Ill-posed problems in thermomechanics

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Abstract: In the literature there exist several thermomechanical models which are proposed from a heuristic point of view. A mathematical analysis should help to clarify the applicability of these models. In recent years several thermal or viscoelastic models have been proposed in which the relaxation time or the delay time plays an important role. Single- and dual-phase-lag heat conduction models can be interpreted as formal expansions of delay equations. The delay equations are shown to be ill-posed, as well as the formal expansions of higher order — in contrast to lower-order expansions leading to Fourier's or Cattaneo's law. The ill-posedness is proved showing the lack of continuous dependence on the data, thus showing that these models (delay, or higher-order expansions) are highly explosive. In this note we shall present conditions when this happens.

1 Introduction

This note presents a mathematical analysis of several thermomechanical models which incorporate delay or relaxation parameters. In particular, we show under which conditions such models are ill-posed.

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Heat conduction is usually described by means of the energy equation

$$\theta_t + \gamma \operatorname{div} q = 0 \tag{1.1}$$

for the temperature θ and the heat flux vector q. With the constitutive law

$$q(t+\tau,\cdot) = -\kappa \nabla \theta(t,\cdot), \tag{1.2}$$

this being a special form of a more general law proposed by Tzou [11, 12] (cf. (1.11) below), where $\gamma, \kappa > 0$, and $\tau > 0$ is a small relaxation parameter, we obtain the delay equation

$$\theta_t(t,\cdot) = \kappa\gamma\Delta\theta(t-\tau,\cdot). \tag{1.3}$$

We shall demonstrate that this problem is ill-posed, namely, the continuous dependence on the initial data is not given. More generally, we look at the problem from an abstract point of view in discussing

$$\frac{d^n}{dt^n} u(t) = Au(t-\tau), \tag{1.4}$$

where $n = 1$, and A essentially is the Laplace operator with appropriate boundary conditions in some bounded domain. Then the abstract result on ill-posedness can be given for any $n \in \mathbb{N}$, and a large class of operators A, including non-homogeneous, anisotropic positive symmetric elliptic operators. We shall prove

Theorem 1.1 *Let A be an operator in a Banach space having a sequence of real eigenvalues $(\lambda_k)_k$ such that $0 > \lambda_k \to -\infty$ as $k \to \infty$. Let $n \in \mathbb{N}$ and $\tau > 0$ be fixed.*
Then there are solutions $(u_l)_l$ to

$$\frac{d^n}{dt^n} u_l(t) = Au_l(t-\tau), \tag{1.5}$$

with norm $\|u_l(t)\|$, for any fixed $t > 0$ tending to infinity (as $l \to \infty$) while the norms of the data $(u_l(0))_l$ remain bounded.

We point out that this result extends the result in [2] in several ways. For connections to Volterra equations cf. [4].

Recently Roy [10] extended the constitutive equation to

$$q(t+\tau_1,\cdot) = -\left(\kappa\nabla\theta(t+\tau_2,\cdot) + \kappa^*\nabla\nu(t+\tau_3,\cdot)\right),$$

where κ, κ^* are positive, ν is the thermal displacement that satisfies $\dot\nu = \theta$, and $\tau_1 > \tau_2 > \tau_3$. This leads to the following heat equation of second order in time with two delay times,

$$\theta_{tt}(t,\cdot) = \kappa\Delta\theta_t(t-\tau,\cdot) + \kappa^*\Delta\theta(t-\tau^*,\cdot), \tag{1.6}$$

where $\tau := \tau_1 - \tau_2 > 0$, and $\tau^* := \tau_1 - \tau_3 > 0$. Again, this equation can be extended to the more general problem

$$\frac{d^n}{dt^n}u(t) = A\frac{d}{dt}u(t-\tau) + \beta A u(t-\tau^*), \tag{1.7}$$

where $n \geq 2$. The constant β is positive and will be assumed to be equal to 1 without loss of generality. Then we also get the ill-posedness of this delay problem, i.e.,

Theorem 1.2 *Let A be an operator in a Banach space having a sequence of real eigenvalues $(\lambda_k)_k$ such that $0 > \lambda_k \to -\infty$ as $k \to \infty$. Let $n \in \mathbb{N}$, $n \geq 2$, and $\tau^*, \tau > 0$ be fixed. Then there are solutions $(u_l)_l$ to*

$$\frac{d^n}{dt^n}u_l(t) = A\frac{d}{dt}u_l(t-\tau) + Au(t-\tau^*), \tag{1.8}$$

with norm $\|u_l(t)\|$, for any fixed $t > 0$ tending to infinity (as $l \to \infty$) while the norms of the data $(u_l(0))_l$ remain bounded.

We remark that it is possible to replace the term $Au(t-\tau^*)$ in (1.8) by $\beta A u(t-\tau^*)$ for any $\beta > 0$.

In view of the Theorems 1.1 and 1.2, a natural way to define a stable theory with a delay is by means of a two-temperature theory as it is proposed in [6].

If we approximate the constitutive equation (1.2) by a formal Taylor expansion with respect to τ of order zero, i.e.,

$$q(t,\cdot) = -\kappa \nabla \theta(t,\cdot), \tag{1.9}$$

we have Fourier's law, and this leads to the classical heat equation

$$\theta_t = \kappa \Delta \theta$$

having the physical paradoxon of infinite propagation speed; this can be interpreted by observing that $\tau = 0$ in (1.2) expresses an instantaneous change in the heat flux for a given temperature gradient.

Formally taking a first-order approximation of (1.2), i.e.,

$$\tau q_t(t,\cdot) + q(t,\cdot) = -\kappa \nabla \theta(t,\cdot) \tag{1.10}$$

yields Cattaneo's law which, inserted in (1.1), leads to a damped wave equation,

$$\tau \theta_{tt} + \theta_t = \gamma \kappa \Delta \theta$$

having finite propagation speed of signals. Both models, Fourier (1.9) and Cattaneo (1.10), augmented by boundary conditions in a bounded domain in $\mathbb{R}^n$ as well as initial conditions, describe an exponentially stable system.

More generally, Tzou [11, 12] proposed a dual-phase-lag theory based on

$$q(t+\tau_q,\cdot) = -\kappa\nabla\theta(t+\tau_\theta,\cdot) \tag{1.11}$$

with two relaxation parameters $\tau_q, \tau_\theta > 0$. The delay time τ_θ is caused by microstructural interactions such as phonon scattering or phonon-electron interactions. The delay τ_q is interpreted as the relaxation time due to fast-transient effects of thermal inertia.

Different formal Taylor approximations like

$$q + \tau_q q_t = -\kappa\nabla\theta - \kappa\tau_\theta\nabla\theta \tag{1.12}$$

(Jeffreys model)

or

$$q + \tau_q q_t + \frac{\tau_q^2}{2} q_{tt} = -\kappa\nabla\theta - \kappa\tau_\theta\nabla\theta_t \tag{1.13}$$

or

$$q + \tau_q q_t = -\kappa\nabla\theta - \kappa\tau_\theta\nabla\theta_t - \kappa\frac{\tau_\theta^2}{2}\nabla\theta_{tt} \tag{1.14}$$

or

$$q + \tau_q q_t + \frac{\tau_q^2}{2} q_{tt} = -\kappa\nabla\theta - \kappa\tau_\theta\nabla\theta_t - \kappa\frac{\tau_\theta^2}{2}\nabla\theta_{tt} \tag{1.15}$$

have been discussed, and exponential stability has been shown for certain parameter domains for (τ_q, τ_θ), see [5, 7, 8].

With Theorem 1.1 we cannot interpret the formal "approximations" through Fourier's law, Cattaneo's law, Jeffreys law, ... in (1.9), (1.10), (1.12) - (1.15) (leading to exponentially stable models) as real approximations of the instable, ill-posed original delay equations (1.2) and (1.11), respectively; cf. [1, 12] for regarding it as a Taylor expansion.

Moreover, the expectation is nourished that formal higher-order expansions – "better approximating" the ill-posed case – lead to ill-posed models as well. Indeed, we consider the more general expansion of (1.11) given by

$$q(t,\cdot) + \cdots + \frac{\tau_q^j}{j!}\frac{\partial^j}{\partial_t^j}q(t,\cdot) = -\kappa\nabla\theta(t,\cdot) - \cdots - \kappa\frac{\tau_0^m}{m!}\frac{\partial^m}{\partial_t^m}\nabla\theta(t,\cdot). \tag{1.16}$$

If we substitute this constitutive equation into the energy equation, we will obtain an equation of the form

$$b_0\theta_t + \cdots + b_j\frac{\partial^{j+1}}{\partial t^{j+1}}\theta = c_0\Delta\theta + \cdots + c_m\Delta\frac{\partial^m}{\partial t^m}\theta, \tag{1.17}$$

with $b_i, c_i > 0$ for $i = 0, \ldots, j$ and $i = 0, \ldots, m$, respectively. A recent study for equations of this type can be found in [9].

It is clear that we could express this in an abstract way in the form

$$b_0 u_t + \cdots + b_j \frac{\partial^{j+1}}{\partial t^{j+1}} u = c_0 A u + \cdots + c_m A \frac{\partial^m}{\partial t^m} u, \tag{1.18}$$

where A is an appropriate operator in a suitable Banach space.

We remark that equations of this type are also present in the study of viscoelasticity. In [3], the following constitutitve relation was proposed,

$$P(\partial/\partial t)\sigma_{ij} = Q(\partial/\partial t)\epsilon_{kk}\delta_{ij} + 2R(\partial/\partial t)\epsilon_{ij}, \tag{1.19}$$

where σ_{ij} and ϵ_{ij} are the stress and the strain tensors, respectively, and P, Q, R are three polynomials. In case that we combine this constitutive equation with the dynamic equilibrium equation and (to make the calculations easier) we restrict our attention to anti-plane shear deformations ($u = u_1(x_2, x_3), u_2 = u_3 = 0$) we obtain again an equation of the form (1.18).

The result on ill-posedness now reads as

Theorem 1.3 *Let A be an operator in a Banach space having a sequence of real eigenvalues $(\lambda_k)_k$ such that $0 > \lambda_k \to -\infty$ as $k \to \infty$. Let $n := j+1 \geq m$ and $k := n - m$. Then (1.18) is ill-posed if $k \geq 3$. There are solutions $(u_l)_l$ with norm $\|u_l(t)\|$, for any fixed $t > 0$ tending to infinity (as $l \to \infty$) while the norms of the data $(u_l(0))_l$ remain bounded.*

In view of Theorem 1.3, we may look again at the – exponentially stable – examples Fourier's law, Cattaneo's law, Jeffreys law, ... in (1.9), (1.10), (1.12) - (1.15) from above where we have $0 \leq k \leq 2$ in each case. In this sense, Theorem 1.3 is sharp. We point out that this theorem also holds in the case that some coefficients vanish, e.g., if $b_0 = b_1 = ... = b_r = 0$, as long as $r < j + 1$.

The paper is organized as follows: In Section 2 we shall prove Theorem 1.1 and Theorem 1.2, and in Section 3 we present the proof of Theorem 1.3. A conclusion is given in Section 4.

2 Proofs of Theorems 1.1 and 1.2

To prove Theorem 1.1, we make the ansatz

$$u_l(t) = e^{\omega_l t}\phi_l \tag{2.1}$$

for a solution, where ϕ_l denotes an eigenfunction to the eigenvalue λ_l with norm one. We shall show the existence of a subsequence such that the real part of ω_l tends to infinity as

$l \to \infty$.
The ansatz (2.1) yields a solution if

$$\omega_{k_l}^n = e^{-\omega_{k_l}\tau}\lambda_{k_l}. \tag{2.2}$$

Dropping the index k_l for simplicity and writing ω with real and imaginary part as

$$\omega = r(\cos\varphi + i\sin\varphi) \equiv \alpha + i\beta,$$

we get from (2.2)

$$r^n(\cos(n\varphi) + i\sin(n\varphi))e^{\alpha\tau}(\cos(\beta\tau) + i\sin(\beta\tau)) = \lambda,$$

or

$$r^n e^{r\tau\cos\varphi}[\cos(n\varphi + \beta\tau) + i\sin(n\varphi + \beta\tau)] = \lambda. \tag{2.3}$$

We look for solutions ω (in polar coordinates) $\equiv (r, \varphi)$ such that

$$r^n e^{r\tau\cos\varphi} = |\lambda|, \tag{2.4}$$

$$n\varphi + r\tau\sin\varphi = \pi. \tag{2.5}$$

(2.5) implies the condition

$$r = \frac{\pi - n\varphi}{\tau\sin\varphi}, \tag{2.6}$$

and we note that $0 \le r < \infty$ if

$$0 \le \varphi \le \frac{\pi}{4n}, \tag{2.7}$$

so we assume (2.7).
Substituting (2.6) into (2.3) we obtain

$$\psi(\varphi) := (\pi - n\varphi)^n \; e^{(\pi-n\varphi)\cot\varphi} - |\lambda|\tau^n\sin^n\varphi = 0. \tag{2.8}$$

Our aim is to show that (2.8) always has a zero in $(0, \pi/(4n))$ whenever $|\lambda|$ is large enough.
We have

$$\lim_{\varphi\downarrow 0}\psi(\varphi) = \infty, \tag{2.9}$$

and

$$\psi\left(\frac{\pi}{4n}\right) = \left(\frac{3\pi}{4}\right)^n \; e^{\frac{3\pi}{4}\cot\left(\frac{\pi}{4n}\right)} - |\lambda|\tau^n\sin^n\left(\frac{\pi}{4n}\right) \to -\infty \text{ as } |\lambda| \to \infty. \tag{2.10}$$

(2.9), (2.10) imply the existence of $\varphi \in (0, \frac{\pi}{4n})$ such that

$$\psi(\varphi) = 0.$$

Hence there is $\omega_{k_l} = (r_{k_l}, \varphi_{k_l})$, a solution to (2.2) such that

$$\Re\omega_{k_l} = r_{k_l}\cos\varphi_{k_l} \to \infty \text{ as } l \to \infty,$$

since

$$\cos\varphi_{k_l} \geq \cos\frac{\pi}{4} > 0,$$

and

$$r_{k_l} \to \infty \text{ as } l \to \infty,$$

because

$$\infty \leftarrow |\lambda_{k_l}| = r_{k_l}^n\, e^{r_{k_l}\tau\cos\varphi_{k_l}}.$$

This proves Theorem 1.1.

Now we prove Theorem 1.2. Again we use the ansatz

$$u_l(t) = e^{\omega_l t}\phi_l \tag{2.11}$$

for a solution, where ϕ_l denotes an eigenfunction to the eigenvalue λ_l with norm one. We shall show the existence of a subsequence such that the real part of ω_l tends to infinity as $l \to \infty$.

Writing

$$x := \omega_l,\ \lambda := \lambda_l$$

for simplicity, the ansatz (2.11) yields a solution if

$$x^n = \lambda x e^{-\tau x} + \lambda e^{-\tau^* x}. \tag{2.12}$$

We have to distinguish the cases I: $\tau > \tau^*$, II: $\tau < \tau^*$ (as in [10]), and III: $\tau = \tau^*$.

First, consider case I: Let ω be the solution to

$$\omega^n = \lambda e^{-\tau^*\omega} \tag{2.13}$$

with $\Re\omega \to \infty$ as $\lambda \to -\infty$ and $0 < \arg(\omega) < \frac{\pi}{4n}$, according to (the proof of) Theorem 1.1. For a solution to (2.12) we look for $\zeta \in \mathbb{C}$ with $|\zeta| < \frac{1}{2}$ such that $x = \omega(1+\zeta)$ satisfies (2.12). Then we have to solve

$$\omega^n(1+\zeta)^n = \lambda x e^{-\tau x} + \lambda e^{-\tau^* x},$$

or, using (2.13),

$$(1+\zeta)^n = x e^{\tau^*\omega} e^{-\tau x} + e^{\tau^*\omega} e^{-\tau^* x},$$

rewritten as

$$\underbrace{\left(e^{\tau^*\omega\zeta} - 1\right)}_{=:f(\zeta)} + \underbrace{\left(e^{\tau^*\omega\zeta}((1+\zeta)^n - 1) - \omega(1+\zeta)e^{(\tau^*-\tau)\omega(1+\zeta)}\right)}_{=:g(\zeta)} = 0. \tag{2.14}$$

Let Ω be the ball with center zero and radius $R_\Omega := \frac{1}{10\tau^*|\omega|}$. Then f has exactly one zero ($\zeta = 0$) in Ω. Moreover, on the boundary of Ω we have

$$|f(\zeta)| \geq \inf_{|z|=\frac{1}{10}} |e^z - 1| > 0, \tag{2.15}$$

independent of $|\omega|$. g is estimated as follows. Writing

$$\begin{aligned} g(\zeta) &= [e^{\tau^*\omega\zeta}((1+\zeta)^n - 1)] - [\omega(1+\zeta)e^{(\tau^*-\tau)\omega(1+\zeta)}] \\ &\equiv [g_1(\zeta)] - [g_2(\zeta)] \end{aligned}$$

we have

$$|g_1(\zeta)| \leq e^{\frac{1}{10}} n R_\Omega (1 + R_\Omega^{n-1}) \leq \frac{c}{|\omega|} \tag{2.16}$$

with some constant $c > 0$, and

$$|g_2(\zeta)| \leq 2|\omega| e^{\Re\{(\tau^*-\tau)\omega(1+\zeta)\}} \leq 2|\omega| e^{(\tau^*-\tau)c|\omega|} \leq \frac{c}{|\omega|} \tag{2.17}$$

since $\tau > \tau^*$. By (2.15), (2.16), and (2.17) we conclude with Rouché's theorem that (2.14) has a solution ζ which gives the desired solution x to (2.12), with $\Re x \to \infty$ as $\lambda \to -\infty$.

The cases II ($\tau < \tau^*$) and III ($\tau = \tau^*$) are treated similarly replacing (2.13) by the following implicit equation for ω:

$$\omega^{n-1} = \lambda e^{-\tau\omega}.$$

This completes the proof of Theorem 1.2.

3 Proof of Theorem 1.3

To prove Theorem 1.3 we again consider the ansatz

$$u_l(t) = e^{\omega_l t}\phi_l, \tag{3.1}$$

where $(\phi_l)_l$ denote again the eigenfunctions (with norm 1) of the operator A. This ansatz yields a solution if, for $\omega := \omega_l$,

$$b_0\omega + \cdots + b_j\omega^{j+1} = c_0\lambda_l + \cdots + c_m\lambda_l\omega^m,$$

or

$$P_n(\omega) = 0,$$

where, assuming without loss of generality: $b_j = 1$,

$$P_n(x) = x^n + a_1\, x^{n-1} + \cdots + a_{n-1}x + a_n, \tag{3.2}$$

with coefficients a_i satisfying

$$a_i > 0, \qquad a_i = a_i(\lambda_l), \tag{3.3}$$
$$a_1, \dots, a_{k-1} \quad \text{remains bounded as } l \to \infty, \tag{3.4}$$
$$a_k, \dots, a_n \quad \sim \quad (-\lambda_l) \text{ as } l \to \infty \tag{3.5}$$

Remember: $n = j+1, k = n-m = j+1-m$.

Let

$$\lambda := -\lambda_l, \quad y := x\lambda^{-\frac{1}{k}}, \quad Q(y) := \frac{P(x)}{\lambda^{\frac{n}{k}}}.$$

Then

$$Q(y) = y^n + a_1\lambda^{-\frac{1}{k}}y^{n-1} + \dots + a_{k-1}\lambda^{-\frac{k-1}{k}}y^{n-k+1} + a_k\lambda^{-1}y^{n-k} + \dots + a_{n-1}\lambda^{-\frac{n-1}{k}}y + a_n\lambda^{-\frac{n}{k}},$$

implying

$$Q(y) = \underbrace{y^n + a_k\lambda^{-1}y^{n-k}}_{=:f(\lambda,y)} + \underbrace{\sum_{l=0}^{n-1}\alpha_l(\lambda)y^l}_{=:R(\lambda,y)},$$

with

$$\alpha_l(\lambda) = \mathcal{O}(\lambda^{-\frac{1}{k}}) \quad \text{as } \lambda \to \infty.$$

Let

$$a^* := \lim_{\lambda\to\infty}\frac{a_k(\lambda)}{\lambda} > 0,$$

and

$$y^* := (-a^*)^{\frac{1}{k}}.$$

where the root with argument π/k is chosen such that

$$\Re y^* > 0.$$

This is possible since, by assumption, $k \geq 3$.

Choosing $r_0 := \frac{1}{10}|y^*|$ we have that $f(\lambda,\cdot)$ has exactly one zero in the ball $B(y^*, r_0)$ of radius r_0 around y^* if $|\lambda|$ is large enough. Since

$$\lim_{\lambda\to\infty}|R(\lambda,y)| = 0,$$

uniformly in $y \in \partial B(y^*, r_0)$, we conclude, by the theorem of Rouché, that also Q has exactly one zero $\hat{y}$ in $B(y^*, r_0)$. Then

$$\hat{x} := \lambda^{\frac{1}{k}}\hat{y},$$

satisfies

$$P(\hat{x}) = 0 \quad \text{and} \quad \Re\hat{x} \to \infty \quad \text{as } \lambda \to \infty.$$

This proves Theorem 1.3.

4 Conclusion

In this note we have investigated the ill-posedness of the problems (1.5), (1.8), (1.18). The results directly apply to several thermomechanical models in heat conduction and in viscoelasticity, but they are proved in a general Banach space setting for further applications.

References

[1] DS. Chandrasekharaiah, Hyperbolic thermoelasticity: A review of recent literature, Appl. Mech. Rev., 51(1998), pp. 705-729.

[2] P. M. Jordan, W. Dai and R. E. Mickens, A note on the delayed heat equation: Instability with respect to initial data. Mech. Res. Comm. (2008), DOI: 10.1016/j.mechrescom.2008.04.001

[3] L. P. Lebedev, G.M.L. Gladwell, On spatial effects of modelling in linear viscoelasticity, J. Elasticity, 47(1997), pp. 241-250.

[4] J. Prüß, Evolutionary integral equations and applications. Monographs Math. 87, Birkhäuser, Basel (1993).

[5] R. Quintanilla, Exponential stability in the dual-phase-lag heat conduction theory, J. Non-Equilibrium Thermodynamics, 27(2002), pp. 217-227.

[6] R. Quintanilla, A well posed problem for the dual-phase-lag heat conduction. J. Thermal Stresses, 31(2008), pp. 260-269.

[7] R. Quintanilla, R. Racke, A note on stability in dual-phase-lag heat conduction, Int. J. Heat Mass Transfer, 49(2006), pp. 1209-1213.

[8] R. Quintanilla, R. Racke, Qualitative aspects in dual-phase-lag heat conduction. Proc. Royal Society London A, 463(2007), pp. 659-674.

[9] R. Quintanilla, R. Racke, A note on stability in three-phase-lag heat conduction. International Journal of Heat and Mass Transfer, 51(2008), pp. 24-29.

[10] S. K. Roy Choudhuri, On a thermoelastic three-phase-lag model. J. Thermal Stresses, 30(2007), pp. 231-238.

[11] D. Y. Tzou, A unified approach for heat conduction from macro to micro-scales, ASME J. Heat Transfer, 117(1995), pp. 8-16.

[12] D. Y. Tzou, The generalized lagging response in small-scale and high-rate heating, Int. J. Heat Mass Transfer, 38(1995), pp. 3231-3240.