Local energy decay estimate of solutions to the thermoelastic plate equations in two- and three-dimensional exterior domains

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Abstract

In this paper we prove frequency expansions of the resolvent and local energy decay estimates for the linear thermoelastic plate equations:

$$u_{tt} + \Delta^2 u + \Delta\theta = 0 \text{ and } \theta_t - \Delta\theta - \Delta u_t = 0 \quad \text{in } \Omega \times (0, \infty),$$

subject to Dirichlet boundary conditions: $u|_\Gamma = D_\nu u|_\Gamma = \theta|_\Gamma = 0$ and initial conditions $(u, u_t, \theta)|_{t=0} = (u_0, v_0, \theta_0)$. Here Ω is an exterior domain (domain with bounded complement) in $\mathbb{R}^n$ with $n = 2$ or $n = 3$, the boundary Γ of which is assumed to be a C^4-hypersurface.

1 Introduction and main results

Let Ω be an exterior domain (domain with bounded complement) in $\mathbb{R}^n$ with $n = 2$ or $n = 3$, the boundary Γ of which is assumed to be a C^4-hypersurface. In this paper, we consider the linear thermoelastic plate equations

$$u_{tt} + \Delta^2 u + \Delta\theta = 0 \text{ and } \theta_t - \Delta\theta - \Delta u_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+ \tag{1.1}$$

subject to the initial conditions

$$u(x, 0) = u_0(x), \;\; u_t(x, 0) = v_0(x), \;\; \theta(x, 0) = \theta_0(x) \quad (x \in \Omega) \tag{1.2}$$

and Dirichlet boundary conditions

$$u|_\Gamma = D_\nu u|_\Gamma = \theta|_\Gamma = 0. \tag{1.3}$$

Here $D_\nu = \sum_{j=1}^n \nu_j D_j$ ($D_j = \partial/\partial x_j$), and $\nu = (\nu_1, \ldots, \nu_n)$ denotes the unit outer normal to Γ.

In (1.1), u stands for a mechanical variable denoting the vertical displacement of the plate, while θ stands for a thermal variable describing the temperature relative to a constant reference temperature $\bar{\theta}$. The thermal effect introduces a damping. In fact, when Ω is a bounded reference configuration, the exponential stability of the associated semigroup under several different kind of boundary conditions have been proved by Kim [5], Munõz Rivera and Racke [18], Liu and Zheng [14], Avalos and Lasiecka [1], Lasiecka and Triggiani [7, 8, 9, 10] and Shibata [22]. Also, the analyticity of the semigroup has been shown, cf. Liu and Renardy [12] and then it has been studied by Russell [20], Liu and Liu [11], Liu and Yong [13], Munõz Rivera and Racke [19] in

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the L_2 or Hilbert space setting (see also the book of Liu and Zheng [15] for a survey). In the L_p-setting this was investigated in our paper [4], where sufficiently strong a priori estimates for the resolvent in L_p-spaces have been proved. Before [4], Denk and Racke [3] studied the Cauchy problem for (1.1) in the whole space $\mathbb{R}^n$, also giving decay rates of solutions, and Naito and Shibata [16] studied the initial boundary value problem for (1.1) with Dirichlet boundary condition in the half-space $\mathbb{R}^n_+$.

There were not yet any decay estimates for exterior domains. The purpose of this paper is to study the local energy decay of solutions to problem (1.1) – (1.3). To formulate the problem (1.1) – (1.3) in the semigroup setting, introducing the unknown function $v = u_t$, we rewrite it in matrix form:

$$U_t = AU \quad \text{in } \Omega \times \mathbb{R}_+, \quad U|_{t=0} = U_0, \quad BU|_\Gamma = 0, \tag{1.4}$$

where we have set

$$U = \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}, \ U_0 = \begin{pmatrix} u_0 \\ v_0 \\ \theta_0 \end{pmatrix}, \ A = \begin{pmatrix} 0 & 1 & 0 \\ -\Delta^2 & 0 & -\Delta \\ 0 & \Delta & \Delta \end{pmatrix}, \ BU = \begin{pmatrix} u \\ D_\nu u \\ \theta \end{pmatrix}. \tag{1.5}$$

To study the initial boundary value problem (1.4), we consider the corresponding resolvent problem:

$$(\lambda I - A)U = F \quad \text{in } \Omega, \quad BU|_\Gamma = 0, \tag{1.6}$$

where I denotes the 3×3 unit matrix. We shall give an expansion of the resolvent with respect to the frequency parameter λ (Theorem 1.3). Then, representing the semigroup via the resolvents (essentially: Laplace transform) will give the local energy decay result (Theorem 1.4).

To state our main results precisely, we introduce several spaces and some symbols at this point. Throughout this paper, let $n \in \{2, 3\}$. For a general domain $\mathcal{O} \subset \mathbb{R}^n$, $p \in (1, \infty)$ and any integer m, $L_p(\mathcal{O})$ and $W^m_p(\mathcal{O})$ stand for the usual Lebesgue space and Sobolev space, respectively. Let $\|\cdot\|_{L_p(\mathcal{O})}$ and $\|\cdot\|_{W^m_p(\mathcal{O})}$ denote their norms. For a general domain $\mathcal{O}$ with C^1 boundary $\partial\mathcal{O}$, we introduce the spaces $W^2_{p,0}(\mathcal{O})$ and $W^m_{p,D}(\mathcal{O})$ $(m = 2, 4)$ as follows:

$$\begin{aligned} W^2_{p,0}(\mathcal{O}) &= \{u \in W^2_p(\mathcal{O}) \mid u|_{\partial\mathcal{O}} = 0\}, \\ W^m_{p,D}(\mathcal{O}) &= \{u \in W^m_p(\mathcal{O}) \mid u|_{\partial\mathcal{O}} = D_\nu u|_{\partial\mathcal{O}} = 0\} \quad (m = 2, 4), \end{aligned} \tag{1.7}$$

where $\nu = (\nu_1, \ldots, \nu_n)$ denotes the unit outer normal to $\partial\mathcal{O}$. Let $\mathcal{H}_p(\mathcal{O})$ and $\mathcal{D}_p(\mathcal{O})$ be the spaces defined by the following formulas:

$$\begin{aligned} \mathcal{H}_p(\mathcal{O}) &= \{F = {}^T(f, g, h) \mid f \in W^2_{p,D}(\mathcal{O}), \ g \in L_p(\mathcal{O}), \ h \in L_p(\mathcal{O})\}, \\ \mathcal{D}_p(\mathcal{O}) &= \{U = {}^T(u, v, \theta) \mid u \in W^4_{p,D}(\mathcal{O}), \ v \in W^2_{p,D}(\mathcal{O}), \ \theta \in W^2_{p,0}(\mathcal{O})\}. \end{aligned} \tag{1.8}$$

Here and hereafter, ${}^T M$ denotes the transposed of M. We define the norms $\|\cdot\|_{\mathcal{H}_p(\mathcal{O})}$ and $\|\cdot\|_{\mathcal{D}_p(\mathcal{O})}$ by the following formulas:

$$\begin{aligned} \|F\|_{\mathcal{H}_p(\mathcal{O})} &= \|f\|_{W^2_p(\mathcal{O})} + \|(g, h)\|_{L_p(\mathcal{O})} \quad (F = {}^T(f, g, h) \in \mathcal{H}_p(\mathcal{O})), \\ \|U\|_{\mathcal{D}_p(\mathcal{O})} &= \|u\|_{W^4_p(\mathcal{O})} + \|(v, \theta)\|_{W^2_p(\mathcal{O})} \quad (U = {}^T(u, v, \theta) \in \mathcal{D}_p(\mathcal{O})). \end{aligned} \tag{1.9}$$

Let $\mathcal{A}_\mathcal{O}$ be the operator whose domain is $\mathcal{D}_p(\mathcal{O})$ and whose operation is defined by the formula:

$$\mathcal{A}_\mathcal{O} U = AU \quad \text{for } U \in \mathcal{D}_p(\mathcal{O}). \tag{1.10}$$

In [4] we proved the following theorem.

Theorem 1.1. *Let $1 < p < \infty$. Let $\rho(\mathcal{A}_\Omega)$ be the resolvent set of $\mathcal{A}_\Omega$. Let*

$$\mathbb{C}_+ = \{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda \geq 0\}$$

where $\mathbb{C}$ denotes the set of all complex numbers. Then, $\rho(\mathcal{A}_\Omega) \supset \mathbb{C}_+ \setminus \{0\}$.

Moreover, for any $\lambda_0 > 0$ there exists a constant C depending on λ_0, p and Ω such that for any $\lambda \in \mathbb{C}_+$ with $|\lambda| \geq \lambda_0$ and $F \in \mathcal{H}_p(\Omega)$ there holds the estimate:

$$|\lambda| \|(\lambda I - \mathcal{A}_p)^{-1} F\|_{\mathcal{H}_p(\Omega)} + \|(\lambda I - \mathcal{A}_p)^{-1} F\|_{\mathcal{D}_p(\Omega)} \leq C \|F\|_{\mathcal{H}_p(\Omega)}.$$

In view of Theorem 1.1, by standard arguments in the theory of analytic semigroups (cf. Vrabie [24]) we know that for any $\sigma > 0$ there exists a $\theta_\sigma \in (0, \pi/2)$ such that

$$\rho(\mathcal{A}_\Omega) \supset \{\lambda \in \Sigma_{\theta_\sigma} \mid |\lambda| > \sigma\}, \tag{1.11}$$

where we have set

$$\Sigma_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| < \pi - \epsilon\}. \tag{1.12}$$

Moreover, there exists a constant C_σ depending on σ such that

$$|\lambda| \|(\lambda I - \mathcal{A}_\Omega)^{-1} F\|_{\mathcal{H}_p(\Omega)} + \|(\lambda I - \mathcal{A}_\Omega)^{-1} F\|_{\mathcal{D}_p(\Omega)} \leq C_\sigma \|F\|_{\mathcal{H}_p(\Omega)} \tag{1.13}$$

for any $\lambda \in \Sigma_{\theta_\sigma}$ with $|\lambda| > \sigma$ and $F \in \mathcal{H}_p(\Omega)$. Let us define a set $\mathcal{U}$ by the formula

$$\mathcal{U} = \bigcup_{\sigma > 0} \{\lambda \in \Sigma_{\theta_\sigma} \mid |\lambda| > \sigma\}. \tag{1.14}$$

From (1.11) we see that

$$\rho(\mathcal{A}_\Omega) \supset \mathcal{U}. \tag{1.15}$$

By (1.13), we have the following theorem.

Theorem 1.2. *Let $1 < p < \infty$. Then, $\mathcal{A}_\Omega$ generates an analytic semigroup $\{T_\Omega(t)\}_{t \geq 0}$ in $\mathcal{H}_p(\Omega)$.*

Let b be a number such that $B_b \supset \mathbb{R}^n \setminus \Omega$, where $B_b = \{x \in \mathbb{R}^n \mid |x| < b\}$. Set $\Omega_b = B_b \cap \Omega$. We introduce the following spaces:

$$\begin{aligned} L_{p,b}(\Omega) &= \{f \in L_p(\Omega) \mid f(x) = 0 \text{ for } |x| > b\}, \\ \mathcal{H}_{p,b}(\Omega) &= \mathcal{H}_p(\Omega) \cap (L_{p,b}(\Omega))^3 \\ &= \{F = {}^T(f, g, h) \mid f \in W^2_{p,D}(\Omega) \cap L_{p,b}(\Omega), \ \ g, h \in L_{p,b}(\Omega)\}. \end{aligned} \tag{1.16}$$

Replacing Ω by $\mathbb{R}^n$, we define $L_{p,b}(\mathbb{R}^n)$ and $\mathcal{H}_{p,b}(\mathbb{R}^n)$. For functions $U = {}^T(u, v, \theta)$ we will write

$$\|U\|_{\mathcal{D}_{p,\mathrm{loc}}(\Omega_b)} := \|U|_{\Omega_b}\|_{\mathcal{D}_p(\Omega_b)}.$$

For Banach spaces X and Y, $\mathcal{L}(X, Y)$ denotes the set of all bounded linear operators from X into Y and $\mathcal{L}(X) = \mathcal{L}(X, X)$. For any domain ω in $\mathbb{C}$, $\mathrm{Anal}\,(\omega, X)$ denotes the set of all holomorphic functions defined on ω with their values in X. We set

$$\omega_\tau := \{\lambda \in \mathbb{C} \mid |\lambda| < \tau\}, \quad \dot{\omega}_\tau := \omega_\tau \setminus (-\infty, 0].$$

The following two theorems are our main results.

Theorem 1.3. *Let* $n \in \{2,3\}$, $1 < p < \infty$ *and let* b *be a number such that* $B_{b-3} \supset \mathbb{R}^n \setminus \Omega$. *Let* $\mathcal{U}$ *be the same set as in* (1.14). *Set* $\mathcal{L}_{p,b}(\Omega) = \mathcal{L}(\mathcal{H}_{p,b}(\Omega), \mathcal{D}_{p,\mathrm{loc}}(\Omega_b))$.

(a) *In the case* $n = 2$ *there exist a constant* $\tau > 0$ *and an operator-valued function* $\mathcal{G} \in \mathrm{Anal}(\dot{\omega}_\tau, \mathcal{L}_{p,b}(\Omega))$ *such that for any* $F \in \mathcal{H}_{p,b}(\Omega)$ *and* $\lambda \in \dot{\omega}_\tau \cap \mathcal{U}$ *there holds the equality:*

$$(\lambda I - \mathcal{A}_\Omega)^{-1} F = \mathcal{G}(\lambda) F \quad \text{in } \Omega_b.$$

Moreover, there exist operators G_1, $G_2 \in \mathcal{L}_{p,b}(\Omega)$ *and an operator-valued function*

$$G_3 \in \mathrm{Anal}(\dot{\omega}_\tau, \mathcal{L}_{p,b}(\Omega))$$

such that

$$\begin{aligned} \mathcal{G}(\lambda) &= G_1 + (\log\lambda)^{-1} G_2 + G_3(\lambda) \quad \text{for any } \lambda \in \dot{\omega}_\tau, \\ \|G_3(\lambda)F\|_{\mathcal{D}_{p,\mathrm{loc}}(\Omega_b)} &\le C|\log\lambda|^{-2} \|F\|_{\mathcal{H}_p(\Omega)} \quad \text{for any } \lambda \in \dot{\omega}_\tau \text{ and } F \in \mathcal{H}_{p,b}(\Omega). \end{aligned} \tag{1.17}$$

(b) *In the case* $n = 3$ *there exist a constant* $\tau > 0$ *and operator-valued functions* $\mathcal{G}_j \in \mathrm{Anal}(\omega_\tau, \mathcal{L}_{p,b}(\Omega))$ $(j = 1, 2)$ *such that for any* $F \in \mathcal{H}_{p,b}(\Omega)$ *and* $\lambda \in \omega_\tau \cap \mathcal{U}$ *there holds the equality:*

$$(\lambda I - \mathcal{A}_\Omega)^{-1} F = \lambda^{\frac{1}{2}} \mathcal{G}_1(\lambda) F + \mathcal{G}_2(\lambda) F \quad \text{in } \Omega_b. \tag{1.18}$$

For wave equations, elasticity or Maxwell equations, a collection of references for results on low frequency asymptotics is given in the work of Pauly [17].

With the expansion of the resolvent in terms of the frequency parameter above, we shall obtain the following local energy decay result.

Theorem 1.4. *Let* $1 < p < \infty$ *and let* b *be the same constant as in Theorem 1.3. Let* $\{T_\Omega(t)\}_{t \ge 0}$ *be the semigroup associated with problem* (1.1) – (1.3) *which is given in Theorem 1.2. Then, we have*

$$\|T_\Omega(t)F\|_{\mathcal{D}_{p,\mathrm{loc}}(\Omega_b)} \le \begin{cases} C_{p,b} t^{-1} (\log t)^{-2} \|F\|_{\mathcal{H}_p(\Omega)} & \text{if } n = 2, \\ C_{p,b} t^{-\frac{3}{2}} \|F\|_{\mathcal{H}_p(\Omega)} & \text{if } n = 3 \end{cases} \tag{1.19}$$

for any $t \ge 1$ *and* $F \in \mathcal{H}_{p,b}(\Omega)$.

The difficulty in proving Theorem 1.3 arises from the facts that the expansion formula of the resolvent operator $(\lambda - \Delta)^{-1}$ in $\mathbb{R}^2$ has the singularity $\log\lambda$ and that of $(\lambda - \Delta^2)^{-1}$ in $\mathbb{R}^n$ has the singularities $\lambda^{-1}\log\lambda$ when $n = 2$ and $\lambda^{-\frac{1}{2}}$ when $n = 3$, respectively. Therefore, we can not use the usual compact perturbation method to obtain the expansion formula in the exterior domain. To prove Theorem 1.3, first of all employing the Seeley argument [21] about the invertibility of $I + K_\lambda$, K_λ being a compact operator valued holomorphic function in λ, we shall show that $(\lambda I - \mathcal{A}_\Omega)^{-1}$ has an expansion formula near $\lambda = 0$ which starts from $\lambda^s (\log\lambda)^\beta$ in two dimensional case and $\lambda^{\frac{s}{2}}$ in three dimensional case for some integers s and β. Then, by a contradiction argument based on the uniqueness theorem we shall show that $s = 0$ and $\beta = 0$. Our strategy of the proof of Theorem 1.3 follows R. Kleinmann and B. Vainberg [6] and W. Dan and Y. Shibata [2], where the low frequency expansions of the Laplace operator and Stokes operator in the two dimensional case were obtained.

We will prove Theorems 1.3 and 1.4 in Sections 2–3 for the (somewhat simpler) case $n = 3$. Modifications for the case $n = 2$ are indicated in Sections 4 and 5.

2 Expansion formulas in three dimensions

We start with the three-dimensional case by showing an expansion formula of the resolvent in the whole-space.

Theorem 2.1. *Let $1 < p < \infty$ and $b > 0$. Let $\mathcal{L}_{p,b}(\mathbb{R}^3)$ be the set of all bounded linear operators from $\mathcal{H}_{p,b}(\mathbb{R}^3)$ into $\mathcal{D}_{p,\mathrm{loc}}(B_b)$ and $\rho(\mathcal{A}_{\mathbb{R}^3})$ the resolvent set of $\mathcal{A}_{\mathbb{R}^3}$. Then, there exist constants $\epsilon \in (0, \pi/2)$ and operator-valued functions $\mathcal{H}_j(\lambda) \in \mathrm{Anal}\,(\mathbb{C}, \mathcal{L}_{p,b}(\mathbb{R}^3))$ $(j = 1, 2)$ such that $\rho(\mathcal{A}_{\mathbb{R}^3}) \supset \Sigma_\epsilon$ and*

$$(\lambda I - \mathcal{A}_{\mathbb{R}^3})^{-1}F = \lambda^{-\frac{1}{2}}\mathcal{E}_0 F + \mathcal{E}_1 F + \lambda^{\frac{1}{2}}\mathcal{H}_1(\lambda)F + \lambda\mathcal{H}_2(\lambda)F \quad \text{in } B_b \tag{2.1}$$

for any $\lambda \in \Sigma_\epsilon$ and $F \in \mathcal{H}_{p,b}(\mathbb{R}^3)$. Here, Σ_ϵ is the set defined in (1.12)*,*

$$\mathcal{E}_0 F = \begin{pmatrix} \alpha \int_{\mathbb{R}^3} g\,dx + \beta \int_{\mathbb{R}^3} h\,dx \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{E}_1 F = \begin{pmatrix} E_3^2 * (-\Delta f + g + h) \\ -f \\ E_3^1 * (h - \Delta f) \end{pmatrix},$$
$$E_3^1(x) = \frac{1}{4\pi|x|}, \quad E_3^2(x) = -\frac{|x|}{8\pi}, \tag{2.2}$$

$*$ stands for the convolution operator, ϵ is given in (2.6)*, and α and β are non-zero constants given in* (2.11) *in the proof below.*

Remark 2.2. $E_3^1(x)$ and $E_3^2(x)$ are fundamental solutions to $-\Delta$ and Δ^2 in $\mathbb{R}^3$, respectively.

Proof. For $F \in \mathcal{H}_p(\mathbb{R}^3)$, we set $U(\lambda) = (\lambda I - \mathcal{A}_{\mathbb{R}^3})^{-1}F$. Let $\hat{U}(\lambda)(\xi) = {}^T(\hat{u}_\lambda(\xi), \hat{v}_\lambda(\xi), \hat{\theta}_\lambda(\xi))$ be the Fourier transform of $U(\lambda)$. Then, from Naito and Shibata [16], we have the following formulas:

$$\begin{aligned}
\hat{u}_\lambda(\xi) &= \sum_{j=1}^{3}\Big[\frac{A_j^0 + A_j^1 + A_j^2}{(\lambda + \gamma_j|\xi|^2)|\xi|^2}|\xi|^2\hat{f}(\xi) + \frac{A_j^0 + A_j^1}{(\lambda + \gamma_j|\xi|^2)|\xi|^2}\hat{g}(\xi) + \frac{A_j^0}{(\lambda + \gamma_j|\xi|^2)|\xi|^2}\hat{h}(\xi)\Big], \\
\hat{v}_\lambda(\xi) &= \sum_{j=1}^{3}\Big[-\frac{(A_j^0 + A_j^1)|\xi|^2}{\lambda + \gamma_j|\xi|^2}\hat{f}(\xi) + \frac{A_j^1 + A_2^1}{\lambda + \gamma_j|\xi|^2}\hat{g}(\xi) + \frac{A_j^1}{\lambda + \gamma_j|\xi|^2}\hat{h}(\xi)\Big], \\
\hat{\theta}_\lambda(\xi) &= \sum_{j=1}^{3}\Big[\frac{A_j^0|\xi|^2}{\lambda + \gamma_j|\xi|^2}\hat{f}(\xi) - \frac{A_j^1}{\lambda + \gamma_j|\xi|^2}\hat{g}(\xi) + \frac{A_j^0 + A_j^2}{\lambda + \gamma_j|\xi|^2}\hat{h}(\xi)\Big].
\end{aligned} \tag{2.3}$$

Here, γ_j $(j = 1, 2, 3)$ are numbers such that

$$\prod_{j=1}^{3}(t + \gamma_j) = t^3 + t^2 + 2t + 1 \quad \text{for any } t \in \mathbb{C}, \tag{2.4}$$

$0 < \gamma_1 < 1$, γ_3 is the complex conjugate of γ_2 and $\mathrm{Re}\,\gamma_2 = (1 - \gamma_1)/2 > 0$; and A_j^0, A_j^1 and A_j^2 $(j = 1, 2, 3)$ are complex numbers such that

$$\frac{\lambda^k}{\prod_{j=1}^{3}(\lambda + \gamma_j|\xi|^2)} = \sum_{j=1}^{3}\frac{A_j^k}{(\lambda + \gamma_j|\xi|^2)|\xi|^{4-2k}} \quad (k = 1, 2, 3)$$

for any $\xi \in \mathbb{R}^3$ and $\lambda \in \mathbb{C}$ with $\lambda + \gamma_j|\xi|^2 \neq 0$ $(j = 1, 2, 3)$. We have the following formulas:

$$\sum_{j=1}^{3} A_j^0 = \sum_{j=1}^{3} A_j^1 = 0, \ \sum_{j=1}^{3} A_j^2 = 1, \ \sum_{j=1}^{3} \frac{A_j^0}{\gamma_j} = 1, \ \sum_{j=1}^{3} \frac{A_j^1}{\gamma_j} = \sum_{j=1}^{3} \frac{A_j^2}{\gamma_j} = 0. \tag{2.5}$$

Since γ_2 and γ_3 are complex conjugate and $\mathrm{Re}\,\gamma_2 > 0$, we may assume that $0 < \arg\gamma_2 < \pi/2$. Let us define ϵ by the formula:

$$\epsilon = \arg\gamma_2. \tag{2.6}$$

Since $\lambda + \gamma_j|\xi|^2 \neq 0$ for any $\lambda \in \Sigma_\epsilon$ and $\xi \in \mathbb{R}^3$, by Fourier multiplier theorem we have $U(\lambda) = {}^T(u_\lambda, v_\lambda, \theta_\lambda) \in \mathcal{D}_p(\mathbb{R}^3)$. Moreover, for any ϵ' with $\epsilon < \epsilon' < \pi/2$ there exists a constant C depending on ϵ' such that

$$\begin{aligned} \sum_{j=0}^{2} |\lambda|^{\frac{2-j}{2}} \|\nabla^j(\nabla^2 u_\lambda, v_\lambda, \theta_\lambda)\|_{L_p(\mathbb{R}^3)} &\leq C\|F\|_{\mathcal{H}_p(\mathbb{R}^3)}, \\ |\lambda|\|\nabla u_\lambda\|_{L_p(\mathbb{R}^3)} + |\lambda|^2\|u_\lambda\|_{L_p(\mathbb{R}^3)} &\leq C\|(|\lambda|f, g, h)\|_{L_p(\mathbb{R}^3)} \end{aligned} \tag{2.7}$$

for any $\lambda \in \Sigma_{\epsilon'}$ (cf. Naito-Shibata [16]), where $\nabla^j w = (D^\alpha w \mid |\alpha| = j)$. From these observations, we see that $\rho(\mathcal{A}_{\mathbb{R}^3}) \supset \Sigma_\epsilon$.

Now, restricting ourselves to the case where $F \in \mathcal{H}_{p,b}(\mathbb{R}^3)$, we shall derive an expansion formula of $(\lambda I - \mathcal{A}_{\mathbb{R}^3})^{-1}F$ by using the formula (2.3). Let $\mathcal{F}_\xi^{-1}$ denote the Fourier inverse transform, and then we have

$$\begin{aligned} \mathcal{F}_\xi^{-1}[(\lambda + |\xi|^2)^{-1}](x) &= \frac{e^{-\sqrt{\lambda}|x|}}{4\pi|x|}, \\ \mathcal{F}_\xi^{-1}[(\lambda + |\xi|^2)^{-1}|\xi|^{-2}](x) &= -\lambda^{-1}\Big(\frac{e^{-\sqrt{\lambda}|x|}}{4\pi|x|} - \frac{1}{4\pi|x|}\Big) \end{aligned} \tag{2.8}$$

for any $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. Since we have $e^{-\sqrt{\lambda}|x|} = \sum_{j=0}^{\infty}(-\sqrt{\lambda}|x|)^j/(j!)$, we have

$$\mathcal{F}_\xi^{-1}[(\lambda + |\xi|^2)^{-1}](x) = \frac{1}{4\pi|x|} - \frac{\lambda^{\frac12}}{4\pi}H_1^1(\lambda|x|^2) + \frac{\lambda|x|}{8\pi}H_2^1(\lambda|x|^2), \tag{2.9}$$

$$\mathcal{F}_\xi^{-1}[(\lambda + |\xi|^2)^{-1}|\xi|^{-2}](x) = \frac{\lambda^{-\frac12}}{4\pi} - \frac{|x|}{8\pi} + \frac{\lambda^{\frac12}|x|^2}{4\pi}H_1^2(\lambda|x|^2) - \frac{\lambda|x|^3}{4\pi}H_2^2(\lambda|x|^2), \tag{2.10}$$

where we have set

$$\begin{aligned} H_1^2(z) &= \sum_{j=0}^{\infty} \frac{z^j}{(2j+3)!}, \quad H_2^2(z) = \sum_{j=0}^{\infty} \frac{z^j}{(2j+4)!}, \\ H_1^1(z) &= 1 + zH_1^2(z), \qquad H_2^1(z) = 1 + 2zH_2^2(z). \end{aligned}$$

Now, we assume that $F \in \mathcal{H}_{p,b}(\mathbb{R}^3)$. Since $\lambda + \gamma_j|\xi|^2 = \gamma_j(\lambda\gamma_j^{-1} + |\xi|^2)$, using (2.10) and (2.5), from (2.3) we have

$$u_\lambda(x) = \Big[\Big(\sum_{j=1}^{3} \frac{A_j^0 + A_j^1}{\sqrt{\gamma_j}}\Big)\frac{1}{4\pi}\int_{\mathbb{R}^3} g\,dx + \Big(\sum_{j=1}^{3} \frac{A_j^0}{\sqrt{\gamma_j}}\Big)\frac{1}{4\pi}\int_{\mathbb{R}^3} h\,dx\Big]\lambda^{-\frac12} + E_3^2 * (-\Delta f + g + h)$$

$$
\begin{aligned}
&+ \lambda^{\frac{1}{2}}\Big[\Big\{\Big(\sum_{j=1}^{3} \frac{A_j^0 + A_j^1 + A_j^2}{\gamma_j^{3/2}} H_1^2(\gamma_j^{-1}\lambda|x|^2)\Big)\frac{|x|^2}{4\pi}\Big\} * (-\Delta f) \\
&+ \Big\{\Big(\sum_{j=1}^{3} \frac{A_j^0 + A_j^1}{\gamma_j^{3/2}} H_1^2(\gamma_j^{-1}\lambda|x|^2)\Big)\frac{|x|^2}{4\pi}\Big\} * g + \Big\{\Big(\sum_{j=1}^{3} \frac{A_j^0}{\gamma_j^{3/2}} H_1^2(\gamma_j^{-1}\lambda|x|^2)\Big)\frac{|x|^2}{4\pi}\Big\} * h\Big] \\
&+ \lambda\Big[\Big\{\Big(\sum_{j=1}^{3} \frac{A_j^0 + A_j^1 + A_j^2}{\gamma_j^{2}} H_2^2(\gamma_j^{-1}\lambda|x|^2)\Big)\frac{|x|^3}{4\pi}\Big\} * (-\Delta f) \\
&+ \Big\{\Big(\sum_{j=1}^{3} \frac{A_j^0 + A_j^1}{\gamma_j^{2}} H_1^2(\gamma_j^{-1}\lambda|x|^2)\Big)\frac{|x|^3}{4\pi}\Big\} * g + \Big\{\Big(\sum_{j=1}^{3} \frac{A_j^0}{\gamma_j^{2}} H_1^2(\gamma_j^{-1}\lambda|x|^2)\Big)\frac{|x|^3}{4\pi}\Big\} * h\Big].
\end{aligned}
$$

Setting

$$
\alpha = \sum_{j=1}^{3} \frac{A_j^0 + A_j^1}{\sqrt{\gamma_j}}, \quad \beta = \sum_{j=1}^{3} \frac{A_j^0}{\sqrt{\gamma_j}}, \tag{2.11}
$$

we have the first line of the formula (2.1) with (2.2). Using the fact that $E_3^1 * (-\Delta f) = f$ to obtain the formula for $v_\lambda(x)$, by (2.3), (2.5) and (2.9) we have

$$
\begin{aligned}
v_\lambda(x) &= -f + \lambda^{\frac{1}{2}}\Big[\Big(\sum_{j=1}^{3} \frac{A_j^0 + A_j^1}{4\pi\gamma_j^{3/2}} H_1^1(\gamma_j^{-1}\lambda|x|^2)\Big) * (-\Delta f) \\
&- \Big(\sum_{j=1}^{3} \frac{A_j^1 + A_j^2}{4\pi\gamma_j^{3/2}} H_1^1(\gamma_j^{-1}\lambda|x|^2)\Big) * g - \Big(\sum_{j=1}^{3} \frac{A_j^1}{4\pi\gamma_j^{3/2}} H_1^1(\gamma_j^{-1}\lambda|x|^2)\Big) * h\Big] \\
&- \lambda\Big[\Big\{\Big(\sum_{j=1}^{3} \frac{A_j^0 + A_j^1}{\gamma_j^{2}} H_2^1(\gamma_j^{-1}\lambda|x|^2)\Big)\frac{|x|}{8\pi}\Big\} * (-\Delta f) \\
&- \Big\{\Big(\sum_{j=1}^{3} \frac{A_j^1 + A_j^2}{\gamma_j^{2}} H_2^1(\gamma_j^{-1}\lambda|x|^2)\Big)\frac{|x|}{8\pi}\Big\} * g - \Big\{\Big(\sum_{j=1}^{3} \frac{A_j^1}{\gamma_j^{2}} H_2^1(\gamma_j^{-1}\lambda|x|^2)\Big)\frac{|x|}{8\pi}\Big\} * h\Big], \\
\theta_\lambda(x) &= E_3^1 * (h - \Delta f) - \lambda^{\frac{1}{2}}\Big[\Big(\sum_{j=1}^{3} \frac{A_j^0}{4\pi\gamma_j^{3/2}} H_1^1(\gamma_j^{-1}\lambda|x|^2)\Big) * (-\Delta f) \\
&- \Big(\sum_{j=1}^{3} \frac{A_j^1}{4\pi\gamma_j^{3/2}} H_1^1(\gamma_j^{-1}\lambda|x|^2)\Big) * g + \Big(\sum_{j=1}^{3} \frac{A_j^0 + A_j^1}{4\pi\gamma_j^{3/2}} H_1^1(\gamma_j^{-1}\lambda|x|^2)\Big) * h\Big] \\
&+ \lambda\Big[\Big\{\Big(\sum_{j=1}^{3} \frac{A_j^0}{\gamma_j^{2}} H_2^1(\gamma_j^{-1}\lambda|x|^2)\Big)\frac{|x|}{8\pi}\Big\} * (-\Delta f) \\
&- \Big\{\Big(\sum_{j=1}^{3} \frac{A_j^1}{\gamma_j^{2}} H_2^1(\gamma_j^{-1}\lambda|x|^2)\Big)\frac{|x|}{8\pi}\Big\} * g + \Big\{\Big(\sum_{j=1}^{3} \frac{A_j^0 + A_j^1}{\gamma_j^{2}} H_2^1(\gamma_j^{-1}\lambda|x|^2)\Big)\frac{|x|}{8\pi}\Big\} * h\Big].
\end{aligned}
$$

This completes the proof of Theorem 2.1. □

The next step in the proof of our main results consists in an expansion formula for the resolvent operator in Ω near $\lambda = 0$. We will show the following theorem.

Theorem 2.3. *Let $1 < p < \infty$ and b be a positive number such that $B_{b-3} \supset \mathbb{R}^3 \setminus \Omega$. Let $\mathcal{U}$ and $\mathcal{L}_{p,b}(\Omega)$ be the same sets as in (1.14) and Theorem 1.3, respectively. Then, there exist a constant $\tau > 0$, an integer s and operators $\mathcal{G}_j(\lambda) \in \mathrm{Anal}\,(\omega_\tau, \mathcal{L}_{p,b}(\Omega))$ $(j = 1, 2)$ such that*

$$(\lambda I - \mathcal{A}_\Omega)^{-1}F = \lambda^{\frac{s}{2}}\mathcal{G}_1(\lambda)F + \lambda^{\frac{s+1}{2}}\mathcal{G}_2(\lambda)F \quad \text{in } \Omega_b$$

for any $\lambda \in \omega_\tau \cap \mathcal{U}$ and $F \in \mathcal{H}_{p,b}(\Omega)$.

In what follows, we shall prove Theorem 2.3. For a given function f defined on Ω, ιf denotes the zero extension of f to the whole space $\mathbb{R}^3$ and rf denotes the restriction of f to the domain $\Omega_b = \Omega \cap B_b$. From Denk, Racke and Shibata [4] (also Simader [23]), we know the unique existence of a solution $U_0 = {}^T(u_0, v_0, \theta_0) \in \mathcal{D}_p(\Omega_b)$ of the equation:

$$-AU_0 = F \quad \text{in } \Omega_b, \quad BU_0|_{\partial\Omega_b} = 0 \tag{2.12}$$

for any $F \in \mathcal{H}_p(\Omega_b)$, Here, $\partial\Omega_b = \Gamma \cup S_b$, $S_b = \{x \in \mathbb{R}^3 \mid |x| = b\}$ and $BU_0|_{\partial\Omega_b} = 0$ means that

$$u_0 = D_\nu u_0 = \theta_0 = 0 \quad \text{on } \Gamma \text{ and } S_b,$$

where $D_\nu = (x/|x|) \cdot \nabla$ on S_b. Let us define the operator S_{Ω_b} by the formula: $S_{\Omega_b}F = U_0$ and write $S_{\Omega_b}F = (u_{\Omega_b}, v_{\Omega_b}, \theta_{\Omega_b})$ as long as no confusion occurs. Let $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{H}_1(\lambda)$ and $\mathcal{H}_2(\lambda)$ be the same operator as in Theorem 2.1 and set

$$\mathcal{H}(\lambda) = \lambda^{-\frac{1}{2}}\mathcal{E}_0 + \mathcal{E}_1 + \lambda^{\frac{1}{2}}\mathcal{H}_1(\lambda) + \lambda\mathcal{H}_2(\lambda). \tag{2.13}$$

In what follows, we write $\mathcal{H}(\lambda)F = (u_{\lambda,\mathbb{R}^3}, v_{\lambda,\mathbb{R}^3}, \theta_{\lambda,\mathbb{R}^3})$. Let φ be a function in $C_0^\infty(\mathbb{R}^3)$ such that $\varphi(x) = 1$ for $|x| < b-2$ and $\varphi(x) = 0$ for $|x| > b-1$. With these preparations, we introduce the operator Φ as follows:

$$\Phi(\lambda)F = (1 - \varphi)\mathcal{H}(\lambda)\iota F + \varphi S_{\Omega_b} rF. \tag{2.14}$$

By Theorem 2.1, we have

$$\Phi(\lambda)F = (1 - \varphi)(\lambda I - \mathcal{A}_{\mathbb{R}^2})^{-1}\iota F + \varphi S_{\Omega_b} rF \tag{2.15}$$

when $\lambda \in \Sigma_\epsilon$. And therefore, applying $\lambda I - A$ to $\Phi(\lambda)F$, we have

$$(\lambda I - A)\Phi(\lambda)F = F + T(\lambda)F \quad \text{in } \Omega, \quad B\Phi(\lambda)F|_\Gamma = 0 \tag{2.16}$$

for any $\lambda \in \Sigma_\epsilon$, where $T(\lambda)F$ is defined by the formula:

$$T(\lambda)F = \begin{pmatrix} 0 \\ -L_\varphi^3(u_{\lambda,\mathbb{R}^2} - u_{\Omega_b}) - L_\varphi^1(\theta_{\lambda,\mathbb{R}^2} - \theta_{\Omega_b}) \\ L_\varphi^1(\theta_{\lambda,\mathbb{R}^2} - \theta_{\Omega_b}) + L_\varphi^1(v_{\lambda,\mathbb{R}^2} - v_{\Omega_b}) \end{pmatrix}, \tag{2.17}$$

$L_\varphi^3(w) = \Delta^2(\varphi w) - \varphi\Delta^2 w$, and $L_\varphi^1(w) = \Delta(\varphi w) - \varphi\Delta w$. If we consider (2.16) only on Ω_b, the operators in both sides of (2.16) are analytic with respect to $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, and therefore by analytic continuation we have

$$(\lambda I - A)\Phi(\lambda)F = F + T(\lambda)F \quad \text{in } \Omega_b, \quad B\Phi(\lambda)F|_\Gamma = 0 \tag{2.18}$$

for any $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. If $(I + T(\lambda))^{-1}$ exists, then $\Phi(\lambda)(I + T(\lambda))^{-1}F$ solves equations (2.16) and (2.18).

Lemma 2.4. *Let $\mathcal{U}$ and Σ_ϵ be the same sets as in (1.14) and Theorem 2.1, respectively. Then, $(I+T(\lambda))^{-1}$ exists as a bounded linear operator on $\mathcal{H}_{p,b}(\Omega)$ for any $\lambda \in \mathcal{U}\cap\Sigma_\epsilon$.*

Proof. Let $\lambda \in \Sigma_\epsilon \cap \mathcal{U}$. Since the second and third components of $T(\lambda)F$ belong to $W^1_p(\Omega)$ and $\operatorname{supp} T(\lambda)F \subset D_{b-2,b-1} = B_{b-1}\setminus B_{b-2}$, by Rellich's compactness theorem $T(\lambda)$ is a compact operator on $\mathcal{H}_{p,b}(\Omega)$. Therefore, to prove the lemma it suffices to show that $I+T(\lambda)$ is injective. Let F be an element of $\mathcal{H}_{p,b}(\Omega)$ such that $(I+T(\lambda))F = 0$. Set $U = \Phi(\lambda)F$, and then by (2.18) we have

$$(\lambda I - A)U = 0 \quad \text{in } \Omega, \quad BU|_\Gamma = 0.$$

Since $S_{\Omega_b} rF \in \mathcal{D}_p(\Omega_b)$ and $(\lambda I - \mathcal{A}_{\mathbb{R}^3})^{-1}\iota F \in \mathcal{D}_p(\mathbb{R}^3)$ for $\lambda \in \Sigma_\epsilon$ (cf. (2.7)), by (2.15) we have $U \in \mathcal{D}_p(\Omega)$. Since $\mathcal{U} \subset \rho(\mathcal{A}_\Omega)$ as follows from (1.15), we have $U = 0$, which implies that

$$(1-\varphi)(\lambda I - \mathcal{A}_{\mathbb{R}^3})^{-1}\iota F + \varphi S_{\Omega_b} rF = 0 \quad \text{in } \Omega. \tag{2.19}$$

Recalling that $\varphi(x) = 1$ for $|x| < b-2$ and $\varphi(x) = 0$ for $|x| > b-1$, by (2.19) we have

$$(\lambda I - \mathcal{A}_{\mathbb{R}^3})^{-1}\iota F = 0 \quad \text{for } |x| > b-1, \quad S_{\Omega_b} rF = 0 \quad \text{for } |x| < b-2.$$

If we set $V(x) = (S_{\Omega_b} rF)(x)$ for $x \in \Omega_b$ and $V(x) = 0$ for $x \notin \Omega$, then $V(x)$ belongs to $\mathcal{D}_p(B_b)$ and satisfies the equation:

$$(\lambda I - A)V = \iota F \quad \text{in } B_b, \quad BV|_{S_b} = 0.$$

Since $(\lambda I - A_{\mathbb{R}^3})^{-1}\iota F$ also satisfies the above equation, by the uniqueness of solutions we have $V = (\lambda I - A_{\mathbb{R}^3})^{-1}\iota F$ in B_b, and therefore $S_{\Omega_b} F = (\lambda I - A_{\mathbb{R}^3})^{-1}\iota F$ in Ω_b, which inserted into (2.19) implies that

$$0 = (\lambda I - \mathcal{A}_{\mathbb{R}^3})^{-1}\iota F + \varphi(S_{\Omega_b} F - (\lambda I - \mathcal{A}_{\mathbb{R}^3})^{-1}\iota F) = (\lambda I - \mathcal{A}_{\mathbb{R}^3})^{-1}\iota F \quad \text{in } \Omega.$$

Therefore, $F = (\lambda I - A)(\lambda I - \mathcal{A}_{\mathbb{R}^3})^{-1}\iota F = 0$ in Ω, which completes the proof of the lemma. □

By Lemma 2.4 we have

$$(\lambda I - \mathcal{A}_\Omega)^{-1} = \Phi(\lambda)(I + T(\lambda))^{-1} \tag{2.20}$$

for $\lambda \in \Sigma_\epsilon \cap \mathcal{U}$.

Now, we shall discuss the invertibility of $(I+T(\lambda))$ for $\lambda \in \dot{\omega}_\sigma$ with some $\sigma > 0$, where we have set

$$\dot{\omega}_\sigma = \{\lambda \in \mathbb{C}\setminus\{0\} \mid |\lambda| < \sigma \text{ and } |\arg\lambda| < \pi\}.$$

For this purpose, we introduce an auxiliary operator:

$$\Phi_0 F = (1-\varphi)\mathcal{E}_1 \iota F + \varphi S_{\Omega_b} rF$$

for $F \in \mathcal{H}_{p,b}(\Omega)$, where $\mathcal{E}_1$ is the same operator as in Theorem 2.1. Note that

$$-A\mathcal{E}_1 \iota F = \iota F \quad \text{in } \mathbb{R}^3.$$

We write $\mathcal{E}_1 \iota F = {}^T(u_{0,\mathbb{R}^3}, v_{0,\mathbb{R}^3}, \theta_{0,\mathbb{R}^3})$ unless any confusion may occur. Applying A to $\Phi_0 F$, we have

$$-A\Phi_0 F = F + T_0 F \quad \text{in } \Omega, \quad B\Phi_0 F|_\Gamma = 0, \tag{2.21}$$

where

$$T_0F = \begin{pmatrix} 0 \\ -L^3_\varphi(u_{0,\mathbb{R}^3} - u_{\Omega_b}) - L^1_\varphi(\theta_{0,\mathbb{R}^3} - \theta_{\Omega_b}) \\ L^1_\varphi(\theta_{0,\mathbb{R}^3} - \theta_{\Omega_b}) + L^1_\varphi(v_{0,\mathbb{R}^3} - v_{\Omega_b}) \end{pmatrix}.$$

Since the second and third members of T_0F belong to $W^1_p(\Omega)$ and $\operatorname{supp} T_0F \subset D_{b-2,b-1}$, by Rellich's compactness theorem T_0 is a compact operator on $\mathcal{H}_{p,b}(\Omega)$. According to Theorem 2.1, we set

$$\begin{aligned} u_{\lambda,\mathbb{R}^3} =& u_{0,\mathbb{R}^3} + \lambda^{-\frac{1}{2}} T(\alpha g + \beta h) + U_{\lambda,\mathbb{R}^3}, \\ v_{\lambda,\mathbb{R}^3} =& v_{0,\mathbb{R}^3} + V_{\lambda,\mathbb{R}^3}, \\ \theta_{\lambda,\mathbb{R}^3} =& \theta_{0,\mathbb{R}^3} + \Theta_{\lambda,\mathbb{R}^3}, \end{aligned}$$

where $Ta = \int_{\mathbb{R}^3} a\, dx$ and

$$^T(U_{\lambda,\mathbb{R}^3}, V_{\lambda,\mathbb{R}^3}, \Theta_{\lambda,\mathbb{R}^3}) = \lambda^{\frac{1}{2}} \mathcal{H}_1(\lambda)\iota F + \lambda \mathcal{H}_2(\lambda)\iota F. \tag{2.22}$$

Then, we have

$$(I + T(\lambda))F = (I + T_0)F + \lambda^{-\frac{1}{2}} (\Delta^2\varphi)^T (0, T(\alpha g + \beta h), 0) + R(\lambda)F \tag{2.23}$$

where

$$R(\lambda)F = \begin{pmatrix} 0 \\ -L^3_\varphi(U_{\lambda,\mathbb{R}^3}) - L^1_\varphi(\Theta_{\lambda,\mathbb{R}^3}) \\ L^1_\varphi(\Theta_{\lambda,\mathbb{R}^3}) + L^1_\varphi(V_{\lambda,\mathbb{R}^3}) \end{pmatrix}. \tag{2.24}$$

In view of (2.22) and (2.24), there exist operators $R_j(\lambda) \in \operatorname{Anal}(\mathbb{C}, \mathcal{L}(\mathcal{H}_{p,b}(\Omega)))$ $(j = 1, 2)$ such that

$$R(\lambda)F = \lambda^{\frac{1}{2}} R_1(\lambda)F + \lambda R_2(\lambda)F \tag{2.25}$$

for any $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. In particular, we have

$$\lim_{\lambda \to 0} \|R(\lambda)\|_{\mathcal{L}(\mathcal{H}_{p,b}(\Omega))} = 0. \tag{2.26}$$

Here, $\|\cdot\|_{\mathcal{L}(\mathcal{H}_{p,b}(\Omega))}$ denotes the operator norm of $\mathcal{L}(\mathcal{H}_{p,b}(\Omega))$. Since T_0 is a compact operator on $\mathcal{H}_{p,b}(\Omega)$, by Seeley's lemma [21] there exists a finite range operator B such that $I+T_0-B$ has an inverse operator $(I+T_0-B)^{-1} \in \mathcal{L}(\mathcal{H}_{p,b}(\Omega))$. Set $G_\lambda = I+T_0-B+R(\lambda)$ and $G_0 = I+T_0-B$, and then

$$(I + T(\lambda))F = G_\lambda F + BF + \lambda^{-\frac{1}{2}} (\Delta^2\varphi)^T (0, T(\alpha g + \beta h), 0) \tag{2.27}$$

$$G_\lambda = (I + R(\lambda)G_0^{-1})G_0. \tag{2.28}$$

By (2.26) there exists a $\tau_0 > 0$ such that $\|R(\lambda)G_0^{-1}\|_{\mathcal{L}(\mathcal{H}_{p,b}(\Omega)} \le 1/2$ for any $\lambda \in \dot{\omega}_{\tau_0}$, and therefore by Neumann series expansion we have

$$G_\lambda^{-1} = G_0^{-1}(I + R(\lambda)G_0^{-1})^{-1} = G_0^{-1} \sum_{j=0}^\infty (-R(\lambda)G_0^{-1})^j \quad (\lambda \in \dot{\omega}_{\tau_0}). \tag{2.29}$$

In view of (2.25), we see that there exist a $\tau_1 > 0$ and operators $G_j(\lambda) \in \operatorname{Anal}(\omega_{\tau_1}, \mathcal{L}(\mathcal{H}_{p,b}(\Omega))$ $(j = 1, 2)$ such that

$$G_\lambda^{-1} = \lambda^{\frac{1}{2}} G_1(\lambda) + G_2(\lambda) \quad \text{for any } \lambda \in \dot{\omega}_{\tau_1}. \tag{2.30}$$

We define the operator $\tilde{B}$ by the formula $\tilde{B}F = (\Delta^2\varphi)^T(0, \int_{\mathbb{R}^3}(\alpha g + \beta h)\,dx, 0)$. As both operators B and $\tilde{B}$ are finite range operators, we can choose $\mathbf{h}_1, \ldots, \mathbf{h}_m \in \mathcal{H}_{p,b}(\Omega)$ which are linearly independent over $\mathbb{C}$ in such a way that

$$BF = \sum_{j=1}^m \beta_j(F)\mathbf{h}_j, \quad \tilde{B}F = \sum_{j=1}^m \tilde{\beta}_j(F)\mathbf{h}_j$$

with $\beta_j(F)$, $\tilde{\beta}_j(F) \in \mathbb{C}$. To represent $\beta_j(F)$, $\tilde{\beta}_j(F) \in \mathbb{C}$ in more convenient way, we introduce $\mathbf{h}_1^*$, ..., $\mathbf{h}_m^* \in \mathcal{H}_{p,b}(\Omega)^*$ such that $< \mathbf{h}_j, \mathbf{h}_k^* >= \delta_{jk}$, where $< \cdot, \cdot >$ is the dual paring between $\mathcal{H}_{p,b}(\Omega)$ and its dual space $\mathcal{H}_{p,b}(\Omega)^*$ and δ_{jk} denote the Kronecker delta symbols. By using these symbols, we write

$$\beta_j(F) =< BF, \mathbf{h}_j^* >=< F, B^*\mathbf{h}_j^* >, \ \ \tilde{\beta}_j(F) =< \tilde{B}F, \mathbf{h}_j^* >=< F, \tilde{B}^*\mathbf{h}_j^* > .$$

Setting $\ell_{aj}^* = B^*\mathbf{h}_j^*$ and $\ell_{bj}^* = \tilde{B}^*\mathbf{h}_j^*$, we have

$$BF + \lambda^{-\frac{1}{2}}(\Delta^2\varphi)^T(0, T(\alpha g + \beta h), 0) = \sum_{j=1}^m < F, \ell_{aj}^* + \lambda^{-\frac{1}{2}}\ell_{bj}^* > \mathbf{h}_j,$$

and therefore we have

$$(I + T(\lambda))F = G_\lambda F + \sum_{j=1}^m < F, \ell_{aj}^* + \lambda^{-\frac{1}{2}}\ell_{bj}^* > \mathbf{h}_j. \tag{2.31}$$

Applying G_λ^{-1} to the both side of (2.31), we have

$$G_\lambda^{-1}(I + T(\lambda))F = F + \sum_{j=1}^m < F, \ell_{aj}^* + \lambda^{-\frac{1}{2}}\ell_{bj}^* > G_\lambda^{-1}\mathbf{h}_j = (I + N_\lambda)F \tag{2.32}$$

where we have defined the operator N_λ by the formula:

$$N_\lambda F = \sum_{j=1}^m < F, \ell_{aj}^* + \lambda^{-\frac{1}{2}}\ell_{bj}^* > G_\lambda^{-1}\mathbf{h}_j. \tag{2.33}$$

Now, we shall show the existence of the inverse operator of $I + N_\lambda$. For the notational simplicity, we set $G_\lambda^{-1}\mathbf{h}_j = \mathbf{v}_{\lambda,j}$ and $\ell_{aj}^* + \lambda^{-\frac{1}{2}}\ell_{bj}^* = A_{\lambda,j}$. Since $\{\mathbf{h}_j\}_{j=1}^m$ is linearly independent, so is $\{\mathbf{v}_{\lambda,j}\}_{j=1}^m$. Let us consider the $m \times m$ matrix: $M(\lambda) = (\delta_{jk}+ < \mathbf{v}_{\lambda,k}, A_{\lambda,j} >)$. By (2.30) the (j,k) component $\delta_{jk}+ < \mathbf{v}_{\lambda,k}, A_{\lambda,j} >$ is of the form: $\lambda^{-\frac{1}{2}}m_{1jk}(\lambda) + m_{2jk}(\lambda)$, where $m_{1jk}(\lambda)$ and $m_{2jk}(\lambda)$ are complex valued holomorphic functions defined on ω_{τ_1}. Let $D(\lambda)$ be the determinant of $M(\lambda)$. In particular, we can say that $D(\lambda) \equiv 0$ on ω_{τ_1} or there exist an integer q_1, and functions $D_j(\lambda)$ $(j = 1, 2)$ such that

$$D(\lambda) = \lambda^{\frac{q_1}{2}}D_1(\lambda) + \lambda^{\frac{q_1+1}{2}}D_2(\lambda) \quad \text{for } \lambda \in \dot{\omega}_{\tau_1}, \tag{2.34}$$

$D_1(0) \neq 0$, and $D_j(\lambda)$ $(j = 1, 2)$ are both holomorphic in ω_{τ_1}. We shall show that

$$D(\lambda) \not\equiv 0 \quad \text{in } \omega_{\tau_1}. \tag{2.35}$$

In fact, let $\lambda \in \mathcal{U} \cap \Sigma_\epsilon \cap \omega_{\tau_1}$ and assume that $D(\lambda) = 0$. Then there exists a vector $x_\lambda = {}^T(x_{\lambda 1}, \ldots, x_{\lambda m}) \in \mathbb{R}^m \setminus \{0\}$ such that

$$0 = \sum_{k=1}^{m} (\delta_{jk} + < \mathbf{v}_{\lambda,k}, A_{\lambda,j} >) x_{\lambda,k} = x_{\lambda,j} + \sum_{k=1}^{m} < \mathbf{v}_{\lambda,k}, A_{\lambda,j} > x_{\lambda,k} \tag{2.36}$$

for $j = 1, \ldots, m$. Set $F_\lambda = \sum_{k=1}^m x_{\lambda,k} \mathbf{v}_{\lambda,k} \in \mathcal{H}_{p,b}(\Omega)$, and then $F_\lambda \neq 0$, because $\{\mathbf{v}_{\lambda,k}\}_{k=1}^m$ is linearly independent. On the other hand, by (2.33) and (2.36)

$$N_\lambda F_\lambda = \sum_{j=1}^{m} < F_\lambda, A_{\lambda,j} > \mathbf{v}_{\lambda,j} = \sum_{j,k=1}^{m} x_{\lambda,k} < \mathbf{v}_{\lambda,k}, A_{\lambda,j} > \mathbf{v}_{\lambda,j} = -\sum_{j=1}^{m} x_{\lambda,j} \mathbf{v}_{\lambda,j} = -F_\lambda,$$

which implies that $(I + N_\lambda)F_\lambda = 0$. And therefore, by (2.32) and (2.31) $(I + T(\lambda))F_\lambda = 0$. On the other hand, by Lemma 2.4 $I + T(\lambda)$ is invertible when $\lambda \in \mathcal{U} \cap \Sigma_\epsilon$, and therefore we have $F_\lambda = 0$. This leads to a contradiction. Therefore, we have (2.35), and then (2.34) holds.

From (2.34), there exist a constant τ_2 $(0 < \tau_2 \leq \tau_1)$ and holomorphic functions $E_j(\lambda)$ $(j = 1, 2)$ defined on ω_{τ_2} such that

$$D^{-1}(\lambda) = \lambda^{-\frac{q_1}{2}} E_1(\lambda) + \lambda^{-\frac{q_1}{2}+\frac{1}{2}} E_2(\lambda) \quad \text{for } \lambda \in \dot{\omega}_{\tau_2}. \tag{2.37}$$

By using this fact, we shall show the existence of $(I + N_\lambda)^{-1}$. We may assume that $D^{-1}(\lambda) \neq 0$ when $\lambda \in \omega_{\tau_2} \setminus \{0\}$. Let us denote the (j, k) cofactor of $M(\lambda)$ by $M_{jk}(\lambda)$, which has the similar formula to $D^{-1}(\lambda)$ in (2.37). We observe that

$$\begin{aligned}
(I + N_\lambda)[G - D(\lambda)^{-1} \sum_{j=1}^{m}\sum_{k=1}^{m} < G, A_{\lambda,k} > M_{jk}(\lambda)\mathbf{v}_{\lambda,j}]& \\
= G - D(\lambda)^{-1} \sum_{j,k=1}^{m} < G, A_{\lambda,k} > M_{jk}(\lambda)\mathbf{v}_{\lambda,j}& \\
+ N_\lambda G - D(\lambda)^{-1} \sum_{j,k=1}^{m} < G, A_{\lambda,k} > M_{j,k}(\lambda) N_\lambda \mathbf{v}_{\lambda,j} &= (*).
\end{aligned}$$

Since $N_\lambda \mathbf{v}_{\lambda,j} = \sum_{\ell=1}^m < \mathbf{v}_{\lambda,j}, A_{\lambda,\ell} > \mathbf{v}_{\lambda,\ell}$ as follows from (2.33) and our short notation: $\ell_{aj}^* + \lambda^{-\frac{1}{2}} \ell_{bj}^* = A_{\lambda,j}$, we can proceed as follows:

$$\begin{aligned}
(*) =& G - D(\lambda)^{-1} \sum_{j,k=1}^{m} < G, A_{\lambda,k} > M_{jk}(\lambda)\mathbf{v}_{\lambda,j} + \sum_{k=1}^{m} < G, A_{\lambda,k} > \mathbf{v}_{\lambda,k} \\
&- D(\lambda)^{-1} \sum_{j,k,\ell=1}^{m} < G, A_{\lambda,k} > M_{jk}(\lambda) < \mathbf{v}_{\lambda,j}, A_{\lambda,\ell} > \mathbf{v}_{\lambda,\ell} \\
=& G + \sum_{k=1}^{m} < G, A_{\lambda,k} > \mathbf{v}_{\lambda,k} - D(\lambda)^{-1} \Big(\sum_{j,k,\ell=1}^{m} (\delta_{\ell j} + < \mathbf{v}_{\lambda,j}, A_{\lambda,\ell} >) M_{jk}(\lambda) < G, A_{\lambda,k} > \Big) \mathbf{v}_{\lambda,\ell} \\
=& G + \sum_{k=1}^{m} < G, A_{\lambda,k} > \mathbf{v}_{\lambda,k} - \sum_{k,\ell=1}^{m} \delta_{\ell k} < G, A_{\lambda,k} > \mathbf{v}_{\lambda,\ell} \\
=& G.
\end{aligned}$$

From this observation and our short notations: $G_\lambda^{-1}\mathbf{h}_j = \mathbf{v}_{\lambda,j}$ and $\ell_{aj}^* + \lambda^{-\frac{1}{2}}\ell_{bj}^* = A_{\lambda,j}$, we have

$$(I + N(\lambda))^{-1}G = G - D(\lambda)^{-1}\sum_{j,k=1}^{m} < G, \ell_{ak}^* + \lambda^{-\frac{1}{2}}\ell_{bk}^* > M_{jk}(\lambda)G_\lambda^{-1}\mathbf{h}_k$$

for $\lambda \in \omega_{\tau_2} \setminus \{0\}$. By (2.35), we see that

$$(I + T(\lambda))^{-1} = (I + N_\lambda)^{-1}G_\lambda^{-1}$$

which combined with (2.30) and (2.37) implies that there exist an integer q_2 and operators $T_j(\lambda) \in \mathrm{Anal}\,(\omega_{\tau_2}, \mathcal{L}(\mathcal{H}_{p,b}(\Omega)))$ $(j = 1, 2)$ such that

$$(I + T(\lambda))^{-1} = \lambda^{\frac{q_2}{2}}T_1(\lambda) + \lambda^{\frac{q_2+1}{2}}T_2(\lambda)$$

for any $\lambda \in \omega_{\tau_2} \setminus \{0\}$. Combining this fact with (2.20), (2.14) and Theorem 2.1 implies Theorem 2.3.

3 The proofs of Theorems 1.3 and 1.4 in the three-dimensional case

In what follows, b denotes a large number such that $B_{b-3} \supset \mathbb{R}^3 \setminus \Omega$. To prove Theorem 1.3, we start with the following lemmas.

Lemma 3.1. *Let ℓ be a positive integer and $n \in \{2, 3\}$. If $u \in \mathcal{S}'(\mathbb{R}^n) \cap L_{1,\mathrm{loc}}(\mathbb{R}^n)$ satisfies the homogeneous equation:*

$$\Delta^\ell u = 0 \quad \text{in } \mathbb{R}^n \tag{3.1}$$

and the radiation condition:

$$u(x) = O(|x|^m) \quad \text{as } |x| \to \infty, \tag{3.2}$$

for some non-negative integer m, then u is a polynomial of order m.

Proof. Since $u \in \mathcal{S}'(\mathbb{R}^n)$, applying the Fourier transform to (3.1) we have $|\xi|^{2\ell}\hat{u}(\xi) = 0$, which implies that $\mathrm{supp}\,\hat{u}(\xi) \subset \{0\}$. By the structure theorem of distributions, $\hat{u}(\xi)$ is represented as follows: $\hat{u}(\xi) = \sum_{|\alpha| \le k} c_\alpha \delta^{(\alpha)}(\xi)$ for some non-negative integer k, where δ denotes the Dirac delta function and c_α are complex numbers. By the Fourier inverse transform, we have

$$u(x) = \sum_{|\alpha| \le k} c_\alpha(-ix)^\alpha,$$

which combined with (3.2) implies that $u = u(x)$ should be a polynomial of order m. This completes the proof of the lemma. □

Lemma 3.2. *Let $\mathcal{E}_1$ be the same operator as in Theorem 2.1. Given $F = {}^T(f, g, h)$, we set $U = \mathcal{E}_1 F = {}^T(u, v, \theta)$. If $F \in \mathcal{H}_{p,b}(\mathbb{R}^3)$ and*

$$\int_{\mathbb{R}^3} (g(x) + h(x))\,dx = 0, \tag{3.3}$$

then

$$u(x) = O(1),\ \nabla u(x) = O(|x|^{-1}), \tag{3.4}$$

$$\theta(x) = O(|x|^{-1}) \tag{3.5}$$

as $|x| \to \infty$.

Proof. Since $\int_{\mathbb{R}^3}(g(y)+h(y)-\Delta f(y))\,dy = 0$ as follows from (3.3), by (2.2) we have

$$u(x) = \frac{-1}{8\pi}\int_{\mathbb{R}^3}(|x-y|-|x|)(g(y)+h(y)-\Delta f(y))\,dy.$$

By Taylor's formula we have

$$|x-y|-|x| = \int_0^1 \frac{d}{d\theta}|x-\theta y|\,d\theta = -\sum_{i=1}^3\int_0^1 (x_i-\theta y_i)y_i|x-\theta y|^{-1}\,d\theta,$$

and therefore

$$u(x) = \sum_{i=1}^3\int_0^1\Big\{\int_{\mathbb{R}^3}\frac{(x_i-\theta y_i)y_i}{|x-\theta y|}(g(y)+h(y)-\Delta f(y))\,dy\Big\}d\theta,$$

which combined with the fact that $g(y)+h(y)-\Delta f(y) = 0$ vanishes for $|y|\geq b$ implies (3.4). Since

$$\theta = E_3^1 * (h-\Delta f) = \frac{1}{4\pi|x|} * (h-\Delta f)$$

and since $h(y)-\Delta f(y)$ vanishes for $|y|\geq b$, we have (3.5), which completes the proof of the lemma. □

Lemma 3.3. *Let* $1<p<\infty$. (1) *If* $\theta\in W^2_{p,\mathrm{loc}}(\overline{\Omega})$ *satisfies the homogeneous equation:*

$$\Delta\theta = 0 \quad in\ \Omega, \quad \theta|_\Gamma = 0 \tag{3.6}$$

and the radiation condition:

$$\theta(x) = O(|x|^{-1}) \tag{3.7}$$

as $|x|\to\infty$, *then* $\theta=0$.
(2) *If* $u\in W^4_{p,\mathrm{loc}}(\overline{\Omega})$ *satisfies the homogeneous equation:*

$$\Delta^2 u = 0 \ \ in\ \Omega, \ \ u|_\Gamma = D_\nu u|_\Gamma = 0 \tag{3.8}$$

and the radiation condition:

$$u(x) = O(1) \tag{3.9}$$

as $|x|\to\infty$, *then* $u=0$.

Proof. (1) By L_p $(1<p<\infty)$ solvability in any C^2 bounded domain for the Dirichlet problem of the Laplace operator (cf. Simader [23]) and Sobolev's imbedding theorem, we see that $\theta\in W^2_{2,\mathrm{loc}}(\overline{\Omega})$. Let ρ be a function in $C_0^\infty(\mathbb{R}^3)$ such that $\rho(x)=1$ for $|x|\leq 1$ and $\rho(x)=0$ for $|x|\geq 2$. Set $\rho_L(x)=\rho(x/L)$ for $L>b$. Then, we have

$$0 = (\Delta\theta,\rho_L\theta)_\Omega = -(\nabla\theta,\rho_L\nabla\theta)_\Omega + (1/2)(\theta,(\Delta\rho_L)\theta)_\Omega \tag{3.10}$$

where $(a,b)_\Omega = \int_\Omega a(x)b(x)\,dx$. Since

$$|(\theta,(\Delta\rho_L)\theta)_\Omega| \leq \|\Delta\rho\|_{L_\infty(\mathbb{R}^3)}L^{-2}\int_{L\leq|x|\leq 2L}|\theta(x)|^2\,dx,$$

and therefore by (3.7) we see that $\lim_{L\to\infty} |(\theta, (\Delta\rho_L)\theta)_\Omega| = 0$. Letting $L \to \infty$ in (3.10), we have $\|\nabla\theta\|^2_{L_2(\Omega)} = 0$, which implies that $\nabla\theta = 0$, that is θ is a constant. But, $\theta|_\Gamma = 0$, which means that $\theta = 0$.
(2) By L_p $(1 < p < \infty)$ solvability in any C^4 bounded domain for the Dirichlet problem of the biharmonic operator (cf. Simader [23]) and Sobolev's imbedding theorem, we see that $u \in W^4_{2,\mathrm{loc}}(\overline{\Omega})$. First, we shall prove that $u = 0$ assuming that u satisfies the radiation condition:

$$u(x) = O(1), \ \ \nabla u(x) = O(|x|^{-1}) \tag{3.11}$$

as $|x| \to \infty$. Let ρ_L be the same function as in the proof of (1), and then we have

$$0 = (\Delta^2 u, \rho_L u)_\Omega = -(\nabla u, (\nabla\Delta\rho_L)u)_\Omega - 2(\nabla u, (\nabla^2\rho_L)\nabla u)_\Omega + (\Delta u, \rho_L \Delta u)_\Omega \tag{3.12}$$

where $\nabla u(\nabla^2\rho_L)\nabla u = \sum_{j,k=1}^3 (D_j D_k \rho_L) D_j u D_k u$. The radiation condition (3.11) implies that

$$\lim_{L\to\infty} (\nabla u, (\nabla\Delta\rho_L)u)_\Omega = 0, \ \ \lim_{L\to\infty} (\nabla u, (\nabla^2\rho_L)\nabla u)_\Omega = 0,$$

and therefore letting $L \to \infty$ in (3.12), we have $\|\Delta u\|_{L_2(\Omega)} = 0$, which implies that $\Delta u = 0$ in Ω. Since $u|_\Gamma = D_\nu u|_\Gamma = 0$, the zero extension u_0 of u to the whole space $\mathbb{R}^3$ satisfies the Laplace equation: $\Delta u_0 = 0$ in $\mathbb{R}^3$. Since $u_0(x) = u(x) = O(1)$ as $|x| \to \infty$, from Lemma 3.1 we see that u_0 is a constant. But, $u_0(x) = 0$ for $x \in \mathbb{R}^3 \setminus \Omega$, which means that $u_0 = 0$.

Finally, we shall show that the condition (3.9) together with (3.8) implies (3.11). Let ψ be a function in $C^\infty(\mathbb{R}^3)$ such that $\psi(x) = 1$ for $|x| \geq b+1$ and $\psi(x) = 0$ for $|x| \leq b$. Then, by (3.8) we have

$$\Delta^2(\psi u) = f \quad \text{in } \mathbb{R}^3, \tag{3.13}$$

where $f(x) = \Delta^2(\psi u) - \psi\Delta^2 u$. Since $\operatorname{supp} f \subset B_{b+1} \setminus B_b$, we have $f \in L_2(\mathbb{R}^3)$. Setting $v(x) = -(8\pi)^{-1}|x| * f$, by (3.13) and the fact that $-(8\pi)^{-1}|x|$ is a fundamental solution to the biharmonic operator Δ^2, we have $\Delta^2(u - v) = 0$ in $\mathbb{R}^3$. Employing the same argument as in the proof of Lemma 3.1, we have $u(x) - v(x) = \sum_{|\alpha|\leq m} c_\alpha x^\alpha$ for some non-negative integer m and complex numbers c_α. If we write

$$v(x) = -\frac{|x|}{8\pi}\int_{\mathbb{R}^3} f(y)\,dy - \frac{1}{8\pi}\int_{\mathbb{R}^3} (|x-y| - |x|) f(y)\,dy,$$

then by (3.9) we have

$$\sum_{|\alpha|\leq m} c_\alpha x^\alpha - \frac{|x|}{8\pi}\int_{\mathbb{R}^3} f(y)\,dy = u(x) + \frac{1}{8\pi}\int_{\mathbb{R}^3} (|x-y| - |x|) f(y)\,dy = O(1)$$

as $|x| \to \infty$, which implies that

$$u(x) = c_0 - \frac{1}{8\pi}\int_{\mathbb{R}^3} (|x-y| - |x|) f(y)\,dy$$

as $|x| \to \infty$, which implies that $|\nabla u(x)| = O(|x|^{-1})$ as $|x| \to \infty$. This completes the proof of the lemma. □

After these preparations, we are now able to prove our main results Theorem 1.3 and Theorem 1.4 in the case $n = 3$.

Proof of Theorem 1.3 for $n = 3$. Let s, $\mathcal{G}_1(\lambda)$ and $\mathcal{G}_2(\lambda)$ be the same as in Theorem 2.3 and set $\mathcal{G}(\lambda) = \lambda^{\frac{s}{2}}\mathcal{G}_1(\lambda) + \lambda^{\frac{s+1}{2}}\mathcal{G}_2(\lambda)$. Let η be a function in $C^\infty(\mathbb{R}^3)$ such that $\eta(x) = 1$ for $|x| \geq b-1$ and $\eta(x) = 0$ for $|x| \leq b-2$. Given $F \in \mathcal{H}_{p,b}(\Omega)$ and $\lambda \in \dot{\omega}_\tau$, we set $U(\lambda) = \mathcal{G}(\lambda)F$. When $\lambda \in \omega_\tau \cap \mathcal{U}$, by (2.20) we have $U(\lambda) = (\lambda I - \mathcal{A}_\Omega)^{-1}F \in \mathcal{D}_p(\Omega)$, and

$$(\lambda I - A)U(\lambda) = F \quad \text{in } \Omega, \quad BU_\lambda|_\Gamma = 0. \tag{3.14}$$

Therefore, $\eta U(\lambda) \in \mathcal{D}_p(\mathbb{R}^3)$ and $\eta U(\lambda)$ satisfies the equation:

$$(\lambda I - A)(\eta U(\lambda)) = \eta F + g(U(\lambda)) \quad \text{in } \mathbb{R}^3, \tag{3.15}$$

where for $U = {}^T(u, v, \theta)$ we have set

$$g(U) = \begin{pmatrix} 0 \\ \Delta^2(\eta u) - \eta\Delta^2 u + \Delta(\eta\theta) - \eta\Delta\theta \\ -(\Delta(\eta\theta) - \eta\Delta\theta) - (\Delta(\eta v) - \eta\Delta v) \end{pmatrix}. \tag{3.16}$$

Note that $\operatorname{supp} g(U) \subset D_{b-2,b-1}$. Since $\Sigma_\epsilon \subset \rho(\mathcal{A}_{\mathbb{R}^3})$ as follows from Theorem 2.1, we have

$$\eta U(\lambda) = (\lambda I - \mathcal{A}_{\mathbb{R}^3})^{-1}(\eta F + g(U(\lambda))) \tag{3.17}$$

whenever $\lambda \in \omega_\tau \cap \mathcal{U} \cap \Sigma_\epsilon$. Let $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{H}_1(\lambda)$ and $\mathcal{H}_2(\lambda)$ be the same operators as in (2.1) of Theorem 2.1 and let $\mathcal{H}(\lambda)$ be the same operator as in (2.13). By (3.17) and Theorem 2.1 we have

$$\eta U(\lambda) = \mathcal{H}(\lambda)(\eta F + g(U(\lambda))) \quad \text{in } \Omega_b \tag{3.18}$$

whenever $\lambda \in \omega_\tau \cap \mathcal{U} \cap \Sigma_\epsilon$. But, the both sides in (3.18) are analytic in $\dot{\omega}_\tau$, and therefore (3.18) holds for any $\lambda \in \dot{\omega}_\tau$. In view of Theorem 2.3, we write

$$U(\lambda) = \lambda^{\frac{s}{2}}V + O(\lambda^{\frac{s+1}{2}}) \quad \text{in } \Omega_b \tag{3.19}$$

as $|\lambda| \to 0$. We shall show that $s = 0$ by contradiction. Since

$$(\lambda I - A)U(\lambda) = F \quad \text{in } \Omega_b, \quad BF|_\Gamma = 0$$

for any $\lambda \in \dot{\omega}_\tau$ as follows from (3.14) and Theorem 2.3, we have

$$\lambda^{\frac{s}{2}}(-AV) + O(\lambda^{\frac{s+1}{2}}) = F \quad \text{in } \Omega_b, \quad (\lambda^{\frac{s}{2}}BV + O(\lambda^{\frac{s+1}{2}}))|_\Gamma = 0. \tag{3.20}$$

If $s > 0$, then letting $\lambda \to 0$, we have $F = 0$, which leads to a contradiction. Therefore, $s \leqq 0$. Assume that $s < 0$. We choose $F \in \mathcal{H}_{p,b}(\Omega)$ such that $V \neq 0$. Multiplying (3.20) by $\lambda^{-\frac{s}{2}}$ and letting $\lambda \to 0$, we have

$$-AV = 0 \quad \text{in } \Omega_b, \quad BV|_\Gamma = 0. \tag{3.21}$$

On the other hand, inserting (3.19) into (3.18) and using (3.16), we have

$$\eta\lambda^{\frac{s}{2}}V + O(\lambda^{\frac{s+1}{2}}) = [\lambda^{-\frac{1}{2}}\mathcal{E}_0 + \mathcal{E}_1 + \lambda^{\frac{1}{2}}\mathcal{H}_1(\lambda) + \lambda\mathcal{H}_2(\lambda)](\eta F + \lambda^{\frac{s}{2}}g(V) + O(\lambda^{\frac{s+1}{2}})),$$

and equating the terms: $\lambda^{\frac{s}{2}}$, $\lambda^{\frac{s}{2}-\frac{1}{2}}$, we have

$$\mathcal{E}_0 g(V) = 0, \tag{3.22}$$

$$\eta V = \mathcal{E}_1 g(V) + \mathcal{E}_0\eta F^1 \quad \text{in } \Omega_b, \tag{3.23}$$

where we have set

$$F^1 = \begin{cases} F & s = -1, \\ 0 & s \leq -2. \end{cases}$$

We extend V by the formula: $V = \mathcal{E}_1 g(V) + \mathcal{E}_0 \eta F$ for $|x| \geq b-1$. By the definitions of $\mathcal{E}_0$ and $\mathcal{E}_1$, we have

$$-AV = g(V) = 0 \quad \text{for } |x| \geq b-1 \tag{3.24}$$

because $\operatorname{supp} g(V) \subset D_{b-2,b-1}$. If we write $V = {}^T(u_0, v_0, \theta_0)$, then noting that $\eta(x) = 1$ for $|x| \geq b-1$, by (3.23) $u_0 \in W^4_{p,\mathrm{loc}}(\overline{\Omega})$, v_0, $\theta_0 \in W^2_{p,\mathrm{loc}}(\overline{\Omega})$. Moreover, by (3.21) and (3.24), V satisfies the homogeneous equation:

$$-AV = 0 \quad \text{in } \Omega, \quad BV|_\Gamma = 0. \tag{3.25}$$

On the other hand, if we set $g(V) = {}^T(0, g_0, h_0)$ and $F^1 = {}^T(f, g, h)$, then by (3.23) and Theorem 2.1 we have

$$V(x) = {}^T(E^2_3 * (g_0 + h_0) + \alpha T\eta g + \beta T\eta h, 0, E^1_3 * h_0) \tag{3.26}$$

for $|x| \geq b-1$. By (3.22) we have

$$\alpha \int_{\mathbb{R}^3} g_0 \, dx + \beta \int_{\mathbb{R}^3} h_0 \, dx = 0. \tag{3.27}$$

In particular, by (3.25) we have $v_0 = 0$.

Now, we shall show that $\theta_0 = u_0 = 0$. By (3.26) we have

$$\theta_0(x) = \frac{1}{4\pi|x|} * h_0 \quad \text{for } |x| > b-1. \tag{3.28}$$

Moreover, by (3.25) we have

$$\Delta\theta_0 = 0 \quad \text{in } \Omega, \quad \theta_0|_\Gamma = 0. \tag{3.29}$$

Since $h_0(x) = 0$ for $|x| \geq b-1$, we have $\theta_0(x) = O(|x|^{-1})$ as $|x| \to \infty$, so that by Lemma 3.3 we see that $\theta_0(x) = 0$. Therefore, we have

$$0 = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{h_0(y)}{|x-y|} \, dy = \frac{1}{4\pi|x|} \int_{\mathbb{R}^3} h_0(y) \, dy + \frac{1}{4\pi} \int_{\mathbb{R}^3} \Big(\frac{1}{|x-y|} - \frac{1}{|x|} \Big) h_0(y) \, dy$$

when $|x| > b$. Since the last term of the right hand side $= O(|x|^{-2})$ as $|x| \to \infty$, we have

$$\int_{\mathbb{R}^3} h_0(y) \, dy = 0 \tag{3.30}$$

Combining (3.30) with (3.27) implies that

$$\int_{\mathbb{R}^3} g_0(y) \, dy = 0 \tag{3.31}$$

because $\alpha \neq 0$. By (3.26), $u_0 = E^2_3 * (g_0 + h_0) + \alpha T\eta g + \beta T\eta h$. By (3.30) and (3.31),

$$\begin{aligned} [E^2_3 * (g_0 + h_0)](x) &= \frac{-1}{8\pi} \int_{\mathbb{R}^3} |x-y| (g_0(y) + h_0(y)) \, dy \\ &= -\frac{1}{8\pi} \int_{\mathbb{R}^3} [|x-y| - |x|] (g_0(y) + h_0(y)) \, dy \\ &= -\frac{1}{8\pi} \sum_{j=1}^{3} \int_0^1 \Big\{ \int_{\mathbb{R}^3} \frac{(x_i - \theta y_i) y_i}{|x - \theta y|} (g_0(y) + h_0(y)) \, dy \Big\} \, d\theta \end{aligned}$$

when $|x| > b$. Since $g_0(y) = h_0(y) = 0$ for $|y| \geq b - 1$ and since $\alpha T\eta g + \beta T\eta h$ is a constant, we have $u_0(x) = O(1)$ as $|x| \to \infty$. Since

$$\Delta^2 u_0 = 0 \quad \text{in } \Omega \quad u_0|_\Gamma = D_\nu u_0|_\Gamma = 0$$

as follows from (3.25), by Lemma 3.3 we have $u_0 = 0$, and therefore $V = 0$, which leads to a contradiction. This implies that $s = 0$, which combined with Theorem 2.3 implies Theorem 1.3. □

Proof of Theorem 1.4 in the case $n = 3$. Let τ, $\mathcal{G}_1(\lambda)$ and $\mathcal{G}_2(\lambda)$ be the same constant and operators in Theorem 1.3. And, let $\mathcal{U}$ be the same domain in $\mathbb{C}$ as in (1.14). Let $\Gamma = \Gamma_+ \cup \Gamma_0 \cup \Gamma_-$ be a path in $\mathbb{C}$ defined by the formulas:

$$\begin{aligned}
\Gamma_+ &: \lambda = se^{i(\pi-\theta)}, & s &: \infty \to (\tau/2)(\cos\theta)^{-1}, \\
\Gamma_0 &: \lambda = (\tau/2)(\cos\theta)^{-1}e^{is}, & s &: \pi - \theta \to -(\pi - \theta), \\
\Gamma_- &: \lambda = se^{-i(\pi-\theta)}, & s &: (\tau/2)(\cos\theta)^{-1} \to \infty,
\end{aligned}$$

where $\theta \in (0, \pi/2)$ is chosen so close to $\pi/2$ that $\Gamma \subset \mathcal{U}$. By (1.11) and (1.13) we have

$$T(t)F = \frac{1}{2\pi}\int_\Gamma (\lambda I - \mathcal{A}_\Omega)^{-1}F\,d\lambda.$$

To estimate $T(t)F$, let us set

$$\begin{aligned}
I_\pm &= \frac{1}{2\pi}\int_{\Gamma_\pm}(\lambda I - \mathcal{A}_\Omega)^{-1}F\,d\lambda, \\
I_0 &= \frac{1}{2\pi}\int_{\Gamma_0}(\lambda I - \mathcal{A}_\Omega)^{-1}F\,d\lambda.
\end{aligned}$$

By (1.13) we have

$$\|I_\pm(t)\|_{\mathcal{D}_p(\Omega)} \leq C\int_{(\tau/2)(\cos\theta)^{-1}}^\infty e^{s\cos\theta(\pi-\theta)t}\,ds\|F\|_{\mathcal{H}_p(\Omega)} = \frac{C}{(\cos\theta)t}e^{-(\tau/2)t}\|F\|_{\mathcal{H}_p(\Omega)}$$

for any $t > 0$ and $F \in \mathcal{H}_p(\Omega)$. To estimate $I_0(t)$, we restrict ourselves to the case where $F \in \mathcal{H}_{p,b}(\Omega)$. Let $C = C_1 \cup C_+ \cup C_- \cup C_2$ be a path defined by the formulas:

$$\begin{aligned}
C_1 &: \lambda = -(\tau/2) + is, & s &: (\tau/2)\tan\theta \to 0, \\
C_+ &: \lambda = e^{\pi i}s, & s &: \tau/2 \to 0, \\
C_- &: \lambda = e^{-\pi i}s, & s &: 0 \to \tau/2, \\
C_2 &: \lambda = -(\tau/2) + is, & s &: 0 \to -(\tau/2)\tan\theta.
\end{aligned}$$

Then, by Theorem 1.3 we have

$$I_0(t) = \frac{1}{2\pi i}\Big\{\int_{C_1} + \int_{C_+} + \int_{C_-} + \int_{C_2}\Big\}e^{\lambda t}(\lambda^{\frac{1}{2}}\mathcal{G}_1(\lambda) + \mathcal{G}_2(\lambda))F\,d\lambda \quad \text{in } \Omega_b \text{ for any } t > 0.$$

We have

$$\begin{aligned}
&\Big\|\frac{1}{2\pi i}\Big\{\int_{C_1} + \int_{C_2}\Big\}e^{\lambda t}(\lambda^{\frac{1}{2}}\mathcal{G}_1(\lambda) + \mathcal{G}_2(\lambda))F\,d\lambda\Big\|_{\mathcal{D}_p(\Omega_b)} \\
&\quad \leq Ce^{-(\tau/2)t}\int_0^{(\tau/2)\tan\theta} d\lambda\|F\|_{\mathcal{H}_p(\Omega)} \leq C(\tau/2)(\tan\theta)e^{-(\tau/2)t}\|F\|_{\mathcal{H}_p(\Omega)}.
\end{aligned}$$

Since $\mathcal{G}_2(\lambda) \in \mathrm{Anal}\,(\omega_\tau, \mathcal{L}(\mathcal{H}_{p,b}(\Omega), \mathcal{D}_{p,\mathrm{loc}}(\Omega_b)))$, we have

$$\Big\{\int_{C_+} + \int_{C_-}\Big\} e^{\lambda t}\mathcal{G}_2(\lambda)F\,d\lambda = 0.$$

On the other hand, we have

$$\begin{aligned}\Big\|\Big\{\int_{C_+} + \int_{C_-}\Big\} e^{\lambda t}\lambda^{\frac{1}{2}}\mathcal{G}_1(\lambda)F\,d\lambda\Big\|_{\mathcal{D}_{p,\mathrm{loc}}(\Omega_b)} &\le C\int_0^{(\tau/2)} s^{\frac{1}{2}}e^{-st}\,ds\,\|F\|_{\mathcal{H}_p(\Omega)} \\ &\le Ct^{-\frac{3}{2}}\int_0^\infty \ell e^{-\ell}\,d\ell\,\|F\|_{\mathcal{H}_p(\Omega)}.\end{aligned}$$

Combining these estimates, we have Theorem 1.4. □

4 Expansion formulas in two dimensions

In the following two sections, we will prove our main results Theorems 1.3 and 1.4 in the two-dimensional case. Although the structure of the proofs is the same as for $n = 3$, the asymptotic expansion is more involved. We will start with the expansion formula for the whole space $\mathbb{R}^2$.

Theorem 4.1. *Let $1 < p < \infty$ and $b > 0$. Let $\mathcal{L}_{p,b}(\mathbb{R}^2)$ be the set of all bounded linear operators from $\mathcal{H}_{p,b}(\mathbb{R}^2)$ into $\mathcal{D}_{p,\mathrm{loc}}(B_b)$ and $\rho(\mathcal{A}_{\mathbb{R}^2})$ the resolvent set of $\mathcal{A}_{\mathbb{R}^2}$. Then, there exist constants $\epsilon \in (0, \pi/2)$ and operator-valued functions $\mathcal{H}_j(\lambda) \in \mathrm{Anal}\,(\mathbb{C}, \mathcal{L}_{p,b}(\mathbb{R}^2))$ $(j = 1, 2)$ such that $\rho(\mathcal{A}_{\mathbb{R}^2}) \supset \Sigma_\epsilon$ and*

$$(\lambda I - \mathcal{A}_{\mathbb{R}^2})^{-1}F = \lambda^{-1}\mathcal{E}_0F + \log\lambda\,\mathcal{E}_1F + \mathcal{E}_2F + \mathcal{E}_3F + \lambda\log\lambda\,\mathcal{H}_1(\lambda)F + \lambda\mathcal{H}_2(\lambda)F \quad \text{in } B_b \tag{4.1}$$

for any $\lambda \in \Sigma_\epsilon$ and $F \in \mathcal{H}_{p,b}(\mathbb{R}^2)$. Here, Σ_ϵ is the set defined in (1.12), $\mathcal{E}_0$, $\mathcal{E}_1$ and $\mathcal{E}_2$ are operators in $\mathcal{L}(\mathcal{H}_{p,b}(\mathbb{R}^2), \mathcal{D}_{p,\mathrm{loc}}(B_b))$ defined by the formulas:

$$\begin{aligned}&\mathcal{E}_0F = \begin{pmatrix}\alpha_2\int_{\mathbb{R}^2} g\,dx + \alpha_3\int_{\mathbb{R}^2} h\,dx \\ 0 \\ 0\end{pmatrix},\ \mathcal{E}_1F = \begin{pmatrix}\frac{|x|^2}{16\pi} * (-\Delta f + g + h) \\ 0 \\ -\frac{1}{4\pi}\int_{\mathbb{R}^2} h\,dx\end{pmatrix}, \\ &\mathcal{E}_2F = \begin{pmatrix}\frac{\beta_1|x|^2}{16\pi} * (-\Delta f) + \frac{\beta_2|x|^2}{16\pi} * g + \frac{\beta_3|x|^2}{16\pi} * h \\ \delta_2^2\int_{\mathbb{R}^2} g\,dx + \delta_3^2\int_{\mathbb{R}^2} h\,dx \\ \delta_2^3\int_{\mathbb{R}^2} g\,dx + \delta_3^3\int_{\mathbb{R}^2} h\,dx\end{pmatrix},\ \mathcal{E}_3F = \begin{pmatrix}E_3^2 * (-\Delta f + g + h) \\ -f \\ E_3^1 * (h - \Delta f)\end{pmatrix}, \\ &E_2^1(x) = -\frac{1}{2\pi}(\log|x| - \log 2 + \gamma),\quad E_2^2(x) = \frac{1}{8\pi}|x|^2\log|x| - \frac{1}{8\pi}(\log 2 - \gamma + 1)|x|^2,\end{aligned} \tag{4.2}$$

$*$ stands for the convolution operator, γ is the Euler number, ϵ is given in (2.6), and α_2, α_3, β_1, β_2, β_3, δ_2^2, δ_3^2, δ_2^3 and δ_3^3 are non-zero constants which will be given in the proof below.

Remark 4.2. $E_2^1(x)$ and $E_2^2(x)$ are fundamental solutions of $-\Delta$ and Δ^2 in $\mathbb{R}^2$, respectively.

Proof. As in the proof for the three-dimensional case (Theorem 2.1), we have the representation formulas (2.3) or $\hat{u}_\lambda$, $\hat{v}_\lambda$, and $\hat{\theta}_\lambda$. But now the inverse Fourier transform is given by

$$\mathcal{F}_\xi^{-1}[(\lambda + |\xi|^2)^{-1}](x) = K_0(\sqrt{\lambda}|x|),$$

for $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, where K_0 stands for a modified Bessel function of order zero. We know that

$$K_0(z) = \frac{1}{2\pi}\Big[(-\log z)\sum_{m=0}^{\infty}\frac{1}{(m!)^2}\Big(\frac{z}{2}\Big)^{2m} + \sum_{m=0}^{\infty}\frac{\psi(m+1)}{(m!)^2}\Big(\frac{z}{2}\Big)^{2m}\Big],$$

where $\psi(z)$ is the psi function and for any integer $m \geq 1$ we have

$$\psi(1) = -\gamma, \ \ \psi(m) = -\gamma + 1 + \cdots + \frac{1}{m-1} \ \ (m \geq 2).$$

Setting

$$h_1(z) = \sum_{m=0}^{\infty}\frac{1}{((m+2)!)^2}\Big(\frac{z}{4}\Big)^m, \quad h_2(z) = \sum_{m=0}^{\infty}\frac{\psi(m+3)}{((m+2)!)^2}\Big(\frac{z}{4}\Big)^m,$$

we have

$$K_0(z) = \frac{1}{2\pi}\big[(-\log z)(1 + \frac{z^2}{4} + \frac{z^4}{16}h_1(z)) + \psi(1) + \psi(2)\frac{z^2}{4} + \frac{z^4}{16}h_2(z)\big]. \tag{4.3}$$

By (4.3) we have

$$\begin{aligned}\mathcal{F}_\xi^{-1}[(\lambda+|\xi|^2)^{-1}](x) &= -\frac{1}{4\pi}\log\lambda + E_2^1(x) - \frac{|x|^2}{16\pi}\lambda\log\lambda \\ &- \lambda E_2^2(x) - \lambda^2\log\lambda\frac{|x|^2}{64\pi}h_1(\lambda|x|^2) - \lambda^2\frac{|x|^2}{32\pi}\big\{(\log|x|+1)h_1(\lambda|x|^2) + h_2(\lambda|x|^2)\big\}.\end{aligned} \tag{4.4}$$

Using the resolvent formula

$$-\lambda^{-1}((\lambda-\Delta)^{-1} - (-\Delta)^{-1}) = (\lambda-\Delta)^{-1}(-\Delta)^{-1},$$

by (4.4) we have

$$\begin{aligned}\mathcal{F}_\xi^{-1}[(\lambda+|\xi|^2)^{-1}|\xi|^{-2}](x) &= -\lambda^{-1}(\mathcal{F}_\xi^{-1}[(\lambda+|\xi|^2)^{-1}](x) - E_2^1(x)) \\ &= \frac{1}{4\pi}\lambda^{-1}\log\lambda + \frac{|x|^2}{16\pi}\log\lambda + E_2^2(x) + \lambda\log\lambda\frac{|x|^2}{16\pi}h_1(\lambda|x|^2) \\ &+ \lambda\frac{|x|}{32\pi}((\log|x|+1)h_1(\lambda|x|^2) + h_2(\lambda|x|^2)).\end{aligned}$$

Therefore, setting

$$\begin{aligned}H_1^2(\lambda,|x|) &= \frac{|x|^2}{64\pi}h_1(\lambda|x|^2), \quad H_2^2(\lambda,|x|) = \frac{|x|^2}{32\pi}((\log|x|+1)h_1(\lambda|x|^2) + h_2(\lambda|x|^2)), \\ H_1^1(\lambda,|x|) &= -\frac{|x|^2}{16\pi} - \lambda H_1^2(\lambda,|x|), \quad H_2^1(\lambda,|x|) = -E_2^2(x) - H_2^2(\lambda,|x|),\end{aligned}$$

we have

$$\mathcal{F}_\xi^{-1}[(\lambda+|\xi|^2)^{-1}](x) = -\frac{1}{4\pi}\log\lambda + E_2^1(x) + \lambda\log\lambda H_1^1(\lambda,|x|) + \lambda H_2^1(\lambda,|x|), \tag{4.5}$$

$$\begin{aligned}\mathcal{F}_\xi^{-1}[(\lambda+|\xi|^2)^{-1}|\xi|^{-2}](x) &= \frac{1}{4\pi}\lambda^{-1}\log\lambda + \frac{|x|^2}{16\pi}\log\lambda + E_2^2(x) \\ &+ \lambda\log\lambda H_1^2(\lambda,|x|) + \lambda H_2^2(\lambda,|x|).\end{aligned} \tag{4.6}$$

Using (4.5) and (2.5), from (2.3) we have

$$\begin{aligned} u_\lambda(x) =& \lambda^{-1}\Big(\alpha_2\int_{\mathbb{R}^2} g\,dx + \alpha_3\int_{\mathbb{R}^2} h\,dx\Big) + \log\lambda\Big(\frac{|x|^2}{16\pi} * (-\Delta f + g + h)\Big) \\ &+ \frac{\beta_1|x|^2}{16\pi} * (-\Delta f) + \frac{\beta_2|x|^2}{16\pi} * g + \frac{\beta_3|x|^2}{16\pi} * h + E_2^2 * (-\Delta f + g + h) \\ &+ \lambda\log\lambda K_1^1(\lambda)F + \lambda K_2^1(\lambda)F, \end{aligned}$$

where we have set

$$\begin{aligned} \alpha_2 &= \sum_{j=1}^{3}\frac{(A_j^0 + A_j^1)\log\gamma_j^{-1}}{4\pi},\ \ \alpha_3 = \sum_{j=1}^{3}\frac{A_j^0\log\gamma_j^{-1}}{4\pi}, \\ \beta_1 &= \sum_{j=1}^{3}\frac{A_j^0 + A_j^1 + A_j^2}{\gamma_j}\log\gamma_j^{-1},\ \ \beta_2 = \sum_{j=1}^{3}\frac{A_j^0 + A_j^1}{\gamma_j}\log\gamma_j^{-1},\ \ \beta_3 = \sum_{j=1}^{3}\frac{A_j^0}{\gamma_j}\log\gamma_j^{-1}, \\ K_1^1(\lambda)F &= \sum_{j=1}^{3}\frac{A_j^0 + A_j^1 + A_j^2}{\gamma_j^2}H_1^2(\gamma_j^{-1}\lambda, |x|) * (-\Delta f) + \sum_{j=1}^{3}\frac{A_j^0 + A_j^1}{\gamma_j^2}H_1^2(\gamma_j^{-1}\lambda, |x|) * g \\ &\quad + \sum_{j=1}^{3}\frac{A_j^0}{\gamma_j^2}H_1^2(\gamma_j^{-1}\lambda, |x|) * h, \\ K_2^1(\lambda)F &= \Big\{\sum_{j=1}^{3}\frac{A_j^0 + A_j^1 + A_j^2}{\gamma_j^2}\log\gamma_j^{-1}H_1^2(\gamma_j^{-1}\lambda, |x|) \\ &\qquad\qquad + \sum_{j=1}^{3}\frac{A_j^0 + A_j^1 + A_j^2}{\gamma_j^2}H_2^2(\gamma_j^{-1}\lambda, |x|)\Big\} * (-\Delta f) \\ &\quad + \Big\{\sum_{j=1}^{3}\frac{A_j^0 + A_j^1}{\gamma_j^2}\log\gamma_j^{-1}H_1^2(\gamma_j^{-1}\lambda, |x|) + \sum_{j=1}^{3}\frac{A_j^0 + A_j^1}{\gamma_j^2}H_2^2(\gamma_j^{-1}\lambda, |x|)\Big\} * g \\ &\quad + \Big\{\sum_{j=1}^{3}\frac{A_j^0}{\gamma_j^2}\log\gamma_j^{-1}H_1^2(\gamma_j^{-1}\lambda, |x|) + \sum_{j=1}^{3}\frac{A_j^0}{\gamma_j^2}H_2^2(\gamma_j^{-1}\lambda, |x|)\Big\} * h. \end{aligned}$$

Since $E_2^1 * (-\Delta f) = f$ and $\int_{\mathbb{R}^2}\Delta f\,dx = 0$, by (2.3), (2.5) and (4.5) we have

$$v_\lambda(x) = -f + \delta_2^2\int_{\mathbb{R}^2} g\,dx + \delta_3^2\int_{\mathbb{R}^2} h\,dx + \lambda\log\lambda K_1^2(\lambda)F + \lambda K_2^2(\lambda)F,$$

where we have set

$$\begin{aligned} \delta_2^2 &= \frac{1}{4\pi}\sum_{j=1}^{3}\frac{A_j^1 + A_j^2}{\gamma_j}\log\gamma_j,\ \ \delta_3^2 = \frac{1}{4\pi}\sum_{j=1}^{3}\frac{A_j^1}{\gamma_j}\log\gamma_j, \\ K_1^2(\lambda)F &= -\sum_{j=1}^{3}\frac{A_j^0 + A_j^1}{\gamma_j^2}H_1^1(\gamma_j^{-1}\lambda, |x|) * (-\Delta f) + \sum_{j=1}^{3}\frac{A_j^1 + A_j^2}{\gamma_j^2}H_1^1(\gamma_j^{-1}\lambda, |x|) * g \\ &\quad + \sum_{j=1}^{3}\frac{A_j^1}{\gamma_j^2}H_1^1(\gamma_j^{-1}\lambda, |x|) * h, \end{aligned}$$

$$K_2^2(\lambda)F = -\Big\{\sum_{j=1}^3 \frac{A_j^0 + A_j^1}{\gamma_j^2}\log\gamma_j^{-1} H_1^1(\gamma_j^{-1}\lambda, |x|) + \sum_{j=1}^3 \frac{A_j^0 + A_j^1}{\gamma_j^2} H_2^1(\gamma_j^{-1}\lambda, |x|)\Big\} * (-\Delta f)$$
$$+ \Big\{\sum_{j=1}^3 \frac{A_j^1 + A_j^2}{\gamma_j^2}\log\gamma_j^{-1} H_1^1(\gamma_j^{-1}\lambda, |x|) + \sum_{j=1}^3 \frac{A_j^1 + A_j^2}{\gamma_j^2} H_2^1(\gamma_j^{-1}\lambda, |x|)\Big\} * g$$
$$+ \Big\{\sum_{j=1}^3 \frac{A_j^1}{\gamma_j^2}\log\gamma_j^{-1} H_1^1(\gamma_j^{-1}\lambda, |x|) + \sum_{j=1}^3 \frac{A_j^1}{\gamma_j^2} H_2^1(\gamma_j^{-1}\lambda, |x|)\Big\} * h.$$

Since $E_2^1 * (-\Delta f) = f$, by (2.3), (2.5) and (4.5) we have

$$\theta_\lambda(x) = -\frac{1}{4\pi}\log\lambda\int_{\mathbb{R}^2} h\,dx + E_2^1 * (h - \Delta h) + \delta_2^3\int_{\mathbb{R}^2} g\,dx + \delta_3^3\int_{\mathbb{R}^2} h\,dx$$
$$+ \lambda\log\lambda K_1^3(\lambda)F + \lambda K_2^3(\lambda)F,$$

where we have set

$$\delta_2^3 = \frac{1}{4\pi}\sum_{j=1}^3 \frac{A_j^1}{\gamma_j}\log\gamma_j, \quad \delta_3^3 = \frac{1}{4\pi}\sum_{j=1}^3 \frac{A_j^0 + A_j^1}{\gamma_j}\log\gamma_j,$$

$$K_1^3(\lambda)F = \sum_{j=1}^3 \frac{A_j^0}{\gamma_j^2} H_1^1(\gamma_j^{-1}\lambda, |x|) * (-\Delta f) - \sum_{j=1}^3 \frac{A_j^1}{\gamma_j^2} H_1^1(\gamma_j^{-1}\lambda, |x|) * g$$
$$+ \sum_{j=1}^3 \frac{A_j^0 + A_j^2}{\gamma_j^2} H_1^1(\gamma_j^{-1}\lambda, |x|) * h,$$

$$K_2^3(\lambda)F = \Big\{\sum_{j=1}^3 \frac{A_j^0}{\gamma_j^2}\log\gamma_j^{-1} H_1^1(\gamma_j^{-1}\lambda, |x|) + \sum_{j=1}^3 \frac{A_j^0 + A_j^1}{\gamma_j^2} H_2^1(\gamma_j^{-1}\lambda, |x|)\Big\} * (-\Delta f)$$
$$- \Big\{\sum_{j=1}^3 \frac{A_j^1}{\gamma_j^2}\log\gamma_j^{-1} H_1^1(\gamma_j^{-1}\lambda, |x|) + \sum_{j=1}^3 \frac{A_j^1 + A_j^2}{\gamma_j^2} H_2^1(\gamma_j^{-1}\lambda, |x|)\Big\} * g$$
$$+ \Big\{\sum_{j=1}^3 \frac{A_j^0 + A_j^2}{\gamma_j^2}\log\gamma_j^{-1} H_1^1(\gamma_j^{-1}\lambda, |x|) + \sum_{j=1}^3 \frac{A_j^1}{\gamma_j^2} H_2^1(\gamma_j^{-1}\lambda, |x|)\Big\} * h.$$

This completes the proof of Theorem 4.1. □

The analogue of Theorem 2.3 for $n = 2$ reads as follows.

Theorem 4.3. *Let $1 < p < \infty$ and let $\mathcal{U}$ be the same set as in (1.14). Then, there exist a constant $\tau > 0$ and an operator valued function $\mathcal{G}(\lambda) \in \mathrm{Anal}\,(\dot{\omega}_\tau, \mathcal{L}_{p,b}(\Omega))$ such that*

$$(\lambda I - \mathcal{A}_\Omega)^{-1}F = \mathcal{G}(\lambda)F \quad \text{in } \Omega_b$$

for any $\lambda \in \omega_\tau \cap \mathcal{U}$ and $F \in \mathcal{H}_{p,b}(\Omega)$.

Moreover, there exist integers s, β, a constant coefficient polynomial $L(t)$, a polynomial $M(t)$ whose coefficients belong to $L_{p,b}(\Omega)$ and a positive constant C such that

$$\|\mathcal{G}(\lambda)F - \lambda^s(M(\log\lambda)/L(\log\lambda))F\|_{\mathcal{D}_{p,\mathrm{loc}}(\Omega_b)} \le C|\lambda^{s+1}(\log\lambda)^\beta|\|F\|_{\mathcal{H}_{p,b}(\Omega)} \tag{4.7}$$

for any $\lambda \in \dot{\omega}_\tau$ and $F \in \mathcal{H}_{p,b}(\Omega)$.

Proof. The proof follows the lines of the proof of Theorem 2.3 but now the expansion formula is more complicated. Instead of (2.13) we now set

$$\mathcal{H}(\lambda) = \lambda^{-1}\mathcal{E}_0 + \log\lambda\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \lambda\log\lambda\mathcal{H}_1(\lambda) + \lambda\mathcal{H}_2(\lambda), \tag{4.8}$$

where the operators $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{H}_1(\lambda)$ and $\mathcal{H}_2(\lambda)$ are given in Theorem 4.1. Defining again $\Phi(\lambda)$ by (2.14), we obtain

$$(\lambda I - A)\Phi(\lambda)F = F + T(\lambda)F \quad \text{in } \Omega, \quad B\Phi(\lambda)F|_\Gamma = 0 \tag{4.9}$$

for any $\lambda \in \Sigma_\epsilon$, where $T(\lambda)F$ is defined by (2.17). The proof of Lemma 2.4 works also for $n = 2$, so $(I + T(\lambda))^{-1}$ exists as a bounded linear operator on $\mathcal{H}_{p,b}(\Omega)$ for any $\lambda \in \mathcal{U} \cap \Sigma_\epsilon$ and we have

$$(\lambda I - \mathcal{A}_\Omega)^{-1} = \Phi(\lambda)(I + T(\lambda))^{-1} \tag{4.10}$$

for $\lambda \in \Sigma_\epsilon \cap \mathcal{U}$.

To discuss the invertibility of $I + T(\lambda)$ for $\lambda \in \dot{\omega}_\sigma$, we consider

$$\Phi_0 F = (1-\varphi)\mathcal{E}_3\iota F + \varphi S_{\Omega_b} rF$$

for $F \in \mathcal{H}_{p,b}(\Omega)$, where $\mathcal{E}_3$ is the same operator as in Theorem 4.1. Note that

$$-A\mathcal{E}_3\iota F = \iota F \quad \text{in } \mathbb{R}^2.$$

We write $\mathcal{E}_3 F = {}^T(u_{0,\mathbb{R}^2}, v_{0,\mathbb{R}^2}, \theta_{0,\mathbb{R}^2})$ to avoid any confusion, if necessary. Applying A to $\Phi_0 F$, we have

$$-A\Phi_0 F = F + T_0 F \quad \text{in } \Omega, \quad B\Phi_0 F|_\Gamma = 0, \tag{4.11}$$

where

$$T_0 F = \begin{pmatrix} 0 \\ -L^3_\varphi(u_{0,\mathbb{R}^2} - u_{\Omega_b}) - L^1_\varphi(\theta_{0,\mathbb{R}^2} - \theta_{\Omega_b}) \\ L^1_\varphi(\theta_{0,\mathbb{R}^2} - \theta_{\Omega_b}) + L^1_\varphi(v_{0,\mathbb{R}^2} - v_{\Omega_b}) \end{pmatrix}.$$

Since the second and third members of $T_0 F$ belong to $W^1_p(\Omega)$ and $\operatorname{supp} T_0 F \subset D_{b-2,b-1}$, by Rellich's compactness theorem, T_0 is a compact operator on $\mathcal{H}_{p,b}(\Omega)$. According to Theorem 4.1, we set

$$\begin{aligned}
u_{\lambda,\mathbb{R}^2} &= u_{0,\mathbb{R}^2} + \lambda^{-1}S_0(\alpha_2 g + \alpha_3 h) + \log\lambda\frac{|x|^2}{16\pi} * (-\Delta f + g + h) + U_{\lambda,\mathbb{R}^2}, \\
v_{\lambda,\mathbb{R}^2} &= v_{0,\mathbb{R}^2} + S_0(\delta^2_2 g + \delta^2_3 h) + V_{\lambda,\mathbb{R}^2}, \\
\theta_{\lambda,\mathbb{R}^2} &= \theta_{0,\mathbb{R}^2} - \log\lambda\frac{1}{4\pi}S_0 h + S_0(\delta^3_2 g + \delta^3_3 h) + \Theta_{\lambda,\mathbb{R}^2},
\end{aligned}$$

where $S_0 a = \int_{\mathbb{R}^2} a\,dx$ and

$${}^T(U_{\lambda,\mathbb{R}^2}, V_{\lambda,\mathbb{R}^2}, \Theta_{\lambda,\mathbb{R}^2}) = \lambda\log\lambda\mathcal{H}_1(\lambda)F + \lambda\mathcal{H}_2(\lambda)F. \tag{4.12}$$

Then, we have

$$(I + T(\lambda))F = (I + T_0)F + \lambda^{-1}R_0 F + \log\lambda R_1 F + R_2 F + R(\lambda)F \tag{4.13}$$

where

$$R_0 F = -(\Delta^2\varphi)\begin{pmatrix} 0 \\ S_0(\alpha_2 g + \alpha_3 h) \\ 0 \end{pmatrix}, \quad R_1 F = \begin{pmatrix} 0 \\ -L^3_\varphi\Big(\frac{|x|^2}{16\pi} * (-\Delta f + g + h)\Big) + \frac{1}{4\pi}(\Delta\varphi)S_0 h \\ -\frac{1}{4\pi}(\Delta\varphi)S_0 h \end{pmatrix},$$

$$R_2F = -(\Delta\varphi)\begin{pmatrix} 0 \\ 0 \\ S_0(\delta_2^2 g + \delta_3^2 h) \end{pmatrix}, \quad R(\lambda)F = \begin{pmatrix} 0 \\ -L^3_\varphi(U_{\lambda,\mathbb{R}^2}) - L^1_\varphi(\Theta_{\lambda,\mathbb{R}^2}) \\ L^1_\varphi(\Theta_{\lambda,\mathbb{R}^2}) + L^1_\varphi(V_{\lambda,\mathbb{R}^2}) \end{pmatrix}. \tag{4.14}$$

In view of (4.12) and (4.14), there exist operators $R_j(\lambda) \in \mathrm{Anal}\,(\mathbb{C}, \mathcal{L}(\mathcal{H}_{p,b}(\Omega)))$ $(j = 1, 2)$ such that

$$R(\lambda)F = \lambda \log \lambda R_1(\lambda)F + \lambda R_2(\lambda)F \tag{4.15}$$

for any $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. In particular, we have

$$\lim_{\lambda \to 0} \|R(\lambda)\|_{\mathcal{L}(\mathcal{H}_{p,b}(\Omega))} = 0. \tag{4.16}$$

Here, $\|\cdot\|_{\mathcal{L}(\mathcal{H}_{p,b}(\Omega))}$ denotes the operator norm of $\mathcal{L}(\mathcal{H}_{p,b}(\Omega))$. Since T_0 is a compact operator on $\mathcal{H}_{p,b}(\Omega)$, by Seeley's lemma [21] there exists a finite range operator B such that $I + T_0 - B$ has an inverse operator $(I + T_0 - B)^{-1} \in \mathcal{L}(\mathcal{H}_{p,b}(\Omega))$. Set $G_\lambda = I + T_0 - B + R(\lambda)$ and $G_0 = I + T_0 - B$, and then

$$(I + T(\lambda))F = G_\lambda F + BF + S_\lambda F, \tag{4.17}$$

$$G_\lambda = (I + R(\lambda)G_0^{-1})G_0. \tag{4.18}$$

By (4.16) there exists a $\tau_0 > 0$ such that $\|R(\lambda)G_0^{-1}\|_{\mathcal{L}(\mathcal{H}_{p,b}(\Omega))} \leq 1/2$ for any $\lambda \in \dot{\omega}_{\tau_0}$, and therefore by Neumann series expansion we have

$$G_\lambda^{-1} = G_0^{-1}(I + R(\lambda)G_0^{-1})^{-1} = G_0^{-1}\sum_{j=0}^\infty (-R(\lambda)G_0^{-1})^j \quad (\lambda \in \dot{\omega}_{\tau_0}). \tag{4.19}$$

In view of (4.15), we have

$$G_\lambda^{-1} = \sum_{j=0}^\infty \Big[\sum_{k=0}^j G_{jk}(\log \lambda)^k\Big]\lambda^j \tag{4.20}$$

where $G_{jk} \in \mathcal{L}(\mathcal{H}_{p,b}(\Omega))$. The right-hand side of (4.20) is absolutely and uniformly convergent with operator norm in $\dot{\omega}_{\tau_0}$, that is

$$\sum_{j=0}^\infty \Big[\sum_{k=0}^j \|G_{jk}\|_{\mathcal{L}(\mathcal{H}_{p,k}(\Omega))} |\log \lambda|^k\Big]|\lambda|^j < \infty \quad (\lambda \in \dot{\omega}_{\tau_0}).$$

Since B is a finite range operator, there exists a finite number of elements $\mathbf{k}_1, \ldots, \mathbf{k}_k \in \mathcal{H}_{p,b}(\Omega)$ such that

$$BF = \sum_{j=1}^k \alpha_j(F)\mathbf{k}_j \quad (\alpha_j(F) \in \mathbb{C}).$$

On the other hand, if we define the operators S_0, S_1 and S_2 by the formula:

$$S_0 k = \int_{\mathbb{R}^2} k(y)\,dy, \quad S_1 k = \int_{\mathbb{R}^2} y k(y)\,dy, \quad S_2 k = \int_{\mathbb{R}^2} |y|^2 k(y)\,dy \tag{4.21}$$

for $k \in \mathcal{H}_{p,b}(\Omega)$ (S_0 was already defined before (4.12)), then we can write

$$L^3_\varphi\Big(\frac{|x|^2}{16\pi} * k\Big) = \frac{\Delta^2\varphi}{16\pi}(|x|^2 S_0 k - 2x \cdot S_1 k + S_2 k) + \frac{\nabla\Delta\varphi}{2\pi} \cdot (x S_0 k - S_1 k) + \frac{\Delta\varphi}{2\pi} S_0 k, \tag{4.22}$$

where $\cdot$ stands for the usual inner product in $\mathbb{R}^2$. For the notational simplicity, now we set

$$S_\lambda = \lambda^{-1}R_0 + \log\lambda R_1 + R_2$$

in the formula (4.13). From above observation we see that there exists a finite number of $\tilde{\mathbf{k}}_j \in \mathcal{H}_{p,b}(\Omega)$ $(j = 1, \ldots, \ell+1)$ such that $S_\lambda F$ is written in the form:

$$S_\lambda F = \lambda^{-1}\beta_1(F)\tilde{\mathbf{k}}_1 + \log\lambda\sum_{j=2}^{\ell}\beta_j(F)\tilde{\mathbf{k}}_j + \beta_{\ell+1}(F)\tilde{\mathbf{k}}_{\ell+1} \quad (\beta_j(F) \in \mathbb{C}).$$

There exist $\mathbf{h}_1, \ldots, \mathbf{h}_m \in \mathcal{H}_{p,b}(\Omega)$ which are linearly independent over $\mathbb{C}$ such that

$$BF + S_\lambda F = \lambda^{-1}W^1F + \log\lambda W^2F + W^3F, \quad W^kF = \sum_{j=1}^{m}\gamma_j^k(F)\mathbf{h}_j \quad (k = 1, 2, 3) \ (\gamma_j^k(F) \in \mathbb{C}).$$

To represent $\gamma_j^k(F)$, we introduce $\mathbf{h}_1^*, \ldots, \mathbf{h}_m^* \in \mathcal{H}_{p,b}(\Omega)^*$ such that $< \mathbf{h}_j, \mathbf{h}_k^* >= \delta_{jk}$ where $< \cdot, \cdot >$ is the dual paring between $\mathcal{H}_{p,b}(\Omega)$ and its dual space $\mathcal{H}_{p,b}(\Omega)^*$ and δ_{jk} denotes the Kronecker delta symbol. Using these symbols, we write

$$\gamma_j^k(F) =< W^kF, \mathbf{h}_j^* >=< F, (W^k)^*\mathbf{h}_j^* > .$$

Setting $\ell_{kj}^* = (W^k)^*\mathbf{h}_j^*$, we have

$$BF + S_\lambda F = \sum_{j=1}^{m} < F, \lambda^{-1}\ell_{1j}^* + \log\lambda\ell_{2j}^* + \ell_{3j}^* > \mathbf{h}_j,$$

and therefore, we have

$$(I + T(\lambda))F = G_\lambda F + \sum_{j=1}^{m} < F, \lambda^{-1}\ell_{1j}^* + \log\lambda\ell_{2j}^* + \ell_{3j}^* > \mathbf{h}_j. \tag{4.23}$$

Applying G_λ^{-1} to the both side of (4.23), we have

$$G_\lambda^{-1}(I + T(\lambda))F = F + \sum_{j=1}^{m} < F, \lambda^{-1}\ell_{1j}^* + \log\lambda\ell_{2j}^* + \ell_{3j}^* > G_\lambda^{-1}\mathbf{h}_j = (I + N_\lambda)F, \tag{4.24}$$

where we have defined the operator N_λ by the formula:

$$N_\lambda F = \sum_{j=1}^{m} < F, \lambda^{-1}\ell_{1j}^* + \log\lambda\ell_{2j}^* + \ell_{3j}^* > G_\lambda^{-1}\mathbf{h}_j. \tag{4.25}$$

Now, we shall show the existence of the inverse operator to $I + N_\lambda$. For the notational simplicity, we set $G_\lambda^{-1}\mathbf{h}_j = \mathbf{v}_{\lambda,j}$ and $\lambda^{-1}\ell_{1j}^* + \log\lambda\ell_{2j}^* + \ell_{3j}^* = A_{\lambda,j}$. Since $\{\mathbf{h}_j\}_{j=1}^m$ is linearly independent, so is $\{\mathbf{v}_{\lambda,j}\}_{j=1}^m$. Let us consider the $m \times m$ matrix: $M(\lambda) = (\delta_{jk}+ < \mathbf{v}_{\lambda,k}, A_{\lambda,j} >)$. By (4.20) the (j, k) component $\delta_{jk}+ < \mathbf{v}_{\lambda,k}, A_{\lambda,j} >$ is of the form: $\lambda^{-1}m_{1jk}(\lambda) + \log\lambda m_{2jk}(\lambda) + +m_{3jk}(\lambda)$. Here, $m_{ijk}(\lambda)$ are usual complex valued holomorphic functions defined on $\dot{\omega_{\tau_0}}$ and have the expansion formulas:

$$m_{ijk}(\lambda) = \sum_{b=0}^{\infty}\Big[\sum_{a=0}^{b}\beta_{ijk}^{a,b}(\log\lambda)^a\Big]\lambda^b \quad (\beta_{ijk}^{a,b} \in \mathbb{C}), \tag{4.26}$$

where the right-hand side is absolutely and uniformly convergent in $\dot{\omega}_{\tau_0}$. Let $D(\lambda)$ be the determinant of $M(\lambda)$. In view of (4.26), we have

$$\det(\lambda M(\lambda)) = \sum_{b=0}^{\infty}\Big[\sum_{a=0}^{b}\delta^{a,b}(\log\lambda)^a\Big]\lambda^b \quad (\delta^{a,b}\in\mathbb{C}),$$

where the right-hand side is absolutely and uniformly convergent in $\dot{\omega}_{\tau_0}$, and therefore we have

$$D(\lambda) = \lambda^{-m}\sum_{b=0}^{\infty}\Big[\sum_{a=0}^{b}\delta^{a,b}(\log\lambda)^a\Big]\lambda^b \tag{4.27}$$

for $\lambda\in\dot{\omega}_{\tau_0}$. In particular, we can say that $D(\lambda)\equiv 0$ on U_{τ_1} or there exists an integer γ such that

$$\sum_{a=0}^{b}\delta^{a,b}(\log\lambda)^a \equiv 0 \ \ (b<\gamma),\ \ \sum_{a=0}^{\gamma}\delta^{a,\gamma}(\log\lambda)^a \not\equiv 0 \tag{4.28}$$

for any $\lambda\in\dot{\omega}_{\tau_0}$. In the latter case, choosing τ_0 smaller if necessary, we may assume that

$$\sum_{a=0}^{\gamma}\delta^{a,\gamma}(\log\lambda)^a \neq 0 \quad \text{for any } \lambda\in\dot{\omega}_{\tau_0}. \tag{4.29}$$

In the same way as for $n=3$, one can show that

$$D(\lambda)\not\equiv 0 \quad \text{in } U_{\tau_1}. \tag{4.30}$$

By (4.27) and (4.28) we write

$$D(\lambda) = \lambda^{-m}\sum_{b=\gamma}^{\infty}\Big[\sum_{a=0}^{b}\delta^{a,b}(\log\lambda)^a\Big]\lambda^b = \lambda^{-m+\gamma}\sum_{b=0}^{\infty}L_b(\log\lambda)\lambda^b,$$

where we have set $L_b(t)=\sum_{a=0}^{b+\gamma}\delta^{a,b+\gamma}t^a$. Since $L_0(\log\lambda)\neq 0$ ($\lambda\in\dot{\omega}_{\tau_0}$) as follows from (4.29), we write

$$D(\lambda) = \lambda^{-m+\gamma}L_0(\log\lambda)\Big[1+\sum_{b=1}^{\infty}\frac{L_b(\log\lambda)}{L_0(\log\lambda)}\lambda^b\Big].$$

Since

$$\lim_{\lambda\to 0}\sum_{b=1}^{\infty}\frac{L_b(\log\lambda)}{L_0(\log\lambda)}\lambda^b = 0,$$

there exists a τ_1 ($0<\tau_1\leq\tau_2$) such that

$$\Big|\sum_{b=1}^{\infty}\frac{L_b(\log\lambda)}{L_0(\log\lambda)}\lambda^b\Big| \leq 1/2 \quad (\lambda\in\dot{\omega}_{\tau_1}),$$

and therefore we have

$$\begin{aligned} D(\lambda)^{-1} &= \lambda^{m-\gamma}L_0(\log\lambda)^{-1}\Big[1+\sum_{j=1}^{\infty}\Big\{\sum_{b=1}^{\infty}\frac{L_b(\log\lambda)}{L_0(\log\lambda)}\lambda^b\Big\}^j\Big] \\ &= \lambda^{m-\gamma}L_0(\log\lambda)^{-1}\Big[1+\sum_{j=1}^{\infty}\Big\{\sum_{b=1}^{\infty}L_b(\log\lambda)L_0(\log\lambda)^{b-1}\Big(\frac{\lambda}{L_0(\log\lambda)}\Big)^b\Big\}^j\Big]. \end{aligned}$$

Since $L_b(t)L_0(t)^{b-1}$ is a polynomial of degree not greater than $b(\gamma+1)$, we can write

$$D(\lambda)^{-1} = \frac{\lambda^{m-\gamma}}{L_0(\log\lambda)}\Big[1 + \sum_{j=1}^{\infty} P_{j(\gamma+1)}(\log\lambda)\Big(\frac{\lambda}{L_0(\log\lambda)}\Big)^j\Big], \tag{4.31}$$

where $P_{j(\gamma+1)}(t)$ is a polynomial of degree not greater than $j(\gamma+1)$.

Similar to the case $n=3$, one can show that the inverse of $I+N(\lambda)$ exists and has the form

$$(I+N(\lambda))^{-1}G = G - D(\lambda)^{-1}\sum_{j,k=1}^{m} < G, \lambda^{-1}\ell_{1j}^* + \log\lambda\ell_{2j}^* + \ell_{3j} > M_{jk}(\lambda)G_\lambda^{-1}\mathbf{h}_k$$

for $\lambda \in \dot{\omega}_{\tau_1}$, which combined with (4.20) and (4.31) implies that there exists an integer s such that

$$(I+T(\lambda))^{-1} = \frac{\lambda^s}{L_0(\log\lambda)}\sum_{j=0}^{\infty} Q_{j(\gamma+1)}(\log\lambda)\Big(\frac{\lambda}{L_0(\log\lambda)}\Big)^j, \tag{4.32}$$

where $Q_{j(\gamma+1)}(t)$ is a polynomial of degree not greater than $j(\gamma+1)$, whose coefficients belong to $\mathcal{L}(\mathcal{H}_{p,b}(\Omega))$. In fact, by (4.20) we have

$$G_\lambda^{-1} = \sum_{j=0}^{\infty}\Big[\sum_{k=0}^{j} G_{jk}(\log\lambda)^k\Big]\lambda^j = \sum_{j=0}^{\infty}\Big\{\Big[\sum_{k=0}^{j} G_{jk}(\log\lambda)^k\Big]L_0(\log\lambda)^j\Big\}\Big(\frac{\lambda}{L_0(\log\lambda)}\Big)^j.$$

If we set $\tilde{G}_j(\gamma+1)(t) = (\sum_{k=0}^{j} G_{jk}t^k)L_0(t)^j$, then $\tilde{G}_j(t)$ is a polynomial of degree not greater than $j(\gamma+1)$ and we have

$$G_\lambda^{-1} = \sum_{j=0}^{\infty}\tilde{G}_{j(\gamma+1)}(\log\lambda)\Big(\frac{\lambda}{L_0(\log\lambda)}\Big)^j.$$

And also, setting $M_{\gamma+1}(t) = tL_0(t)\ell_{2j}^* + L_0(t)\ell_{3j}^*$, we can write

$$\lambda^{-1}\ell_{1j}^* + \log\lambda\ell_{2j}^* + \ell_{3j}^* = \lambda^{-1}\Big[\ell_{1j}^* + M_{\gamma+1}(\log\lambda)\frac{\lambda}{L_0(\lambda)}\Big],$$

where $M_{\gamma+1}(t)$ is a polynomial of degree not greater than $\gamma+1$. Therefore, we have (4.32). Combining (4.32) with (2.20), (2.14) and Theorem 4.1 implies Theorem 4.3. □

5 The proofs of Theorems 1.3 and 1.4 for $n = 2$

To prove Theorem 1.3, we start with the following lemmas.

Lemma 5.1. *Let E_2^1 and E_2^2 be the fundamental solutions of $-\Delta$ and Δ^2 given in Theorem 4.1, respectively. Given $g, h \in L_{p,b}(\mathbb{R}^2)$, we set $u = E_2^2 * g$ and $\theta = E_2^1 * h$. If*

$$S_0 g = S_1 g = S_0 h = 0 \tag{5.1}$$

then

$$u(x) = O(\log|x|),\ \nabla u(x) = O(|x|^{-1}),\ \nabla^2 u(x) = O(|x|^{-2}),\ \nabla^3 u(x) = O(|x|^{-3}), \tag{5.2}$$

$$\theta(x) = O(|x|^{-1}),\ \nabla\theta(x) = O(|x|^{-2}) \tag{5.3}$$

as $|x| \to \infty$, where S_0, S_1 and S_2 are the same operators as in (4.21).

Proof. From (4.2) we have

$$u(x) = \frac{1}{8\pi}\int_{\mathbb{R}^2}(|x-y|^2\log|x-y| - c_1|x-y|^2)g(y)\,dy,$$
$$\theta(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2}(\log|x-y| - c_2)h(y)\,dy$$

where $c_1 = \log 2 - \gamma + 1$ and $c_2 = -\log 2 + \gamma$. By Taylor expansion, we have

$$\begin{aligned}|x-y|^2\log|x-y| - c_1|x-y|^2 = |x|^2\log|x| - c_1|x|^2 - 2\log|x|(x\cdot y) \\ -(1-2c_1)(x\cdot y) + (\log|x|)|y|^2 + O(1)\end{aligned} \tag{5.4}$$

as $|x|\to\infty$ when $|y|\le b$, and therefore,

$$\begin{aligned}u(x) =&(8\pi)^{-1}((|x|^2\log|x|)S_0g - c_1|x|^2S_0g - 2(x\log|x|)\cdot(S_1g) \\ &-(1-2c_1)x\cdot(S_1g) + (\log|x|)S_2g + u_1(x)\end{aligned}$$

where $u_1(x)$ is the function which has the asymptotic behaviour:

$$u_1(x) = O(1),\ \nabla u_1(x) = O(|x|^{-1}),\ \nabla^2u(x) = O(|x|^{-2}),\ \nabla^3u(x) = O(|x|^{-3})$$

as $|x|\to\infty$, and S_j are the same operators as in (4.21). By (5.1) we have $u(x) = (\log|x|)(S_2g) + u_1(x)$, which implies (5.2).

By (5.1) we have

$$\theta(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2}(\log|x-y| - \log|x|)h(y)\,dy.$$

Since

$$\log|x-y| - \log|x| = \int_0^1 \frac{d}{d\theta}\log|x-\theta y|\,d\theta = \int_0^1 \frac{\sum_{i=1}^3(x_i-\theta y_i)y_i}{|x-\theta y|^2}\,d\theta,$$

we have

$$\log|x-y| - \log|x| = O(|x|^{-1}),\ \frac{\partial}{\partial x_k}(\log|x-y| - \log|x|) = O(|x|^{-2})\ \ (k=1,2)$$

as $|x|\to\infty$ when $|y|\le b$, and therefore we have (5.3). This completes the proof of the lemma. □

Lemma 5.2. *Let $1<p<\infty$. (1) If $\theta\in W^2_{p,\mathrm{loc}}(\overline{\Omega})$ satisfies the homogeneous equation:*

$$\Delta\theta = 0 \quad in\ \Omega, \quad \theta|_\Gamma = 0 \tag{5.5}$$

and the radiation condition:

$$\theta(x) = O(1) \tag{5.6}$$

as $|x|\to\infty$, then $\theta = 0$.
(2) *If $u\in W^4_{p,\mathrm{loc}}(\overline{\Omega})$ satisfies the homogeneous equation:*

$$\Delta^2u = 0\ \ in\ \Omega,\ \ u|_\Gamma = D_\nu u|_\Gamma = 0 \tag{5.7}$$

and the radiation condition:

$$u(x) = O(|x|) \tag{5.8}$$

as $|x|\to\infty$, then $u = 0$.

Proof. (1) By L_p $(1 < p < \infty)$ solvability in any C^2 bounded domain for the Dirichlet problem of the Laplace operator (cf. Simader [23]) and Sobolev's imbedding theorem, we see that $\theta \in W^2_{2,\mathrm{loc}}(\overline{\Omega})$. Let $\psi(t)$ be a function in $C_0^\infty(\mathbb{R})$ such that $\psi(t) = 1$ for $t \leq 1/2$ and $\psi(t) = 0$ for $t \geq 1$ and set $\rho_L(x) = \psi(\log(\log|x|)(\log(\log L))^{-1})$ for large L. Then, we have

$$0 = (\Delta\theta, \rho_L\theta)_\Omega = -(\nabla\theta, \rho_L\nabla\theta)_\Omega + (1/2)(\theta, (\Delta\rho_L)\theta)_\Omega \tag{5.9}$$

where $(a, b)_\Omega = \int_\Omega a(x)b(x)\,dx$. Since

$$|\Delta\rho_L(x)| \leq C(\log(\log L))^{-1}(\log|x|)^{-2}|x|^{-2} \ \ (L \to \infty)$$

and $\mathrm{supp}\,\Delta\rho_L \subset \{x \in \mathbb{R}^2 \mid e^{\sqrt{\log L}} \leq |x| \leq L\}$, by (5.6) we have

$$|(\theta, (\Delta\rho_L)\theta)_\Omega| \leq C(\log(\log L))^{-1}\int_{e^{\sqrt{\log L}}}^L (\log r)^{-2}r^{-1}\,dr \leq C(\log(\log L))^{-1}(\log L)^{-\frac{1}{2}} \to 0$$

as $L \to \infty$. Letting $L \to \infty$ in (5.9), we have $\|\nabla\theta\|^2_{L_2(\Omega)} = 0$, which implies that $\nabla\theta = 0$, that is θ is a constant. But, $\theta|_\Gamma = 0$, which means that $\theta = 0$.
(2) By L_p $(1 < p < \infty)$ solvability in any C^4 bounded domain for the Dirichlet problem of the biharmonic operator (cf. Simader [23]) and Sobolev's imbedding theorem, we see that $u \in W^4_{2,\mathrm{loc}}(\overline{\Omega})$. First, we shall show that $u = 0$, assuming that

$$u(x) = O(|x|),\ \nabla^2 u(x) = o(1) \tag{5.10}$$

as $|x| \to \infty$. Let ρ_L be the same function as in the proof of (1), and then we have

$$0 = (\Delta^2 u, \rho_L u)_\Omega = -(1/2)(u, (\Delta^2\rho_L)u)_\Omega + 2\sum_{j,k=1}^2 (u, (D_jD_k\rho_L)D_jD_ku)_\Omega + (\Delta u, \rho_L\Delta u)_\Omega. \tag{5.11}$$

Since

$$|\Delta^2\rho_L(x)| \leq C(\log(\log L))^{-1}(\log|x|)^{-2}|x|^{-4},\ \ |D_jD_k\rho_L(x)| \leq C(\log(\log L))^{-1}(\log|x|)^{-1}|x|^{-2}$$

as $L \to \infty$ and $\mathrm{supp}\,\Delta^2\rho_L$, $\mathrm{supp}\,D_jD_k\rho_L \subset \{x \in \mathbb{R}^2 \mid e^{\sqrt{\log L}} \leq |x| \leq L\}$, by (5.10) we have

$$|(u, (\Delta^2\rho_L)u)_\Omega| \leq C(\log(\log L))^{-1}\int_{e^{\sqrt{\log L}}}^L (\log r)^{-2}r^{-1}\,dr \leq C(\log(\log L))^{-1}(\log L)^{-\frac{1}{2}} \to 0,$$

$$\begin{aligned}|(u, (D_jD_k\rho_L)D_jD_ku)_\Omega| &\leq C\{\sup_{e^{\sqrt{\log L}} \leq |x| \leq L} |D_jD_ku(x)|\}(\log(\log L))^{-1}\int_{e^{\sqrt{\log L}}}^L (\log r)^{-1}r^{-1}\,dr \\ &\leq C\sup_{e^{\sqrt{\log L}} \leq |x| \leq L} |D_jD_ku(x)| \to 0\end{aligned}$$

as $L \to \infty$, letting $L \to \infty$ in (5.11) we have $\|\Delta u\|_{L_2(\Omega)} = 0$, which implies that $\Delta u = 0$ in Ω. Since $u|_\Gamma = D_\nu u|_\Gamma = 0$, the zero extension u_0 of u to the whole space $\mathbb{R}^2$ satisfies the Laplace equation: $\Delta u_0 = 0$ in $\mathbb{R}^2$. Since $u_0(x) = u(x) = O(|x|)$ as $|x| \to \infty$, from Lemma 3.1 we see that u_0 is a polynomial of degree 1. But, $u_0(x) = 0$ for $x \in \mathbb{R}^2 \setminus \Omega$, which means that $u_0 = 0$.

Finally, we shall show that the radiation condition (5.8) together with (5.7) implies that the radiation condition (5.10) holds. Let η be a function in $C^\infty(\mathbb{R}^2)$ such that $\eta(x) = 1$ for $|x| \geq b + 1$ and $\eta(x) = 0$ for $|x| \leq b$, where b is a large number such that $B_b \supset \mathbb{R}^3 \setminus \Omega$. Then, by (5.7) we have

$$\Delta^2(\eta u) = 0 \quad \text{in } \mathbb{R}^2, \tag{5.12}$$

where $f(x) = \Delta^2(\eta u) - \eta\Delta^2 u$. Since $\operatorname{supp} f \subset B_{b+1} \setminus B_b$, we have $f \in L_2(\mathbb{R}^2)$. Setting $v(x) = E_2^2 * f$, by (5.10) and the fact that E_2^2 is a fundamental solution to the biharmonic operator Δ^2, we have $\Delta^2(u - v) = 0 \quad \text{in } \mathbb{R}^2$. Employing the same argument as in the proof of Lemma 3.1, we have $u(x) - v(x) = \sum_{|\alpha|\le m} c_\alpha x^\alpha$ for some non-negative integer m and complex numbers c_α. If we write

$$v(x) = E_2^2(x)\int_{\mathbb{R}^2} f(y)\,dy + \int_{\mathbb{R}^2}(E_2^2(x-y) - E_2^2(x))f(y)\,dy,$$

we have

$$\sum_{|\alpha|\le m} c_\alpha x^\alpha - E_2^2(x)\int_{\mathbb{R}^2} f(y)\,dy = u(x) - \int_{\mathbb{R}^2}(E_2^2(x-y) - E_2^2(x))f(y)\,dy = O(|x|\log|x|)$$

as $|x| \to \infty$, which implies that

$$u(x) = \sum_{|\alpha|\le 1} c_\alpha x^\alpha + \int_{\mathbb{R}^2}(E_2^2(x-y) - E_2^2(x))f(y)\,dy.$$

Therefore, $\nabla^2 u(x) = o(1)$ as $|x| \to \infty$. This completes the proof of the lemma. □

Now, we shall show Theorem 1.3 in the two-dimensional case.

Proof of Theorem 1.3 for $n = 2$. Let s and $\mathcal{G}(\lambda)$ be the same as in Theorem 4.3. Let η be a function in $C^\infty(\mathbb{R}^2)$ such that $\eta(x) = 1$ for $|x| \ge b - 1$ and $\eta(x) = 0$ for $|x| \le b - 2$. Given $F \in \mathcal{H}_{p,b}(\Omega)$ and $\lambda \in \dot{\omega}_\tau$, we set $U(\lambda) = \mathcal{G}(\lambda)F$. We have $U(\lambda) = (\lambda I - \mathcal{A}_\Omega)^{-1}F \in \mathcal{D}_p(\Omega)$ for $\lambda \in \dot{\omega}_\tau \cap \mathcal{U}$ and $U(\lambda) = \mathcal{G}(\lambda)F \in \mathcal{D}_{p,\mathrm{loc}}(\Omega_b)$ for $\lambda \in \dot{\omega}_\tau$. Moreover, by (4.10) we have

$$(\lambda I - A)U(\lambda) = F \quad \text{in } \Omega, \quad BU(\lambda)|_\Gamma = 0, \quad (\lambda \in \dot{\omega}_\tau \cap \mathcal{U}). \tag{5.13}$$

Since $U(\lambda) \in \operatorname{Anal}(\dot{\omega}_\tau, \mathcal{D}_{p,\mathrm{loc}}(\Omega_b))$, it follows from (5.13) that

$$(\lambda I - A)U(\lambda) = F \quad \text{in } \Omega_b, \quad BU(\lambda)|_\Gamma = 0, \quad (\lambda \in \dot{\omega}_\tau). \tag{5.14}$$

From (5.13) it follows that $\eta U(\lambda)$ satisfies the equation:

$$(\lambda I - A)(\eta U(\lambda)) = \eta F + g(U(\lambda)) \quad \text{in } \mathbb{R}^2 \tag{5.15}$$

for $\lambda \in \dot{\omega}_\tau \cap \mathcal{U}$, where for $U = {}^T(u, v, \theta)$ we have set

$$g(U) = \begin{pmatrix} 0 \\ \Delta^2(\eta u) - \eta\Delta^2 u + \Delta(\eta\theta) - \eta\Delta\theta \\ -(\Delta(\eta\theta) - \eta\Delta\theta) - (\Delta(\eta v) - \eta\Delta v) \end{pmatrix}. \tag{5.16}$$

Note that $\operatorname{supp} g(U) \subset D_{b-2,b-1}$. Since $\Sigma_\epsilon \subset \rho(\mathcal{A}_{\mathbb{R}^2})$ as follows from Theorem 4.1, we have

$$\eta U(\lambda) = (\lambda I - \mathcal{A}_{\mathbb{R}^2})^{-1}(\eta F + g(U(\lambda))) \tag{5.17}$$

whenever $\lambda \in \dot{\omega}_\tau \cap \mathcal{U} \cap \Sigma_\epsilon$. Let $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{H}_1(\lambda)$ and $\mathcal{H}_2(\lambda)$ be the same operators as in (4.1) of Theorem 4.1 and let $\mathcal{H}(\lambda)$ be the same operator as in (4.8). By (5.17) and Theorem 4.1 we have

$$\eta U(\lambda) = \mathcal{H}(\lambda)(\eta F + g(U(\lambda))) \tag{5.18}$$

whenever $\lambda \in \dot\omega_\tau \cap \mathcal{U} \cap \Sigma_\epsilon$. But, both sides in (5.18) are analytic in $\dot\omega_\tau$, and therefore (5.18) holds for any $\lambda \in \dot\omega_\tau$.

In view of Theorem 4.3, we write

$$U(\lambda) = \lambda^s V_1(s) + \lambda^{s+1} V_2(s) + O(|\lambda|^{s+2}|\log\lambda|^\gamma) \quad (\lambda \to 0) \tag{5.19}$$

where s and γ are integers, $V_1(\lambda)$, $V_2(\lambda) \in \mathcal{D}_{p,\mathrm{loc}}(\Omega_b)$ and $\|V_j(\lambda)\|_{\mathcal{D}_{p,\mathrm{loc}}(\Omega_b)} \le C|\log\lambda|^{\gamma_j}\|F\|_{\mathcal{H}_p(\Omega)}$ for some integer γ_j $(j = 1, 2)$. We shall show that $s = 0$ by contradiction. Since

$$(\lambda I - A)U(\lambda) = F \quad \text{in } \Omega_b, \quad BU(\lambda)|_\Gamma = 0 \tag{5.20}$$

as follows from (5.14), we have

$$\lambda^s(-AV_1(\lambda)) + O(|\lambda^{s+1}(\log\lambda)^{\gamma_2}|) = F \ \text{ in } \Omega_b, \ \{\lambda^s BV_1(\lambda) + O(|\lambda^{s+1}(\log\lambda)^{\gamma_2}|)\}|_\Gamma = 0. \tag{5.21}$$

If $s > 0$, letting $\lambda \to 0$ in (5.21), we have $F = 0$, which leads to a contradiction. Therefore, we may assume that $s \le 0$. By contradiction, we shall prove that $s = 0$, so that we assume that s is a negative integer. Equating the term λ^s in (5.21), we have

$$-AV_1(\lambda) = 0 \quad \text{in } \Omega_b, \quad BV_1(\lambda)|_\Gamma = 0. \tag{5.22}$$

On the other hand, inserting the formula (5.19) into (5.18) and using Theorem 4.1 we have

$$\begin{aligned}&\eta\lambda^s V_1(\lambda) + O(|\lambda^{s+1}(\log\lambda)^{\gamma_2}|)\\ &= (\lambda\mathcal{E}_0 + \log\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + O(|\lambda\log\lambda|))(\eta F + \lambda^s g(V_1(\lambda)) + \lambda^{s+1}g(V_2(\lambda)) + O(|\lambda(\log\lambda)^\gamma|)).\end{aligned}$$

Equating the terms of λ^s, $\lambda^s\log\lambda$ and λ^{s-1}, we have

$$\eta V(\lambda) = \mathcal{E}_0(\eta F^1) + \mathcal{E}_0 g(V_2(\lambda)) + \mathcal{E}_2 g(V_1(\lambda)) + \mathcal{E}_3 g(V_1(\lambda)), \tag{5.23}$$

$$\mathcal{E}_0 g(V_1(\lambda)) = 0, \ \ \mathcal{E}_1 g(V_1(\lambda)) = 0, \tag{5.24}$$

where

$$F^1 = \begin{cases} F & \text{when } s = -1,\\ 0 & \text{when } s \le -2.\end{cases}$$

Since $\eta = 1$ for $|x| \ge b - 1$, we extend $V_1(\lambda)$ to the domain $B^b = \{x \in \mathbb{R}^2 \mid |x| > b\}$ by the formula:

$$V_1(\lambda) = \mathcal{E}_0(\eta F^1) + \mathcal{E}_0 g(V_2(\lambda)) + \mathcal{E}_2 g(V_1(\lambda)) + \mathcal{E}_3 g(V_1(\lambda)) \quad \text{in } B^b. \tag{5.25}$$

Set $V_1(\lambda) = {}^T(u, v, \theta)$, $\eta F^1 = {}^T(f_0, g_0, h_0)$, $g(V_1(\lambda)) = {}^T(0, g_1, h_1)$ and $g(V_2(\lambda)) = {}^T(0, g_2, h_2)$. Then, by Theorem 4.1 we have

$$\begin{aligned} u &= \alpha_2 S_0 g_0 + \alpha_3 S_0 h_0 + \alpha_2 S_0 g_2 + \alpha_3 S_0 h_2\\ &\quad + \frac{\beta_2}{16\pi}|x|^2 * g_1 + \frac{\beta_3}{16\pi}|x|^2 * h_1 + E_3^2 * (g_1 + h_1),\\ v &= \delta_2^2 S_0 g_1 + \delta_3^2 S_0 h_1,\\ \theta &= \delta_2^3 S_0 g_1 + \delta_3^3 S_0 h_1 + E_3^1 * h_1 \end{aligned} \tag{5.26}$$

for $|x| \ge b$, where $S_0 k = \int_{\mathbb{R}^2} k\,dx$ (cf. (4.21)). On the other hand, by (5.24) we have

$$\begin{aligned} &\alpha_2 S_0 g_1 + \alpha_3 S_0 h_1 = 0,\\ &|x|^2 * (g_1 + h_1) = 0 \quad \text{for } x \in \Omega_b,\\ &S_0 h_1 = 0. \end{aligned} \tag{5.27}$$

Since $|x|^2 * (g_1 + h_1) = |x|^2 S_0(g_1 + h_1) - 2x \cdot S_1(g_1 + h_1) + S_2(g_1 + h_1)$, $|x|^2 * (g_1 + h_1)$ is a polynomial of degree 2 and vanishes identically in Ω_b, so that we have

$$S_0(g_1 + h_1) = S_1(g_1 + h_1) = S_2(g_1 + h_1) = 0. \tag{5.28}$$

Since $S_0 h_1 = 0$, we have

$$S_0 g_1 = S_0 h_1 = 0. \tag{5.29}$$

Since

$$\frac{\beta_2}{16\pi}|x|^2 * g_1 + \frac{\beta_3}{16\pi}|x|^2 * h_1 = -\frac{\beta_2}{8\pi} x \cdot (S_1 g_1) - \frac{\beta_3}{8\pi} x \cdot (S_1 h_1) + \frac{\beta_2}{16\pi} S_2 g_1 + \frac{\beta_3}{16\pi} S_2 h_1$$

as follows from (5.29), from (5.26) and (5.29) we have

$$u = c_1(x) + E_2^2 * (g_1 + h_1), \;\; v = 0, \;\; \theta = E_2^1 * h_1 \tag{5.30}$$

for $x \in B^b$, where $c_1(x)$ is a constant coefficient polynomial of degree 1 which is given by the formula:

$$\begin{aligned} c_1(x) = & - x \cdot (\frac{\beta_2}{8\pi} S_1 g_1 + \frac{\beta_3}{8\pi} S_1 h_1) \\ & + \alpha_1 S_0 g_0 + \alpha_2 S_0 h_0 + \alpha_1 S_0 g_2 + \alpha_2 S_0 h_2 + \frac{\beta_2}{16\pi} S_2 g_1 + \frac{\beta_3}{16\pi} S_2 h_1. \end{aligned}$$

Noting that E_2^2 and E_2^1 are fundamental solutions of Δ^2 and $-\Delta$, respectively, we have

$$-AV_1(\lambda) = \begin{pmatrix} 0 \\ \Delta^2 u + \Delta\theta \\ -\Delta\theta \end{pmatrix} = \begin{pmatrix} 0 \\ g_1 \\ h_1 \end{pmatrix} = 0 \quad \text{in } B^b, \tag{5.31}$$

because $g_1 = h_1 = 0$ for $|x| > b - 1$. Combining (5.31) and (5.22) implies that

$$\begin{aligned} \Delta^2 u = 0 \quad & \text{in } \Omega, \quad u|_\Gamma = D_\nu u|_\Gamma = 0 \\ v = 0 \quad & \text{in } \Omega, \\ -\Delta\theta = 0 \quad & \text{in } \Omega, \quad \theta|_\Gamma = 0. \end{aligned} \tag{5.32}$$

Now, we shall show that $u = \theta = 0$ by using Lemmas 5.1 and 5.2. By (5.28), (5.29), (5.30) and Lemma 5.1 we have

$$\begin{gathered} u(x) = O(|x|), \;\; \nabla u(x) = O(1), \;\; \nabla^2 u(x) = O(|x|^{-2}), \;\; \nabla^3 u(x) = O(|x|^{-3}), \\ \theta(x) = O(|x|^{-1}), \;\; \nabla\theta(x) = O(|x|^{-2}) \end{gathered}$$

as $|x| \to \infty$, which combined with (5.32) and Lemma 5.2 implies that $u = \theta = 0$. Therefore, we have $V_1(\lambda) = 0$, which leads to a contradiction. Namely, we have shown that $s = 0$.

Now, in view of Theorem 4.3, we can write

$$U(\lambda) = (\log\lambda)^d V_1 + (\log\lambda)^{d-1} V_2 + O(|\log\lambda|^{d-2}) \tag{5.33}$$

as $\lambda \to 0$, where $V_j \in \mathcal{D}_{p,\mathrm{loc}}(\Omega_b)$ and $\|V_j\|_{\mathcal{D}_{p,\mathrm{loc}}(\Omega_b)} \leq C\|F\|_{\mathcal{H}_p(\Omega)}$ $(j = 1, 2)$. We may assume that $V_1 \neq 0$. Employing the contradiction argument again, we shall show that $d = 0$. From (5.14) we have

$$(\log\lambda)^d(-AV_1) + O(|\log\lambda|^{d-1}) = F \;\text{ in } \Omega_b, \;\; \{(\log\lambda)^d BV_1 + O(|\log\lambda|^{d-1})\}|_\Gamma = 0. \tag{5.34}$$

If $d < 0$, then letting $\lambda \to 0$ in (5.34), we have $F = 0$, which leads to a contradiction. Therefore, we may assume that $d \geq 0$. Assume that d is a positive integer. Multiplying (5.34) by $(\log\lambda)^{-d}$ and letting $\lambda \to 0$, we have

$$-AV_1 = 0 \quad \text{in } \Omega_b, \quad BV_1|_\Gamma = 0. \tag{5.35}$$

On the other hand, inserting the formula (5.33) into (5.19) and using Theorem 4.1, we have

$$\begin{aligned}
&\eta(\log\lambda)^d V_1 + O(|\log\lambda|^{d-1})\\
&\quad = (\lambda^{-1}\mathcal{E}_0 + \log\lambda\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + O(|\lambda\log\lambda|))\\
&\qquad (\eta F + (\log\lambda)^d g(V_1) + (\log\lambda)^{d-1} g(V_2) + O(|\log\lambda|^{d-2}))\\
&\quad = \lambda^{-1}(\mathcal{E}_0(\eta F) + (\log\lambda)^d\mathcal{E}_0 g(V_1) + (\log\lambda)^{d-1}\mathcal{E}_0 g(V_2) + O(|\log\lambda)|^{d-2}))\\
&\qquad + \log\lambda\mathcal{E}_1(\eta F) + (\log\lambda)^{d+1}\mathcal{E}_1 g(V_1) + (\log\lambda)^d\mathcal{E}_1 g(V_2) + (\log\lambda)^d\mathcal{E}_2 g(V_1)\\
&\qquad + (\log\lambda)^d\mathcal{E}_3 g(V_1) + O(|\log\lambda|^{d-1}).
\end{aligned}$$

Equating the terms of λ^{-1}, $\lambda^{-1}(\log\lambda)^d$, $\lambda^{-1}(\log\lambda)^{d-1}$, $(\log\lambda)^{d+1}$ and $(\log\lambda)^d$, we have

$$\mathcal{E}_0 g(V_1) = \mathcal{E}_0(\eta F_1 + g(V_2)) = \mathcal{E}_1 g(V_1) = 0, \tag{5.36}$$

$$\eta V_1 = \mathcal{E}_1(\eta F_1 + g(V_2)) + \mathcal{E}_2 g(V_1) + \mathcal{E}_3 g(V_1), \tag{5.37}$$

where

$$F_1 = \begin{cases} 0 & \text{when } d \geq 2,\\ F & \text{when } d \geq 1. \end{cases}$$

Note that now $\mathcal{E}_1$ appears and $\mathcal{E}_0$ disappears in (5.37), while $\mathcal{E}_1$ disappears and $\mathcal{E}_0$ appears in (5.25). Again we set $V_1 = {}^T(u, v, \theta)$, $\eta F_1 = (f_0, g_0, h_0)$, $g(V_1) = {}^T(0, g_1, h_1)$ and $g(V_2) = {}^T(0, g_2, h_2)$. By Theorem 4.1 and (5.37), we have

$$\begin{aligned}
u &= \frac{1}{16\pi}|x|^2 * (-\Delta(\eta f_0) + \eta g_0 + g_2 + \eta h_0 + h_2) + \frac{\beta_2}{16\pi}|x|^2 * g_1\\
&\quad + \frac{\beta_3}{16\pi}|x|^2 * h_1 + E_2^2 * (g_1 + h_1),\\
v &= \delta_2^2 S_0 g_1 + \delta_3^2 S_0 h_1,\\
\theta &= -\frac{1}{4\pi}S_0(\eta h_0 + h_2) + \delta_2^3 S_0 g_1 + \delta_3^3 S_0 h_1 + E_2^1 * h_1
\end{aligned} \tag{5.38}$$

for $x \in B^b$. By (5.36) and (4.2) we have

$$\begin{aligned}
&\alpha_2 S_0 g_1 + \alpha_3 S_0 h_1 = 0,\\
&\alpha_2 S_0(\eta g_0 + g_2) + \alpha_3 S_0(\eta h_0 + h_2) = 0,\\
&|x|^2 * (g_1 + h_1) = 0 \quad (x \in \Omega_b),\\
&S_0 h_1 = 0.
\end{aligned} \tag{5.39}$$

The first and last formulas in (5.39) implies that

$$S_0 g_1 = S_0 h_1 = 0. \tag{5.40}$$

Moreover, the third formula in (5.39) implies that

$$S_0(g_1 + h_1) = S_1(g_1 + h_1) = S_2(g_1 + h_1) = 0. \tag{5.41}$$

By (5.38) and (5.40) we have $v = 0$ for $x \in B^b$, which combined with (5.35) implies that

$$v = 0 \quad \text{in } \Omega. \tag{5.42}$$

Since $\Delta^2|x|^2 = 0$, and $S_0(\eta h_0 + h_2)$, $S_0 g_1$ and $S_0 h_1$ are constants, and since E_2^2 and E_2^1 are fundamental solutions of Δ^2 and $-\Delta$, respectively, from (5.38) we have

$$\Delta^2 u = g_1 + h_1 = 0, \ \ -\Delta\theta = h_1 = 0 \tag{5.43}$$

for $x \in B^b$, because $g_1 = h_1 = 0$ for $|x| \geq b - 1$. Combining (5.43) with (5.35) implies that

$$\begin{aligned} \Delta^2 u =& 0 \quad \text{in } \Omega, \quad u|_\Gamma = D_\nu u|_\Gamma = 0, & (5.44)\\ -\Delta\theta =& 0 \quad \text{in } \Omega, \quad \theta|_\Gamma = 0. & (5.45) \end{aligned}$$

Since $S_0 h_1 = 0$, by Lemma 5.1 we have $\theta(x) = (|x|^{-1})$ as $|x| \to \infty$, which combined with (5.45) and Lemma 5.2 implies that $\theta = 0$. Since

$$\theta = -\frac{1}{4\pi} S_0(\eta h_0 + h_2) - \frac{1}{2\pi} \int_{\mathbb{R}^2} (E_2^1(x-y) - E_2^1(x)) h_1(y)\, dy$$

as $|x| \to \infty$ as follows from the third formula in (5.38) and (5.40), we have

$$S_0(\eta h_0 + h_2) = 0, \tag{5.46}$$

because $\int_{\mathbb{R}^2} (E_2^1(x-y) - E_2^1(x)) h_1(y)\, dy = O(|x|^{-1})$ as $|x| \to \infty$. Combining (5.46) and the second formula of (5.39), we have

$$S_0(\eta g_0 + g_2) = S_0(\eta h_0 + h_2) = 0. \tag{5.47}$$

From the first formula of (5.38), we have $u = c_0 + c_1 + u_0$, where we have set

$$\begin{aligned} u_0 &= E_2^2 * (g_1 + h_1)\\ c_0 &= \frac{|x|^2}{16\pi}(S_0(-\Delta(\eta f_0)) + S_0(\eta g_0 + g_2) + S_0(\eta h_0 + h_2))\\ c_1 &= -\frac{x}{8\pi} \cdot (S_1(-\Delta(\eta f_0)) + S_1(\eta g_0 + g_2) + S_1(\eta h_0 + h_2) + \beta_2 S_1 g_1 + \beta_3 S_1 h_1)\\ &\quad + S_2(-\Delta(\eta f_0)) + S_2(\eta g_0 + g_2) + S_2(\eta h_0 + h_2) + \beta_2 S_2 g_1 + \beta_3 S_2 h_1 \end{aligned}$$

By (5.41) and Lemma 5.1 we have

$$u_0(x) = O(\log|x|), \ \ \nabla u_0(x) = O(|x|^{-1}), \ \ \nabla^2 u_0(x) = O(|x|^{-2}) \tag{5.48}$$

as $|x| \to \infty$. Noting that $S_0(-\Delta(\eta f_0)) = 0$ as follows from the divergence theorem of Gauss, by (5.47) we have $c_0 = 0$. Since c_1 is a polynomial of degree 1, by (5.48) we have $u(x) = O(|x|)$ as $|x| \to \infty$, which combined with (5.44) and Lemma 5.2 implies that $u = 0$. Therefore, we have $V_1 = 0$, which leads to a contradiction, and then we have $d = 0$. This completes the proof of Theorem 1.3 for $n = 2$. □

Proof of Theorem 1.4 for $n = 2$. Let τ, G_1, G_2 and $G_3(\lambda)$ be the same as in Theorem 1.3. And, let $\mathcal{U}$ be the same as in (1.14). Let $\Gamma = \Gamma_+ \cup \Gamma_0 \cup \Gamma_-$ be a path in $\mathbb{C}$ defined by the formulas:

$$\begin{aligned} &\Gamma_+ : \lambda = s e^{i(\pi-\theta)}, && s : \infty \to (\tau/2)(\cos\theta)^{-1},\\ &\Gamma_0 \ : \lambda = (\tau/2)(\cos\theta)^{-1} e^{is}, && s : \pi - \theta \to -(\pi - \theta),\\ &\Gamma_- : \lambda = s e^{-i(\pi-\theta)}, && s : (\tau/2)(\cos\theta)^{-1} \to \infty, \end{aligned}$$

where $\theta \in (0, \pi/2)$ is chosen so close to $\pi/2$ that $\Gamma \subset \mathcal{U}$. By (1.11) and (1.13) we have

$$T(t)F = \frac{1}{2\pi i}\int_\Gamma (\lambda I - \mathcal{A}_\Omega)^{-1}F\,d\lambda.$$

To estimate $T(t)F$, let us set

$$I_\pm = \frac{1}{2\pi i}\int_{\Gamma_\pm} (\lambda I - \mathcal{A}_\Omega)^{-1}F\,d\lambda,$$
$$I_0 = \frac{1}{2\pi i}\int_{\Gamma_0} (\lambda I - \mathcal{A}_\Omega)^{-1}F\,d\lambda.$$

By (1.13) we have

$$\|I_\pm(t)\|_{\mathcal{D}_p(\Omega)} \leq C\int_{(\tau/2)(\cos\theta)^{-1}}^\infty e^{(s\cos(\pi-\theta))t}\,ds\|F\|_{\mathcal{H}_p(\Omega)} = \frac{C}{(\cos\theta)t}e^{-(\tau/2)t}\|F\|_{\mathcal{H}_p(\Omega)}$$

for any $t > 0$ and $F \in \mathcal{H}_p(\Omega)$. To estimate $I_0(t)$, we restrict ourselves to the case where $F \in \mathcal{H}_{p,b}(\Omega)$. Let $C = C_1 \cup C_+ \cup C_- \cup C_2$ be a path defined by the formulas:

$$\begin{aligned}
&C_1\ : \lambda = -(\tau/2) + s, && s : (\tau/2)\tan\theta \to 0,\\
&C_+ : \lambda = e^{\pi i}s, && s : \tau/2 \to 0,\\
&C_- : \lambda = e^{-\pi i}s, && s : 0 \to \tau/2,\\
&C_2\ : \lambda = -(\tau/2) + s, && s : 0 \to -(\tau/2)\tan\theta.
\end{aligned}$$

Then, by (1.17) in Theorem 1.3 we have

$$I_0(t) = \frac{1}{2\pi i}\Big\{\int_{C_1} + \int_{C_+} + \int_{C_-} + \int_{C_2}\Big\}e^{\lambda t}(G_1F + (\log\lambda)^{-1}G_2F + G_3(\lambda)F)\,d\lambda$$

in Ω_b for any $t > 0$. Setting

$$J_0(t) = \frac{1}{2\pi i}\Big\{\int_{C_1} + \int_{C_2}\Big\}e^{\lambda t}(G_1F + (\log\lambda)^{-1}G_2F + G_3(\lambda)F)\,d\lambda,$$

we have

$$\|J_0(t)\|_{\mathcal{D}_{p,\mathrm{loc}}(\Omega_b)} \leq Ce^{-\tau/2)t}\int_0^{(\tau/2)\tan\theta} ds\|F\|_{\mathcal{H}_p(\Omega)} \leq C(\tau/2)(\tan\theta)e^{-(\tau/2)t}\|F\|_{\mathcal{H}_p(\Omega)}.$$

Obviously, $\Big\{\int_{C_+} + \int_{C_-}\Big\}e^{\lambda t}G_1F\,d\lambda = 0$. Setting

$$J_1(t) = \frac{1}{2\pi i}\Big\{\int_{C_+} + \int_{C_-}\Big\}e^{\lambda t}(\log\lambda)^{-1}\,d\lambda\,G_2F,$$

we observe that

$$\begin{aligned}
J_1(t) &= \frac{1}{2\pi i}\int_{\tau/2}^0 (\log se^{i\pi})^{-1}e^{-st}e^{i\pi}\,ds\,G_2F + \frac{1}{2\pi i}\int_0^{\tau/2}(\log se^{-i\pi})^{-1}e^{-st}e^{-i\pi}\,ds\,G_2F\\
&= \frac{1}{2\pi i}\int_0^{\tau/2}\Big(\frac{1}{\log s + i\pi} - \frac{1}{\log s - i\pi}\Big)e^{-st}\,ds\,G_2F = -\int_0^{\tau/2}\frac{e^{-st}}{(\log s)^2 + \pi^2}\,ds\,G_2F.
\end{aligned}$$

Therefore, for $t \geq 1$ we have

$$\begin{aligned}
\|J_1(t)\|_{\mathcal{D}_{p,\mathrm{loc}}(\Omega_b)} &\leq C\int_0^\infty \frac{e^{-st}}{(\log s)^2+\pi^2}\,ds\|F\|_{\mathcal{H}_p(\Omega)} \\
&= Ct^{-1}\int_0^\infty \frac{e^{-\ell}}{(\log t-\log \ell)^2+\pi^2}\,d\ell\,\|F\|_{\mathcal{H}_p(\Omega)} \\
&\leq Ct^{-1}\Big\{\int_0^{\sqrt{t}} \frac{e^{-\ell}}{(\log t-\log \ell)^2}\,d\ell + \frac{1}{\pi^2}\int_{\sqrt{t}}^\infty e^{-\ell}\,d\ell\Big\}\,\|F\|_{\mathcal{H}_p(\Omega)} \\
&\leq Ct^{-1}\Big\{4(\log t)^{-2}\int_0^\infty e^{-\ell}\,d\ell + \frac{e^{-\sqrt{t}/2}}{\pi^2}\int_0^\infty e^{-\ell/2}\,d\ell\Big\}\,\|F\|_{\mathcal{H}_p(\Omega)} \\
&\leq Ct^{-1}(\log t)^{-2}\,\|F\|_{\mathcal{H}_p(\Omega)}.
\end{aligned}$$

Finally, setting

$$J_2(t) = \frac{1}{2\pi i}\Big\{\int_{C_+} + \int_{C_-}\Big\}e^{\lambda t}G_3(\lambda)F\,d\lambda,$$

by (1.17) in Theorem 1.3 we have

$$\|J_2(t)\|_{\mathcal{D}_{p,\mathrm{loc}}(\Omega_b)} \leq C\int_0^{\tau/2} \frac{e^{-st}}{(\log s)^2+\pi^2}\,ds\|F\|_{\mathcal{H}_p(\Omega)},$$

and therefore employing the same argument as in the estimate of $J_1(t)$ we have

$$\|J_2(t)\|_{\mathcal{D}_{p,\mathrm{loc}}(\Omega_b)} \leq Ct^{-1}(\log t)^{-2}\|F\|_{\mathcal{H}_p(\Omega)}$$

for $t \geq 1$. Combining these estimations, we have Theorem 1.4 for $n = 2$. □

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