U = F \quad \text{in } \Omega, \tag{6.3}$$

for $F = {}^T(0, g, h)$, then $\lambda u = v$ implies that (u, θ) satisfies the equation (6.1). On the other hand, if (u, θ) satisfies (6.1) then setting $\lambda u = v$, we see that $U = {}^T(u, v, \theta)$ and $F = {}^T(0, g, h)$ satisfy the equation (6.3). Therefore, when $f = 0$, equations (6.1) and (6.3) are equivalent under the substitution for $\lambda u = v$. In particular, we can use Theorem 1.2 (1), Theorems 2.1, 2.2 and 4.1, and Lemma 5.3 with $f = 0$ and $\lambda u = v$ in what follows.

To prove Theorem 6.1 we shall construct a parametrix. As a preparation for this, we consider the resolvent problems in $\mathbb{R}^n$ and $\Omega_{b+4} = \Omega \cap B_{b+4} = \{x \in \Omega \mid |x| < b + 4\}$. First, we consider the resolvent problem: $(\lambda I - \mathcal{A}_p)U = F$ in $\mathbb{R}^n$. Let Σ_ϵ be the same set as in (2.1) of Theorem 2.1. Then, by Theorem 2.1 we have

$$(\lambda I - \mathcal{A}_p)^{-1} \in \mathrm{Anal}(\Sigma_\epsilon, \mathcal{L}(\mathcal{H}_p(\mathbb{R}^n), \mathcal{D}_p(\mathbb{R}^n))). \tag{6.4}$$

Here and hereafter, Anal $(\mathcal{O}, X)$ denotes the set of all holomorphic functions defined on a domain $\mathcal{O}$ of the complex field $\mathbb{C}$ with their values in X, and $\mathcal{L}(X, Y)$ the set of all bounded linear operators from X into Y for two Banach spaces X and Y. For short, we write $\mathcal{L}(X) = \mathcal{L}(X, X)$ as usual. The assertion (6.4) follows from the resolvent equation:

$$(\lambda_1 I - A)^{-1} - (\lambda_2 I - A)^{-1} = (\lambda_2 - \lambda_1)(\lambda_1 I - A)^{-1}(\lambda_2 I - A)^{-1}$$

and estimate (2.3).

Next, we consider the resolvent problem: $(\lambda I - \mathcal{A}_p)U = F$ in Ω_{b+4}. Let $\partial\Omega_{b+4}$ be the boundary of Ω_{b+4}, and then it consists of Γ and $S_{b+4} = \{x \in \mathbb{R}^n \mid |x| = b + 4\}$. In particular, $\partial\Omega_{b+4}$ is a C^4 hypersurface if Γ is assumed to be a C^4 hypersurface. Therefore, using Theorem 1.2 (1) which was already proved in section 5 and employing a standard argument (cf. Vrabie [21, A Proof of Theorem 7.1.1]), we see that there exists an $\epsilon' \in (0, \pi/2)$ such that

$$(\lambda I - \mathcal{A}_p)^{-1} \in \mathrm{Anal}\,(\Sigma_{\epsilon'}, \mathcal{L}(\mathcal{H}_p(\Omega_{b+4}), \mathcal{D}_p(\Omega_{b+4}))) \tag{6.5}$$

where $\Sigma_{\epsilon'}$ is defined by the formula in (2.1), replacing ϵ by ϵ'.

Under above preparations, we shall construct a parametrix for equation (6.1). In what follows, given function $k(x)$ defined on Ω, $k_0(x)$ denotes the zero extension of $k(x)$ to the whole space, that is $k_0(x) = k(x)$ for $x \in \Omega$ and $k_0(x) = 0$ for $x \notin \Omega$. Given $\lambda \in \Sigma_\epsilon$ and $(g, h) \in L_{p,b+3}(\Omega)^2$, by Theorem 2.1 there exists a unique solution $(U, \Theta) \in W_p^4(\mathbb{R}^n) \times W_p^2(\mathbb{R}^n)$ to the equation:

$$\lambda^2 U + \Delta^2 U + \Delta\Theta = g_0 \ \text{ and } \ \lambda\Theta - \Delta\Theta - \lambda\Delta U = h_0 \ \text{ in } \mathbb{R}^n, \tag{6.6}$$

which satisfies the estimate:

$$\sum_{j=0}^{4} |\lambda|^{\frac{4-j}{2}} \|U\|_{L_p(\mathbb{R}^n)} + \sum_{j=0}^{2} |\lambda|^{\frac{2-j}{2}} \|\nabla^j \Theta\|_{L_p(\mathbb{R}^n)} \leq C\|(g, h)\|_{L_p(\mathbb{R}^n)}, \tag{6.7}$$

where C is a constant independent of $\lambda \in \Sigma_\epsilon$, g, h, U and Θ. Let us define the maps: $A_1(\lambda) : L_{p,b+3}(\Omega)^2 \to W_p^4(\mathbb{R}^n)$ and $A_2(\lambda) : L_{p,b+3}(\Omega)^2 \to W_p^2(\mathbb{R}^n)$ by the relations: $A_1(\lambda)(g, h) = U$ and $A_2(\lambda)(g, h) = \Theta$, and set $A(\lambda)(g, h) = {}^T(A_1(\lambda)(g, h), A_2(\lambda)(g, h))$. By (6.4) we have

$$A(\lambda) \in \mathrm{Anal}\,(\Sigma_\epsilon, \mathcal{L}(L_{p,b+3}(\Omega)^2, W_p^4(\mathbb{R}^n) \times W_p^2(\mathbb{R}^n))). \tag{6.8}$$

For an element $(g, h) \in L_{p,b+3}(\Omega)^2$, we consider it also an element of $L_p(\Omega_{b+4}) \times L_p(\Omega_{b+4})$. Applying Theorem 1.2 (1) with $f = 0$ in case of Ω_{b+4}, we see that there exists a unique solution $(v, \tau) \in W^4_{p,D}(\Omega_{b+4}) \times W^2_{p,0}(\Omega_{b+4})$ to equations:

$$\lambda^2 v + \Delta^2 v = g \ \text{ and } \ \lambda\tau - \Delta\tau - \lambda\Delta v = h \ \text{ in } \Omega_{b+4}, \tag{6.9}$$

which satisfies the estimate:

$$|\lambda|(\|v\|_{W^2_p(\Omega_{b+4})} + \|\tau\|_{L_p(\Omega_{b+4})}) + \|v\|_{W^4_p(\Omega_{b+4})} + \|\tau\|_{W^2_p(\Omega_{b+4})} \le C\|(g, h)\|_{L_p(\Omega)}, \tag{6.10}$$

where C depends on Ω, b and p, but independent of $\lambda \in \mathbb{C}_+$, g, h, v and τ. Let us define the maps $B_1(\lambda) : L_{p,b+3}(\Omega)^2 \to W^4_{p,D}(\Omega_{b+4})$ and $B_2(\lambda) : L_{p,b+3}(\Omega)^2 \to W^2_{p,0}(\Omega_{b+4})$ by the relations: $B_1(\lambda)(g, h) = v$ and $B_2(\lambda)(g, h) = \tau$, and set $B(\lambda)(g, h) = (B_1(\lambda)(g, h), B_2(\lambda)(g, h))$. By (6.5) we have

$$B(\lambda) \in \text{Anal}\,(\Sigma_{\epsilon'}, \mathcal{L}(L_{p,b+3}(\Omega)^2, W^4_{p,D}(\Omega_{b+4}) \times W^2_{p,0}(\Omega_{b+4}))). \tag{6.11}$$

Let $\varphi(x)$ be a function in $C^\infty_0(\mathbb{R}^n)$ such that $\varphi(x) = 1$ for $|x| \le b+1$ and $\varphi(x) = 0$ for $|x| \ge b+2$. Set $\Xi = \Sigma_\epsilon \cap \Sigma_{\epsilon'}$. Note that $\mathbb{C}_+ \setminus \{0\} \subset \Xi$. Set

$$\begin{aligned} R_j(\lambda)(g, h) &= (1 - \varphi)A_j(\lambda)(g, h) + \varphi B_j(\lambda)(g, h), \\ R(\lambda)(g, h) &= (R_1(\lambda)(g, h), R_2(\lambda)(g, h)). \end{aligned} \tag{6.12}$$

Then, by (6.8) and (6.11)

$$R(\lambda) \in \text{Anal}\,(\Xi, \mathcal{L}(L_{p,b+3}(\Omega)^2, \mathcal{J}_p(\Omega))). \tag{6.13}$$

If we set $(w, \tau) = R(\lambda)(g, h)$, then (w, π) satisfies the equations:

$$\lambda^2 w + \Delta^2 w + \Delta\pi = g + S_1(\lambda)(g, h) \ \text{ and } \ \lambda\pi - \Delta\pi - \lambda\Delta w = h + S_2(\lambda)(g, h) \ \text{ in } \Omega, \tag{6.14}$$

where we have set

$$\begin{aligned} S_1(\lambda)(g, h) &= G_\varphi(B_1(\lambda)(g, h) - A_1(\lambda)(g, h), B_2(\lambda)(g, h) - A_2(\lambda)(g, h)) \\ S_2(\lambda)(g, h) &= H_\varphi(\lambda(B_1(\lambda)(g, h) - A_1(\lambda)(g, h)), B_2(\lambda)(g, h) - A_2(\lambda)(g, h)) \end{aligned} \tag{6.15}$$

and G_φ and H_φ are the same symbols as in (4.3). Set $S(\lambda)(g, h) = (S_1(\lambda)(g, h), S_2(\lambda)(g, h))$. Since $\text{supp}\, S(\lambda)(g, h) \subset D_{b+1,b+2} = \{x \in \mathbb{R}^n \mid b+1 \le |x| \le b+2\}$, by (6.8), (6.11) and Rellich's compactness theorem we see that

$$S(\lambda) \in \text{Anal}\,(\Xi, \mathcal{L}_C(L_{p,b+3}(\Omega)^2) \tag{6.16}$$

where $\mathcal{L}_C(L_{p,b+3}(\Omega)^2)$ denotes the set of all compact operators on $L_{p,b+3}(\Omega)^2$. By the Harazov-Seeley theorem (cf. Seeley [18]) we know that $(I + S(\lambda))^{-1}$ does not exist for any $\lambda \in \Xi$ or it does exist as a finitely meromorphic function defined on Ξ with its value in $\mathcal{L}(L_{p,b+3}(\Omega)^2)$. Now, we shall show that $(I + S(\lambda))^{-1}$ does exist for all $\lambda \in \mathbb{C}_+ \setminus \{0\}$. Since $S(\lambda)$ is a compact operator, by the Fredholm alternative principle (or the Riesz-Schauder theory), to prove the existence of bounded inverse $(I + S(\lambda))^{-1}$ it suffices to prove the injectivity of $I + S(\lambda)$. Let (g, h) be an element in $L_{p,b+3}(\Omega)^2$ such that $(I + S(\lambda))(g, h) = (0, 0)$. Set $(w, \pi) = R(\lambda)(g, h)$, and then by (6.14) we see that $(w, \pi) \in \mathcal{J}_p(\Omega)$ satisfies the homogeneous equation:

$$\lambda^2 w + \Delta^2 w + \Delta\pi = 0, \ \ \lambda\pi - \Delta\pi - \lambda\Delta w = 0 \ \text{ in } \Omega.$$

Then, it follows that $w = \pi = 0$ from the following lemma.

Lemma 6.2. *Let $1 < p < \infty$ and $\lambda \in \mathbb{C}_+ \setminus \{0\}$. Let $\mathcal{J}_p(\Omega)$ be the space defined in Theorem 6.1. If $(u, \theta) \in \mathcal{J}_p(\Omega)$ satisfies the homogeneous equation:*

$$\lambda^2 u + \Delta^2 u + \Delta\theta = 0, \;\; \lambda\theta - \Delta\theta - \lambda\Delta u = 0 \;\; \text{in } \Omega, \tag{6.17}$$

then $u = \theta = 0$.

Postponing the proof of Lemma 6.2, we continue the proof of Theorem 6.1. Combining the fact that $w = \pi = 0$ in Ω with (6.12) implies that

$$(1 - \varphi)A(\lambda)(g, h) + \varphi B(\lambda)(g, h) = (0, 0) \;\; \text{in } \Omega. \tag{6.18}$$

Since $\varphi(x) = 1$ for $|x| \leq b$ and $\varphi(x) = 0$ for $|x| \geq b + 2$, we have

$$A(\lambda)(g, h) = (0, 0) \quad \text{for } |x| \geq b + 2, \quad B(\lambda)(g, h) = (0, 0) \quad \text{for } |x| \leq b + 1. \tag{6.19}$$

If we define (z, τ) by the relation: $(z(x), \tau(x)) = (w(x), \pi(x))$ for $x \in \Omega_{b+4}$ and $(z(x), \tau(x)) = (0, 0)$ for $x \notin \Omega$, then by (6.19) $(z, \tau) \in W^4_{p,D}(B_{b+4}) \times W^2_{p,0}(B_{b+4})$ and (z, τ) satisfies the equations:

$$\lambda^2 z + \Delta^2 z + \Delta\tau = g_0 \;\text{ and }\; \lambda\tau - \Delta\tau - \lambda\Delta z = h_0 \;\text{ in } B_{b+4}. \tag{6.20}$$

On the other hand, if we restrict $A(\lambda)(g, h)$ on B_{b+4} and represent it also by $A(\lambda)(g, h)$, then by (6.19) $A(\lambda)(g, h)$ belongs to $W^4_{p,D}(B_{b+4}) \times W^2_{p,0}(B_{b+4})$ and satisfies the equation (6.20) too. Therefore, the uniqueness result given in Lemma 5.3 implies that $A(\lambda)(g, h) = (z, \tau)$ in B_{b+4}, and therefore $A(\lambda)(g, h) = B(\lambda)(g, h)$ in Ω_{b+4}. Combining this and (6.18) implies that

$$0 = A(\lambda)(g, h) + \varphi(B(\lambda)(g, h) - A(\lambda)(g, h)) = A(\lambda)(g, h) \;\text{ in } \Omega,$$

which combined with (6.6) implies that $(g, h) = (0, 0)$ in Ω. Therefore, we have the injectivity of $I + S(\lambda)$ for all $\lambda \in \mathbb{C}_+ \setminus \{0\}$, from which it follows that $(I + S(\lambda))^{-1}$ exists for all $\lambda \in \mathbb{C}_+ \setminus \{0\}$.

The Harazov-Seeley theorem tells us that there exists an open set G such that $\mathbb{C}_+ \setminus \{0\} \subset G \subset \Xi$ and $(I + S(\lambda))^{-1} \in \mathrm{Anal}\,(G, \mathcal{L}(L_{p,b+3}(\Omega)^2))$, from which it follows that $(I + S(\lambda))^{-1}$ depends continuously on $z \in \mathbb{C}_+ \setminus \{0\}$. Therefore for any λ_0 and λ_1 with $0 < \lambda_0 < \lambda_1 < \infty$ there exists a constant $K(\lambda_0, \lambda_1)$ such that

$$\|(I + S(\lambda))^{-1}\|_{\mathcal{L}(L_{p,b+3}(\Omega)^2)} \leq K(\lambda_0, \lambda_1) \tag{6.21}$$

for any $\lambda \in \mathbb{C}_+ \setminus \{0\}$ with $\lambda_0 \leq |\lambda| \leq \lambda_1$. If we set $\Phi(\lambda) = R(\lambda)(I + S(\lambda))^{-1}$, then $\Phi(\lambda) \in \mathrm{Anal}\,(G, \mathcal{L}(L_{p,b+3}(\Omega)^2, \mathcal{J}_p(\Omega)))$. Moreover, setting $(u, \theta) = \Phi(\lambda)(g, h)$ for $(g, h) \in L_{p,b+3}(\Omega)^2$, then by (6.7), (6.10), (6.14) and (6.21) we see that $(u, \theta) \in \mathcal{J}_p(\Omega)$ solves (6.1) uniquely and satisfies the estimate:

$$\|u\|_{W^4_p(\Omega)} + \|\theta\|_{W^2_p(\Omega)} \leq C(\lambda_0, \lambda_1)\|(g, h)\|_{L_p(\Omega)} \tag{6.22}$$

whenever $\lambda \in \mathbb{C}_+$ and $\lambda_0 \leq |\lambda| \leq \lambda_1$, where $C(\lambda_0, \lambda_1)$ is a constant depending essentially on p, Ω, b, λ_0 and λ_1 only. This completes the proof of Theorem 6.1. □

Now, we shall give

A proof of Lemma 6.2. First, we consider the case where $(u, \theta) \in \mathcal{J}_2(\Omega)$. Multiplying (6.17) by $\bar\lambda\bar u$ and $\bar\theta$ and integrating the resultant formula over Ω, by the divergence theorem of Gauss and the Green formula we have

$$\lambda|\lambda|^2\|u\|^2_{L_2(\Omega)} + \bar\lambda\|\Delta u\|^2_{L_2(\Omega)} + \bar\lambda(\theta, \Delta u)_\Omega + \lambda\|\theta\|^2_{L_2(\Omega)} + \|\nabla\theta\|^2_{L_2(\Omega)} - \lambda(\Delta u, \theta)_\Omega = 0,$$

where $(a,b)_\Omega = \int_\Omega a(x)\overline{b(x)}\,dx$. Since $\lambda \in \mathbb{C}_+$, taking the real part we have $\|\nabla\theta\|_{L_2(\Omega)} = 0$, which combined with $\theta|_\Gamma = 0$ implies that $\theta = 0$. Inserting this fact into the second equation in (6.17) and using the fact that $\lambda \neq 0$, we have $\Delta u = 0$ in Ω. Since $u|_\Gamma = 0$, we have $0 = -(\Delta u, u)_\Omega = \|\nabla u\|^2_{L_2(\Omega)}$, which combined with the fact that $u|_\Gamma = 0$ also implies that $u = 0$. Therefore, the lemma holds when $p = 2$.

When $p \neq 2$, we shall show that $(u,\theta) \in \mathcal{J}_p(\Omega)$ also belongs to $\mathcal{J}_2(\Omega)$. Applying the same argument as in the proof of Lemma 5.3 and using Theorem 1.2 (1), we see that $(u,\theta) \in W^4_{2,\mathrm{loc}}(\overline{\Omega}) \times W^2_{2,\mathrm{loc}}(\overline{\Omega})$. To show that $(u,\theta) \in \mathcal{J}_2(\Omega)$, we use the cut-off technique. Let φ be a function in $C^\infty(\mathbb{R}^n)$ such that $\varphi(x) = 1$ for $|x| \geq b+2$ and $\varphi(x) = 0$ for $|x| \leq b+1$, where b is a large number such that $\mathbb{R}^n \setminus \Omega \subset B_b = \{x \in \mathbb{R}^n \mid |x| < b\}$. Using the symbols defined in (4.3) and equation (6.17) we have

$$\lambda^2(\varphi u) + \Delta^2(\varphi u) + \Delta(\varphi\theta) = g \ \text{ and } \ \lambda(\varphi\theta) - \Delta(\varphi\theta) - \lambda\Delta(\varphi u) = h \quad \text{in } \mathbb{R}^n \tag{6.23}$$

where $g = G_\varphi(u,\theta)$ and $h = H_\varphi(\lambda u, \theta)$. Since $(u,\theta) \in W^4_{2,\mathrm{loc}}(\overline{\Omega}) \times W^2_{2,\mathrm{loc}}(\overline{\Omega})$, we see that $(g,h) \in L_2(\mathbb{R}^n) \times L_2(\mathbb{R}^n)$. By Theorem 2.1 there exists a unique $(w,\tau) \in W^4_2(\mathbb{R}^n) \times W^2_2(\mathbb{R}^n)$ which solves the equation:

$$\lambda^2 w + \Delta^2 w + \Delta\tau = g \ \text{ and } \ \lambda\tau - \Delta\tau - \lambda\Delta w = h \quad \text{in } \mathbb{R}^n \tag{6.24}$$

Since the uniqueness holds for the $\mathcal{S}'(\mathbb{R}^n)$ solutions in the whole space case, we have $(\varphi u, \varphi\theta) = (w,\tau)$. In fact, setting $z = \varphi u - w$ and $\pi = \varphi\theta - \tau$, by (6.23) and (6.24) we see that $(z,\pi) \in \mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n)$ satisfies the homogeneous equation:

$$\lambda^2 z + \Delta^2 z + \Delta\pi = 0 \ \text{ and } \ \lambda\pi - \Delta\pi - \lambda\Delta z = 0 \ \text{ in } \mathbb{R}^n, \tag{6.25}$$

where $\mathcal{S}'(\mathbb{R}^n)$ is the class of tempered distributions. Applying the Fourier transform to (6.25) we have

$$\begin{pmatrix} \lambda^2 + |\xi|^4 & -|\xi|^2 \\ \lambda|\xi|^2 & \lambda + |\xi|^2 \end{pmatrix} \begin{pmatrix} \hat{z} \\ \hat{\pi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{in } \mathbb{R}^n \tag{6.26}$$

in the sense of distribution of class $\mathcal{S}'(\mathbb{R}^n)$. If we denote the determinant of the matrix $\begin{pmatrix} \lambda^2 + |\xi|^4 & -|\xi|^2 \\ \lambda|\xi|^2 & \lambda + |\xi|^2 \end{pmatrix}$ by $D(\lambda,\xi)$, then $D(\lambda,\xi) = \lambda^3 + |\xi|^2\lambda^2 + 2|\xi|^4\lambda + |\xi|^6$. Let $f(t)$ be a polynomial defined by the formula: $f(t) = t^3 + t^2 + 2t + 1$. We see that $f(\lambda/|\xi|^2)|\xi|^6 = D(\lambda,\xi)$. Since $f(0) = 1$, $f(-1) = -1$ and $f(t)$ is a strictly increasing function, there exists only one real root α with $-1 < \alpha < 0$. Since the coefficients of $f(t)$ are real numbers, other two roots are complex number β and its complex conjugate $\bar{\beta}$. If we write $f(t) = (t-\alpha)(t-\beta)(t-\bar{\beta})$, then $\alpha + 2\mathrm{Re}\beta = -1$, from which it follows that $\mathrm{Re}\beta = -(1+\alpha)/2$. Since $-1 < \alpha < 0$, we have $\mathrm{Re}\beta < 0$. Since

$$D(\lambda,\xi) = |\xi|^6(\lambda/|\xi|^2 - \alpha)(\lambda/|\xi|^2 - \beta)(\lambda/|\xi|^2 - \bar{\beta}) = (\lambda - |\xi|^2\alpha)(\lambda - |\xi|^2\beta)(\lambda - |\xi|^2\bar{\beta}),$$

$D(\lambda,\xi) \neq 0$ whenever $\lambda \in \mathbb{C}_+ \setminus \{0\}$ and $\xi \in \mathbb{R}^n$. Therefore, it follows from (6.26) that $\hat{z} = \hat{\pi} = 0$. Applying the Fourier inverse transform, we have $z = \pi = 0$, which means that $\varphi u = w \in W^4_2(\mathbb{R}^n)$ and $\varphi\theta = \tau \in W^2_2(\mathbb{R}^n)$. Since $\varphi(x) = 1$ for $|x| \geq b+2$ and since $(u,\theta) \in W^4_{2,\mathrm{loc}}(\overline{\Omega}) \times W^2_{2,\mathrm{loc}}(\overline{\Omega})$, we have $(u,\theta) \in W^4_2(\Omega) \times W^2_2(\Omega)$, which implies that $u = \theta = 0$. This completes the proof of Lemma 6.2. □

Now, we shall give

A proof of Theorem 1.2 (2). Let λ_0 and λ_1 be arbitrary positive numbers such that $\lambda_0 < \lambda_1$ and let $\lambda \in \mathbb{C}_+$ such that $\lambda_0 \le |\lambda| \le \lambda_1$. Given $F = {}^T(f, g, h) \in \mathcal{H}_p(\Omega)$, we shall look for a solution $U = {}^T(u, v, \theta) \in \mathcal{D}_p(\Omega)$ of the equation:

$$(\lambda I - A)U = F \quad \text{in } \Omega. \tag{6.27}$$

Let φ and ψ be two cut-off functions in $C^\infty(\mathbb{R}^n)$ such that $\varphi(x) = 1$ for $|x| \ge b+3$ and $\varphi(x) = 0$ for $|x| \le b+2$, and $\psi(x) = 1$ for $|x| \ge b+2$ and $\psi(x) = 0$ for $|x| \le b+1$, where b is a large number such that $\mathbb{R}^n \setminus \Omega \subset B_b$. Note that $\varphi(x)\psi(x) = \varphi(x)$. Let $U_0 = (u_0, v_0, \theta_0) \in \mathcal{D}_p(\mathbb{R}^n)$ be a solution to the equation:

$$(\lambda I - A)U_0 = \psi F \quad \text{in } \mathbb{R}^n. \tag{6.28}$$

By Theorem 2.1 we know the unique existence of U_0 and moreover U_0 satisfies the estimate:

$$\|u_0\|_{W^4_p(\mathbb{R}^n)} + \|(v_0, \theta_0)\|_{W^2_p(\mathbb{R}^n)} \le C(\lambda_0, \lambda_1)(\|f\|_{W^2_p(\Omega)} + \|(g, h)\|_{L_p(\Omega)}) \tag{6.29}$$

with some constant $C(\lambda_0, \lambda_1)$ whenever $\lambda \in \mathbb{C}_+$ and $\lambda_0 \le |\lambda| \le \lambda_1$. Here and hereafter, $C(\lambda_0, \lambda_1)$ denotes a generic constant which depends essentially only on p, Ω, φ, ψ, λ_0 and λ_1. If we set $U = \varphi U_0 + V$ ($V = {}^T(u_1, v_1, \theta_1)$), then new unknown vector V should solve the equation:

$$(\lambda I - A)V = (1 - \varphi)F - R \quad \text{in } \Omega, \tag{6.30}$$

where $R = {}^T(0, G_\varphi(u_0, \theta_0), H_\varphi(v_0, \theta_0))$, and G_φ and H_φ are symbols defined by (4.3). By (6.29) and (4.3) we have

$$\operatorname{supp} R \subset B_{b+3}, \;\; \|R\|_{L_p(\Omega)} \le C(\lambda_0, \lambda_1)(\|f\|_{W^2_p(\Omega)} + \|(g, h)\|_{L_p(\Omega)}). \tag{6.31}$$

Now, we shall solve (6.30). By (1.5), the first equation of (6.30) is $\lambda u_1 - v_1 = (1 - \varphi)f$, so that we set

$$v_1 = \lambda u_1 - (1 - \varphi)f. \tag{6.32}$$

Inserting this formula into the second and third lines of the equation (6.30), we have

$$\left.\begin{aligned} &\lambda^2 u_1 + \Delta^2 u_1 - \Delta\theta_1 = (1 - \varphi)(g + \lambda f) - G_\varphi(u_0, \theta_0), \\ &\lambda\theta_1 - \Delta\theta_1 - \lambda\Delta u_1 = (1 - \varphi)h - \lambda\Delta((1 - \varphi)f) - H_\varphi(v_0, \theta_0) \end{aligned}\right\} \text{ in } \Omega. \tag{6.33}$$

From (6.31) and the fact that $1 - \varphi(x) = 0$ for $|x| \ge b+3$ it follows that the right members of (6.33) are supported by B_{b+3}, and therefore by Theorem 6.1 there exists a $(u_1, \theta_1) \in \mathcal{J}_p(\Omega)$ which solves (6.33) uniquely and satisfies the estimate:

$$\begin{aligned} &\|u_1\|_{W^4_p(\Omega)} + \|\theta_1\|_{W^2_p(\Omega)} \\ &\quad \le C(\lambda_0, \lambda_1)(\|f\|_{W^2_p(\Omega)} + \|(g, h)\|_{L_p(\Omega)} + \|(G_\varphi(u_0, \theta_0), H_\varphi(v_0, \theta_0))\|_{L_p(\Omega)}), \end{aligned} \tag{6.34}$$

whenever $\lambda \in \mathbb{C}_+$ and $\lambda_0 \le |\lambda| \le \lambda_1$. Combining (6.29), (6.32) and (6.34), we see that equation (6.30) admits a solution $V \in \mathcal{D}_p(\Omega)$ which satisfies the estimate:

$$\|V\|_{\mathcal{D}_p(\Omega)} \le C(\lambda_0, \lambda_1)(\|f\|_{W^2_p(\Omega)} + \|(g, h)\|_{L_p(\Omega)}),$$

which combined with (6.28) and (6.29) implies that that $U = \varphi U_0 + V \in \mathcal{D}_p(\Omega)$ solves equation (6.27) and satisfies the estimate:

$$\|U\|_{\mathcal{D}_p(\Omega)} \le C(\lambda_0, \lambda_1)(\|f\|_{W^2_p(\Omega)} + \|(g, h)\|_{L_p(\Omega)}), \tag{6.35}$$

whenever $\lambda \in \mathbb{C}_+$ and $\lambda_0 \le |\lambda| \le \lambda_1$.

Since λ_0 and λ_1 are chosen arbitrarily, for any $\lambda \in \mathbb{C}_+ \setminus \{0\}$ equation (6.27) admits a solution $U \in \mathcal{D}_p(\Omega)$ which satisfies the estimate (6.35) whenever $\lambda \in \mathbb{C}_+$ and $\lambda_0 \le |\lambda| \le \lambda_1$. The uniqueness of solutions to (6.27) follows from Lemma 6.2 for $\lambda \in \mathbb{C}_+ \setminus \{0\}$. Therefore, we have $\mathbb{C}_+ \setminus \{0\} \subset \rho(\mathcal{A}_p)$. Moreover, by Theorem 4.1 and (6.35) we have

$$|\lambda| \|(\lambda I - \mathcal{A}_p)^{-1}F\|_{\mathcal{H}_p(\Omega)} + \|(\lambda I - \mathcal{A}_p)^{-1}F\|_{\mathcal{D}_p(\Omega)} \le C(\|f\|_{W^2_p(\Omega)} + \|(g,h)\|_{L_p(\Omega)})$$

whenever $\lambda \in \mathbb{C}_+$ and $|\lambda| \ge \lambda_0$, where λ_0 is arbitrary positive number and C depends essentially only on p, Ω and λ_0. This completes the proof of Theorem 1.2 (2).

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