Decay rates and global existence for semilinear dissipative Timoshenko systems*

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Abstract

The main goal of this paper is to prove optimal decay estimates for the dissipative Timoshenko system in the one-dimensional whole space, and to prove a global existence theorem for semilinear systems. More precisely, if we restrict the initial data $((\varphi_0,\psi_0),(\varphi_1,\psi_1)) \in \left(H^{s+1}\left(\mathbb{R}^N\right)\cap L^{1,\gamma}\left(\mathbb{R}^N\right)\right)\times \left(H^{s}\left(\mathbb{R}^N\right)\cap L^{1,\gamma}\left(\mathbb{R}^N\right)\right)$ with $\gamma\in[0,1]$, then we can derive faster decay estimates than those given in [8]. Then, we use these decay estimates of the linear problem combined with the weighted energy method introduced by Todorova and Yordanov [35] with the special weight given in [11], to tackle a semilinear problem.

1 Introduction

In this paper, we are concerned with the one dimensional Timoshenko system in the whole space $\mathbb{R}$. Namely, we consider

$$\begin{cases} \varphi_{tt}(t,x)-(\varphi_x-\psi)_x(t,x)=0, & (t,x)\in\mathbb{R}^+\times\mathbb{R}, \\ \psi_{tt}(t,x)-a^2\psi_{xx}(t,x)-(\varphi_x-\psi)(t,x)+\mu\psi_t(t,x)=f(\psi(t,x)), & (t,x)\in\mathbb{R}^+\times\mathbb{R}, \\ (\varphi,\varphi_t,\psi,\psi_t)(0,x)=(\varphi_0,\varphi_1,\psi_0,\psi_1), & x\in\mathbb{R}, \end{cases} \tag{1.1}$$

where t denotes the time variable and x is the space variable, the function φ and ψ are the displacement and the rotation angle of the beam respectively, a and μ are positive constants and $f(\psi(t,x))=|\psi(t,x)|^p$ with $p>1$.

Before stating and proving our results, let us recall some other results related to our work.

A Timoshenko system goes back to Timoshenko [34] in 1921 who proposed a coupled hyperbolic system which is similar to (1.1), describing the transverse vibration of a beam, but without the presence of any

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damping. More precisely, he introduced the following system

$$\begin{cases} \rho\varphi_{tt} = (K(\varphi_x - \psi))_x, & \text{in } (0,L)\times(0,+\infty) \\ I_\rho \psi_{tt} = (EI\psi_x)_x + K(\varphi_x - \psi)), & \text{in } (0,L)\times(0,+\infty), \end{cases} \tag{1.2}$$

where t denotes the time variable, x is the space coordinate along the beam of length L, in its equilibrium configuration. The function $\varphi = \varphi(t,x)$ is the transverse displacement of the beam from an equilibrium state and $\psi = \psi(t,x)$ is the rotation angle of the filament of the beam. The coefficients ρ, I_ρ, E, I and K are respectively the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus. For a physical derivation of Timoshenko's system, we refer the reader to [5].

System (1.2), together with boundary conditions of the form

$$EI\psi_x \mid_{x=0}^{x=L} = 0, \qquad K(\varphi_x - \psi) \mid_{x=0}^{x=L} = 0$$

is conservative, and so the total energy of the beam remains constant along the time.

The subject of stability of Timoshenko-type systems has received a lot of attention in the last years, and quite a number of results concerning uniform and asymptotic decay of energy have been established.

An important issue of research is to look for a minimum dissipation by which solutions of system (1.2) decay uniformly to zero as time goes to infinity. In this regard, several types of dissipative mechanisms have been introduced, such as dissipative mechanism of frictional type, of viscoelastic type and thermal dissipation.

System (1.2) together with two boundary controls of the form

$$\begin{aligned} &K\psi(L,t) - K\varphi_x(L,t) = \alpha\varphi_t(L,t), && \forall t \geq 0, \\ &EI\psi_x(L,t) = -\beta\psi_t(L,t), && \forall t \geq 0, \end{aligned} \tag{1.3}$$

has been considered in [14]. The authors used the multiplier techniques to establish an exponential decay result for the total energy of (1.2)-(1.3). They also provided numerical estimates to the eigenvalues of the operator associated with system (1.2)-(1.3).

Subsequently, extensive attention was paid to the problem of obtaining an explicit decay rate of system (1.2).

Soufyane and Wehbe [33] showed that it is possible to stabilize uniformly (1.2) by using a unique locally distributed feedback of the form $b(x)\psi_t$ in the left hand side of the second equation in (1.2), where b is a positive and continuous function, which satisfies

$$b(x) \geq b_0 > 0, \quad \forall\, x \in [a_0, a_1] \subset [0,L]$$

and proved that the uniform stability holds if and only if the wave speeds are equal, that is $\frac{K}{\rho} = \frac{EI}{I_\rho}$[1]. Otherwise only the asymptotic stability has been proved.

Muñoz Rivera and Racke [26] obtained a similar result in a work where the damping function $b = b(x)$ is allowed to change its sign. Also, Muñoz Rivera and Racke [25] treated a nonlinear Timoshenko-type system of the form

$$\begin{cases} \rho_1\varphi_{tt} - \sigma_1(\varphi_x, \psi)_x = 0, \\ \rho_2\psi_{tt} - \chi(\psi_x)_x + \sigma_2(\varphi_x, \psi) + d\psi_t = 0 \end{cases}$$

[1] This condition is significant only from the mathematical point of view since in practice the velocities of waves propagations are always different, see [15].

in a bounded interval. The dissipation is produced here through the frictional damping $d\psi_t$, $d > 0$ which is only present in the equation for the rotation angle. The authors gave an alternative proof for a necessary and sufficient condition for exponential stability in the linear case and then proved a polynomial stability in general. Moreover, they investigated the global existence of small smooth solutions and exponential stability in the nonlinear case.

Ammar-Khodja *et al.* [2] considered a linear Timoshenko-type system with memory of the form

$$\begin{aligned} &\rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ &\rho_2\psi_{tt} - b\psi_{xx} + \int_0^t g(t-s)\psi_{xx}(s)ds + K(\varphi_x + \psi) = 0 \end{aligned} \tag{1.4}$$

in $(0,L) \times (0,+\infty)$, together with homogeneous boundary conditions. They used the multiplier techniques and proved that the system (1.4) is uniformly stable if and only if the wave speeds are equal

$$\frac{K}{\rho_1} = \frac{b}{\rho_2} \tag{1.5}$$

and g decays uniformly. Precisely, they proved an exponential decay if g decays in an exponential rate and polynomially if g decays in a polynomial rate. They also required some extra technical conditions on both g' and g'' to obtain their result. Guesmia and Messaoudi [6] proved the same result without imposing the extra technical conditions of [2]. Recently, Messaoudi and Mustafa [17] improved the results of [2] by allowing more general decaying relaxation functions and showed that the decay rate of the solution energy is exactly the rate of decay of the relaxation function. Alabau-Boussouira [1] considered the following system

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, & \text{in } (0,L) \times (0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \alpha(\psi_t) = 0, & \text{in } (0,L) \times (0,+\infty), \end{cases} \tag{1.6}$$

associated with two different types of boundary conditions. Under no growth assumption on the nonlinear function α near the origin, the author established a semi-explicit formula for the decay of the energy in the case of equal wave speeds. In the case of different wave speeds, a polynomial decay has been established for both linear and nonlinear globally Lipschitz feedbacks. System (1.6), with $\alpha(t)g(\psi_t)$ instead of $\alpha(\psi_t)$, has been considered by Messaoudi and Mustafa [18]. An explicit formula for the decay rate, depending on α and g, has been given under no growth condition on g at the origin. Also, Muñoz Rivera and Fernández Sare [27], considered Timoshenko type system with past history acting only in one equation. More precisely they looked at the following problem:

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \int_0^{+\infty} g(t)\psi_{xx}(t-s,.)ds + K(\varphi_x + \psi) = 0, \end{cases} \tag{1.7}$$

together with homogenous boundary conditions, and showed that the dissipation given by the history term is strong enough to stabilize the system exponentially if and only if the wave speeds are equal. They also proved that the solution decays polynomially for the case of different wave speeds. This work has been improved recently by Messaoudi and Said-Houari [22], where the authors considered system (1.7) for g decaying polynomially, and proved polynomial stability results for the equal and nonequal wave-speed propagation under some conditions on the relaxation function weaker than those in [27].

Very recently, Said-Houari and Laskri [29] have considered the following Timoshenko system with a delay term in the feedback

$$\begin{cases} \rho_1 \varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) = 0, \\ \rho_2 \psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x + \psi)(x,t) + \mu_1 \psi_t(x,t) + \mu_2 \psi_t(x,t-\tau) = 0. \end{cases} \tag{1.8}$$

Under the assumption $\mu_1 \geq \mu_2$ on the weights of the two feedbacks, they proved the well-posedness of the system. They also established for $\mu_1 > \mu_2$ an exponential decay result for the case of equal-speed wave propagation.

For similar problems dealing with the stability theory of the Timoshenko systems with thermal dissipation, we refer to [19, 20, 21, 24, 30].

All the above papers treated the Timoshenko systems in a bounded domain in which the Poincaré inequality and the type of the boundary conditions play a decisive role. But in the whole space $\mathbb{R}$ there are almost no results, to our knowledge, except the two papers of Kawashima and his collaborators in [8] and [9].

In [8], Ide, Haramoto and Kawashima investigated problem (1.1) with $f = 0$ and proved that if $a = 1$, then the solution of (1.1) decays like:

$$\left\|\partial_x^k U(t)\right\|_2 \leq C(1+t)^{-1/4-k/2} \|U_0\|_1 + Ce^{-ct}\left\|\partial_x^k U_0\right\|_2, \tag{1.9}$$

where $U = (\varphi_x - \psi, \varphi_t, a\psi_x, \psi_t)^T$. While if $a \neq 1$, then system (1.1) is of regularity-loss type and the solutions decay as:

$$\left\|\partial_x^k U(t)\right\|_2 \leq C(1+t)^{-1/4-k/2} \|U_0\|_1 + C(1+t)^{-l/2}\left\|\partial_x^{k+l} U_0\right\|_2, \tag{1.10}$$

where the parameters k and l in (1.9) and (1.10) are non-negative integers, and C and c are positive constants. The work in [8] was followed by [9] where Ide and Kawashima generalized the above decay results to a nonlinear version of the form

$$\begin{cases} \varphi_{tt}(t,x) - (\varphi_x - \psi)_x(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \psi_{tt}(t,x) - \sigma(\psi_x)_x(t,x) - (\varphi_x - \psi)(t,x) + \mu\psi_t(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ (\varphi, \varphi_t, \psi, \psi_t)(0,x) = (\varphi_0, \varphi_1, \psi_0, \psi_1), & x \in \mathbb{R}, \end{cases} \tag{1.11}$$

where $\sigma(\eta)$ is a smooth function of η such that $\sigma(\eta) > 0$. In fact, they showed the existence of global solutions and the asymptotic decay of these solutions under the smallness condition on the initial data in $H^s \cap L^1$ with suitably large s. In both papers [8] and [9] the authors have found the diffusion phenomenon of systems (1.10) and (1.11). In other words, they showed that the solutions approach the diffusion wave expressed in terms of the superposition of the heat kernels as time tends to infinity.

The purpose of this paper is twofold:

- First, we extend the decay results obtained in [8]. In fact, by restricting ourselves to initial data $U_0 \in H^s(\mathbb{R}) \cap L^{1,\gamma}(\mathbb{R})$ with a suitably large s and $\gamma \in [0,1]$, then we can derive faster decay estimates than those given in [8]. Indeed, by transforming our problem in the Fourier space, using the pointwise estimates derived in [8] and adapting the devise introduced by Ikehata in [10], to treat the Fourier transform in the low frequency region, we succeed to improve the decay rate given in [8] by $t^{-\gamma/2}$, $\gamma \in [0,1]$ especially in the case of equal wave speeds, i.e. $a = 1$. Also, for $a \neq 1$, a refinement of the decay estimates is given which improves the decay rate in [8, Theorem 5.1, Corollary 5.1]. (See Theorem 4.2 below). Moreover, we give a more general proof for the large time approximation given in [8, Theorem 5.2].

- Second, we analyze the asymptotic behavior of the semilinear problem (1.1) with the power type nonlinearity $|u|^p$ satisfying
$$p > 12. \tag{1.12}$$
Here, we use the decay estimates obtained for the linear problem combined with the weighted energy method introduced by Todorova and Yordanov [35] with the special weight given in [11] to obtain the small data global existence and some optimal decay estimates for the semilinear problem. A restriction like (1.12) seems to be justified since the damping is acting only on the second equation of (1.1); see Remark 6.5 for more details. We recall that our result has been proved without assuming the compactness assumption of the support on the initial data.

The rest of the paper is organized as follows. In section 2 we introduce some notations and some useful tools that we will use throughout this paper. Section 3 is devoted to the analysis of the asymptotic behavior of the linear hyperbolic system (3.1), the main result of this section is Theorem 3.1, in which we have proved better decay estimates than those given in [8]. Since in the case where $a \neq 1$ and as it was shown in [8] our system (3.1) is of regularity loss type. Therefore, the goal of section 4, is to give a refinement of the decay estimates in the case $a \neq 1$. Still our estimate in this section better than those proved in [8]. In section 5, we prove the asymptotic profile of the solution of our problem (3.1) as t tends to infinity. In fact we show that the solution of system (3.1) behaves asymptotically like the one of the parabolic system (5.1). Our proof is more general than the one given [8] and [9], including all the values of $\gamma \in [0,1]$. We also extend the result obtained by Ikehata [10] for the hyperbolic wave equation to some parabolic systems (Lemma 5.1), to our knowledge this result is new. In section 6 we investigate the semilinear problem (6.1). More precisely, in subsection 6.1, by combining the semigroup approach with the fixed point theorem and using some weighted estimates, we show that our system is well-posed. Furthermore in subsection 6.2 we investigate the global existence and the asymptotic behavior of the semilinear problem (6.1). Our result is carried out by making use of our estimates for the linear problem in section 3 and the Todorova-Yordanov weighted energy method with the a special weight. The result of this subsection (Theorem 6.4) shows that for small initial data, the solution of the semilinear problem decays with the same rate as the one of the linear problem. As far as we know, this is the first result dealing with this type of nonlinearity in the Timoshenko systems. Finally, in section 7, we conclude by making some comments.

2 Preliminaries

In this section, we introduce some notations and some technical lemmas to be used throughout this paper.

Throughout this paper, $\|.\|_q$ and $\|.\|_{H^l}$ stand for the $L^q(\mathbb{R})$-norm $(1 \leq q \leq \infty)$ and the $H^l(\mathbb{R})$-norm and some times for $L^q(\mathbb{R}^+)$-norm and the $H^l(\mathbb{R}^+)$-norm, respectively. Also, for $\gamma \in [0,+\infty)$, we define the weighted function space $L^{1,\gamma}(\mathbb{R})$ as follows: $u \in L^{1,\gamma}(\mathbb{R})$ iff $u \in L^1(\mathbb{R})$ and

$$\|u\|_{1,\gamma} = \int_{\mathbb{R}} (1+|x|)^{\gamma} |u(x)| dx < +\infty.$$

Similarly, we can define the space $L^{1,\gamma}(\mathbb{R}^+)$.

Let us also denote by $\hat{f} = \mathscr{F}(f)$ the Fourier transform of f with inverse $\mathscr{F}^{-1}$:

$$\hat{f}(\xi) = \mathscr{F}(f)(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x} dx,$$

Next, we introduce the following interpolation inequality which will be used in this paper.

Lemma 2.1 *([23]) Let $N \geq 1$. Let $1 \leq p, q, r \leq \infty$, and let k be a positive integer. Then for any integer j with $0 \leq j \leq k$, we have*

$$\left\|\partial_x^j u\right\|_{L^p} \leq C \left\|\partial_x^k u\right\|_{L^q}^a \|u\|_{L^r}^{1-a} \tag{2.1}$$

where

$$\frac{1}{p} = \frac{j}{N} + a\left(\frac{1}{q} - \frac{k}{N}\right) + (1-a)\frac{1}{r}$$

for a satisfying $j/k \leq a \leq 1$ and C is a positive constant; there are the following exceptional cases:

1. *If $j = 0$, $qk < N$ and $r = \infty$, then we made the additional assumption that either $u(x) \to 0$ as $|x| \to \infty$ or $u \in L^{q'}$ for some $0 < q' < \infty$.*
2. *If $1 < r < \infty$ and $k - j - N/r$ is nonnegative integer, then (2.1) holds only for $j/k \leq a < 1$.*

Furthermore, we introduce the following lemma, which can be found, for example in [16, 31].

Lemma 2.2 *Let $a > 0$ and $b > 0$ be constants. If $\max(a,b) > 1$, then*

$$\int_0^t (1+t-s)^{-a}(1+s)^{-b}\,ds \leq C(1+t)^{-\min(a,b)}. \tag{2.2}$$

If $\max(a,b) = 1$, then

$$\int_0^t (1+t-s)^{-a}(1+s)^{-b}\,ds \leq C(1+t)^{-\min(a,b)}\ln(2+t). \tag{2.3}$$

If $\max(a,b) < 1$, then

$$\int_0^t (1+t-s)^{-a}(1+s)^{-b}\,ds \leq C(1+t)^{1-a-b}. \tag{2.4}$$

3 Decay estimates

Our goal now is to write system (1.1) as a first-order system of the form

$$\begin{cases} U_t + AU_x + LU = 0, \\ U(x,0) = U_0, \end{cases} \tag{3.1}$$

where A is a real symmetric matrix and L is non-negative definite matrix. To this end, we introduce the following variables:

$$v = \varphi_x - \psi,\ u = \varphi_t,\ z = a\psi_x,\ y = \psi_t.$$

Consequently, system (1.1) can be rewritten as the following first-order hyperbolic system (see [8])

$$U = \begin{pmatrix} v \\ u \\ z \\ y \end{pmatrix},\quad A = -\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & a & 0 \end{pmatrix},\quad L = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \mu \end{pmatrix}. \tag{3.2}$$

and $U_0 = (v_0, u_0, z_0, y_0)^T$. It is clear that A is real symmetric and since $U^T LU = \mu y^2 > 0$ for all non-zero vector U, then L is positive definite, but it is not real symmetric.

System (3.1) can be seen as a particular case of a general hyperbolic system of balance laws. We point out that Shizuta and Kawashima [32] have introduced the so-called algebraic condition (SK), namely

(SK) $\qquad Ker(L) \cap \{\text{eigenvectors of} A\} = \{0\},$

which is satisfied in many examples and sufficient to establish a general result of global existence for small perturbations of constant-equilibrium state. Our system (3.1) satisfies the condition (SK), but the general theory on the dissipative structure established in [32] is not applicable since the matrix L is not real symmetric. Consequently, to treat the global existence and asymptotic stability of (3.1), new ideas have to be implemented. See [9] for more details.

Recently, Beauchard and Zuazua [3] have showed that the condition (SK) is equivalent to the classical Kalman rank condition in control theory for the pair (A, L).

The the semigroup $e^{t\Phi}$ associated with system (3.1) can be represented as

$$\left(e^{t\Phi}w\right)(x) = \mathscr{F}^{-1}\left(e^{t\hat{\Phi}(i\xi)}\hat{w}(\xi)\right)(x)$$

where

$$\hat{\Phi}(i\xi) = -(i\xi A + L) \tag{3.3}$$

and $e^{t\hat{\Phi}(i\xi)}$ satisfies $\hat{U}(\xi,t) = e^{t\hat{\Phi}(i\xi)}\hat{U}_0$ and $\hat{U}(\xi,t)$ is the solution of the problem

$$\begin{cases} \hat{U}_t + i\xi A\hat{U}_x + L\hat{U} = 0, \\ \hat{U}(\xi,0) = \hat{U}_0. \end{cases} \tag{3.4}$$

Of course, problem (3.4) is obtained by taking the Fourier transform of (3.1).

Our first main result reads as follows:

Theorem 3.1 *Let $\gamma \in [0,1]$, and let $e^{t\Phi}$ be the semigroup associated with the system (3.1). Then, if w is an odd function, we have the following sharp decay estimates:*

- *When $a = 1$, we have*

$$\left\|\partial_x^k e^{t\Phi}w\right\|_2 \leq C(1+t)^{-1/4-k/2-\gamma/2}\|w\|_{1,\gamma} + Ce^{-ct}\left\|\partial_x^k w\right\|_2 \tag{3.5}$$

- *When $a \neq 1$, we have*

$$\left\|\partial_x^k e^{t\Phi}w\right\|_2 \leq C(1+t)^{-1/4-k/2-\gamma/2}\|w\|_{1,\gamma} + C(1+t)^{-l/2}\left\|\partial_x^{k+l} w\right\|_2 \tag{3.6}$$

where k and l are non-negative integers, and C and c are two positive constants.

In order to proof Theorem 3.1, we recall the following result from [8]. The proof of the following lemma is carried out by using the energy method in the Fourier space.

Lemma 3.2 *Let $\hat{\Phi}(i\xi)$ be the matrix defined in (3.3). Then the corresponding matrix $e^{i\hat{\Phi}(i\xi)}$ satisfies the following estimates for any $t \geq 0$ and $\xi \in \mathbb{R}$:*

- *When $a = 1$, we have*

$$\left|e^{t\hat{\Phi}(i\xi)}\right| \leq Ce^{-c\rho_1(\xi)t}. \tag{3.7}$$

- *When $a \neq 1$, we have*

$$\left|e^{t\hat{\Phi}(i\xi)}\right| \leq Ce^{-c\rho_2(\xi)t} \tag{3.8}$$

where $\rho_1(\xi) = \xi^2/\left(1+\xi^2\right)$, $\rho_2(\xi) = \xi^2/\left(1+\xi^2\right)^2$, *and* C *and* c *are positive constants.*

Proof of Theorem 3.1. The prove of the above Theorem is reduced through the Fourier transform to the analysis of the behavior of the spectral parameter ξ near the origin $\xi = 0$. That is to say, in order to get a better decay estimates, we have to improve the decay estimate of the low frequency part.

First, let us assume that $a = 1$. It is clear that Plancherel's theorem leads to

$$\left\|\partial_x^k e^{t\Phi} w\right\|_2^2 = \frac{1}{2\pi}\int_{\mathbb{R}} |\xi|^{2k} \left|e^{t\hat{\Phi}(i\xi)}\hat{w}(\xi)\right|^2 d\xi$$

and therefore, exploiting (3.7), to obtain

$$\begin{aligned}\left\|\partial_x^k e^{t\Phi} w\right\|_2^2 &\leq C\int_{\mathbb{R}} |\xi|^{2k} e^{-c\rho_1(\xi)t} |\hat{w}(\xi)|^2 d\xi \\ &= C\int_{|\xi|\leq 1} |\xi|^{2k} e^{-c\rho_1(\xi)t} |\hat{w}(\xi)|^2 d\xi + C\int_{|\xi|\geq 1} |\xi|^{2k} e^{-c\rho_1(\xi)t} |\hat{w}(\xi)|^2 d\xi \\ &= I_1 + I_2. \end{aligned} \tag{3.9}$$

From [8], for the high frequency part, we have

$$I_2 \leq Ce^{-ct}\left\|\partial_x^k w\right\|_2^2. \tag{3.10}$$

For the low frequency part, we have the following estimate:

Lemma 3.3 *Let us suppose that* $\gamma \in [0,1]$. *If* w *is an odd function with respect to* $x = 0$, *then the following estimate holds*

$$I_1 \leq C(1+t)^{-1/2-(k+\gamma)}\|w\|_{L^{1,\gamma}(\mathbb{R})}^2. \tag{3.11}$$

Proof. From (3.9) we have

$$I_1 = C\int_{|\xi|\leq 1} |\xi|^{2k} e^{-c\rho_1(\xi)t} |\hat{w}(\xi)|^2 d\xi.$$

Since w is an odd function, then we get

$$\mathscr{F}(w(\xi)) = -2i\int_0^\infty w(x)\sin(x\xi)d\xi.$$

Consequently, it's clear that

$$|\mathscr{F}(w(\xi))| \leq 2\int_0^\infty |w(x)|\sin(x\xi)dx. \tag{3.12}$$

Since $\rho_1(\xi) \geq c|\xi|^2$, for $|\xi| \leq 1$, then the above inequality takes the form

$$\begin{aligned}|I_1| &\leq C\int_{|\xi|\leq 1} |\xi|^{2k} e^{-ct|\xi|^2}|\mathscr{F}(w(\xi))|^2 d\xi \\ &= C\int_{-1}^0 |\xi|^{2k} e^{-ct|\xi|^2}|\mathscr{F}(w(\xi))|^2 d\xi \\ &\quad + C\int_0^1 |\xi|^{2k} e^{-ct|\xi|^2}|\mathscr{F}(w(\xi))|^2 d\xi \\ &= C(I_{1^-} + I_{1^+}). \end{aligned} \tag{3.13}$$

We will estimate I_{1^+}, the same arguments work for I_{1^-}, we omit the details. Indeed

$$I_{1^+} = \int_0^1 |\xi|^{2k} e^{-ct|\xi|^2} |\mathscr{F}(w(\xi))|^2 d\xi.$$

The inequality (3.12) implies

$$I_{1^+} \leq 4 \int_0^1 |\xi|^{2k} e^{-ct|\xi|^2} \left(\int_0^\infty |w(x)||\sin(x\xi)| \right)^2 dx. \tag{3.14}$$

Let us fix $\varepsilon > 0$, then for each $\xi > 0$, we obtain

$$\begin{aligned} \int_\varepsilon^\infty |w(x)||\sin(x\xi)|dx &= \int_\varepsilon^\infty (x\xi)^\gamma |w(x)| \frac{|\sin(x\xi)|}{(x\xi)^\gamma} dx \\ &\leq \xi^\gamma \int_\varepsilon^\infty (1+x)^\gamma |w(x)| M_\gamma dx \\ &\leq M_\gamma \xi^\gamma \int_\varepsilon^\infty (1+x)^\gamma |w(x)| dx \end{aligned} \tag{3.15}$$

where

$$M_\gamma = \sup_{\theta > 0} \frac{|\sin\theta|}{\theta^\gamma}$$

is a constant independent of ε. It's Clear that $M_\gamma < +\infty$ since $\gamma \in [0,1]$.
Once (3.15) holds for any $\varepsilon > 0$, then letting ε tends to 0, therefore (3.15) implies

$$\int_0^\infty |w(x)||\sin(x\xi)|dx \leq M_\gamma \xi^\gamma \|w\|_{1,\gamma}.$$

Consequently, for any $\varepsilon > 0$ (3.14) gives

$$\begin{aligned} & \int_\varepsilon^1 |\xi|^{2k} e^{-ct|\xi|^2} \left(\int_0^\infty |w(x)||\sin(x\xi)| \right)^2 dx \\ \leq \quad & M_\gamma^2 \|w\|_{1,\gamma}^2 \int_\varepsilon^1 |\xi|^{2k} e^{-ct|\xi|^2} d\xi. \end{aligned} \tag{3.16}$$

Similarly, letting $\varepsilon \to 0$ once again, we conclude

$$I_{1^+} \leq M_\gamma \|w\|_{1,\gamma}^2 \int_0^1 |\xi|^{2k+2\gamma} e^{-ct|\xi|^2} d\xi.$$

By exploiting the following inequality

$$\int_0^1 |\xi|^\sigma e^{-ct|\xi|^2} d\xi \leq C(1+t)^{-(\sigma+1)/2} \tag{3.17}$$

then, we deduce

$$\int_0^1 |\xi|^{2k+2\gamma} e^{-ct|\xi|^2} d\xi \leq C(1+t)^{-(k+\gamma)-1/2}.$$

Consequently, we have

$$I_{1^+} \leq C_\gamma (1+t)^{-(k+\gamma)-1/2} \|w\|_{1,\gamma}^2.$$

By carrying the same calculations of I_{1^-}, then our desired result holds. This completes the proof of Lemma 3.3. □

Now, going back to the proof of Theorem 3.1, and from the estimates (3.10) and (3.11), the desired estimate (3.5) holds.

Now, let us assume that $a \neq 1$. Using (3.8) we have exactly as above

$$\begin{aligned} \left\| \partial_x^k e^{t\Phi} w \right\|_2^2 &\le C \int_{\mathbb{R}} |\xi|^{2k} e^{-c\rho_2(\xi)t} |\hat{w}(\xi)|^2 d\xi \\ &= C \int_{|\xi|\le 1} |\xi|^{2k} e^{-c\rho_2(\xi)t} |\hat{w}(\xi)|^2 d\xi + C \int_{|\xi|\ge 1} |\xi|^{2k} e^{-c\rho_2(\xi)t} |\hat{w}(\xi)|^2 d\xi \\ &= J_1 + J_2. \end{aligned} \tag{3.18}$$

Since $\rho_2(\xi) \ge c\xi^2$, then the low frequency part J_1 can be estimated as I_1, so we find

$$J_1 \le C(1+t)^{-1/2-(k+\gamma)} \|w\|^2_{L^{1,\gamma}(\mathbb{R})}. \tag{3.19}$$

Concerning the term J_2, and since for $a \neq 1$, the dissipative structure of system (1.1) is too weak to produce and exponential decay for J_2. Thus, we have (see [8])

$$J_2 \le C(1+t)^{-l} \left\| \partial_x^{k+l} w \right\|_2^2. \tag{3.20}$$

Finally, the estimate (3.6) is a direct consequence of the inequalities (3.18), (3.19) and (3.20).

Remark 3.4 *In Theorem 3.1, the condition on the function w to be an odd function is not restrictive, and is imposed for the sake of brevity. In fact our results hold under the condition $\int_{\mathbb{R}} w(x)dx = 0$, (see Theorem 4.2) or without this condition. See Remark 4.8 for more details.*

4 Refinement of the decay estimates

In this section, we will give a refinement of our decay estimates (3.5) and (3.6). To this end, we recall first from [8] the asymptotic expressions of $e^{t\hat{\Phi}(i\xi)}$ for $\xi \to 0$ and $|\xi| \to \infty$.

By using Sylvester's formula (see [7]), the matrix exponential $e^{t\hat{\Phi}(i\xi)}$ can be represented in the following form

$$e^{t\hat{\Phi}(i\xi)} = \sum_{j=1}^{4} e^{\lambda_j(i\xi)} P_j(i\xi), \tag{4.1}$$

where $\lambda_j(i\xi)$, $j = 1,..,4$ are the four eigenvalues of the matrix $\hat{\Phi}(i\xi) = -(i\xi A + L)$ and the matrices $P_j(i\xi)$, $j = 1,..,4$ are the corresponding Frobenius covariants of $\hat{\Phi}(i\xi)$ defined by

$$P_j(i\xi) = \prod_{\substack{j=1\\ i\neq j}}^{4} \frac{\left(\hat{\Phi}(i\xi) - \lambda_j(i\xi) I\right)}{\lambda_i(i\xi) - \lambda_j(i\xi)}. \tag{4.2}$$

The matrix

$$\hat{\Phi}(i\xi) = -L - i\xi A$$

looks like the matrix $-L$ subjected to a small perturbation. So, $-L$ is the unperturbed matrix and $i\xi A$ the perturbation. According to the perturbation theory (see [13]), in the neighborhood of $\xi = 0$, the eigenvalues of the matrix $\hat{\Phi}(i\xi)$ can be expressed as power series in $i\xi$, that is

$$\lambda_j(i\xi) = \sum_{k=0}^{\infty} \lambda_j^{(k)}(i\xi)^k \text{ and } P_j(i\xi) = \sum_{k=0}^{\infty} P_j^{(k)}(i\xi)^k.$$

Let us assume that $a \neq 1$, $\mu \neq 2a$ and $\mu \neq 2$. We introduce now two semigroups $e^{tD\partial_x^2}$ and $e^{t\Psi_\infty}$ as approximations of the semigroup $e^{t\Phi}$ in the low and high frequency regions, respectively:

$$\left(e^{tD\partial_x^2} w\right)(x) = \mathscr{F}^{-1}\left(e^{-D\xi^2}\hat{w}(\xi)\right)(x) \tag{4.3}$$

$$\left(e^{t\Psi_\infty} w\right)(x) = \mathscr{F}^{-1}\left(e^{t\hat{\Psi}_\infty(i\xi)}\hat{w}(\xi)\right)(x) \tag{4.4}$$

where

$$D = \sum_{j=1}^{2} \kappa_j \Pi_j^0 = \begin{pmatrix} \mu & -a \\ a & 0 \end{pmatrix}, \qquad \hat{\Psi}_\infty(i\xi) = \sum_{j=1}^{2} \lambda_j^\infty(i\xi)\Pi_j^\infty, \tag{4.5}$$

where

$$\begin{aligned} \kappa_j &= \frac{1}{2}(\mu \pm \beta), \qquad \beta = \sqrt{\mu^2 - 4a^2},\ j = 1,2, \\ \lambda_j^\infty(i\xi) &= \pm i\xi \pm \frac{\sigma}{2}(i\xi)^{-1} + \sigma^2\mu(i\xi)^{-2}, \qquad \sigma = \frac{1}{(a^2-1)},\ j = 1,2 \end{aligned}$$

and for $j = 1,2$, the matrices Π_j^0 and Π_j^∞ are defined as follows (see [8] for more details)

$$\begin{aligned} \Pi_1^0 &= \frac{1}{\beta}\begin{pmatrix} \kappa_1 & -a \\ a & -\kappa_2 \end{pmatrix}, \qquad \Pi_1^0 = \frac{1}{\beta}\begin{pmatrix} -\kappa_2 & a \\ -a & \kappa_1 \end{pmatrix} \\ \Pi_1^\infty &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \Pi_2^\infty = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{aligned}$$

Let $\mathscr{R}_0$ and $\mathscr{R}_\infty$ be the matrices

$$\mathscr{R}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \mathscr{R}_\infty = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{4.6}$$

Now, by using the material above, we define the following operators:

$$\begin{cases} S_0(t) = \mathscr{R}_0^T e^{tD\partial_x^2}\mathscr{R}_0, \\ S_\infty(t) = \mathscr{R}_\infty^T e^{t\Psi_\infty}\mathscr{R}_\infty. \end{cases} \tag{4.7}$$

The when $\xi \to 0$ and $|\xi| \to \infty$, the matrix exponential $e^{t\hat{\Phi}(i\xi)}$ can be represented as follows

$$\begin{cases} e^{t\hat{\Phi}(i\xi)} = \hat{S}_0(i\xi,t) + \hat{R}_0(i\xi,t), \\ e^{t\hat{\Phi}(i\xi)} = \hat{S}_\infty(i\xi,t) + \hat{R}_\infty(i\xi,t), \end{cases} \tag{4.8}$$

respectively. In (4.8) $\hat{R}_0(i\xi,t)$ and $\hat{R}_\infty(i\xi,t)$ are the remainder terms when $\xi \to 0$ and $|\xi| \to \infty$ respectively. According to [8] these terms can be estimated as follows:

Lemma 4.1 *([8]) Let $a \neq 1$ and let $\mu \neq 2a, 2$, then we get:*

- *There is a small positive constant r_0 such that for $|\xi| \le r_0$, we have*

$$\left|\hat{R}_0(i\xi, t)\right| \le C|\xi| e^{-c\xi^2 t} + Ce^{-ct}. \tag{4.9}$$

- *There is a large positive constant K_0 such that for $|\xi| \ge K_0$, we have*

$$\left|\hat{R}_\infty(i\xi, t)\right| \le C|\xi|^{-1} e^{-c|\xi|^{-2} t} + Ce^{-ct}. \tag{4.10}$$

where C and c are positive constants.

Now, instead of Theorem 5.1 in [8], we have the following extended estimates:

Theorem 4.2 *Let $a \neq 1$ and let $\mu \neq 2a, 2$. Let $e^{t\Phi}$ be the semigroup associated with (3.1), and let S_0 and S_∞ be the operators defined above. Assume further that $\int_{\mathbb{R}} w(x)\,dx = 0$. Then we have the following estimates:*

$$\begin{aligned} &\left\|\partial_x^k \left(e^{t\Phi} - S_0(t)\right) w\right\|_2 \\ &\le C(1+t)^{-3/4-k/2-\gamma/2} \|w\|_{1,\gamma} + C(1+t)^{-l/2} \left\|\partial_x^{k+l} w\right\|_2, \end{aligned} \tag{4.11}$$

$$\begin{aligned} &\left\|\partial_x^k \left(e^{t\Phi} - S_\infty(t)\right) w\right\|_2 \\ &\le C(1+t)^{-1/4-k/2-\gamma/2} \|w\|_{1,\gamma} + C(1+t)^{-l/2} \left\|\partial_x^{k+l-1} w\right\|_2, \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} &\left\|\partial_x^k \left(e^{t\Phi} - S_0(t) - S_\infty(t)\right) w\right\|_2 \\ &\le C(1+t)^{-3/4-k/2-\gamma/2} \|w\|_{1,\gamma} + C(1+t)^{-l/2} \left\|\partial_x^{k+l-1} w\right\|_2. \end{aligned} \tag{4.13}$$

Moreover, for $w = (0,0,0,w_4)$ such that $\int_{\mathbb{R}} w(x)\,dx = 0$, we have

$$\left\|\partial_x^k e^{t\Phi} w\right\|_2 \le C(1+t)^{-3/4-k/2-\gamma/2} \|w\|_{1,\gamma} + C(1+t)^{-l/2} \left\|\partial_x^{k+l-1} w\right\|_2 \tag{4.14}$$

where $k, l \ge 0$ with $k + l \ge 1$ in (4.12), (4.13) and (4.14), and C is a positive constant.

Remark 4.3 *The estimates in Theorem 3.1 and Theorem 4.2 show that by taking the initial data w in $L^{1,\gamma}(\mathbb{R})$, then the decay rates given in [8] can be improved by $t^{-\gamma/2}$, $\gamma \in [0,1]$.*

Proof of Theorem 4.2. Let us first prove the estimate (4.11). By exploiting the Plancherel theorem, we have

$$\left\|\partial_x^k \left(e^{t\Phi} - S_0(t)\right) w\right\|_2 = \frac{1}{2\pi} \int_{\mathbb{R}} |\xi|^{2k} \left|\left(e^{t\hat{\Phi}(i\xi)} - \hat{S}_0(i\xi)\right) \hat{w}(\xi)\right|^2 d\xi.$$

Let r_0 be as in Lemma 4.1. Following the same strategy as in the proof of Theorem 3.1, we divide the above integral in two parts: the low frequency part ($|\xi| \le r_0$) and the high frequency part ($|\xi| \ge r_0$). Indeed

$$\begin{aligned} \left\|\partial_x^k \left(e^{t\Phi} - S_0(t)\right) w\right\|_2 &\le C \int_{|\xi| \le r_0} |\xi|^{2k} \left|\left(e^{t\hat{\Phi}(i\xi)} - \hat{S}_0(i\xi)\right) \hat{w}(\xi)\right|^2 d\xi \\ &\quad + C \int_{|\xi| \ge r_0} |\xi|^{2k} \left|\left(e^{t\hat{\Phi}(i\xi)} - \hat{S}_0(i\xi)\right) \hat{w}(\xi)\right|^2 d\xi \\ &= J_1 + J_2. \end{aligned}$$

As, we have said before, in order to get better decay estimate, we have to improve the decay rate of the low frequency part J_1. In order to do this let us first prove the following crucial Lemma. A similar one was shown in [10] for the linear wave equation.

Lemma 4.4 *Let us suppose that $\gamma \in [0,1]$. Assume that $\int_{\mathbb{R}} w(x)\,dx = 0$, then the following estimate holds*

$$J_1 \le C(1+t)^{-3/2-(k+\gamma)} \|w\|^2_{L^{1,\gamma}(\mathbb{R})}. \tag{4.15}$$

Proof. From (4.8), J_1 takes the form

$$J_1 = C\int_{|\xi|\le r_0} |\xi|^{2k} \left|\hat{R}_0(i\xi,t)\right|^2 |\hat{w}(\xi)|^2\, d\xi. \tag{4.16}$$

Our goal now is to estimate $\hat{w} = \mathscr{F} w$ in the above formula. From [10], we have the following estimate:

Lemma 4.5 *Let us suppose that $\gamma \in [0,1]$. Assume that $\int_{\mathbb{R}} w(x)\,dx = 0$, then we have*

$$|\mathscr{F}(w(\xi))| \le C_\gamma |\xi|^\gamma \|w\|_{L^{1,\gamma}(\mathbb{R})}$$

with some constant $C_\gamma > 0$, which depends only on γ.

With the result of Lemma 4.5, formula (4.16) takes the form

$$J_1 \le C\|w\|^2_{L^{1,\gamma}(\mathbb{R})} \int_{|\xi|\le r_0} |\xi|^{2(k+\gamma)} \left|\hat{R}_0(i\xi,t)\right|^2 d\xi$$

where C is a positive constant, which will may vary from line to another.

Next, inequality (4.9) in Lemma 4.1 gives

$$J_1 \le C\|w\|^2_{L^{1,\gamma}(\mathbb{R})} \int_{|\xi|\le r_0} |\xi|^{2(k+\gamma)+2} \left(e^{-c\xi^2 t} + Ce^{-ct}\right) d\xi. \tag{4.17}$$

The last inequality (4.17) together with (3.17) imply

$$J_1 \le C(1+t)^{-\frac{3}{2}-(k+\gamma)} \|w\|^2_{L^{1,\gamma}(\mathbb{R})}. \tag{4.18}$$

The estimate of J_2 can be proved by the same method as in the paper [8]. Thus, we have

$$J_2 \le C(1+t)^{-l/2} \left\|\partial_x^{k+l} w\right\|_2. \tag{4.19}$$

Consequently, the estimate (4.11) follows immediately from (4.18) and (4.19).

Our goal now is to prove the estimate (4.13). By the same procedure as in [8], we can write

$$e^{t\hat{\Phi}(i\xi)} = \hat{S}_0(i\xi,t) + \hat{S}_\infty(i\xi,t) + \hat{R}(i\xi,t) \tag{4.20}$$

where the remainder part $\hat{R}(i\xi,t)$ satisfies the following estimates.

Lemma 4.6 *([8, Lemma 4.3]) Under the same condition of Lemma 4.1, we have*

$$\left|\hat{R}(i\xi,t)\right| \le \begin{cases} C|\xi|e^{-c\xi^2 t} + Ce^{-ct} & \text{for } |\xi| \le r_0, \\ Ce^{-ct} & \text{for } r_0 \le |\xi| \le K_0, \\ C|\xi|^{-1} e^{-c|\xi|^{-2} t} + Ce^{-ct} & \text{for } |\xi| \ge K_0, \end{cases}$$

where C and c are positive constants.

Consequently, from (4.20) and Plancherel theorem, we may write

$$\begin{aligned}
&\left\|\partial_x^k\left(e^{t\Phi}-S_0(t)-S_\infty(t)\right)w\right\|_2\\
&=\tfrac{1}{2\pi}\int_{\mathbb{R}}|\xi|^{2k}\left|\left(e^{t\hat{\Phi}(i\xi)}-\hat{S}_0(i\xi,t)-\hat{S}_\infty(i\xi,t)\right)\hat{w}(\xi)\right|^2 d\xi\\
&\leq \tfrac{1}{2\pi}\int_{\mathbb{R}}|\xi|^{2k}\left|\hat{R}(i\xi,t)\right|^2|\hat{w}(\xi)|^2 d\xi\\
&=\tfrac{1}{2\pi}\left\{\int_{|\xi|\leq r_0}+\int_{r_0\leq|\xi|\leq K_0}+\int_{|\xi|\geq K_0}\right\}:=K_1+K_2+K_3.
\end{aligned}\tag{4.21}$$

By using Lemma 4.4, we can estimate the term K_1 in the same way as J_1. So we have

$$K_1\leq C(1+t)^{-\frac{3}{2}-(k+\gamma)}\|w\|^2_{L^{1,\gamma}(\mathbb{R})}.\tag{4.22}$$

The other two terms K_2 and K_3 satisfy the same estimates as in [8]. Namely,

$$K_2\leq Ce^{-ct}\|w\|^2_{L^1(\mathbb{R})}\tag{4.23}$$

and

$$K_3\leq C(1+t)^{-l}\left\|\partial_x^{k+l-1}w\right\|_2^2\tag{4.24}$$

where $k+l\geq 1$. Finally, our result (4.13) holds by inserting (4.22)-(4.24) into (4.21).

To prove (4.14), it's suffices to see that from (4.7), we deduce that for $w=(0,0,0,w_4)$, $S_0(t)w=S_\infty(t)w=0$, consequently, the estimate (4.13) implies (4.14).

Remark 4.7 *(The particular case $\gamma=1$)*

In the particular case $\gamma=1$, the proof of the better decay estimates in Theorem 4.2 as well in Theorem 3.1 are carried out simply as follows:

Let us prove (3.5) for example. Indeed, for the high frequency part, the same estimate (3.10) holds. For the low frequency part, we have from (3.9)

$$I_1=C\int_{|\xi|\leq 1}|\xi|^{2k}e^{-c\rho_1(\xi)t}|\hat{w}(\xi)|^2 d\xi$$

and since $\rho_1(\xi)\geq c|\xi|^2$, for $|\xi|\leq 1$, we obtain

$$|I_1|\leq C\int_{|\xi|\leq 1}|\xi|^{2k}e^{-ct|\xi|^2}|\hat{w}(\xi)|^2 d\xi.$$

It is clear that $\int_{\mathbb{R}}w(x)\,dx=0$ implies $\hat{w}(0)=0$. Therefore, by using the mean value theorem, we get

$$\begin{aligned}
|I_1| &\leq C\int_{|\xi|\leq 1}|\xi|^{2k}e^{-ct|\xi|^2}|\hat{w}(\xi)-\hat{w}(0)|^2 d\xi\\
&\leq C\left\|\partial_\xi\hat{w}\right\|^2_{L^\infty_\xi}\int_{|\xi|\leq 1}|\xi|^{2k+2}e^{-ct|\xi|^2}d\xi\\
&\leq C\|w\|^2_{L^{1,1}(\mathbb{R})}(1+t)^{-3/2-k},
\end{aligned}$$

which is exactly the estimate (3.11) for $\gamma=1$.

Remark 4.8 *The assumption*

$$\int_{\mathbb{R}} w(x)\,dx = 0 \tag{4.25}$$

in Theorem 4.2 is only a technical condition in order to make our proof simple. If (4.25) does'nt hold, the estimates (4.11)-(4.13) in Theorem 4.2 take the following form, respectively:

$$\left\|\partial_x^k\left(e^{t\Phi}-S_0(t)\right)w\right\|_2 \le C(1+t)^{-3/4-k/2-\gamma/2}\|w\|_{1,\gamma}$$
$$+C(1+t)^{-3/4-k/2}\left(\int_{\mathbb{R}} w(x)\,dx\right)+C(1+t)^{-l/2}\left\|\partial_x^{k+l}w\right\|_2$$

$$\left\|\partial_x^k\left(e^{t\Phi}-S_\infty(t)\right)w\right\|_2 \le C(1+t)^{-1/4-k/2-\gamma/2}\|w\|_{1,\gamma}$$
$$+C(1+t)^{-1/4-k/2}\left(\int_{\mathbb{R}} w(x)\,dx\right)+C(1+t)^{-l/2}\left\|\partial_x^{k+l-1}w\right\|_2,$$

and

$$\left\|\partial_x^k\left(e^{t\Phi}-S_0(t)-S_\infty(t)\right)w\right\|_2 \le C(1+t)^{-3/4-k/2-\gamma/2}\|w\|_{1,\gamma}$$
$$+C(1+t)^{-3/4-k/2}\left(\int_{\mathbb{R}} w(x)\,dx\right)+C(1+t)^{-l/2}\left\|\partial_x^{k+l-1}w\right\|_2.$$

Of course, in this case a slight modification in the proof is needed.

5 Large time approximation

The purpose of this section is to show that the asymptotic profile of the solution $U=(v,u,z,y)^T$ of problem (3.1) is given by $\overline{U}=\mathscr{R}_0^T W(t,x)$ in the sense that the estimate

$$\left\|\partial_x^k\left(U-\overline{U}\right)\right\|_{L^2(\mathbb{R})} = O\left(t^{-3/4-k/2}\right), \qquad \text{as } t\to\infty$$

holds for suitably small smooth initial data U_0, where $W=(u,z)^T$ is the solution of the corresponding parabolic system (5.2) and $\mathscr{R}_0$ is the matrix defined in (4.6). This result indicates that problem (3.1) has an asymptotically parabolic structure.

Let us consider the problem

$$\begin{cases} u_t = \mu u_{xx} - a z_{xx}, & (t,x)\in\mathbb{R}^+\times\mathbb{R} \\ z_t = a u_{xx}, & (t,x)\in\mathbb{R}^+\times\mathbb{R} \\ u(0,x)=u_0(x), \qquad z(0,x)=z_0(x), & x\in\mathbb{R}. \end{cases} \tag{5.1}$$

System (5.1) can be rewritten in vector notation as

$$\begin{cases} W_t = DW_{xx}, \\ W(0,x)=W_0(x) \end{cases} \tag{5.2}$$

where D is the matrix defined in (4.5). Each solution of the Cauchy problem (5.2) can be written as

$$W(t,x) = \left(e^{tD\partial_x^2}W_0\right)(x) = G(t,x)*W_0(x) \tag{5.3}$$

with the heat kernel

$$G(x,t) = \sum_{j=1}^{2} H(t,x,\kappa_j)\,\Pi_j^0$$

where

$$H(t,x,\kappa_j) = \frac{1}{\sqrt{4\pi\kappa_j t}} e^{-x^2/4\kappa_j t}.$$

Now, we are going to prove the decay rate of the L^p-norms of (5.3). The following result extends the well know decay estimate written in [8, lemma 5.1]

Lemma 5.1 *Let $\gamma \in [0,1]$ and let $W_0 \in L^{1,\gamma}(\mathbb{R}) \cap H^l(\mathbb{R})$ with $l \geq 1$. Then for any $1 \leq p \leq \infty$, we have*

$$\left\| \partial_x^k e^{tD\partial_x^2} W_0 \right\|_p \leq Ct^{-\alpha-\frac{k+\gamma}{2}} \|W_0\|_{1,\gamma} + C\left| \int_{\mathbb{R}} W_0(x)\,dx \right| t^{-\alpha-\frac{k}{2}},\ k = 0,1,2,..,l-1 \tag{5.4}$$

where $\alpha = 1/2(1-1/p)$.

Proof. Let us prove (5.4) for the L^∞ and L^2 norms. By using the Fourier transform, we have by using (5.3)

$$\hat{W}(t,\xi) = e^{-D\xi^2 t}\hat{W}_0(\xi).$$

Consequently, we have

$$\begin{aligned} \left\| \widehat{\partial_x^k W} \right\|_{L^1(\mathbb{R})} &= \left\| (i\xi)^k \hat{W} \right\|_{L^1(\mathbb{R})} \leq C \left\| |\xi|^k e^{-D\xi^2 t} \hat{W}_0 \right\|_{L^1(\mathbb{R})} \\ &\leq C \int_{\mathbb{R}} |\xi|^k e^{-c\xi^2 t} \left| \hat{W}_0(\xi) \right| d\xi \end{aligned} \tag{5.5}$$

where we have used the relation (4.5). Our goal now is to estimate $\left|\hat{W}_0\right|$. Indeed, we have (see [10])

$$\begin{aligned} \left|\hat{W}_0(\xi)\right| &= \left| \int_{\mathbb{R}} e^{-ix\xi} W_0(x)\,dx \right| \\ &\leq \int_{\mathbb{R}} |\cos(x\xi)-1|\,|W_0(x)|\,dx + \int_{\mathbb{R}} |\sin(x\xi)|\,|W_0(x)|\,dx + \left| \int_{\mathbb{R}} W_0(x)\,dx \right|. \end{aligned}$$

Since

$$\begin{cases} K_\gamma = \sup_{\theta\neq 0} \frac{|1-\cos\theta|}{|\theta|^\gamma} < +\infty, \\ M_\gamma = \sup_{\theta\neq 0} \frac{\sin\theta}{|\theta|^\gamma} < +\infty \end{cases}$$

for $0 \leq \gamma \leq 1$. Then we deduce

$$\left|\hat{W}_0(\xi)\right| \leq C_\gamma |\xi|^\gamma \|W_0\|_{1,\gamma} + \left| \int_{\mathbb{R}} W_0(x)\,dx \right| \tag{5.6}$$

with $C_\gamma = K_\gamma + M_\gamma$. Consequently, inserting (5.6) in (5.5) yields

$$\left\| \widehat{\partial_x^k W} \right\|_{L^1(\mathbb{R})} \leq C\|W_0\|_{1,\gamma} \int_{\mathbb{R}} |\xi|^{k+\gamma} e^{-c\xi^2 t} d\xi + C\left| \int_{\mathbb{R}} W_0(x)\,dx \right| \int_{\mathbb{R}} |\xi|^k e^{-c\xi^2 t} d\xi. \tag{5.7}$$

By using the inequality

$$\|f\|_{L^p(\mathbb{R})} \leq \|\hat{f}\|_{L^q(\mathbb{R})}, \qquad \frac{1}{p}+\frac{1}{q}=1, \qquad 1\leq q\leq 2$$

we have

$$\left\|\partial_x^k W\right\|_{L^\infty(\mathbb{R})} \leq \left\|\widehat{\partial_x^k W}\right\|_{L^1(\mathbb{R})},$$

and therefore, this last inequality together with the estimate (5.7) imply

$$\left\|\partial_x^k W\right\|_{L^\infty(\mathbb{R})} \leq Ct^{-\frac{1}{2}-\frac{(k+\gamma)}{2}}\|W_0\|_{1,\gamma} + C\left|\int_{\mathbb{R}} W_0(x)\,dx\right| Ct^{-\frac{1}{2}-\frac{k}{2}},$$

which is equivalent to (5.4), for $p=\infty$.

By the same method, and Plancherel theorem, we can easily show the L^2 decay estimate. Once (5.4) is true for $p=2$ and $p=\infty$, then (5.4) for $2<p<\infty$ follows from the interpolation inequality (2.1) by choosing $j=k$, $q=2$ and $r=\infty$.

Now, to complete the proof of (5.4) for $1\leq p<2$, we have only to prove (5.4) for $p=1$, then the interpolation inequality fills the gap for $1<p<2$.

Let us first prove the estimate (5.4) for $k=0$. Indeed, (5.3), can be written as

$$W(t,x) = \int_{\mathbb{R}} \sum_{j=1}^{2} \frac{1}{\sqrt{4\pi\kappa_j t}} e^{-(x-y)^2/4\kappa_j t}\Pi_j^0 W_0(y)\,dy \tag{5.8}$$

Then (5.8) easily takes the form

$$\begin{aligned} W(t,x) &= \int_{\mathbb{R}} \sum_{j=1}^{2} \frac{1}{\sqrt{4\pi\kappa_j t}} \left(e^{-(x-y)^2/4\kappa_j t} - e^{-x^2/4\kappa_j t}\right)\Pi_j^0 W_0(y)\,dy \\ &\quad + \int_{\mathbb{R}} \sum_{j=1}^{2} \frac{e^{-x^2/4\kappa_j t}}{\sqrt{4\pi\kappa_j t}}\Pi_j^0 W_0(y)\,dy \\ &= W_1(t,x) + W_2(t,x). \end{aligned}$$

It is clear that

$$\int_{\mathbb{R}} |W_2(t,x)|\,dx \leq Ct^{-1/2}\int_{\mathbb{R}}\left|\int_{\mathbb{R}}\sum_{j=1}^{2} e^{-x^2/4\kappa_j t}dx\right||W_0(y)|\,dy \leq C\left|\int_{\mathbb{R}} W_0(x)\,dx\right|. \tag{5.9}$$

On the other hand,

$$\begin{aligned} |W_1(t,x)| &\leq C\int_{\mathbb{R}} \sum_{j=1}^{2} \frac{1}{\sqrt{4\pi\kappa_j t}} \left|e^{-(x-y)^2/4\kappa_j t} - e^{-x^2/4\kappa_j t}\right| |W_0(y)|\,dy \\ &= C\int_{\mathbb{R}} (\sum_{j=1}^{2} \frac{1}{\sqrt{4\pi\kappa_j t}} \left|e^{-(x-y)^2/4\kappa_j t} - e^{-x^2/4\kappa_j t}\right|^{\gamma} \\ &\quad \times \left|e^{-(x-y)^2/4\kappa_j t} - e^{-x^2/4\kappa_j t}\right|^{1-\gamma} |W_0(y)|)dy \\ &\leq C_\gamma \int_0^1 \int_{\mathbb{R}} \sum_{j=1}^{2} \frac{1}{\sqrt{4\pi\kappa_j t}} \left|\frac{y(x-\theta y)}{2t\kappa_j} e^{-|x-\theta y|^2/4\kappa_j t}\right|^{\gamma} |W_0(y)|\,dy d\theta \\ &\leq C_\gamma \int_0^1 \int_{\mathbb{R}} \sum_{j=1}^{2} \frac{1}{(\kappa_j t)^{\gamma/2}\sqrt{4\pi\kappa_j t}} \left|\frac{y(x-\theta y)}{2\sqrt{\kappa_j t}} e^{-|x-\theta y|^2/4\kappa_j t}\right|^{\gamma} |W_0(y)|\,dy d\theta. \end{aligned} \tag{5.10}$$

By putting $z=\frac{(x-\theta y)}{2\sqrt{\kappa_j t}}$, and since $\int_{\mathbb{R}}|z|^{\gamma}e^{-|z|^{2\gamma}}dz$ is bounded, then (5.10) implies

$$\begin{aligned}\|W_1(t,x)\|_1 &\leq Ct^{-\gamma/2}\int_{\mathbb{R}^N}|y|^{\gamma}|W_0(y)|\,dy\\ &\leq Ct^{-\gamma/2}\|W_0\|_{1,\gamma}.\end{aligned}\tag{5.11}$$

Therefore, (5.9) together with (5.11) imply the inequality (5.4) for $p=1$ and $k=0$. It is sufficient to use the induction on k to obtain higher order estimates for W_1 and W_2. (This higher order estimates of W_2 are sharp for k even). In order to let the reader understand the core of the argument, we prove here the estimate of W_1 (the most difficult term) for $k=1$. For simplicity, let us take $\kappa_j=1$, $j=1,2$, and take $j=1$, then we have

$$\partial_x W_1=\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}}\left(-\frac{2(x-y)}{4t}e^{-(x-y)^2/4t}+\frac{x}{2t}e^{-x^2/4t}\right)\Pi_1^0W_0(y)\,dy,$$

this implies that

$$\begin{aligned}|\partial_x W_1| \leq\ & \frac{C}{\sqrt{t}}\int_{\mathbb{R}}\left|\frac{2(x-y)}{4t}e^{-(x-y)^2/4t}-\frac{x}{2t}e^{-x^2/4t}\right|^{1-\gamma}\\ &\times\left|\frac{(x-y)}{2t}e^{-(x-y)^2/4t}-\frac{x}{2t}e^{-x^2/4t}\right|^{\gamma}|W_0(y)|\,dy.\end{aligned}\tag{5.12}$$

However, for the first line in (5.12), setting first $z=|x-y|/\left(2\sqrt{t}\right)$ for the first term and $z=|x|/\left(2\sqrt{t}\right)$ for the second term and since the function ze^{-z^2} is a bounded function in $z\geq 0$, then using the fact that

$$\left|\frac{x}{2t}e^{-x^2/4t}\right|\leq\frac{1}{\sqrt{t}}ze^{-z^2},$$

we get

$$\begin{aligned}|\partial_x W_1| &\leq \frac{Ct^{(\gamma-1)/2}}{\sqrt{t}}\int_{\mathbb{R}}\left|\frac{(x-y)}{2t}e^{-(x-y)^2/4t}-\frac{x}{2t}e^{-x^2/4t}\right|^{\gamma}|W_0(y)|\,dy\\ &\leq \frac{Ct^{(\gamma-1)/2}}{\sqrt{t}}\int_{\mathbb{R}}\left|\int_0^1\frac{d}{d\theta}\left\{\frac{(x-\theta y)}{2t}e^{-(x-\theta y)^2/4t}\right\}d\theta\right|^{\gamma}|W_0(y)|\,dy\\ &= \frac{Ct^{(\gamma-1)/2}}{\sqrt{t}}\int_{\mathbb{R}}\left|\int_0^1\frac{-y}{2t}e^{-(x-\theta y)^2/4t}+\frac{(x-\theta y)^2}{4t^2}e^{-(x-\theta y)^2/4t}d\theta\right|^{\gamma}|W_0(y)|\,dy.\end{aligned}$$

Now, putting $z=\frac{(x-\theta y)}{2\sqrt{t}}$, and since $\int_{\mathbb{R}}e^{-|z|^{2\gamma}}dz$ and $\int_{\mathbb{R}}|z|^{2\gamma}e^{-|z|^{2\gamma}}dz$ are bounded, then we get

$$\begin{aligned}\int_{\mathbb{R}}|\partial_x W_1|\,dx &\leq \frac{Ct^{(\gamma-1)/2}t^{\gamma/2}}{\sqrt{t}}t^{-\gamma}\int_{\mathbb{R}}(1+y)^{\gamma}|W_0(y)|\,dy\\ &\leq Ct^{-\frac{\gamma+1}{2}}\|W_0\|_{1,\gamma}.\end{aligned}$$

Thus (5.4) is fulfilled for $k=1$; the rest follows inductively.

Remark 5.2 *If $\gamma=1$ and $\int_{\mathbb{R}}W_0(x)\,dx=0$, then our estimate (5.4) will be the same as the estimate (5.14) in [8]. So, once again our Lemma 5.1 extends the early L^p-L^q decay estimates for the heat equation.*

Now, as in [8], we define the linear diffusion wave $\overline{U}(t,x)$ by

$$\overline{U}(t,x) = \mathscr{R}_0^T G(x,t+1) M_0, \qquad M_0 = \int_{\mathbb{R}} \mathscr{R}_0 U_0(x) \tag{5.13}$$

Now, instead of Theorem 5.1 in [8], we have the following result.

Theorem 5.3 *Let $a \neq 1$, $\mu \neq 2a, 2$ and let $\gamma \in [0,1]$. Suppose that the initial data $U_0 \in H^s(\mathbb{R}) \cap L^{1,\gamma}(\mathbb{R})$ for $s \geq 2$. Then we have*

$$\left\| \partial_x^k \left(U(t) - \overline{U}(t) \right) \right\|_2 \leq C I_0 (1+t)^{-1/2-(k+\gamma)/2}$$

for $0 \leq k \leq [s/2] - 1$, where $U(t,x)$ is the solution of (3.1), $\overline{U}(t,x)$ is defined in (5.13), C is a positive constant and $I_0 = \|U_0\|_{H^s(\mathbb{R})} + \|U_0\|_{L^{1,\gamma}(\mathbb{R})}$.

Theorem 5.3 can be proved by the same method as in [8], we have only to use our estimates in Theorem 4.2 and Lemma 5.1 instead of the estimates used in [8]. We omit the details.

6 The semilinear problem

In this section, we consider the problem

$$\begin{cases} \varphi_{tt}(t,x) - (\varphi_x - \psi)_x(t,x) = 0 & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \psi_{tt}(t,x) - a^2 \psi_{xx}(t,x) - (\varphi_x - \psi)(t,x) + \mu \psi_t(t,x) = |\psi(t,x)|^p & (t,x) \in \mathbb{R}^+ \times \mathbb{R} \\ (\varphi, \varphi_t, \psi, \psi_t)(0,x) = (\varphi_0, \varphi_1, \psi_0, \psi_1) & x \in \mathbb{R} \end{cases} \tag{6.1}$$

where $p > 1$. We will use Duhamel's principle to express the solution to the nonlinear problem (6.1) with the help of solution to the linear problem (1.1). The basic idea in our proof is based on the weighted energy estimate used in [35] and [11]. In order to use the better decay estimates of the above sections, let us take $\gamma = 1$, and for simplicity, we take $\mu = 1$.

As in section 3 for the linear problem, problem (6.1) can be rewritten as

$$\begin{cases} U_t + AU_x + LU = G(U), \\ U(0,x) = U_0, \end{cases} \tag{6.2}$$

where A, L, U are defined by (3.2), and $G(U)(t,\cdot) = \left(0,0,0, \left|\psi_0 + \int_0^t U_4(s,\cdot)\,ds\right|^p\right)^T$.

6.1 The well-posedness

In this subsection, we state and prove the local well-posedness result of problem (6.2).

Theorem 6.1 *Let $(\phi_0, \phi_1, \psi_0, \psi_1)$ satisfy $U_0 \in H^1(\mathbb{R})$, $\psi_0 \in L^2(\mathbb{R})$, and*

$$J := \|e^{\phi(0,.)} U_0\|_2 + \|e^{\phi(0,.)} \psi_0\|_2 < \infty,$$

then there exists a maximal existence time $T_m = T_m(J) > 0$ such that problem (6.2) has a unique solution $U \in C\left([0,T_m), H^1(\mathbb{R})\right)$ satisfying

$$\sup_{[0,T]} \left\{ \|e^{\phi(t,.)} U(t,.)\|_2 + \|e^{\phi(t,.)} \psi(t,.)\|_2 \right\} < +\infty, \tag{6.3}$$

for any $T < T_m$. If in particular $T_m < +\infty$, then the following holds:

$$\lim_{t\to T_m} \sup \left\{ \|e^{\phi(t,.)} U(t,.)\|_2 + \|e^{\phi(t,.)} \psi(t,.)\|_2 \right\} = +\infty. \tag{6.4}$$

Our goal in the next steps, is to prove Theorem 6.1. In order to do so, we use the contraction mapping theorem and it suffices to prove Theorem 6.1 on $[0,T]$ for small $T > 0$. For simplicity, we take $a = \mu = 1$. Let us first consider the mixed problem with a fixed nonlinear term $|\hat{\psi}(t,x)|^p$

$$\begin{cases} \varphi_{tt}(t,x) - (\varphi_x - \psi)_x(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \psi_{tt}(t,x) - \psi_{xx}(t,x) - (\varphi_x - \psi)(t,x) + \psi_t(t,x) = |\hat{\psi}(t,x)|^p, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ (\varphi, \varphi_t, \psi, \psi_t)(0,x) = (\varphi_0, \varphi_1, \psi_0, \psi_1), & x \in \mathbb{R}. \end{cases} \tag{6.5}$$

Then, we have the following result.

Proposition 6.2 *Let us assume that $U_0 = (\varphi_x(0,x) - \psi_0, \varphi_1, \psi_x(0,x), \psi_1) \in \left(L^2(\mathbb{R})\right)^4$ satisfy*

$$\int_{\mathbb{R}} e^{\left(2+\rho|x|^2\right)/2\rho} \left\{ \varphi_1^2(x) + \psi_1^2(x) + \psi_x(0,x) + (\varphi_x(0,x) - \psi_0)^2 + \psi_0^2 \right\} dx < +\infty$$

and the function $\hat{\psi} \in C\left([0,T], H^1(\mathbb{R})\right) \cap C^1\left([0,T], L^2(\mathbb{R})\right)$ satisfies

$$M = \sup_{[0,T]} \left\{ \|e^{\phi(t,.)} \hat{\psi}_x(t,.)\|_2 + \|e^{\phi(t,.)} \hat{\psi}_t(t,.)\|_2 + \|e^{\phi(t,.)} \hat{\psi}(t,.)\|_2 \right\} < +\infty.$$

Then, problem (6.5) has a weak solution (φ, ψ) such that

$$(\varphi_x - \psi, \varphi_t, \psi_x, \psi_t) \in \left(C\left([0,T], L^2(\mathbb{R})\right)\right)^4 \tag{6.6}$$

and satisfying

$$\begin{aligned} &\sup_{[0,T]} \left\{ \|e^{\phi(t,.)} \varphi_t(t,.)\|_2 + \|e^{\phi(t,.)} \psi_t(t,.)\|_2 + \|e^{\phi(t,.)} \psi_x(t,.)\|_2 \right. \\ &\left. + \|e^{\phi(t,.)} (\varphi_x(t,.) - \psi(t,.))\|_2 + \|e^{\phi(t,.)} \psi(t,.)\|_2 \right\} < +\infty. \end{aligned}$$

If $U_0 \in H^1(\mathbb{R})$, then it is the unique classical solution.

In order to define the notion of weak and classical solution and to prove Proposition 6.2, we first study for any $T > 0$ and a fixed forcing term $g \in C\left([0,T], L^2(\mathbb{R})\right)$ the following problem:

$$\begin{cases} \varphi_{tt}(t,x) - (\varphi_x - \psi)_x(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \psi_{tt}(t,x) - \psi_{xx}(t,x) - (\varphi_x - \psi)(t,x) + \psi_t(t,x) = g(t,x), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ (\varphi, \varphi_t, \psi, \psi_t)(0,x) = (\varphi_0, \varphi_1, \psi_0, \psi_1), & x \in \mathbb{R}. \end{cases} \tag{6.7}$$

Then, we have the following Lemma.

Lemma 6.3 *Let* $(\varphi_x(0,x)-\psi_0,\varphi_1,\psi_x(0,x),\psi_1)\in\left(L^2(\mathbb{R})\right)^4$ *and* $g\in C\left([0,T],L^2(\mathbb{R})\right)$ *then problem (6.7) has a weak solution* (φ,ψ) *such that*

$$(\varphi_x-\psi,\varphi_t,\psi_x,\psi_t)\in\left(C\left([0,T],L^2(\mathbb{R})\right)\right)^4.$$

By a weak solution, we mean the following: Rewrite (6.7) again as first-order system

$$U_t+\underbrace{(A\partial_x+L)}_{=:\hat{A}}U=G,\quad G=(0,0,0,g),\quad U(0,\cdot)=U_0. \tag{6.8}$$

Then $\hat{A}:D(\hat{A}):=(H^1(\mathbb{R})^4\subset(L^2(\mathbb{R})^4\to(L^2(\mathbb{R})^4$ is the generator of a contraction semigroup $(e^{-t\hat{A}})_{t\geq 0}$, and, for $U_0\in D(A)$ and $g\in C^1([0,\infty),L^2(\mathbb{R}))$, we have a *classical* solution $U\in C^1([0,\infty),L^2(\mathbb{R}))\cap C^0([0,\infty),H^1(\mathbb{R}))$ satisfying

$$U(t)=e^{-t\hat{A}}U_0+\int_0^t e^{-(t-r)\hat{A}}G(r)dr. \tag{6.9}$$

Fixing a Dirac sequence of mollifiers (j_n^1) in x and (j_n^2) in t, we define, for $U_0\in L^2(\mathbb{R})$ and $G\in C^0([0,\infty),L^2(\mathbb{R}))$ — now fixed — approximations $U_{0,n}:=j_n^1*U_0$ and $G_n:=j_n^2*G$ satisfying

$$U_{0,n}\to U_0\quad\text{in } L^2(\mathbb{R}),\qquad G_n\to G\quad\text{in } C^0([0,\infty),L^2(\mathbb{R})).$$

We conclude from (6.9) applied to the solution U_n corresponding to $(U_{0,n},G_n)$, that (U_n) converges to some U in $C^0([0,\infty),L^2(\mathbb{R}$, and U satisfies (6.9). This U is called a (the) weak solution.

Proof of Proposition 6.2

First, we observe that, using $\hat{\psi}\in C^0([0,T],H^1(\mathbb{R}))$, we conclude

$$|\hat{\psi}|^p\in C^1([0,T],L^2(\mathbb{R})). \tag{6.10}$$

The prove of (6.10) will be given in the Appendix. Now, we approximate U_0 by $U_{0,n}\in H^1(\mathbb{R})$ as above, and obtain a classical solution U_n, for which we can proceed as in Lemma 6.6. We get — dropping the index n —

$$E_{\varphi,\psi}^{\phi}(t)\leq E_{\varphi,\psi}^{\phi}(0)+\int_0^t\int_{\mathbb{R}}e^{2\phi(s,x)}|\hat{\psi}|^p\psi_s dxds \tag{6.11}$$

where

$$E_{\varphi,\psi}^{\phi}(t)=\int_{\mathbb{R}}\frac{e^{2\phi(t,x)}}{2}\left(\varphi_t^2+\psi_t^2+\psi_x^2+(\varphi_x-\psi)^2\right)dx.$$

The Cauchy&Schwarz inequality and Hölder's inequality imply

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}e^{2\phi(s,x)}|\hat{\psi}|^p\psi_s dxds &= \int_0^t\int_{\mathbb{R}}\left(e^{\phi(s,x)}|\hat{\psi}|^p\right)\left(e^{\phi(s,x)}\psi_s\right)dxds\\ &\leq \int_0^t\|e^{\phi(s)}|\hat{\psi}|^p\|_2\|e^{\phi(s)}\psi_s\|_2 ds. \end{aligned}\tag{6.12}$$

Consequently, from (6.11) and (6.12) we get

$$\begin{aligned}E_{\varphi,\psi}^{\phi}(t) &\leq E_{\varphi,\psi}^{\phi}(0)+\int_0^t\|e^{\phi(s)}|\hat{\psi}|^p\|_2\|e^{\phi(s)}\psi_s\|_2 ds\\ &\leq E_{\varphi,\psi}^{\phi}(0)+\int_0^t\|e^{\phi(s)}|\hat{\psi}|^p\|_2\left(E_{\varphi,\psi}^{\phi}(s)\right)^{1/2}ds,\end{aligned}$$

which implies by Gronwall inequality

$$\left(E^{\phi}_{\varphi,\psi}(t)\right)^{1/2} \leq \left(E^{\phi}_{\varphi,\psi}(0)\right)^{1/2} + C\int_0^t \|e^{\phi(s)}|\hat{\psi}|^p\|_2 ds \tag{6.13}$$

Now, using assumption of Proposition 6.2, we deduce that $\hat{\psi} \in H^1_{\phi(t,.)(\mathbb{R})}$, so, we can apply the result of Lemma 6.7, to get

$$\|e^{\phi(s)}|\hat{\psi}|^p\|_2^2 \leq \int_{\mathbb{R}} \left(e^{\phi(s)}|\hat{\psi}|\right)^{2p} = \|e^{\phi(s)}\hat{\psi}\|_{2p}^{2p} \leq C(1+s)^{p(\rho+2)(1-\theta(2p))}\|e^{\phi(s)}\hat{\psi}_x\|_2^p,$$

By the assumption on $\hat{\psi}$, we deduce that

$$\|e^{\phi(s)}\hat{\psi}_x\|_2 \leq M,$$

and this leads to

$$\begin{aligned}\int_0^t \|e^{\phi(s)}|\hat{\psi}|^p\|_2 ds &\leq CM^p\int_0^t C(1+s)^{\frac{p(\rho+2)(1-\theta(2p))}{2}} ds \\ &\leq CM^p(1+T)^{\frac{p(\rho+2)(1-\theta(2p))}{2}} T.\end{aligned} \tag{6.14}$$

From (6.13) and (6.14), we get

$$\left(E^{\phi}_{\varphi,\psi}(t)\right)^{1/2} \leq \left(E^{\phi}_{\varphi,\psi}(0)\right)^{1/2} + CM^p(1+T)^{\frac{p(\rho+2)(1-\theta(2p))}{2}} T. \tag{6.15}$$

On the other hand, we have

$$\psi(t,x) = \psi_0(x) + \int_0^t \psi_s(s,x)\,ds,$$

and then

$$e^{\phi(t,x)}\psi(t,x) = e^{\phi(t,x)}\psi_0(x) + \int_0^t e^{\phi(t,x)}\psi_s(s,x)\,ds.$$

Since the function $t \mapsto \phi(t,x)$ is monotone decreasing, we get by using (6.15)

$$\begin{aligned}\|e^{\phi(t,.)}\psi(t)\|_2 &\leq \|e^{\phi(t,.)}\psi_0\|_2 + \int_0^t \|e^{\phi(t,.)}\psi_s(s)\|_2 ds \\ &\leq \|e^{\phi(0,.)}\psi_0\|_2 + \int_0^t \|e^{\phi(t,.)}\psi_s(s)\|_2 ds \\ &\leq \|e^{\phi(0,.)}\psi_0\|_2 + \int_0^t \left(\left(E^{\phi}_{\varphi,\psi}(0)\right)^{1/2} + CM^p(1+T)^{\frac{p(\rho+2)(1-\theta(2p))}{2}} T\right) ds \\ &\leq \|e^{\phi(0,.)}\psi_0\|_2 + \left(E^{\phi}_{\varphi,\psi}(0)\right)^{1/2} T + CM^p(1+T)^{\frac{p(\rho+2)(1-\theta(2p))}{2}} T^2.\end{aligned} \tag{6.16}$$

From (6.15) and (6.16), we get the desired result for a classical solution U_n. Now, let $n \to \infty$, and we obtain the estimates for the weak solution U.

Proof of Theorem 6.1.

Let us define

$$B^{\phi}_{T,R} = \left\{V = (\hat{\varphi},\hat{\psi})' : (\hat{\varphi}_x - \hat{\psi}, \hat{\varphi}_t, \hat{\psi}_x, \hat{\psi}_t) \in \left(C\left([0,T], L^2(\mathbb{R})\right)\right)^4 \text{ and } \|V\|^{\phi}_T \leq R\right\}$$

where $R>0$, $T>0$ and

$$\|V\|_T^\phi = \|(\hat{\varphi},\hat{\psi})\|_T^\phi = \sup_{[0,T]}\Big\{\|e^{\phi(t,.)}(\hat{\varphi}_x-\hat{\psi})(t,.)\|_2+\|e^{\phi(t,.)}\hat{\varphi}_t(t,.)\|_2 + \|e^{\phi(t,.)}\hat{\psi}_x(t,.)\|_2+\|e^{\phi(t,.)}\hat{\psi}_t(t,.)\|_2+\|e^{\phi(t,.)}\hat{\psi}(t,.)\|_2\Big\}.$$

Let us define

$$X=\Big\{(\varphi,\psi):(\varphi_x-\psi,\varphi_t,\psi_x,\psi_t)\in\left(C\left([0,T],L^2(\mathbb{R})\right)\right)^4\Big\}.$$

Then X is a Banach space with norm $\|\cdot\|_T^\phi$. From now on, we fix the intial data to satisfy

$$U_0\in H^1(\mathbb{R}),\ \psi_0\in L^2(\mathbb{R}).$$

For a fixed $V=(0,\hat{\psi})'\in B_{T,R}^\phi$, we define a mapping $\Phi: B_{T,R}^\phi\to X$ such that $\begin{pmatrix}\varphi\\ \psi\end{pmatrix}=\Phi\begin{pmatrix}0\\ \hat{\psi}\end{pmatrix}$ is the weak solution of problem (6.5) defined via approximation of $|\hat{\psi}|^p$ as above.

Our goal now is to show that, for a suitable $T>0$, Φ is a contractive map satisfying $\Phi\left(B_{T,R}^\phi\right)\subset B_{T,R}^\phi$. Proving the following estimates first for the approximations in the class of classical solutions, we finally get the same also for the weak solution.

Using the same method as in the proof of Lemma 6.6, we deduce from (6.39) that

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{e^{2\phi}}{2}\left(\varphi_t^2+\psi_t^2+\psi_x^2+(\varphi_x-\psi)^2\right)\right\}\\ &-\frac{d}{dx}\left\{e^{2\phi}(\varphi_x-\psi)\varphi_t\right\}-\frac{d}{dx}\left\{e^{2\phi}(\psi_x\psi_t)\right\}\le e^{2\phi}|\hat{\psi}|^p\psi_t.\end{aligned}\tag{6.17}$$

Which gives by integrating (6.17) over $[0,t]\times\mathbb{R}$,

$$E_{\varphi,\psi}^\phi(t)\le E_{\varphi,\psi}^\phi(0)+\int_0^t\int_{\mathbb{R}}e^{2\phi(s,x)}|\hat{\psi}|^p\psi_t dxds.$$

The Cauchy&Schwarz inequality gives

$$E_{\varphi,\psi}^\phi(t)\le E_{\varphi,\psi}^\phi(0)+\sqrt{2}\int_0^t\left(\int_{\mathbb{R}}e^{2\phi(s,x)}|\hat{\psi}(s,x)|^{2p}dx\right)^{1/2}\left(E_{\varphi,\psi}^\phi(s)\right)^{1/2}ds.\tag{6.18}$$

Applying Gronwall type inequality to (6.18), we arrive at

$$\left(E_{\varphi,\psi}^\phi(t)\right)^{1/2}\le\left(E_{\varphi,\psi}^\phi(0)\right)^{1/2}+\frac{1}{\sqrt{2}}\int_0^t\left(\int_{\mathbb{R}}e^{2\phi(s,x)}|\hat{\psi}(s,x)|^{2p}dx\right)^{1/2}ds.\tag{6.19}$$

Next, applying Lemma 6.7 to (6.19), we obtain

$$\begin{aligned}\int_{\mathbb{R}}e^{2\phi(s,x)}|\hat{\psi}(s,x)|^{2p}dx &\le C(1+t)^{p(2+\rho)(1-\theta(2p))}\|\hat{\psi}_x\|_2^{2(p-1)}\|e^{\phi(t,.)}\hat{\psi}_x\|_2^2\\ &\le C(1+t)^{p(2+\rho)(1-\theta(2p))}R^{2p}.\end{aligned}$$

Therefore, (6.19) implies

$$\left(E_{\varphi,\psi}^\phi(t)\right)^{1/2}\le\left(E_{\varphi,\psi}^\phi(0)\right)^{1/2}+C(1+T)^{((p+1)(2+\rho)+2)/4}R^p.\tag{6.20}$$

On the other hand and as in (6.16), we have

$$\begin{aligned}
\|e^{\phi(t,.)}\psi(t)\|_2 &\leq \|e^{\phi(t,.)}\psi_0\|_2 + \int_0^t \|e^{\phi(t,.)}\psi_s(s)\|_2 ds \\
&\leq \|e^{\phi(0,.)}\psi_0\|_2 + \int_0^t \|e^{\phi(t,.)}\psi_s(s)\|_2 ds \\
&\leq \|e^{\phi(0,.)}\psi_0\|_2 + \int_0^t \left(E^{\phi}_{\varphi,\psi}(0)^{1/2} + C(1+T)^{((p+1)(2+\rho)+2)/4}R^p\right) ds \\
&\leq \|e^{\phi(0,.)}\psi_0\|_2 + E^{\phi}_{\varphi,\psi}(0)^{1/2}T + CR^p(1+T)^{((p+1)(2+\rho)+2)/4}T. \qquad (6.21)
\end{aligned}$$

Consequently, from (6.20) and (6.21), we get

$$\begin{aligned}
\|(\varphi,\psi)\|^{\phi}_T &\leq \left(E^{\phi}_{\varphi,\psi}(0)\right)^{1/2} + C(1+T)^{((p+1)(2+\rho)+2)/4}R^p \\
&\quad + \|e^{\phi(0,.)}\psi_0\|_2 + E^{\phi}_{\varphi,\psi}(0)^{1/2}T + CR^p(1+T)^{((p+1)(2+\rho)+2)/4}T.
\end{aligned}$$

By taking R large enough such that

$$\left(E^{\phi}_{\varphi,\psi}(0)\right))^{1/2} + \|e^{\phi(0,.)}\psi_0\|_2 < R/2,$$

and choosing T sufficiently small such that

$$RT + C(1+T)^{((p-2)(2+\rho)+4)/4}R^p + CR^p(1+T)^{((p+1)(2+\rho)+2)/4}T < R/2,$$

then, we get

$$\|(\varphi,\psi)\|^{\phi}_T \leq R.$$

This shows that $(\varphi,\psi) \in B^{\phi}_{T,R}$.

Next, we have to verify that $\Phi : B^{\phi}_{T,R} \to B^{\phi}_{T,R}$ is a contraction. To this end, we set $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \Phi\begin{pmatrix} 0 \\ \hat{\psi} \end{pmatrix} = \Phi(V)$ and $\begin{pmatrix} \overline{\varphi} \\ \overline{\psi} \end{pmatrix} = \Phi\begin{pmatrix} 0 \\ \hat{\overline{\psi}} \end{pmatrix} = \Phi(\overline{V})$, where $(\overline{\varphi},\overline{\psi})$ is the solution of the following system

$$\begin{cases}
\overline{\varphi}_{tt}(t,x) - (\overline{\varphi}_x - \overline{\psi})_x(t,x) = 0 & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\
\overline{\psi}_{tt}(t,x) - \overline{\psi}_{xx}(t,x) - (\overline{\varphi}_x - \overline{\psi})(t,x) + \overline{\psi}_t(t,x) = |\hat{\overline{\psi}}(t,x)|^p & (t,x) \in \mathbb{R}^+ \times \mathbb{R} \\
(\overline{\varphi},\overline{\varphi}_t,\overline{\psi},\overline{\psi}_t)(0,x) = (\varphi_0,\varphi_1,\psi_0,\psi_1) & x \in \mathbb{R}
\end{cases}$$

Then by setting $\tilde{\varphi} = \varphi - \overline{\varphi}$ and $\tilde{\psi} = \psi - \overline{\psi}$, we arrive

$$\begin{cases}
\tilde{\varphi}_{tt}(t,x) - (\tilde{\varphi}_x - \tilde{\psi})_x(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\
\tilde{\psi}_{tt}(t,x) - \tilde{\psi}_{xx}(t,x) - (\tilde{\varphi}_x - \tilde{\psi})(t,x) + \tilde{\psi}_t(t,x) = |\hat{\psi}(t,x)|^p - |\hat{\overline{\psi}}(t,x)|^p, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\
(\tilde{\varphi},\tilde{\varphi}_t,\tilde{\psi},\tilde{\psi}_t)(0,x) = (0,0,0,0), & x \in \mathbb{R},
\end{cases}$$

where $\tilde{\varphi} = \varphi - \overline{\varphi}$ and $\tilde{\psi} = \psi - \overline{\psi}$. Now, proceeding as in the proof of (6.40), we obtain after an integration over $[0,t] \times \mathbb{R}$,

$$E^{\phi}_{\tilde{\varphi},\tilde{\psi}}(t) \leq \int_0^t \int_{\mathbb{R}} e^{2\phi(t,x)} \left(|\hat{\psi}(s,x)|^p - |\hat{\overline{\psi}}(s,x)|^p\right) \tilde{\psi}_t(s,x)\, dxds.$$

Applying the inequality

$$|\hat{\psi}(t,x)|^p - |\overline{\hat{\psi}}(t,x)|^p \le p|\hat{\psi}(t,x) - \overline{\hat{\psi}}(t,x)|\left(|\hat{\psi}(t,x)| + |\overline{\hat{\psi}}(t,x)|\right)^{p-1},$$

then Cauchy&Schwarz inequality implies

$$\begin{aligned}
E^{\phi}_{\tilde{\varphi},\tilde{\psi}}(t) &\le p\int_0^t\int_{\mathbb{R}} e^{2\phi(s,x)}|\hat{\psi}(s,x) - \overline{\hat{\psi}}(s,x)|\left(|\hat{\psi}(s,x)| + |\overline{\hat{\psi}}(s,x)|\right)^{p-1}|\tilde{\psi}_t(s,x)|dxds \\
&\le p\int_0^t\left(\int_{\mathbb{R}} e^{2\phi(s,x)}|\tilde{\psi}_t(s,x)|^2dx\right)^{1/2} \\
&\quad\times\left(\int_{\mathbb{R}} e^{2\phi(s,x)}|\hat{\psi}(s,x) - \overline{\hat{\psi}}(s,x)|^2\left(|\hat{\psi}(s,x)| + |\overline{\hat{\psi}}(s,x)|\right)^{2(p-1)}\right)^{1/2} ds \qquad (6.22)\\
&\le C\int_0^t\left(E^{\phi}_{\tilde{\varphi},\tilde{\psi}}(s)\right)^{1/2} \\
&\quad\times\left(\int_{\mathbb{R}} e^{2\phi(s,x)}|\hat{\psi}(s,x) - \overline{\hat{\psi}}(s,x)|^2\left(|\hat{\psi}(s,x)| + |\overline{\hat{\psi}}(s,x)|\right)^{2(p-1)}\right)^{1/2} ds.
\end{aligned}$$

Our objective now, is to handel the last term in the right-hand side of (6.22). Indeed, exploiting Hölder's inequality, we get

$$\begin{aligned}
&\int_{\mathbb{R}} e^{2\phi(t,x)}|\hat{\psi}(s,x) - \overline{\hat{\psi}}(s,x)|^2\left(|\hat{\psi}(s,x)| + |\overline{\hat{\psi}}(s,x)|\right)^{2(p-1)}dx \\
\le\ & \left\|e^{\phi(s)/2}(\hat{\psi}(s) - \overline{\hat{\psi}}(s))\right\|_{2p}^2\left\|e^{\phi(s)/2(p-1)}\left(|\hat{\psi}(s)| + |\overline{\hat{\psi}}(s)|\right)\right\|_{2p}^{2(p-1)}.
\end{aligned}$$

Inserting this last estimate into (6.22), we obtain

$$\begin{aligned}
&E^{\phi}_{\tilde{\varphi},\tilde{\psi}}(t) \\
\le\ & C\int_0^t\left(E^{\phi}_{\tilde{\varphi},\tilde{\psi}}(s)\right)^{1/2}\|e^{\phi(s)/2}(\hat{\psi}(s) - \overline{\hat{\psi}}(s))\|_{2p}\|e^{\phi(s)/2(p-1)}\left(|\hat{\psi}(s)| + |\overline{\hat{\psi}}(s)|\right)\|_{2p}^{p-1}.
\end{aligned}$$

Gronwall inequality and Minkowski inequality imply

$$\left(E^{\phi}_{\tilde{\varphi},\tilde{\psi}}(t)\right)^{1/2} \le C\int_0^t\|e^{\phi(s)/2}(\hat{\psi}(s) - \overline{\hat{\psi}}(s))\|_{2p}\left(\|e^{\phi(s)/2(p-1)}\hat{\psi}(s)\|_{2p} + \|e^{\phi(s)/2(p-1)}\overline{\hat{\psi}}(s)\|_{2p}\right)^{p-1}. \qquad (6.23)$$

Applying Lemma 6.7 for $\nu = 1/(2(p-1))$ and $q = 2p$, we get

$$\begin{aligned}
\|e^{\phi(s)/2(p-1)}\hat{\psi}(s)\|_{2p} &\le C(1+s)^{(2+\rho)(p+1)/(4p)}\|\hat{\psi}_x(s)\|_2^{1-1/(2(p-1))}\|e^{\phi(s)}\hat{\psi}_x(s)\|_2^{1/(2(p-1))} \\
&\le C(1+s)^{(2+\rho)(p+1)/(4p)}R. \\
&\le C(1+T)^{(2+\rho)(p+1)/(4p)}R.
\end{aligned}$$

Similarly, we get for $\nu = 1/2$ and $q = 2p$

$$\|e^{\phi(s)/2}(\hat{\psi}(s) - \overline{\hat{\psi}}(s))\|_{2p} \le C(1+T)^{(2+\rho)(p+1)/(4p)}\|e^{\phi(s)}(\hat{\psi}_x(s) - \overline{\hat{\psi}}_x(s))\|_2.$$

Consequently, (6.23) becomes

$$\begin{aligned}
\left(E^{\phi}_{\tilde{\varphi},\tilde{\psi}}(t)\right)^{1/2} &\le C(1+T)^{(2+\rho)(p-1)(p+1)/4}R^{p-1}\int_0^t\|e^{\phi(s)/2}\left(\hat{\psi}(s) - \overline{\hat{\psi}}(s)\right)\|_{2p} \\
&\le C(1+T)^{(2+\rho)(p+1)/(4)}TR^{p-1}\left\|V - \overline{V}\right\|_T^{\phi}. \qquad (6.24)
\end{aligned}$$

Also, since

$$\tilde{\psi}(t,x)=\int_0^t \tilde{\psi}_s(s,x)\,ds,$$

then, we get as in (6.16)

$$\begin{aligned}\|e^{\phi(t,.)}\tilde{\psi}(t,x)\|_2 &\leq \int_0^t \|e^{\phi(t,.)}\tilde{\psi}_s(s,x)\|_2 ds \leq \int_0^t \|e^{\phi(s,.)}\tilde{\psi}_s(s,x)\|_2 ds \\ &\leq \int_0^t \left(E^{\phi}_{\tilde{\varphi},\tilde{\psi}}(s)\right)^{1/2} ds \\ &\leq C(1+T)^{(2+\rho)(p+1)/(4)}T^2R^{p-1}\left\|V-\overline{V}\right\|_T^{\phi},\end{aligned} \tag{6.25}$$

where we have used (6.24). Therefore, (6.24) together with (6.25) imply

$$\|(\varphi-\overline{\varphi},\psi-\overline{\psi})\|_T^{\phi}\leq C(1+T)^{(2+\rho)(p+1)/(4)}TR^{p-1}(1+T)\left\|V-\overline{V}\right\|_T^{\phi}. \tag{6.26}$$

By choosing T small enough in order to have

$$C(1+T)^{(2+\rho)(p+1)/(4)}TR^{p-1}(1+T)<\frac{1}{2}, \tag{6.27}$$

estimate (6.26) shows that Φ is a contraction. Consequently the contraction mapping theorem guarantees the existence of a unique (φ,ψ) satisfying $\begin{pmatrix}\varphi\\ \psi\end{pmatrix}=\Phi(V)$.

Using the representation formula (6.9), we observe that, for $U_0\in H^1(\mathbb{R})$ the solution is the unique classical solution.

Our last goal now, is to prove (6.3). To accomplish this, we adapt the method introduced in [12] for the wave equation. We need to show that the norm $\|e^{\phi(t,.)}U(t,.)\|_2+\|e^{\phi(t,.)}\psi(t,.)\|_2$ is bounded for all $t\in[0,T]$ and for any $T<T_m$.

Let $V^{(0)}(t,x)=\begin{pmatrix}0\\ \psi_0(x)\end{pmatrix}$ with $V^{(0)}\in B^{\phi}_{T,R}$, and we define the sequence $\left(\varphi^{(n)},\psi^{(n)}\right)$ satisfying

$$\begin{pmatrix}\varphi^{(n)}\\ \psi^{(n)}\end{pmatrix}(t,x)=\Phi\left(V^{(n-1)}\right)(t,x),\qquad n=1,2,3,...,$$

and $\begin{pmatrix}\varphi^{(n)}\\ \psi^{(n)}\end{pmatrix}$ is the solution of the problem

$$\begin{cases}\varphi^{(n)}_{tt}(t,x)-\left(\varphi^{(n)}_x-\psi^{(n)}\right)_x(t,x)=0, & (t,x)\in\mathbb{R}^+\times\mathbb{R},\\ \psi^{(n)}_{tt}(t,x)-\psi^{(n)}_{xx}(t,x)-\left(\varphi^{(n)}_x-\psi^{(n)}\right)(t,x)+\psi^{(n)}_t(t,x)=|\psi^{(n-1)}(t,x)|^p, & (t,x)\in\mathbb{R}^+\times\mathbb{R},\\ \left(\varphi^{(n)},\varphi^{(n)}_t,\psi^{(n)},\psi^{(n)}_t\right)(0,x)=(\varphi_0,\varphi_1,\psi_0,\psi_1), & x\in\mathbb{R}.\end{cases}$$

Using the estimate (6.26) with the condition (6.27), we deduce that there exists two functions $U=(\varphi_x-\psi,\varphi_t,\psi_x,\psi_t)\in C\left([0,T_m),L^2(\mathbb{R})\right)$ and $\psi\in C\left([0,T_m),L^2(\mathbb{R})\right)$ such that for all $T<T_m$

$$U^{(n)}=\left(\varphi^{(n)}_x-\psi^{(n)},\varphi^{(n)}_t,\psi^{(n)}_x,\psi^{(n)}_t\right)\to U\quad \text{in } C\left([0,T],L^2(\mathbb{R})\right),$$

and

$$\psi^{(n)} \to \psi \quad \text{in } C\left([0,T], L^2(\mathbb{R})\right),$$

as $n \to \infty$ and U becomes the weak solution of (6.1). Obviously, from Proposition 6.2, we have for all $t \in [0,T]$,

$$\|e^{\phi(t,.)}U^{(n)}(t,.)\|_2 + \|e^{\phi(t,.)}\psi^{(n)}(t,.)\|_2 < R. \tag{6.28}$$

Now, let $\vartheta \in \left(C_0^\infty(\mathbb{R})\right)^4$ and $\tilde{\vartheta} \in C_0^\infty(\mathbb{R})$ be fixed, then we have

$$\begin{aligned}
\left|\left(e^{\phi(t)}U(t), \vartheta\right)\right| &= \left|\left(U(t), e^{\phi(t)}\vartheta\right)\right| \\
&\leq \left|\left(U(t) - U^{(n)}(t), e^{\phi^*(t)}\vartheta\right)\right| + \left|\left(U^{(n)}(t), e^{\phi^*(t)}\vartheta\right)\right| \\
&\leq \left|\left(U(t) - U^{(n)}(t), e^{\phi^*(t)}\vartheta\right)\right| + \|e^{\phi(t)}U^{(n)}(t)\|_2 \|\vartheta\|_2,
\end{aligned}$$

where $(f,g) = \int_{\mathbb{R}} f(x).g(x)\,dx$. Consequently, passing to the limit in the above inequality, we deduce from (6.28) that

$$\left|\left(e^{\phi(t)}U(t), \vartheta\right)\right| \leq \limsup_{n\to\infty} \|e^{\phi(t)}U^{(n)}(t)\|_2 \|\vartheta\|_2 \leq R\|\vartheta\|_2.$$

Similarly, we can show that

$$\left|\left(e^{\phi(t)}\psi(t), \tilde{\vartheta}\right)\right| \leq \limsup_{n\to\infty} \|e^{\phi(t)}\psi^{(n)}(t)\|_2 \|\tilde{\vartheta}\|_2 \leq R\|\tilde{\vartheta}\|_2.$$

By density argument, we deduce that

$$e^{\phi(t)}U(t) \in L^2(\mathbb{R}), \qquad e^{\phi(t)}\psi(t) \in L^2(\mathbb{R}), \qquad \forall t \in [0,T]$$

and

$$\|e^{\phi(t,.)}U(t,.)\|_2 \leq R, \qquad \|e^{\phi(t,.)}\psi(t,.)\|_2 \leq R$$

and therefore,

$$\|e^{\phi(t,.)}U(t,.)\|_2 + \|e^{\phi(t,.)}\psi(t,.)\|_2 \leq 2R, \qquad \forall t \in [0,T].$$

This completes the proof of Theorem 6.1.

6.2 Global existence and asymptotic behavior

In this subsection, we show the global existence and the asymptotic behavior of problem (6.2). We investigate only the case $a = 1$, the case $a \neq 1$ can be proved with the same method.

Our main result in this subsection is the following Theorem.

Theorem 6.4 *Let $a = 1$. Under the same condition of Theorem 6.1 and assume further that $U_0 \in L^2(\mathbb{R}) \cap L^{1,1}(\mathbb{R})$*[2] *such that $\int_{\mathbb{R}} U_0 = 0$. Suppose also that $p > 12$. Then there exists a positive number $\varepsilon > 0$ such that if*

$$I_0 + \|U_0\|_{1,1} + \|U_0\|_2 < \varepsilon, \tag{6.29}$$

then problem (6.1) has a unique global solution U satisfying

$$\|U\|_2 \leq C(1+t)^{-3/4}\left(I_0 + \|U_0\|_{1,1} + \|U_0\|_2\right). \tag{6.30}$$

[2] In fact these estimates hold for any $\gamma \in [0,1]$, so we take $\gamma = 1$ because in this case we have fastes decay rate.

where $U = (\varphi_x - \psi, \varphi_t, \psi_x, \psi_t)^T$, $U_0 = (\varphi_x(0) - \psi(0), \varphi_1, \psi_x(0), \psi_1)^T$ *and*

$$I_0^2 = \int_{\mathbb{R}^N} e^{\left(2+\rho|x|^2\right)/2\rho} U_0^2 dx.$$

Remark 6.5 *The restriction on the parameter* $p > 12$ *in Theorem 6.4 is depending on the weighted function* ϕ *defined in (6.31). Moreover, this condition is quite reasonable since the damping* ψ_t *is not strong enough to stabilize the whole system for all* $p > 1$. *But if we add a damping term of the form* φ_t *to the left hand side of the first equation in (6.1), then the result of Theorem 6.4 holds for all* $p > 1$. *Of course in this case, we choose the weighted function*

$$\hat{\phi}(t,x) = \frac{|x|^2}{4(t+1)},$$

and a slight modification in the proof will give the desired result.

To obtain the decay result of our problem (6.1), we shall proceed with our proof based on the (modified) weighted energy method originally developed by Todorova and Yordanov [35]. Now, we define a weight function similar to the one introduced by Ikehata and Inoue [11]. Indeed, we define the function

$$\phi(t,x) = \frac{2(t+1)^2 + \rho|x|^2}{2\rho(t+1)^{2+\rho}} \tag{6.31}$$

as a weight function where ρ is a small positive constant to be fixed later. It is clear that the function ϕ satisfying: $\phi(t,x) \in C^1([0,+\infty) \times \mathbb{R})$ and

$$\begin{cases} \phi_t(t,x) = -\dfrac{1}{(t+1)^{1+\rho}} - \dfrac{\rho+2}{2} \dfrac{|x|^2}{(t+1)^{3+\rho}} < 0, \\ \phi_x(t,x) = \dfrac{x}{(1+t)^{2+\rho}}. \end{cases} \tag{6.32}$$

Also a simple computation shows that

$$-\phi_t(t,x) \le \frac{C_\rho}{1+t}\phi(t,x) \tag{6.33}$$

and

$$\begin{aligned} & \phi_x^2(t,x) - \phi_t(t,x)\phi_x^2(t,x) - \phi_t^2(t,x) \\ = & \frac{(2+\rho)|x|^4}{2(1+t)^{7+3\rho}} \left(1 - \frac{2+\rho}{2}(1+t)^{1+\rho}\right) \\ & + \frac{|x|^2}{(1+t)^{5+3\rho}} \left(1 - \frac{2+\rho}{2}(1+t)^{1+\rho}\right) - \frac{1}{(1+t)^{2+2\rho}} \\ \le & \ 0. \end{aligned} \tag{6.34}$$

Consequently (6.34) implies

$$\frac{\phi_x^2(t,x)}{-\phi_t(t,x)} \le -\phi_x^2(t,x) - \phi_t(t,x). \tag{6.35}$$

Lemma 6.6 *Let (φ,ψ) be a local solution of problem (6.1) on $[0,T_m)$, then the following estimate holds: for all $t\in[0,T_m)$,*

$$\|e^{2\phi}U\|_2^2 \le CI_0^2 + C\left(\sup_{[0,t]}(1+s)^{\delta}\|e^{\lambda\phi(s,.)}\psi(s,.)\|_{p+1}\right)^{p+1} \tag{6.36}$$

where $1\ge\lambda>2/(p+1)$, $\delta>0$ and $C=C_{\delta,\lambda}>0$ is a constant, which depends on δ and λ.

Proof. To prove Lemma 6.6, we multiply the first equation in (6.1) by φ_t and the second equation by ψ_t, we get respectively

$$\frac{1}{2}\frac{d}{dt}\varphi_t^2 - \frac{d}{dx}(\varphi_x-\psi)\varphi_t + (\varphi_x-\psi)\varphi_{tx} = 0 \tag{6.37}$$

and

$$\frac{d}{dt}\left(\frac{1}{2}\psi_t^2+\psi_x^2-\frac{|\psi|^p\psi}{p+1}\right) - \frac{d}{dx}(\psi_x\psi_t) - (\varphi_x-\psi)\psi_t + \psi_t^2 = 0. \tag{6.38}$$

Summing up (6.37) and (6.38), we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\left(\varphi_t^2+\psi_t^2+\psi_x^2+(\varphi_x-\psi)^2\right)-\tfrac{|\psi|^p\psi}{p+1}\right)\\&-\frac{d}{dx}\{(\varphi_x-\psi)\varphi_t\} - \frac{d}{dx}(\psi_x\psi_t) + \psi_t^2 = 0.\end{aligned} \tag{6.39}$$

Multiplying (6.39) by $e^{2\phi}$, we get

$$\begin{aligned}&\frac{d}{dt}\left(\frac{e^{2\phi}}{2}\left(\varphi_t^2+\psi_t^2+\psi_x^2+(\varphi_x-\psi)^2\right)-\frac{e^{2\phi}|\psi|^p\psi}{p+1}\right)\\
=\;& e^{2\phi}\left(\phi_t-\frac{\phi_x^2}{\phi_t}\right)\varphi_t^2 - \frac{2\phi_t}{p+1}e^{2\phi}|\psi|^p\psi + e^{2\phi}\left(1-+\phi_t-\frac{\phi_x^2}{\phi_t}\right)\psi_t^2\\
&+\frac{e^{2\phi}}{\phi_t}(\phi_t\psi_x-\psi_t\phi_x)^2 + \frac{e^{2\phi}}{\phi_t}(\phi_t(\varphi_x-\psi)-\phi_x\varphi_t)^2\\
&-\frac{d}{dx}\left\{e^{2\phi}(\varphi_x-\psi)\varphi_t\right\} - \frac{d}{dx}\left\{e^{2\phi}(\psi_x\psi_t)\right\}.\end{aligned}$$

Recalling (6.32) and (6.35), we get a useful identity

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{e^{2\phi}}{2}\left(\varphi_t^2+\psi_t^2+\psi_x^2+(\varphi_x-\psi)^2\right)-\frac{e^{2\phi}|\psi|^p\psi}{p+1}\right\}\\
&-\frac{d}{dx}\left\{e^{2\phi}(\varphi_x-\psi)\varphi_t\right\} - \frac{d}{dx}\left\{e^{2\phi}(\psi_x\psi_t)\right\} \le -\frac{2\phi_t}{p+1}e^{2\phi}|\psi|^p\psi.\end{aligned} \tag{6.40}$$

Integrating (6.40) over $[0,t]\times\mathbb{R}$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}}\frac{e^{2\phi}}{2}\left(\varphi_t^2+\psi_t^2+\psi_x^2+(\varphi_x-\psi)^2\right)dx\\
\le\;& \int_{\mathbb{R}}\frac{e^{2\phi(0,x)}}{2}\left(\varphi_0^2+\psi_0^2+\psi_x^2(0)+(\varphi_x(0)-\psi_0)^2\right)dx\\
&+\frac{1}{p+1}\int_{\mathbb{R}}e^{2\phi(t,x)}|\psi(t,x)|^{p+1}dx + \frac{2}{p+1}\int_0^t\int_{\mathbb{R}}(-\phi_s)e^{2\phi(s,x)}|\psi(s,x)|^{p+1}dxds.\end{aligned} \tag{6.41}$$

Our goal now is to estimate the last term in the right hand side of (6.41). Indeed, we have from (6.33) and for $\lambda > 2/(p+1)$ (see [35] and [12]):

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} (-\phi_s)\, e^{2\phi(s,x)} |\psi(s,x)|^{p+1}\, dxds \\
\leq\ & C\int_0^t \frac{1}{s+1} \int_{\mathbb{R}} \phi(s,x)\, e^{(2-\lambda(p+1))\phi(s,x)} e^{\lambda(p+1)\phi(s,x)} |\psi(s,x)|^{p+1}\, dsdx.
\end{aligned}$$

Since $\sup_{r\geq 0}\left(re^{2-\lambda(p+1)r}\right) < +\infty$, then we get from the above estimate

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} (-\phi_s)\, e^{2\phi(s,x)} |\psi(s,x)|^{p+1}\, dxds \\
\leq\ & C\int_0^t \frac{1}{s+1} \|e^{\lambda\phi(s,.)}\psi(s,.)\|_{p+1}^{p+1} ds \\
\leq\ & C\int_0^t \frac{1}{(s+1)^{1+\delta(p+1)}} \left\{ \sup_{[0,t]} (1+s)^{\delta} \|e^{\lambda\phi(s,.)}\psi(s,.)\|_{p+1} \right\}^{p+1} ds \\
\leq\ & C\left\{ \sup_{[0,t]} (1+s)^{\delta} \|e^{\lambda\phi(s,.)}\psi(s,.)\|_{p+1} \right\}^{p+1}.
\end{aligned} \tag{6.42}$$

Also, it is clear that for any $s \in [0,t]$, we have

$$\begin{aligned}
\int_{\mathbb{R}} e^{2\phi(t,x)} |\psi(t,x)|^{p+1}\, dx &= \|e^{\frac{2}{p+1}\phi(s)}\psi(s,.)\|_{p+1}^{p+1} \\
&\leq \|e^{\lambda\phi(s)}\psi(s,.)\|_{p+1}^{p+1} \\
&\leq \left\{ \sup_{[0,t]} (1+s)^{\delta} \|e^{\lambda\phi(s,.)}\psi(s,.)\|_{p+1} \right\}^{p+1}.
\end{aligned} \tag{6.43}$$

Consequently, the above estimates (6.42) and (6.43) give

$$\|e^{2\phi}U\|_2^2 \leq CI_0^2 + C\left\{ \sup_{[0,t]} (1+s)^{\delta} \|e^{\lambda\phi(s,.)}\psi(s,.)\|_{p+1} \right\}^{p+1}.$$

This implies the desired inequality (6.36). □

For $\nu > 0$ and $t \geq 0$, we define a family of weighted function spaces $H^1_{\nu\phi(t,.)}(\mathbb{R})$ as:

$$f \in H^1_{\nu\phi(t,.)}(\mathbb{R}) \Leftrightarrow f \in H^1(\mathbb{R}),\ \|e^{\nu\phi(t,.)}f\|_2^2 + \|e^{\nu\phi(t,.)}f_x\|_2^2 < +\infty, \forall t \geq 0.$$

We recall the Gagliardo-Nirenberg type inequality, which can be easily proved by following [11].

Lemma 6.7 *Let $\theta(q) = 1/2 - 1/q$ with $q > 0$ and $0 \leq \theta(q) \leq 1$ and $0 \leq \nu \leq 1$. If $v \in H^1_{\nu\phi(t,.)}(\mathbb{R})$ then*

$$\|e^{\nu\phi(t,.)}v\|_q \leq C_\nu (1+t)^{(2+\rho)(1-\theta(q))/2} \|v_x\|_2^{1-\nu} \|e^{\phi(t,.)}v_x\|_2^{\nu},$$

with some constant $C_\nu > 0$.

Therefore, the following result holds: (see [12, Lemma 2.5])

Lemma 6.8 *For each $\delta > 0$, there exists a constant $C = C_\delta$ such that*

$$\int_{\mathbb{R}} e^{-2p\delta\phi(t,x)} dx \leq C(1+t)^{(\rho+2)/2}. \tag{6.44}$$

The following lemma is crucial in our argument.

Lemma 6.9 *There exists a constant $C = C_\delta$ such that for all $t \geq 0$, we have*

$$\int_{\mathbb{R}} e^{-2p\delta\phi(t,x)} (1+|x|)^2 dx \leq C(1+t)^{3(\rho+2)/2}. \tag{6.45}$$

Proof. First, let us assume that $x \geq 0$ then it follows from (6.31) that

$$\begin{aligned}
\int_0^{+\infty} e^{-2p\delta\phi(t,x)} (1+x)^2 dx &= \int_0^{+\infty} e^{-p\delta x^2/(t+1)^{(\rho+2)}} (1+x)^2 dx \\
&\leq 4\int_0^1 e^{-p\delta x^2/(t+1)^{(\rho+2)}} + 4\int_1^{+\infty} e^{-p\delta x^2/(t+1)^{(\rho+2)}} x^2 dx \\
&\leq 4 + 4\tilde{K}_1.
\end{aligned}$$

Concerning the integral $\tilde{K}_1$, we have, by making the change of variable $r = \delta x^2/(t+1)^{(\rho+2)}$

$$\begin{aligned}
\tilde{K}_1 &= \frac{(t+1)^{(\rho+2)}}{p\delta} \int_1^{+\infty} e^{-p\delta x^2/(t+1)^{(\rho+2)}} \left(\frac{p\delta x^2}{(t+1)^{(\rho+2)}}\right) dx \\
&= \frac{(t+1)^{(\rho+2)}}{p\delta} \frac{1}{2\sqrt{p\delta}} (t+1)^{-(\rho+2)/2} (1+t)^{(\rho+2)} \int_{\delta/(t+1)^3}^{+\infty} e^{-r} r^{1/2} dr \\
&\leq C_\delta (t+1)^{3(\rho+2)/2} \int_0^{+\infty} e^{-r} r^{3/2-1} dr \\
&= C_\delta \Gamma(3/2)(t+1)^{3(\rho+2)/2},
\end{aligned}$$

where $\Gamma(s) = \int_0^{+\infty} e^{-r} r^{s-1} dr$ is the Gamma function of $s > 0$. We can use the same strategy for $x < 0$. This concludes the proof of Lemma 6.9. $\square$

The following lemma is crucial in the proof of Theorem 6.4

Lemma 6.10 *Let $U(t,x)$ be the solution of problem (6.2), then the following estimate holds:*

$$(1+t)^{3/4} \|U(t)\|_2 \leq C(\|U_0\|_{1,1} + \|U_0\|_2) + C\left(\sup_{[0,t]} (1+\tau)^\beta \|e^{\delta\phi(\tau,x)}\psi(\tau)\|_{2p}\right)^p \tag{6.46}$$

for any $\varepsilon > 0$, $\beta = (3(\rho+2)/4 + 1 + \varepsilon)/p$ and $\delta > 0$.

Proof. Let us first assume that $a = 1$. By virtue of the Duhamel principle, we transform the problem (6.2) into the integral equation as

$$U(t) = e^{t\Phi} U_0 + \int_0^t e^{(t-\tau)\Phi} G(U)(\tau) d\tau. \tag{6.47}$$

Taking the L^2 norm of (6.2), we conclude

$$\begin{aligned}\|U(t)\|_2 &\leq \left\|e^{t\Phi}U_0\right\|_2 + \int_0^t \left\|e^{(t-\tau)\Phi}G(U)\right\|_2 d\tau \\ &= \tilde{I}_1 + \tilde{I}_2\end{aligned}$$

Since $\tilde{I}_1$ is the L^2 norm of the solution of problem (3.1), then $\tilde{I}_1$ satisfies the decay estimate (3.5), consequently we have, for $\gamma = 1$,

$$\tilde{I}_1 \leq C(1+t)^{-3/4}\|U_0\|_{1,1} + Ce^{-ct}\|U_0\|_2, \tag{6.48}$$

where from now on we will denote by C various positive constants which may be different at different occurrences

Our main task now is to estimate the term $\tilde{I}_2$. To do this, we split the integral $\tilde{I}_2$ into two parts:

$$\tilde{I}_2 = \int_0^{t/2}\left\|e^{(t-\tau)\Phi}G(U)\right\|_2 d\tau + \int_{t/2}^t \left\|e^{(t-\tau)\Phi}G(U)\right\|_2 d\tau = \tilde{J}_1 + \tilde{J}_2.$$

For the first integral, and since $G(U) = (0,0,0,|\psi|^p)^T$, we apply (3.5) with $k = 0$ and $\gamma = 1$, and obtain

$$\begin{aligned}\tilde{J}_1 &\leq C\int_0^{t/2}(1+t-\tau)^{-3/4}\|G(U)(\tau)\|_{1,1}d\tau \\ &\quad + C\int_0^{t/2}e^{-c(t-\tau)}\|G(U)(\tau)\|_2 d\tau \\ &= C\int_0^{t/2}(1+t-\tau)^{-3/4}\|\psi(\tau)\|_{p,1}^p d\tau + C\int_0^{t/2}e^{-c(t-\tau)}\|\psi(\tau)\|_{2p}^p d\tau.\end{aligned} \tag{6.49}$$

To estimate the term $\|\psi(\tau)\|_{p,1}^p$, we have from the Cauchy&Schwarz inequality and (6.45)

$$\begin{aligned}\|\psi(\tau)\|_{p,1}^p &= \int_{\mathbb{R}} e^{-p\delta\phi(\tau,x)}|\psi(\tau,x)|^p(1+|x|)e^{p\delta\phi(\tau,x)}dx \\ &\leq \left(\int_{\mathbb{R}} e^{2p\delta\phi(\tau,x)}|\psi(\tau,x)|^{2p}\right)^{1/2}\left(\int_{\mathbb{R}} e^{-2p\delta\phi(\tau,x)}(1+|x|)^2\right)^{1/2} \\ &\leq C(1+\tau)^{3(\rho+2)/4}\|e^{\delta\phi(\tau,x)}\psi(\tau)\|_{2p}^p.\end{aligned} \tag{6.50}$$

On the other hand, since ϕ is a positive function, then the function $e^{-\delta\phi(t,x)}$ is bounded and therefore we may estimate the norm $\|\psi(s)\|_{2p}^p$ as follows

$$\|\psi(s)\|_{2p}^p = \left\|e^{-\delta\phi(t,x)}.\psi(s)e^{\delta\phi(t,x)}\right\|_{2p}^p \leq C(1+\tau)^{(\rho+2)/4}\|e^{\delta\phi(t,x)}\psi(s)\|_{2p}^p. \tag{6.51}$$

Consequently, from (6.49), (6.50) and (6.51) we obtain

$$\begin{aligned}\tilde{J}_1 &\leq C\int_0^{t/2}(1+t-\tau)^{-3/4}(1+\tau)^{3(\rho+2)/4}\|e^{\delta\phi(\tau,x)}\psi(\tau)\|_{2p}^p d\tau \\ &\quad + C\int_0^{t/2}e^{-c(t-\tau)}(1+\tau)^{(\rho+2)/4}\|e^{\delta\phi(\tau,x)}\psi(\tau)\|_{2p}^p d\tau.\end{aligned}$$

This gives,

$$\tilde{J}_1 \leq C\int_0^{t/2}(1+t-\tau)^{-3/4}(1+\tau)^{3(\rho+2)/4}\|e^{\delta\phi(\tau,x)}\psi(\tau)\|_{2p}^p d\tau.$$

Now, for any $\varepsilon > 0$, we may write

$$\begin{aligned} \tilde{J}_1 &\leq C \int_0^{t/2} (1+\tau)^{-1-\varepsilon} (1+t-\tau)^{-3/4} \left\{ (1+\tau)^{(3(\rho+2)/4+1+\varepsilon)/p} \| e^{\delta\phi(\tau,x)} \psi(\tau) \|_{2p} \right\}^p d\tau \\ &\leq C \left(\sup_{[0,t]} (1+\tau)^{\beta} \| e^{\delta\phi(\tau,x)} \psi(\tau) \|_{2p} \right)^p \int_0^{t/2} (1+\tau)^{-1-\varepsilon} (1+t-\tau)^{-3/4} ds, \end{aligned}$$

where

$$\beta = \frac{3(\rho+2)/4+1+\varepsilon}{p}.$$

Using Lemma 2.2, we get

$$\int_0^{t/2} (1+\tau)^{-1-\varepsilon} (1+t-\tau)^{-3/4} ds \leq C (1+t)^{-3/4}.$$

Consequently,

$$\tilde{J}_1 \leq C (1+t)^{-3/4} \left(\sup_{[0,t]} (1+\tau)^{\beta} \| e^{\delta\phi(\tau,x)} \psi(\tau) \|_{2p} \right)^p. \tag{6.52}$$

By the same strategy, we get

$$\tilde{J}_2 \leq C (1+t)^{-3/4} \left(\sup_{[0,t]} (1+\tau)^{\beta} \| e^{\delta\phi(\tau,x)} \psi(\tau) \|_{2p} \right)^p. \tag{6.53}$$

Exploiting the estimates (6.48), (6.52) and (6.53), we find (6.46). This completes the proof of Lemma 6.10. □

Proof of Theorem 6.4

To prove Theorem 6.4, let us now define the functional

$$W(t) = \| e^{2\phi} U \|_2 + (1+t)^{3/4} \| U(t) \|_2. \tag{6.54}$$

Then, it follows from Lemma 6.6 and Lemma 6.10 that

$$\begin{aligned} W(t) &\leq CI_0 + C \left(\sup_{[0,t]} (1+\tau)^{\delta} \| e^{\lambda\phi(\tau,.)} \psi(\tau,.) \|_{p+1} \right)^{(p+1)/2} \\ &\quad + C (\|U_0\|_{1,1} + \|U_0\|_2) + C \left(\sup_{[0,t]} (1+\tau)^{\beta} \| e^{\delta\phi(\tau,x)} \psi(\tau) \|_{2p} \right)^p \end{aligned} \tag{6.55}$$

for $2/(p+1) < \lambda < 1$ and $\delta > 0$.

Applying Lemma 6.7 for $q = 2p$ and $\nu = \delta$, we get (see [12])

$$\begin{aligned} \| e^{\delta\phi(\tau,x)} \psi(\tau) \|_{2p} &\leq C (1+\tau)^{(2+\rho)(1-\theta(2p))/2} \| \psi_x(\tau) \|^{1-\delta} \left\| e^{\delta\phi(\tau,x)} \psi_x(\tau) \right\|^{\delta} \\ &\leq C (1+\tau)^{(2+\rho)(1-\theta(2p))/2} \left\{ (1+\tau)^{3/4} \| \psi_x(\tau) \| \right\}^{1-\delta} \\ &\quad \times (1+\tau)^{-(1-\delta)3/4} W(\tau)^{\delta} \\ &= C (1+\tau)^{(2+\rho)(1-\theta(2p))/2-(1-\delta)3/4} W(\tau). \end{aligned} \tag{6.56}$$

Similarly, we have for $q = p+1$ and $\nu = \lambda$

$$\|e^{\lambda\phi(s,.)}\psi(s,.)\|_{p+1} \leq C(1+s)^{(2+\rho)(1-\theta(p+1))/2-(1-\lambda)3/4} W(s). \tag{6.57}$$

Consequently, from (6.55), (6.56) and (6.57), we obtain

$$\begin{aligned} W(t) &\leq C(I_0 + \|U_0\|_{1,1} + \|U_0\|_2) \\ &\quad + C\left\{ \sup_{[0,t]} (1+\tau)^{\delta} (1+\tau)^{(2+\rho)(1-\theta(p+1))/2-(1-\lambda)3/4} W(\tau) \right\}^{(p+1)/2} \\ &\quad + C\left\{ \sup_{[0,t]} (1+\tau)^{\beta} (1+\tau)^{(2+\rho)(1-\theta(2p))/2-(1-\delta)3/4} W(\tau) \right\}^{p} \end{aligned}$$

Since $\lambda > 2/(p+1)$, then, we can choose λ as $\lambda = 2/(p+1) + \varepsilon_1$. Now, by the definition of $\theta(2p)$ and $\theta(p+1)$ in Lemma 6.7, we obtain

$$\begin{aligned} \kappa_1 &= \beta + \frac{(2+\rho)(1-\theta(2p))}{2} - \frac{(1-\delta)3}{4} \\ &= \left(\frac{3}{p} - \frac{1}{4}\right) + \rho\left(\frac{1}{p} + \frac{1}{4}\right) + \frac{\varepsilon}{p} + \frac{3\delta}{4}, \end{aligned}$$

and

$$\begin{aligned} \kappa_2 &= \delta + \frac{(2+\rho)(1-\theta(p+1))}{2} - \frac{(1-\lambda)3}{4} \\ &= \left(-\frac{1}{4} + \frac{10}{4(p+1)}\right) + \frac{\rho}{2}\left(\frac{1}{2} + \frac{1}{p+1}\right) + \frac{3\varepsilon_1}{4} + \delta. \end{aligned}$$

It is clear that for $12 < p$ and by choosing $\varepsilon, \delta, \rho$ and ε_1 small enough, we get $\kappa_1 < 0$ and $\kappa_2 < 0$. Consequently, we have

$$\sup_{[0,t]} W(t) \leq C(I_0 + \|U_0\|_{1,1} + \|U_0\|_2) + C\left(\sup_{[0,t]} W(\tau)\right)^{(p+1)/2} + C\left(\sup_{[0,t]} W(\tau)\right)^{p}. \tag{6.58}$$

Define

$$M(t) = \sup_{[0,t]} W(\tau),$$

Consequently, inequality (6.58) can be rewritten as

$$M(t) \leq C\left(I_1 + M(t)^p + M(t)^{p+1}\right) \tag{6.59}$$

where

$$I_1 = I_0 + \|U_0\|_{1,1} + \|U_0\|_2.$$

Then, we conclude by standard arguments (cf. [28]) that for sufficiently small I_1, we have

$$M(t) \leq I_1, \qquad \forall t \geq 0. \tag{6.60}$$

This yields

$$\|e^{2\phi}U\|_2+(1+t)^{3/4}\|U(t)\|_2\leq I_{1,}, \qquad \forall t\geq 0. \tag{6.61}$$

In addition, since

$$\psi(t)=\psi_0+\int_0^t\psi_s(s)\,ds$$

we get

$$e^{\phi(t,.)}\psi(t)=e^{\phi(t,.)}\psi_0+\int_0^t e^{\phi(t,.)}\psi_s(s)\,ds,$$

which implies by using the first inequality in (6.32)

$$\begin{aligned}\|e^{\phi(t,.)}\psi(t)\|_2 &\leq \|e^{\phi(t,.)}\psi_0\|_2+\int_0^t\|e^{\phi(t,.)}\psi_s(s)\|_2ds\\ &\leq \|e^{\phi(0,.)}\psi_0\|_2+\int_0^t\|e^{\phi(t,.)}\psi_s(s)\|_2ds.\end{aligned}$$

Then, using (6.61), we get

$$\|e^{\phi(t,.)}\psi(t)\|_2\leq\|e^{\phi(0,.)}\psi_0\|_2+I_1t \tag{6.62}$$

on $[0,T_m)$. Therefore, if $T_m<+\infty$, then the two estimates (6.61) and (6.62) imply that

$$\lim_{t\to T_m}\sup\left\{\|e^{\phi(t,.)}U(t,.)\|_2+\|e^{\phi(t,.)}\psi(t,.)\|_2\right\}<+\infty,$$

which contradicts (6.4). This gives $T_m=+\infty$. Consequently, the proof of Theorem 6.4 is thus completed.

7 Concluding remarks

In this section, we conclude with a few remarks, and future directions worth pursuing.

Remark 7.1 *We can also deal with other nonlinearities. For example* $-|\psi|^p$, $\pm|\psi|^{p-1}\psi$.

Remark 7.2 *The restriction* $p>12$ *in Theorem 6.4 is not optimal. It is an interesting open problem to study the case* $p\leq 12$. *In the case of a damped wave equation of the form*

$$u_{tt}(x,t)-\Delta u(x,t)+u_t(x,t)=|u(x,t)|^p, \qquad (x,t)\in\mathbb{R}^N\times\mathbb{R}^+, \tag{7.1}$$

Todorova and Yordanov [35] showed that the value $p_c=1+2/N$ *is the critical number. In other words, they proved that if* $p>p_c$, *then global solutions exist for small initial data. While if* $p\leq p_c$, *solutions blow up in finite time. We point out that* p_c *is the same critical exponent obtained by Fujita [4] for the problem of a nonlinear parabolic equation with negative initial data. In fact this is obvious since the solution of the linear damped wave equation behaves as* $t\to+\infty$ *like the one of the related heat equation. See [36] for more details.*

Remark 7.3 *We may apply the techniques used in the above sections to establish the optimal decay estimate for (1.1) in* $\mathbb{R}^+\times\mathbb{R}^+$ *with the boundary conditions*

$$\varphi_x(t,0)=\psi(t,0)=0, \qquad t\in\mathbb{R}^+.$$

In order to use the argument developed in the above sections, we extend our problem to the whole domain $\mathbb{R}$*. To do this, we extend the solution* φ *as an even function, and* ψ *as an odd function with respect to* $x = 0$*. That is*

$$\tilde{\varphi}(t,x) := \begin{cases} \varphi(t,x), & x \geq 0 \\ \varphi(t,-x), & x < 0 \end{cases}, \qquad \tilde{\psi}(t,x) := \begin{cases} \psi(t,x), & x \geq 0 \\ -\psi(t,-x), & x < 0 \end{cases}, \tag{7.2}$$

and

$$\tilde{\varphi}_0 \ : \ = \begin{cases} \varphi_0(x), & x \geq 0 \\ \varphi_0(-x), & x < 0 \end{cases}, \qquad \tilde{\psi}_0 := \begin{cases} \psi_0(x), & x \geq 0 \\ -\psi_0(-x), & x < 0 \end{cases}$$
$$\tilde{\varphi}_1 \ : \ = \begin{cases} \varphi_1(x), & x \geq 0 \\ \varphi_1(-x), & x < 0 \end{cases}, \qquad \tilde{\psi}_1 := \begin{cases} \psi_1(x), & x \geq 0 \\ -\psi_1(-x), & x < 0 \end{cases}$$

Consequently, we extend our problem to the following system in the whole space $\mathbb{R}$

$$\begin{cases} \tilde{\varphi}_{tt}(t,x) - (\tilde{\varphi}_x - \tilde{\psi}_x)(t,x) = 0 & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \tilde{\psi}_{tt}(t,x) - a^2 \tilde{\psi}_{xx}(t,x) - (\tilde{\varphi}_x - \tilde{\psi})(t,x) + \mu \tilde{\psi}_t(t,x) = 0 & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ (\tilde{\varphi}, \tilde{\varphi}_t, \tilde{\psi}, \tilde{\psi}_t)(0,x) = (\tilde{\varphi}_0, \tilde{\varphi}_1, \tilde{\psi}_0, \tilde{\psi}_1) & x \in \mathbb{R}. \end{cases} \tag{7.3}$$

It is clear that the unique solution $(\tilde{\varphi}, \tilde{\psi})$ *of problem (7.3) satisfies:* $\tilde{\varphi}$ *is an even function and* $\tilde{\psi}$ *is an odd function. In fact, we can easily see that* $(\tilde{\varphi}_1, \tilde{\psi}_1)$ *such that*

$$\tilde{\varphi}_1(t,x) := \begin{cases} \tilde{\varphi}(t,x), & x \geq 0 \\ \tilde{\varphi}(t,-x), & x < 0 \end{cases}, \qquad \tilde{\psi}_1(t,x) := \begin{cases} \tilde{\psi}(t,x), & x \geq 0 \\ -\tilde{\psi}(t,-x), & x < 0 \end{cases},$$

is also a solution of (7.3) and $\tilde{\varphi}_1$ *is even and* $\tilde{\psi}_1$ *is odd. Thus, the uniqueness of solutions gives us* $(\tilde{\varphi}_1, \tilde{\psi}_1) = (\tilde{\varphi}, \tilde{\psi})$*.*

A Appendix

In this Appendix, we prove the property (6.10).

Let us first show that $\hat{\psi} \in C^0\left([0,T], H^1(\mathbb{R})\right)$, then $|\hat{\psi}|^p \in C^0\left([0,T], L^2(\mathbb{R})\right)$. Indeed, by using the algebraic inequality

$$|a^p - b^p| \leq p\,|a-b|\left(|a|^{p-1} + |b|^{p-1}\right), \qquad a, b \geq 0,$$

we get

$$\begin{aligned} & \int_{\mathbb{R}} ||\hat{\psi}(t_1,x)|^p - |\hat{\psi}(t_2,x)|^p|^2\,dx \\ \leq \ & p\int_{\mathbb{R}} (|\hat{\psi}(t_1,x)| - |\hat{\psi}(t_2,x)|)^2 \underbrace{\left(|\hat{\psi}(t_1,x)|^{p-1} + |\hat{\psi}(t_2,x)|^{p-1}\right)^2}_{\leq 4\|\hat{\psi}\|^{2(p-1)}_{C^0([0,T],H^1(\mathbb{R}))}} \end{aligned}$$

Consequently, we get

$$\int_{\mathbb{R}} ||\hat{\psi}(t_1,x)|^p - |\hat{\psi}(t_2,x)|^p|^2\,dx \leq C\|\hat{\psi}\|^{2(p-1)}_{C^0([0,T],H^1(\mathbb{R}))}\|\hat{\psi}(t_1,x) - \hat{\psi}(t_2,x)\|_2^2$$

and using the fact that $\hat{\psi} \in C^0\left([0,T], L^2(\mathbb{R})\right)$, we deduce that $|\hat{\psi}|^p \in C^0\left([0,T], L^2(\mathbb{R})\right)$.

Next, we want to show that $\partial_t(|\hat{\psi}|^p) \in C^0\left([0,T], L^2(\mathbb{R})\right)$. To do this, we have first $\partial_t(|\hat{\psi}|^p) = p(\partial_t \hat{\psi})|\hat{\psi}|^{p-2}\hat{\psi}$. Consequently, applying the same argument as before, we obtain

$$
\begin{aligned}
& \int_{\mathbb{R}} \left\{ \left| \partial_t \hat{\psi}(t_1,x) |\hat{\psi}(t_1,x)|^{p-1} - \partial_t \hat{\psi}(t_2,x) |\hat{\psi}(t_2,x)|^{p-1} \right|^2 \right\} dx \\
\leq \; & 2\int_{\mathbb{R}} |\partial_t \hat{\psi}(t_1,x)|^2 \left(|\hat{\psi}(t_1,x)|^{p-1} - |\hat{\psi}(t_2,x)|^{p-1} \right) dx \\
& +2\int_{\mathbb{R}} |\partial_t \hat{\psi}(t_1,x) - \partial_t \hat{\psi}(t_1,x)|^2 \underbrace{|\hat{\psi}(t_2,x)|^{2(p-1)}}_{\leq C\|\hat{\psi}\|^{2(p-1)}_{C^0([0,T],H^1(\mathbb{R}))}} dx \\
= \; & I_1 + I_2.
\end{aligned}
$$

It is clear that

$$
I_2 \leq C\|\hat{\psi}\|^{2(p-1)}_{C^0([0,T],H^1(\mathbb{R}))} \|\hat{\psi}(t_1,x) - \hat{\psi}(t_2,x)\|_2^2
$$

and, as above, we get

$$
\begin{aligned}
I_1 \; & \leq \; C\int_{\mathbb{R}} |\partial_t \hat{\psi}(t_1,x)|^2 \left(|\hat{\psi}(t_1,x)| - |\hat{\psi}(t_2,x)| \right)^2 \|\hat{\psi}\|^{2(p-1)}_{C^0([0,T],H^1(\mathbb{R}))} dx \\
& \leq \; C\|\hat{\psi}\|^2_{C^1([0,T],L^2(\mathbb{R}))} \|\hat{\psi}\|^{2(p-1)}_{C^0([0,T],H^1(\mathbb{R}))} \|\hat{\psi}(t_1,x) - \hat{\psi}(t_2,x)\|_2^2,
\end{aligned}
$$

which gives the desired result.

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