Instability of coupled systems with delay[1]

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Abstract: We consider linear initial-boundary value problems that are a coupling like second-order thermoelasticity, or the thermoelastic plate equation or its generalization (the α-β-system introduced in [1, 26]). Now, there is a delay term given in part of the coupled system, and we demonstrate that the expected inherent damping will not prevent the system from not being stable; indeed, the systems will shown to be ill-posed: a sequence of bounded initial data may lead to exploding solutions (at any fixed time).

1 Introduction

It is well-known that delay equations like the simplest one of parabolic type,

$$\theta_t(t,x) = \Delta\theta(t-\tau), \tag{1.1}$$

with a delay parameter $\tau > 0$, or of hyperbolic type,

$$u_{tt}(t,x) = \Delta u(t-\tau), \tag{1.2}$$

are not well-posed. Their instability is given in the sense that there is a sequence of initial data remaining bounded, while the corresponding solutions, at a fixed time, go to infinity in an exponential manner, see Jordan, Dai & Mickens [12] and Dreher, Quintanilla & Racke [9], or Prüß [29], in particular for connections to Volterra equations. Indeed, it was shown in [9] that the same phenomenon of instability is given for a general class of problems of the type

$$\frac{d^n}{dt^n}u(t) = Au(t-\tau), \tag{1.3}$$

$n \in \mathbb{N}$ fixed, whenever $(-A)$ is linear operator in a Banach space having a sequence of real eigenvalues $(\lambda_k)_k$ such that $0 < \lambda_k \to \infty$ as $k \to \infty$.

Delay equations are well motivated from the applications, cf. [29], Chandrasekharaiah [5], Bátkai & Piazzera [4]. For example, to have an alternative to the classical heat equation, which

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corresponds to $\tau = 0$ and shows the physically not justified phenomenon of infinite propagation speed of signals, introducing a delay in the constitutive law can be done as follows: Heat conduction is usually described by means of the energy equation

$$\theta_t + \gamma \operatorname{div} q = 0 \tag{1.4}$$

for the temperature θ and the heat flux vector q. With the constitutive law

$$q(t+\tau, \cdot) = -\kappa \nabla \theta(t, \cdot), \tag{1.5}$$

where $\gamma, \kappa > 0$, which expresses that a change of the temperature gradient at time t is effective in the heat flux only with a delay $\tau > 0$, we obtain the delay equation

$$\theta_t(t, \cdot) = \kappa\gamma \Delta\theta(t-\tau, \cdot). \tag{1.6}$$

Adding certain non-delay terms, e.g. $\Delta\theta(t,x)$ om the right-hand side of (1.1), is already sufficient to obtain a well-posed problem, cf. [29, 4]. We consider coupled systems where there is an expected damping effect of a second differential equation in comparison to the first one, and we ask if there is still instability (ill-posedness) or whether we can have well-posedness. We shall answer this with several ill-posedness results.

For *well-posedness results for wave equations* equations with delay terms (in the interior),there are a number of papers by Nicaise and co-authors Ammari, Fridman, Pignotti, and Valein [27, 28, 10, 2], in [27] also with instability results. For works with delay terms in the *boundary* conditions, see the references in the papers [14].

For the system of *coupled wave equations* of Timoshenko type with delay terms of the type

$$\begin{aligned} \rho_1\phi_{tt}(t,x) - K(\phi_x + \psi)_x(t,x) &= 0, & (1.7)\\ \rho_2\psi_{tt}(t,x) - b\psi_{xx}(t,x) + K(\phi_x+\psi) + \mu_1\psi_t(t,x) + \mu_2\psi_t(t-\tau,x) &= 0 & (1.8) \end{aligned}$$

the *well-posedness* (under certain conditions on μ_1, μ_2) was investigated by Said-Houari & Laskri [34] and extended to a time-varying delay term — replacing $\psi_t(t-\tau,x)$ by $\psi_t(t-\tau(t),x)$ — in the work of Kirane, Said-Houari & Anwar [14].

Here we consider *coupled systems of different types.* A typical first example is the coupling arising in thermoelasticity. In one dimension, we have the hyperbolic-parabolic system

$$\begin{aligned} u_{tt}(t,x) - au_{xx}(t-\tau,x) + b\theta_x(t,x) &= 0, & (1.9)\\ \theta_t(t,x) - d\theta_{xx}(t,x) + bu_{tx}(t,x) &= 0, & (1.10) \end{aligned}$$

where u describes the displacement, and θ is the temperature difference, and where $t \geq 0$ and $x \in (0;L) \subset \mathbb{R}$ with $L > 0$. To complete the initial-boundary value problem, we consider the boundary conditions

$$u(t,x) = \theta_x(t,x) = 0, \tag{1.11}$$

for $t \geq 0$ and $x \in \{0, L\}$, and initial conditions for $u(0,\cdot)$, $u_t(0,\cdot)$, $u(s,\cdot)$, $u_t(s,\cdot)$ for $-\tau \leq s \leq 0$, and for $\theta(0,\cdot)$. The damping through heat conduction essentially given in (1.10) is for classical thermoelasticity — corresponding to $\tau = 0$ — strong enough to compose an exponentially stable system (modulo constant functions θ due to the boundary condition), thus strongly impacting the oscillating part of the (pure) wave equation ($u_{tt} - au_{xx} = 0$), see Racke [32] or Jiang & Racke [11] for extensive surveys, or, more specific, Racke [30, 31].

Here, we shall prove that the system with delay (1.9), (1.10) is not well-posed and instable, that is, the damping through heat conduction turns out to be not strong enough now; the instable system part ($u_{tt}(t,\cdot) - au_{xx}(t-\tau,\cdot) = 0$) will predominate.

Then even more expected, the same will happen for the system, where one delay is given in the equation for the temperature, i.e. for

$$u_{tt}(t,x) - au_{xx}(t,x) + b\theta_x(t,x) = 0, \tag{1.12}$$

$$\theta_t(t,x) - d\theta_{xx}(t-\tau,x) + bu_{tx}(t,x) = 0. \tag{1.13}$$

The classical thermoelastic plate equation, a coupling of the plate equation (with the Schrödinger equation behind) with heat conduction will then be investigated in the same way. Here we have the system

$$u_{tt}(t,x) + a\Delta^2 u(t-\tau,x) + b\Delta\theta(t,x) = 0, \tag{1.14}$$

$$\theta_t(t,x) - d\Delta\theta(t,x) - b\Delta u_t(t,x) = 0, \tag{1.15}$$

where $x \in G \subset \mathbb{R}^n$, $n \in \mathbb{N}$ now, G bounded, and the corresponding one with the delay term as in (1.12), (1.13). Initial conditions are given as usual, and we consider the boundary conditions

$$u(t,x) = \Delta u(t,x) = \theta(t,x) = 0. \tag{1.16}$$

These thermoelastic plate equations (1.14), (1.15) — in comparison to the thermoelastic system above being, for $\tau = 0$, exponentially stable also in space dimension $n \geq 2$ — will also turn out to be ill-posed now.

The thermoelastic plate system (1.14), (1.15) has been widely discussed in particular for bounded reference configurations $G \ni x$, see the work of Kim [13], Muñoz Rivera & Racke [25], Liu & Zheng [23], Avalos & Lasiecka [3], Lasiecka & Triggiani [16, 17, 18, 19] for the question of exponential stability of the associated semigroup (for various boundary conditions), and Russell [33], Liu & Renardy [20], Liu & Liu [21], Liu & Yong [22] for proving its analyticity, see also the book of Liu & Zheng [24] for a survey. For results in exterior domains see, for example, Muñoz Rivera & Racke [26], Denk, Racke & Shibata [7, 8].

The thermoelastic plate system (1.14), (1.15) is a special case of the so-called α-β-system, now with delay,

$$u_{tt}(t) + aAu(t-\tau) - bA^\beta\theta(t) = 0, \tag{1.17}$$

$$\theta_t(t) + dA^\alpha\theta(t) + bA^\beta u_t(t) = 0, \tag{1.18}$$

for functions $u, \theta : [0,\infty) \to \mathcal{H}$, with A being a self-adjoint operator in the Hilbert space H, having a countable complete orthonormal system of eigenfunctions $(\phi_j)_j$ with corresponding eigenvalues $0 < \lambda_j \to \infty$ as $j \to \infty$. The thermoelastic plate equations appear with $\alpha = \beta = \frac{1}{2}$ and $A = (-\Delta_D)^2$, where $-\Delta_D$ denotes the Laplace operator realized in $L^2(G)$ on some bounded domain G in $\mathbb{R}^n$ with Dirichlet boundary conditions. This original α-β-system without delay ($\tau = 0$) was introduced by Muñoz Rivera & Racke [26] and, independently, by Ammar Khodja & Benabdallah [1], and investigated with respect to exponential stability and analyticity of the associated semigroup, the latter also for the Cauchy problem, where $\Omega = \mathbb{R}^n$, and, more general, $A = (-\Delta)^\eta$ for $\eta > 0$ arbitrary, and in arbitrary L^p-spaces for $1 < p < \infty$, see Denk & Racke [6]. It was shown that we have a strong smoothing property for parameters (β, α) in the region $\mathcal{A}_{sm}$ (see Figure 1.1), where

$$\mathcal{A}_{sm} := \{(\beta, \alpha) \,|\, 1 - 2\beta < \alpha < 2\beta, \alpha > 2\beta - 1\}, \qquad (\tau = 0), \tag{1.19}$$

and that the analyticity (in $L^p(\mathbb{R}^n)$) is given in the region $\mathcal{A}_{an}$ (see Figure 1.2), where

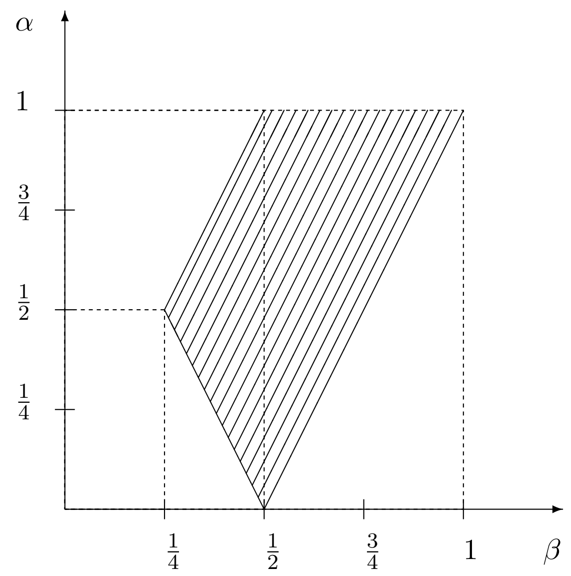

Figure 1.1: Area of smoothing $\mathcal{A}_{sm}$ (without delay)

$$\mathcal{A}_{an} := \{(\beta, \alpha) \,|\; \alpha \geq \beta,\; \alpha \leq 2\beta - 1/2\}, \qquad (\tau = 0). \tag{1.20}$$

Here, we shall show that the α-β-system with delay (1.17), (1.18) is not well-posed in the region $\mathcal{A}^1_{in}$ (see Figure 1.3), where

$$\mathcal{A}^1_{in} := \{(\beta, \alpha) \,|\, 0 \leq \beta \leq \alpha \leq 1,\, \alpha \geq \frac{1}{2},\, (\beta, \alpha) \neq (1, 1)\}, \tag{1.21}$$

A similar result will hold for the related system

$$\begin{aligned} u_{tt}(t) + aAu(t) - bA^\beta \theta(t) &= 0, & (1.22)\\ \theta_t(t) + dA^\alpha \theta(t - \tau) + bA^\beta u_t(t) &= 0, & (1.23) \end{aligned}$$

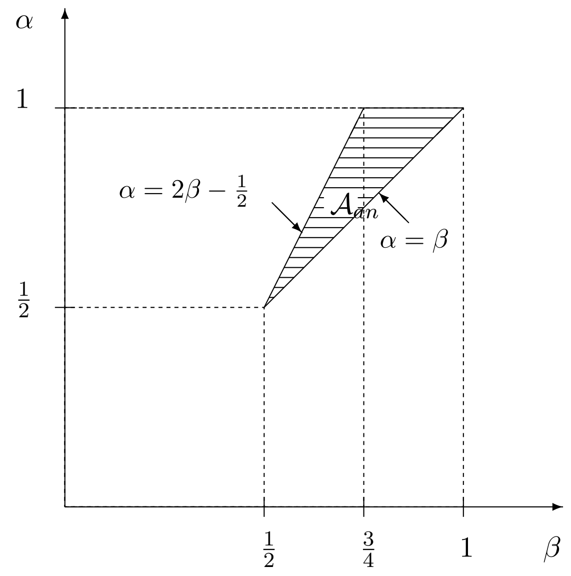

Figure 1.2: Area of analyticity $\mathcal{A}_{an}$ (without delay)

in the region $\mathcal{A}_{in}^2$ (see Figure 1.4), where

$$\mathcal{A}_{in}^2 := \{(\beta,\alpha)\,|\,0 < \beta \le \alpha \le 1,\ (\beta,\alpha) \ne (1,1)\}. \tag{1.24}$$

It is interesting to notice that there are differences comparing the regions $\mathcal{A}_{in}^1$ and $\mathcal{A}_{in}^1$. For the former, the damping through the main equation for θ has to be weak enough ("$\alpha \ge \frac{1}{2}$") to still guarantee the ill-posedness suggested by the main equation with delay for u.

The behavior in the regions outside $\mathcal{A}_{in}^j$, $j = 1,2$, is an open question.

We remark that the α-β-system (without delay) has been recognized to possibly describe also viscoelastic systems, and, with respect to smoothing properties, even the second-order thermoelastic system from above, although in the latter case it is (first) formally not of this type; but after deriving a single differential equation of third order in time for u (or θ) only, the α-β-formalism applies, see [26].

The methods to prove the results mentioned up to now will be to construct exponentially growing solutions with the help of an ansatz trough eigenfunctions, and then modifying and extending ideas from [9] to the situation of coupled systems given here.

We remark that it would be possible to study the delay term at different places, actually to discuss systems like

$$u_{tt}(t) + aAu(t - g_1\tau) + bA^\beta\theta(t - g_2\tau) = 0, \tag{1.25}$$

$$\theta_t(t) + dA^\alpha\theta(t - g_3\tau) + bA^\beta u_t(t - g_4\tau) = 0, \tag{1.26}$$

where $g_j \in \{0,1\}$, $j = 1,2,3,4$. Here we studied the cases $g_1 - 1 = g_2 = g_3 = g_4 = 0$ and $g_1 = g_2 = g_3 - 1 = g_4 = 0$ only, for simplicity of the presentation.

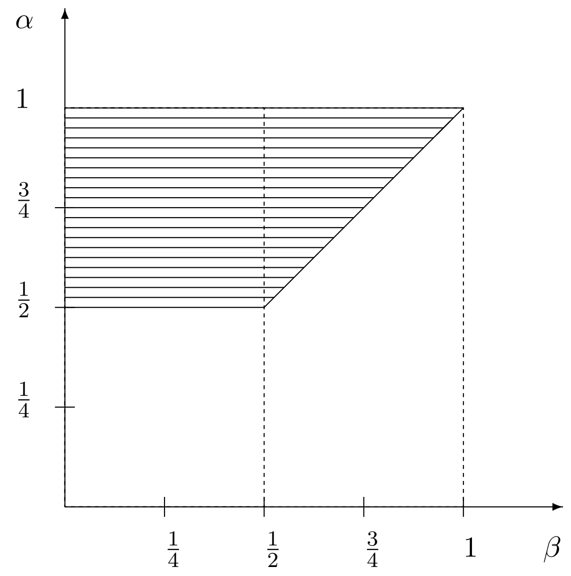

Figure 1.3: Area of instability $\mathcal{A}_{in}^1$ (with delay)

The paper is organized as follows: In Section 2 we shall discuss the second-order thermoelastic systems and prove that the systems with delay are not well-posed, and in Section 3 the thermoelastic plate equations are discussed in a similar manner. In Section 4 the α-β-system with delay will be studied proving the ill-posedness in a certain parameter region. In the appendix, we recall some arguments from [9].

L^p denotes the usual L^p-space of Lebesgue-integrable functions, $\|\cdot\|$ and $\|\cdot\|_{\mathcal{H}}$ denote the norm in L^2 and in a Hilbert space $\mathcal{H}$, respectively, and $\frac{d}{dt}$ or subscripts t or x denote (partial) derivatives.

2 Second-order thermoelasticity with delay terms

We consider the thermoelastic system with delay given in (1.9), (1.10), i.e.

$$u_{tt}(t,x) - au_{xx}(t-\tau,x) + b\theta_x(t,x) = 0, \tag{2.1}$$
$$\theta_t(t,x) - d\theta_{xx}(t,x) + bu_{tx}(t,x) = 0, \tag{2.2}$$

and the related system (1.12), (1.13), i.e.

$$u_{tt}(t,x) - au_{xx}(t,x) + b\theta_x(t,x) = 0, \tag{2.3}$$
$$\theta_t(t,x) - d\theta_{xx}(t-\tau,x) + bu_{tx}(t,x) = 0, \tag{2.4}$$

Here $u,\theta : [0,\infty]x(0,L) \to \mathbb{R}, L > 0$, and a,b,d are positive constants, $\tau > 0$ is the — in applications often relatively small — relaxation parameter.
Both complex systems are completed with the boundary condition

$$u(t,x) = \theta_x(t,x) = 0, \tag{2.5}$$

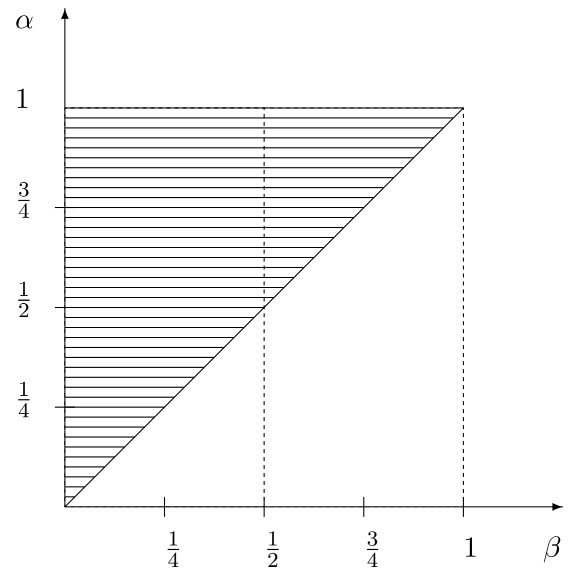

Figure 1.4: Area of instability $\mathcal{A}^2_{in}$ (with delay)

for $t \geq 0$ and $x \in \{0, L\}$, and with initial conditions

$$u(s,\cdot) = u^0(s), u_t(s,\cdot) = u^1(s), (-\tau \leq s \leq 0),\ \theta(0,\cdot) = \theta^0, \tag{2.6}$$

and

$$u(0,\cdot) = u^0, u_t(0,\cdot) = u^1, \theta(s,0) = \theta^0(s), (-\tau \leq s \leq 0), \tag{2.7}$$

respectively.
These systems are shown to be not well-posed, i.e. we prove

Theorem 2.1 *(i) The initial-boundary value problem with delay (2.1), (2.2), (2.5), (2.6) is not well-posed. There exists a sequence* $\big((u_j, \theta_j)\big)_j$ *of solutions with* $L^2 - norm \|u_j(t,\cdot)\|$ *tending to infinity* ($as\, j \to \infty$) *for any fixed* $t > 0$, *while for the initial data the norms* $\sup_{-\tau \leq s \leq 0} \|(u_j^0(s), u_j^1(s), \theta_j^0)\|$ *remain bounded.*

(ii) The corresponding statement on ill-posedness also holds for the initial-boundary value problem with delay (2.3), (2.4), (2.5), (2.7).

We already remark that we can also manage to keep the spatial gradient of u^0 bounded, see the detailed remarks following the proof.

Proof of (i): We make the ansatz

$$u = u_j(t,x) = \underbrace{\sqrt{\frac{2}{L}}\sin(\frac{j\pi}{L}x)}_{=:\varphi_j(x)} h_j(t), \tag{2.8}$$

$$\theta = \theta_j(t,x) = \sqrt{\frac{2}{L}}\cos(\frac{j\pi}{L}x)g_j(t), \tag{2.9}$$

and try to find h_j (and g_j) such that $h_j(t) \to \infty$ as $j \to \infty$, while the initial data remain bounded. Actually, h_j will be of the form $h_j(t) = c_j e^{\omega_j t}$ with $\Re\omega_j \to \infty$ as $j \to \infty$, see below. Let $\lambda_j := \frac{j\pi}{L}$. Plugging the ansatz (2.8), (2.9) into the differential equations (2.1), (2.2) we conclude that (h_j, g_j) should satisfy (as necessary and sufficient condition)

$$h_j''(t) + a\lambda_j^2 h_j(t-\tau) - b\lambda_j g_j(t) = 0, \tag{2.10}$$

$$g_j'(t) + d\lambda_j^2 g_j(t) + b\lambda_j h_j'(t) = 0, \tag{2.11}$$

where a prime "$'$" denotes a one-dimensional derivative.
Additionally we have initial conditions for h_j and for g_j that will be specified below.
(2.10) implies

$$b\lambda_j g_j(t) = h_j''(t) + a\lambda_j^2 h_j(t-\tau) \tag{2.12}$$

and

$$b\lambda_j g_j'(t) = h_j'''(t) + a\lambda_j^2 h_j'(t-\tau). \tag{2.13}$$

(2.11) implies

$$b\lambda_j g_j'(t) + d\lambda_j^2 b\lambda_j g_j'(t) + b^2\lambda_j^2 h_j'(t) = 0. \tag{2.14}$$

Combining (2.12) - (2.14) we obtain

$$h_j'''(t) + d\lambda_j^2 h_j''(t) + b^2\lambda_j^2 h_j'(t) + a\lambda_j^2 h_j'(t-\tau) + ad\lambda_j^4 h_j(t-\tau) = 0. \tag{2.15}$$

Conversely, if (h_j, g_j) satisfy (2.15) and

$$g_j'(t) + d\lambda_j^2 g_j(t) = -b\lambda_j h_j'(t), \tag{2.16}$$

with

$$g_j(0) = \frac{1}{b\lambda_j}(h_j''(0) + a\lambda_j^2 h_j(-\tau)), \tag{2.17}$$

then (h_j, g_j) solve (2.10), (2.11). This can be seen as follows: Let

$$w_\tau(t) := h_j''(t) + a\lambda_j^2 h_j(t-\tau) - b\lambda_j g_j(t), \tag{2.18}$$

then we have, using (2.15), (2.16),

$$\begin{aligned} \frac{d}{dt}w_\tau(t) + d\lambda_j^2 w_\tau(t) &= h_j'''(t) + a\lambda_j^2 h_j'(t-\tau) - b\lambda_j g_j'(t) \\ &\quad + a\lambda_j^2 h_j''(t) + ad\lambda_j^4 h_j(t-\tau) - b\lambda_j d_j^2 g_j(t) \\ &= h_j'''(t) + d\lambda_j^2 h_j''(t) + a\lambda_j^2 h_j'(t-\tau) \\ &\quad + ad\lambda_j^4 h_j(t-\tau) - b\lambda_j \underbrace{(g_j'(t) + d\lambda_j^2 g_j(t))}_{=-b\lambda_j h_j'(t)} \\ &= 0. \end{aligned} \tag{2.19}$$

Moreover,

$$w_\tau(0) = h_j''(0) + a\lambda_j^2 h_j(-\tau) - b\lambda_j g_j(0) = 0, \tag{2.20}$$

by (2.17). Thus, by (2.19), (2.20), we conclude

$$w_\tau(t) = 0, \qquad t \geq 0,$$

hence (2.10) is satisfied, while (2.11) is given by (2.16) (which was to be proved). Now we can make the following ansatz for h_j;

$$h_j(t) = \frac{1}{\omega_j^2} e^{\omega_j t}, \tag{2.21}$$

where $(\omega_j)_j$ will be determined such that $\Re\omega_j \to \infty$ as $j \to \infty$.
The initial data for h_j will remain bounded as $j \to \infty$,

$$h_j(s) = \frac{1}{\omega_j^2} e^{\omega_j s}, h_j'(s) = \frac{1}{\omega_j} e^{\omega_j s}, -\tau \leq s \leq 0. \tag{2.22}$$

Then g_j will be determined as solution to (2.16) with initial value (2.17), i.e.

$$g_j(0) := \frac{1}{b\lambda_j}\big(1 + \frac{a\lambda_j^2}{\omega_j^2} e^{-\omega_j \tau}\big), \tag{2.23}$$

and $(g_j(0))_j$ will also be shown to be a bounded sequence. In order to satisfy the equation (2.15) with the ansatz (2.21) it is sufficient and necessary that ω_j satisfies

$$\omega_j^3 + d\lambda_j^2\omega_j^2 + (b^2\lambda_j^2 + a\lambda_j^2 e^{\omega_j\tau})\omega_j = -ad\lambda_j^4 e^{-\omega_j\tau}. \tag{2.24}$$

If we can find $(\omega_j)_j$ such that the following three conditions (2.25) - (2.27) are satisfied, then part (i) of Theorem (2.1) will be proved.
For a subsequence $(\omega_{j_k})_k, j_k \to \infty$ as $k \to \infty$,

$$\Re\omega_{j_k} \to \infty \qquad \text{as } k \to \infty, \tag{2.25}$$

$$\sup_k |\frac{\lambda_{j_k}^2}{\omega_{j_k}^2} e^{-\omega_{j_k}\tau}| < \infty, \tag{2.26}$$

(to assure the boundedness of $(g_j(0))_j$),

$$|\frac{e^{\omega_{j_k} t}}{\omega_{j_k}^2}| \to \infty \qquad \text{as } k \to \infty. \tag{2.27}$$

We shall now prove the solvability of (2.24) and the properties (2.25) - (2.27). For simplicity we (first for a while) drop the index j, i.e. we write $\omega = \omega_j, \lambda = \lambda_j$, and so on. Then (2.24) is equivalent to

$$\omega^2\big(1 + \frac{\omega}{d\lambda^2} + \frac{b^2/d + a/de^{-\omega\tau}}{\omega}\big) = -a\lambda^2 e^{-\tau\omega}. \tag{2.28}$$

To solve this we make the ansatz (as in [9])

$$\omega = \mu(1 + \zeta) \tag{2.29}$$

where $|\zeta| < \frac{1}{2}$, and where $\mu = \mu_j$ solves

$$\mu^2 = -a\lambda^2 e^{-\tau\mu}. \tag{2.30}$$

This problem (2.30) has solutions $\mu = \mu_{j_k}$, for a subsequence $j_k \to \infty$, with $\Re\mu_{j_k} \to \infty$ as $k \to \infty$, according to the proof of Theorem 2.1 in [9], which we repeat in the appendix for the reader's convenience. Then (2.28) is equivalent to solving

$$(1+\zeta)^2(1+q(\zeta)) = e^{-\tau\mu\zeta}, \tag{2.31}$$

where

$$q(\zeta) := \frac{\mu(1+\zeta)}{d\lambda^2} + \frac{b^2/d + a/de^{-\tau\mu(1+\zeta)}}{\mu(1+\zeta)}. \tag{2.32}$$

(2.31) is equivalent to

$$\underbrace{(1-e^{-\tau\mu\zeta})}_{=:f(\zeta)} + \underbrace{(g(\zeta) + (2\zeta+\zeta^2)(1+q(\zeta))}_{=:g(\zeta)} = 0. \tag{2.33}$$

f and g are holomorphic in $\Omega := \{\zeta \mid |\zeta| < \frac{1}{10\tau|\mu|} \equiv B(0, \frac{1}{10\tau|\mu|})$. f has in Ω exactly one zero ($\zeta = 0$) since w.l.o.g. $\frac{1}{10\tau|\mu|} < \frac{1}{2}$. f satisfies on $\partial\Omega$:

$$|f(\zeta)| \geq \inf_{10\tau|\mu||\zeta|=1} |e^{-\tau\mu\zeta} - 1| = \inf_{10|z|=1} |e^{-z} - 1| =: \bar{f} > 0, \tag{2.34}$$

$\bar{f}$ being of $\mu = \mu_j$. Moreover, we have on $\partial\Omega$

$$|\zeta| \leq \frac{C}{|\mu|}, \tag{2.35}$$

where $C = \frac{1}{10\tau}$ here, but will denote constructs being independent of j. We notice that, w.l.o.g., $10\tau|\mu| < 1/2$, and that

$$|q(\zeta)| \leq C\big(|\frac{\mu}{\lambda^2}| + \frac{1}{|\mu|}\big) \leq \frac{C}{|\mu|}, \tag{2.36}$$

where we used (2.30) and $|e^{-\tau\mu\zeta}| \leq e^{1/10}$ as well as

$$|\frac{\mu}{\lambda^2}| = \frac{1}{|\mu|} a e^{-\tau\Re\mu} \leq \frac{C}{|\mu|}. \tag{2.37}$$

We conclude from (2.36) and (2.35)

$$|g(\zeta)| \leq \frac{C}{|\mu|}. \tag{2.38}$$

Combining (2.33), (2.34) and (2.38) we get with Rouche's theorem that there is exactly one solution $\zeta = \zeta_{j_k}$ in $\Omega = \Omega_j = B(0, \frac{1}{10\tau|\mu_{j_k}|})$, and $\omega_{j_k} = \mu_{j_k}(1+\zeta_{j_k})$ solves (2.24). To prove (2.25) we observe that

$$\arg(\omega_{j_k}) = \arg(\mu_{j_k}) + \arg(1+\zeta_{j_k}) \leq \frac{\pi}{8} + \frac{\pi}{4} < \frac{\pi}{2}, \tag{2.39}$$

where we used that $|\zeta_{j_k}| < \frac{1}{2}$ and the fact that $\arg(\mu_{j_k}) \le \frac{\pi}{8}$ which arises from the construction of μ_{j_k} in the proof of Theorem 1.1 in [9]. As a consequence we conclude the validity of (2.25). Using (2.30) we get

$$|\frac{\lambda_{j_k}^2}{\omega_{j_k}^2}\, e^{-\tau\omega_{j_k}}| \le C|e^{-\tau\mu_{j_k}\zeta_{j_k}}| \le C$$

which assures (2.26).

Finally, we prove (2.27) as follows. For $0 < \varepsilon < 1$ we have

$$\begin{aligned} |\frac{e^{\omega_{j_k}t}}{\omega_{j_k}^2}| &\ge C|\frac{e^{\mu_{j_k}t}}{\mu_{j_k}^2}||e^{\mu_{j_k}\zeta_{j_k}t}| \\ &= C|\frac{e^{\varepsilon\mu_{j_k}t}}{\mu_{j_k}^2}||e^{\mu_{j_k}(1-\varepsilon+\zeta_j)t}|. \end{aligned} \tag{2.40}$$

Observing

$$|\frac{e^{\varepsilon\mu_{j_k}t}}{\mu_{j_k}^2}| \ge \frac{e^{\varepsilon\cos(\frac{\pi}{8})|\mu_{j_k}|t}}{|\mu_{j_k}|^2} \to \infty \text{ as } k \to \infty, \tag{2.41}$$

(cf. [9]), and

$$\arg\left(\mu_{j_k}(1-\varepsilon+\zeta_{j_k})\right) \le \frac{\pi}{8} + \arg(1-\varepsilon+\zeta_{j_k}) < \frac{\pi}{2}$$

for ε small enough, implying

$$|e^{\mu_{j_k}(1-\varepsilon\zeta_{j_k})t}| \ge 1, \tag{2.42}$$

we get, combining (2.40) - (2.42) the desired relation (2.27), and the proof of part (i) is finished. We remark that the behavior of g_j, being determined through (2.16), (2.17), is open.

Proof of (ii): Making the same ansatz for (u_j, θ_j) as in (2.8), (2.9) we derive the equations

$$h_j''(t) + a\lambda_j^2 h_j(t) = b\lambda_j g_j(t), \tag{2.43}$$

$$g_j'(t) + a\lambda_j^2 g_j(t-\tau) + b\lambda_j h_j'(t) = 0, \tag{2.44}$$

from which the differential equation

$$g_j'''(t) + d\lambda_j^2 g_j''(t-\tau) + b^2\lambda_j^2 g_j'(t) a\lambda_j^2 g_j'(t) + ad\lambda_j^4 g_j(t-\tau) = 0 \tag{2.45}$$

follows (cp. with (2.15)).

If g_j solves (2.45) (with given initial conditions) and if h_j solves (2.43) with initial conditions satisfying

$$h_j(0) = \frac{1}{ba\lambda_j^3}\left(b\lambda_j g_j(0) + g_j''(0) + d\lambda_j^2 g_j'(-\tau)\right), \tag{2.46}$$

$$h_j'(0) = -\frac{1}{b\lambda_j}\left(g_j'(0) + a\lambda_j^2 g_j(-\tau)\right), \tag{2.47}$$

then (h_j, g_j) solves (2.43), (2.44), which can be seen looking at

$$w_\tau(t) := g_j'(t) + d\lambda_j^2 g_j(t-\tau) + b\lambda_j h_j'(t) \tag{2.48}$$

and deriving the relation

$$\frac{d^2}{dt^2}w_\tau(t) + a\lambda_j^2 w_\tau(t) = 0, \tag{2.49}$$

$$w_\tau(0) = 0, \quad \frac{d}{dt}w_\tau(0) = 0, \tag{2.50}$$

the latter given by (2.46), (2.47). This implies $w_\tau(t) = 0, \quad t \geq 0$, which is equivalent to (2.44). For g_j we make the ansatz

$$g_j(t) = \frac{1}{\omega_j^2}e^{\omega_j t} \tag{2.51}$$

implying the boundedness of the initial data $g_j(s), g'(s), g_j''(s), -\tau \leq s \leq 0$, as $j \to \infty$ since $\Re\omega_{j_k} \to \infty$ will be shown ($k \to \infty, j_k \to \infty$). h_j will then be determined by (2.43), (2.46), (2.47), and the data prescribed in (2.46), (2.47) will be shown to be bounded too.
In order to satisfy the equation (2.45) with the ansatz (2.51) it is sufficient (and necessary) that ω_j satisfies

$$\omega_j^3 + d\lambda_j^2\omega_j^2 e^{-\tau\omega_j} + b^2\lambda_j^2\omega_j + a\lambda_j^2\omega_j = -ad\lambda_j^4 e^{-\tau\omega_j}. \tag{2.52}$$

If we can find $(\omega_j)_j$ such that the following three conditions (2.53) - (2.55) are satisfied, then part (ii) will be proved.
For a subsequence $(\omega_{j_k})_k, j_k \to \infty$ as $k \to \infty$,

$$\Re\omega_{j_k} \to \infty \text{ as } k \to \infty, \tag{2.53}$$

$$\sup_k |\frac{\lambda_{j_k}e}{\omega_{j_k}^2}e^{-\tau\omega_j}| < \infty, \tag{2.54}$$

(to assure the boundedness of $(hj_k(0), hj_k'(0))_k$),

$$|\frac{e^{\omega_{jn}t}}{\omega_{j_k}^2}| \to \infty \text{ as } k \to \infty. \tag{2.55}$$

Dropping the index j again, (2.52) is equivalent to

$$\omega\big(1 + \frac{\omega^2}{(a+b^2)\lambda_j^2} + \frac{d}{(a+b^2)}\omega e^{-\tau\omega}\big) = -\frac{ad}{a+b^2}\lambda_j^2 e^{-\tau\omega}. \tag{2.56}$$

Remark: Wether we choose ω^2 (as in part (i), cp.(2.28)) or ω (in (2.56)) as a factor depends on the powers of λ_j and the position of $e^{-\tau\omega}$ on the left-hand side. The aim is to get the right estimate (2.63) for q below (cp. (2.36)).
Let $\mu = \mu_{j_k}$ be the solution(s) to

$$\mu = -\frac{ad}{a+b^2}\lambda_j^2 e^{-\tau\mu} \tag{2.57}$$

which exist according to Theorem 1.1 (proof) in [9], satisfying $\Re\mu_{j_k} \to \infty$ for a subsequence as $j_k \to \infty$ for $k \to \infty$. Then, with the ansatz

$$\omega = \mu(1+\zeta), \quad |\zeta| < 1/2, \tag{2.58}$$

(2.56) is equivalent to solving

$$\underbrace{(1-e^{-\tau\mu\zeta})}_{=:f(\zeta)}+\underbrace{(g(\zeta)+\zeta+\zeta g(\zeta))}_{=:g(\zeta)}=0, \tag{2.59}$$

where

$$q(\zeta):=\frac{\mu(1+\zeta)}{(a+b^2)\lambda_j^2}+\frac{d}{a+b^2}\mu(1+\zeta)e^{-\tau\mu(1+\zeta)}. \tag{2.60}$$

f and g are holomorphic in $\Omega:=B(0,\frac{1}{10\tau|\mu|})$. f has exactly one zero ($\zeta=0$) in Ω since w.l.o.g. $\frac{1}{10\tau|\mu|}<\frac{1}{2}$. On $\partial\Omega$ we have again (cp. (2.34))

$$|f(\zeta)|\geq\bar{f}>0, \tag{2.61}$$

moreover, using (2.57),

$$|\zeta|\leq\frac{C}{|\mu|}, \tag{2.62}$$

$$|q(\zeta)|\leq\frac{C}{|\mu|}|\mu e^{-\tau\mu}|\leq\frac{C}{|\mu|} \tag{2.63}$$

since $|\mu|e^{-\tau\Re\mu}=|\mu|e^{-\tau|\mu|\cos(\arg(\mu))\to 0}$.
Hence

$$|g(\zeta)|\leq\frac{C}{|\mu|} \tag{2.64}$$

and (2.61), (2.64) combined with Rouché's theorem gives exactly one solution $\zeta=\zeta_{j_k}$ in Ω to (2.59), and $\omega_{j_k}=\mu_{j_k}(1+\zeta_{j_k})$ solves (2.52). The relation (2.53) is proved as in (2.39) (replacing $\pi/8$ by $\pi/4$).
Using (2.58) we get

$$|\frac{\lambda_{j_k}}{\omega_{j_k}^2}e^{-\tau\omega_{j_k}}|=\frac{1}{|\lambda_{j_k}\omega_{j_k}|}\frac{1}{|1+\zeta_{j_k}|}|\,e^{-\tau\mu_{j_k}\zeta_{j_k}}|\leq C,$$

proving (2.54).
Finally, (2.55) follows as in the proof of part (i), see (2.40) - (2.42) (replacing $\pi/8$ by $\pi/4$).
Thus, Theorem 2.1 is proved.
Q.e.d.
Remark: We have proved the exploding of the solution in L^2 for bounded data in L^2. Coming from semigroup theory in the case $\tau=0$ (without delay), one might argue that usually $V:=(u_x,u_t,\theta)$ in L^2 or $V:=(u,u_t,\theta)$ with gradient norm for u is considered, and, hence, one should prove the exploding of u_x in L^2 for data with bounded norm u_x^0. But this can also be achieved replacing in part (i) — for example — the ansatz (2.21) by

$$h_j(t)=\frac{1}{\lambda_j\omega_j^2}\,e^{\omega_j t}. \tag{2.65}$$

Then everything (but one thing) carries over literally; only the arguments to prove now

$$|\frac{e^{\omega_{j_k}t}}{\lambda_{j_k}\omega_{j_k}^2}|\to 0,\quad k\to\infty$$

given before in (2.40) - (2.42) have to be slightly modified in (2.41) to

$$|\frac{e^{\varepsilon\mu_{j_k}t}}{\lambda_{j_k}\omega_{j_k}^2}|^2 = |\frac{e^{2\varepsilon\mu_{j_k}t}}{\lambda_j^2\omega_{j_k}^4}| = d\frac{e^{(2\varepsilon t-\tau)|\mu_{j_k}|\cos(\arg(\mu_{j_k}))}}{|\omega_{j_k}|^6} \tag{2.66}$$

where we used (2.57) again, hence

$$|\frac{e^{\varepsilon\mu_{j_k}t}}{\lambda_{j_k}\omega_{j_k}^2}| \to \infty \text{ as } k \to \infty \tag{2.67}$$

if

$$t > \frac{\tau}{2\varepsilon}. \tag{2.68}$$

That is, we obtain the instability for $t > 0$ satisfying (2.68).

3 Thermoelastic plates with delay terms

With the same methods as for the second-order thermoelastic systems in Section 2, we can deal with the following systems for thermoelastic plates with delay,

$$u_{tt}(t,x) + a\Delta^2 u(t-\tau,x) + b\Delta\theta(t,x) = 0, \tag{3.1}$$

$$\theta_t(t,x) - d\Delta\theta(t,x) - b\Delta u_t(t,x) = 0, \tag{3.2}$$

and the related system

$$u_{tt}(t,x) + a\Delta^2 u(t,x) + b\Delta\theta(t,x) = 0, \tag{3.3}$$

$$\theta_t(t,x) - d\Delta\theta(t-\tau,x) - b\Delta u_t(t,x) = 0, \tag{3.4}$$

where $u, \theta : [o,\infty) \times G \to \mathbb{R}$, and $a, b, d, \tau > 0$ as before, and G is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$.
Additionally, one has boundary conditions,

$$u(t,x) = \Delta u(t,x) = \theta(t,x) = 0, \tag{3.5}$$

for $t \geq 0$ and $x \in \partial G$, and initial conditions,

$$u(s,\cdot) = u^0(s), u_t(s,\cdot) = u^1(s), (-\tau \leq s \leq 0),\ \theta(0,\cdot) = 0^0, \tag{3.6}$$

and

$$u(0,\cdot) = u^0, u_t(0,\cdot) = u^1, \theta(s,\cdot) = \theta^0(s), (-\tau \leq s \leq 0), \tag{3.7}$$

respectively.
Replacing $\lambda_j = \frac{j\pi}{L}$ and φ_j from (2.8) by the eigenvalues $(\lambda_j)_j$ of the Laplace operator $(-\Delta)$ in $L^2(G)$ with Dirichlet boundary conditions, we can make the ansatz (cp. (2.8), (2.9)).

$$u = u_j(t,x) = \varphi_j(x)h_j(t), \tag{3.8}$$

$$\theta = \theta_j(t,x) = \varphi_j(x)g_j(t), \tag{3.9}$$

Then the methods of the proof of Theorem (2.1) carry over, and we have

Theorem 3.1 *(i) The initial-boundary value problem with delay (3.1), (3.2), (3.5), (3.6) is not well-posed. There exists a sequence $(u_j, \theta_j)_j$ of solutions with L^2-norm $\| u_j(t, \cdot) \|$ tending to infinity (as $j \to \infty$) for any fixed $t > 0$, while for the initial data the norms*

$$\sup_{-\tau \le j \le 0} \| (u_j^0(s), u_j^1(s), \theta^0) \|$$

remain bounded.

(ii) The corresponding statement on ill-posedness also holds for the initial-boundary value problem with delay (3.3), (3.4), (3.5), (3.7).

The remarks at the end of Section 2 including (here) the L^2-norm of Δu_j carry over *mutatis mutandis.*

We do not give details of the proof since the thermoelastic plate systems with delay (3.1), (3.2), resp. (3.3), (3.4) are a special case of the general α-β-system with delay following in the next section.

4 The α-β-system with delay terms

As a generalization of the thermoelastic plate system — well discussed for $\tau = 0$ — we study the following *α-β-system with delay,*

$$u_{tt}(t) + aAu(t - \tau) - bA^\beta \theta(t) = 0, \tag{4.1}$$

$$\theta_t(t) + dA^\alpha \theta(t) + bA^\beta u_t(t) = 0, \tag{4.2}$$

and the related system

$$u_{tt}(t) + aAu(t) - bA^\beta \theta(t) = 0, \tag{4.3}$$

$$\theta_t(t) + dA^\alpha \theta(t - \tau) + bA^\beta u_t(t) = 0, \tag{4.4}$$

where $u, \theta : [0, \infty) \to \mathcal{H}$, $\mathcal{H}$ a separable Hilbert space, A being a linear self-adjoint operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$, having a complete orthonormal system of eigenfunctions $(\varphi_j)_j$ with corresponding eigenvalues $0 < \lambda_j \to \infty$ as $j \to \infty$.

$a, b, d, \tau > 0$ are as before, and

$$0 \le \beta \le \alpha \le 1 \tag{4.5}$$

are parameters. The thermoelastic plate system from Section 3 is given by $\alpha = \beta = \frac{1}{2}$ and $A = (-\Delta_\Delta)^2$, where $-\Delta_\Delta$ denotes the Laplace operator realized in $L^2(G)$ for a bounded domain $G \subset \mathbb{R}^n$. As usual, we have the conditions

$$u(t) \in D(A), \theta(t) \in D(A^\alpha), t \ge 0, \tag{4.6}$$

and initial conditions,

$$u(s) = u^0(s), u_t(s) = u^1(s), (-\tau \le s \le 0),\ \theta(0) = 0^0, \tag{4.7}$$

and

$$u(0) = u^0, u_t(0) = u^1, \theta(s) = \theta^0(s), (-\tau \le s \le 0), \tag{4.8}$$

respectively.
For (β, α) in the region

$$\mathcal{A}_{in} = \{(\beta, \alpha) \,|\, 0 \le \beta \le \alpha \le 1,\, \alpha \ge \frac{1}{2},\, (\beta, \alpha) \ne (1,1)\} \tag{4.9}$$

see Figure 1.3, we get the following ill-posedness result for the delay problem (4.1), (4.2):

Theorem 4.1 *Let $(\beta, \alpha) \in \mathcal{A}_{in}^1$. Then the delay problem (4.1), (4.2), (4.6), (4.7) is not well-posed. There exists a sequence $(u_j, \theta_j)_j$ of solutions with norm $||u_j(t)||_{\mathcal{H}}$ tending to infinity (as $j \to \infty$) for any fixed t, while for the initial data the norms $\sup_{-\tau \le s \le 0} \parallel (u_j^0(s), u_j^1(s), \theta^0) \parallel_{\mathcal{H}}$ remain bounded.*

For (β, α) in the region

$$\mathcal{A}_{in}^2 := \{(\beta, \alpha) \,|\, 0 < \beta \le \alpha \le 1,\, (\beta, \alpha) \ne (1,1)\}, \tag{4.10}$$

see Figure 1.4, we get the following ill-posedness result for the delay problem (4.3), (4.4):

Theorem 4.2 *Let $(\beta, \alpha) \in \mathcal{A}_{in}^2$. Then the delay problem (4.3), (4.4), (4.6), (4.8) is not well-posed. There exists a sequence $(u_j, \theta_j)_j$ of solutions with norm $\parallel \theta_j(t) \parallel_{\mathcal{H}}$ tending to infinity (as $j \to \infty$) for any fixed t, while for the initial data the norms*

$$\sup_{-\tau \le s \le 0} \parallel (u_j^0(s), u_j^1(s), \theta_j^0(s)) \parallel_{\mathcal{H}}$$

remain bounded.

Proof of Theorem 4.1: We make the ansatz

$$u = u_j(t) = h_j(t)\varphi_j, \tag{4.11}$$

$$\theta = \theta_j(t) = g_j(t)\varphi_j. \tag{4.12}$$

As in the specific examples in Sections 2 and 3, we look for a solution h_j to the third-order equation

$$h_j'''(t) + d\lambda_j^\alpha h_j''(t) + b^2\lambda_j^{2\beta} h_j'(t) + a\lambda_j h_j'(t-\tau) + ad\lambda_j^{1+\alpha} h_j(t-\tau) = 0, \tag{4.13}$$

derived from the ansatz (4.11), (4.12), and, then, for g_j satisfying

$$g_j'(t) + d\lambda_j^\alpha g_j(t) = -b\lambda_j^\beta h_j'(t), \tag{4.14}$$

with

$$g_j(0) := \frac{1}{b\lambda_j^\beta}(h_j''(0) + a\lambda_j^2 h_j(-\tau)). \tag{4.15}$$

Then (u_j, θ_j) satisfy (4.1), (4.2), cp. the arguments in (2.18) - (2.20).
Making the ansatz

$$h_j(t) = \frac{1}{\omega_j^2}\, e^{\omega_j t}, \tag{4.16}$$

we shall obtain $(\omega_j)_j$ such that $\Re\omega_j \to \infty$ as $j \to \infty$, at least for a subsequence $(\omega_{j_k})_k$. In order to satisfy the equations (4.13) with the ansatz (4.16) it is sufficient (and necessary) that ω_j satisfies

$$\omega_j^3 + d\lambda_j^\alpha \omega_j^2 + (b^2\lambda_j^{2\beta} + a\lambda_j\, e^{-\tau\omega_j})\omega_j = -ad\lambda_j^{1+\alpha}\, e^{-\tau\omega_j}. \tag{4.17}$$

If we find $(\omega_j)_j$ resp. a subsequence $(\omega_{j_k})_k$ such that the following three conditions (4.18) - (4.20) are satisfied, then Theorem 4.1 will be proved.

$$\Re\omega_{j_k} \to \infty \text{ as } k \to \infty, \tag{4.18}$$

$$\sup_k \left|\frac{\lambda_{j_k}^{2-\beta}\, e^{-\tau\omega_{j_k}}}{\omega_{j_k}^2}\right| < \infty \tag{4.19}$$

(to assure the boundedness of the data),

$$\text{For } t > 0 : \left|\frac{e^{\omega_{j_k} t}}{\omega_{j_k}^2}\right| \to \infty \text{ as } k \to \infty. \tag{4.20}$$

To get (4.17) - (4.20) we have to distinguish two cases (in order to guarantee the boundedness of the corresponding functions q below).

Case 1:

$$\alpha < 2\beta. \tag{4.21}$$

Now, (4.17) is equivalent to

$$\omega_j\big(1 + \frac{\omega_j^2}{b^2\lambda_j^{2\beta}} + \frac{d\omega_j}{b^2\lambda_j^{2\beta-\alpha}} + \frac{a}{b^2}\,\lambda_j^{1-2\beta}\, e^{-\tau\omega_j}\big) = \frac{-ad}{b^2}\lambda_j^{1+\alpha-2\beta}\, e^{-\tau\omega_j}. \tag{4.22}$$

We make the ansatz (dropping the index j again)

$$\omega = \mu(1+\zeta), \quad |\zeta| < 1/2, \tag{4.23}$$

where μ is the solution to

$$\mu = \frac{-ad}{b^2}\lambda^{1+2\alpha-\beta}\, e^{-\tau\mu} \tag{4.24}$$

which exists according to Theorem 1.1 in [9] since $1 + \alpha - 2\beta > 0$ due to $(\beta, \alpha) \neq (1, 1)$.
Then, solving (4.22) is equivalent to solving

$$\underbrace{(1 - e^{-\tau\mu\zeta})}_{=:f(\zeta)} + \underbrace{(q(\zeta) + \zeta + \zeta q(\zeta))}_{=:g(\zeta)} = 0, \tag{4.25}$$

where

$$q(\zeta) := \frac{\mu^2(1+\zeta)^2}{b^2\lambda^{2\beta}} + \frac{d\mu(1+\zeta)}{b^2\lambda^{2\beta-\alpha}} + \frac{a}{b^2}\,\lambda^{1-2\beta}\, e^{-\tau\mu(1+\zeta)}. \tag{4.26}$$

f and g are holomorphic in $\Omega := B(0, \frac{1}{10\tau|\mu|})$, where $10\tau|\mu| > 2$ w.l.o.g. f has exactly one zero ($\zeta = 0$) in Ω and satisfies on $\partial\Omega$ (cp. (2.34)).

$$f(\zeta) \geq \bar{f} > 0. \tag{4.27}$$

On $\partial\Omega$ we also have

$$|\zeta| \leq \frac{C}{|\mu|}, \tag{4.28}$$

and

$$|q(\zeta)| \leq \frac{C}{|\mu|}\Big(\underbrace{|\frac{\mu^3}{\lambda^{2\beta}}| + |\frac{\mu^2}{\lambda^{2\beta-\alpha}}| + |\frac{\mu^2}{\lambda^{2\alpha+\beta}}|}_{=:p_1(\mu)}\Big) \tag{4.29}$$

where we used (4.24) which also yields, for $\varrho \in \{2\beta, 2\beta-\alpha, 2\alpha+\beta\}$, and for $\delta_\varrho := \varrho/(1+2\alpha-\beta)$,

$$\frac{1}{\lambda^\varrho} = \big(\frac{ad}{b^2}\big)^{\delta_\varrho} \, \frac{e^{-\tau\delta_\varrho \Re\mu}}{|\mu|^{\delta_\varrho}} \tag{4.30}$$

which implies

$$|p_1(\mu)| \leq C \sum_{m=1}^{3} |\mu|^{3-\delta_{\varrho m}} \; e^{-\tau\delta_{\varrho m}\Re\mu} \tag{4.31}$$

where

$$\varrho_1 := 2\beta, \varrho_2 := 2\beta - \alpha, \varrho_3 := 2\alpha + \beta$$

such that $\delta_{\varrho m} > 0,\ m = 1, 2, 3$, by our assumption (4.21).
Remark: $2 - \delta_{\varrho m} > 0,\ m = 1, 2, 3$.
We conclude for $m = 1, 2, 3$

$$|\mu_{j_k}|^{3-\delta_{\varrho m}} \; e^{-\tau\delta_{\varrho m}\Re\mu_{j_k}} \to 0 \text{ as } k \to \infty \tag{4.32}$$

(since $\cos\arg(\mu_{j_k}) \geq c_0 > 0$ as before). Combining (4.29) - (4.32) we conclude

$$|g(\zeta)| \leq \frac{C}{|\mu|} \tag{4.33}$$

and, hence, using again Rouché's theorem, that there is exactly one solution ζ_{j_k} in Ω to (4.25), and $\omega_{j_k} = \mu_{j_k}(1+\zeta_{j_k})$ solves (4.17). The relation (4.18) is proved as in (2.39). Regarding (4.19) we estimate, using (4.24),

$$\Big|\frac{\lambda_{j_k}^{2-\beta} \; e^{-\tau\omega_{j_k}}}{\omega_{j_k}^2}\Big| \leq \frac{C}{\lambda_{j_k}^{2\alpha-1}} \leq C$$

since $\alpha \geq 1/2$.
Finally, (4.20) follows as in the proof of part (i) of Theorem (2.1), see (2.40) - (2.42).
(Q.e.d. (case 1))

Case 2:

$$\alpha \geq 2\beta. \tag{4.34}$$

Proceeding in the same spirit as before, we recognize (4.17) to be equivalent to

$$\omega^2\big(1+\frac{\omega}{d\lambda^\alpha}+\frac{b^2}{\lambda^{\alpha-2\beta}\omega}+\frac{a\lambda^{1-\alpha}}{d\omega}\; e^{-\tau\omega}\big)=-a\lambda e^{-\tau\omega} \tag{4.35}$$

for which we make the ansatz

$$\omega=\mu(1+\zeta),\quad |\zeta|<\frac{1}{2}, \tag{4.36}$$

μ solving

$$\mu^2=-a\lambda e^{-\tau\mu} \tag{4.37}$$

according to the proof of Theorem 1.1 from [9], see the appendix. Then (4.35) is equivalent to

$$\underbrace{(1-e^{-\tau\mu\zeta})}_{=:f(\zeta)}+\underbrace{(q(\zeta)+2\zeta+\zeta^2+(2\zeta+\zeta^2)q(\zeta))}_{=:g(\zeta)}=0, \tag{4.38}$$

where

$$q(\zeta):=\frac{\mu(1+\zeta)}{d\lambda^\alpha}+\frac{b^2}{\lambda^{\alpha-2\beta}\mu(1+\zeta)}+\frac{a\lambda^{1-\alpha}}{d\mu(1+\zeta)}\; e^{-\tau\mu(1+\zeta)}. \tag{4.39}$$

f is the same as in case 1. We estimate

$$|q(\zeta)|\le\frac{C}{|\mu|}\Big(|\frac{\mu^2}{\lambda^\alpha}|+1+|\lambda^{1-\alpha}\; e^{-\tau\mu}\; e^{-\tau\mu\zeta}|\Big)\le\frac{C}{|\mu|} \tag{4.40}$$

as before, since $\alpha\ge 2\beta$ and $\alpha>0$.
This implies

$$|g(\zeta)|\le\frac{C}{|\mu|} \tag{4.41}$$

and, with Rouché's theorem, we get a unique solution ζ_{j_k} in Ω to (4.38), and $\omega_{j_k}=\mu_{j_k}(1+\zeta_{j_k})$ solves (4.22). The remaining relations (4.18) -(4.20) follows as in case 1.
Q.e.d.
Proof of Theorem 4.2: Making the same ansatz (4.11), (4.12) as in the proof of Theorem 4.1 we look for a solution g_j to

$$g_j'''(t)+d\lambda_j^\alpha g_j''(t-\tau)+a\lambda_j g_j'(t)+b^2\lambda_j^2 g_j'(t)+ad\lambda_j^{1+\alpha}g_j(t-\tau)=0, \tag{4.42}$$

and, then, for h_j satisfying

$$h_j''(t)+a\lambda_j h_j(t)=b\lambda_j^\beta g_j(t), \tag{4.43}$$

with

$$h_j(0)=\frac{1}{a\lambda_j^{1+b}}\big(b^2\lambda_j^{2\beta}g_j(0)+g_j''(0)+d\lambda_j^\alpha g_j'(-\tau)\big), \tag{4.44}$$

$$h_j'(0)=-\frac{1}{b\lambda_j^\beta}\big(g_j'(0)+a\lambda_j^\alpha g(-\tau)\big), \tag{4.45}$$

Then (u_j,θ_j) satisfy (4.3), (4.4), cp. the arguments in (2.48) - (2.50).
For g_j we make the ansatz

$$g_j(t)=\frac{1}{\omega_j^2}\; e^{\omega_j t}. \tag{4.46}$$

We shall obtain $(\omega_{j_k})_k$ such that $\Re\omega_{j_k} \to \infty$ as $j_k \to \infty$ as $k \to \infty$. To satisfy (4.42) by (4.46) it is sufficient (and necessary) that ω_j satifies

$$\omega_j^3 + d\lambda_j^\alpha\, e^{-\tau\omega_j}\omega_j^2 + (a\lambda_j + b^2\lambda_j^{2\beta})\omega_j = -ad\lambda_j^{1+\alpha}\, e^{-\tau\omega_j}. \tag{4.47}$$

We have to find $(\omega_j)_j$ resp. a subsequence $(\omega_{j_k})_k$ such that the following conditions (4.48) -(4.50) are satisfied; then the theorem will be proved.

$$\Re\omega_{j_k} \to \infty \quad \text{ as } k \to \infty, \tag{4.48}$$

$$\sup_k |\lambda_{j_k}^{\alpha-\beta}\, e^{-\tau\omega_{j_k}}| < \infty, \tag{4.49}$$

(to assure the boundedness of the data),

$$\text{For } t > 0 : |\frac{e^{\omega_{j_k} t}}{\omega_{j_k}^2}| \to \infty \quad \text{ as } k \to \infty. \tag{4.50}$$

Case 1:

$$0 < \beta \le 1/2. \tag{4.51}$$

Then (4.47) is equivalent to

$$\omega_j\big(1 + \frac{b^2}{a\lambda_j^{1-2\beta}} + \frac{\omega_j^2}{a\lambda_j} + \frac{d}{a}\lambda_j^{\alpha-2\beta}\, e^{-\tau\omega_j}\omega_j\big) = -d\lambda_j^\alpha\, e^{-\tau\omega_j} \tag{4.52}$$

The ansatz — dropping the index j again —

$$\omega = \mu(1+\zeta), \quad |\zeta| < 1/2, \tag{4.53}$$

where μ is the solution to

$$\mu = -d\lambda_j^\alpha\, e^{-\tau\mu}, \tag{4.54}$$

according to Theorem 1.1 in [9], observe $\alpha > 0$, yields that solving (4.52) is equivalent to solving

$$\underbrace{(1 - e^{-\tau\mu\zeta})}_{=:f(\zeta)}\underbrace{(q(\zeta) + \zeta + \zeta q(\zeta))}_{=:g(\zeta)} = 0, \tag{4.55}$$

where

$$q(\zeta) := \frac{b^2}{a\lambda^{1-2\beta}} + \frac{\mu^2(1+\zeta)^2}{a\lambda} + \frac{d}{a}\lambda^{\alpha-2\beta}\mu(1+\zeta)\, e^{-\tau\mu(1+\zeta)}. \tag{4.56}$$

With (4.27), (4.28) analogously, we additionally get on $\partial\Omega$ (with $\Omega = B(0, \frac{1}{10\tau|\mu|})$ again)

$$|q(\zeta)| \le \frac{C}{|\mu|}\Big(\underbrace{|\frac{\mu}{\lambda^{1-2\beta}}| + |\frac{\mu^3}{\lambda}| + |\lambda^{\alpha-2\beta}\mu^2\, e^{-\tau\mu}|}_{=:p_2(\mu)}\Big) \tag{4.57}$$

W.l.o.g. we assume $b < 1/2$ since for $b = 1/2$ we may replace (4.24) by

$$\mu = \frac{-ad}{a+b^2}\lambda^\alpha\, e^{-\tau\mu} \tag{4.58}$$

and (4.56) by

$$q(\zeta) = \frac{\mu^2(1+\zeta)^2}{(a+b^2)\lambda} + \frac{d}{a+b^2}\lambda^{\alpha-1}\, e^{-\tau\mu(1+\zeta)}\mu(1+\zeta), \tag{4.59}$$

and then (4.57) by

$$|q(\zeta)| \le \frac{C}{|\mu|}\Big(|\frac{\mu^3}{\lambda}| + |\lambda^{\alpha-2\beta}\mu^2\, e^{-\tau\mu}|\Big) \tag{4.60}$$

and finally continue as follows, now for $b < 1/2$.
By (4.54) we have for $\varrho > 0$

$$\frac{1}{\lambda^\varrho} = d^{\delta_\varrho}\frac{e^{-\tau\delta_\varrho\Re\mu}}{|\mu|^{\delta_\varrho}} \tag{4.61}$$

with

$$\delta_\varrho := \frac{\varrho}{\alpha}, \tag{4.62}$$

as well as

$$e^{-\tau\mu} = -\frac{1}{d}\,\frac{\mu}{\lambda^\alpha}. \tag{4.63}$$

Hence, using (4.54) again,

$$|p_2(\mu)| \le C\Big(|\frac{\mu}{\lambda^{1-2\beta}}| + |\frac{\mu^3}{\lambda}| + |\frac{\mu^3}{\lambda^{2\beta}}|\Big) \le C\sum_{m=1}^{3}|\mu|^{3-\varrho_m}\, e^{-\tau\delta_{\varrho m}\Re\mu}, \tag{4.64}$$

where

$$\varrho_1 := 1-2\beta,\ \varrho_2 := 1,\ \varrho_3 := 2\beta \tag{4.65}$$

assuring $\varrho_m > 0,\ m = 1,2,3$, by our assumptions.
As in the proofs before, we successively conclude

$$|p_2(\mu)| \le C, \quad |q|(\zeta)| \le \frac{C}{|\mu|},$$

hence

$$|g|(\zeta)| \le \frac{C}{|\mu|} \tag{4.66}$$

This implies again the existence of exactly one solution ζ_{j_k} in Ω to (4.55), and $\omega_{j_k} = \mu_{j_k}(1+\zeta_{j_k})$ solves (4.47).
The relation (4.48) is proved as in (2.39). Regarding (4.49) we estimate, using $\beta > 0$ and (4.54),

$$|\lambda_{j_k}^{\alpha-\beta}\, e^{-\tau\omega_{j_k}}| \le C|\frac{\mu_{j_k}}{\lambda_{j_k}^{\beta}}| \le C|\mu_{j_k}|^{1-\frac{\beta}{\alpha}}\, e^{-\frac{\tau\beta}{\alpha}\Re\mu_{j_k}} \le C. \tag{4.67}$$

Finally, (4.50) follows as in the proof of part (i) of Theorem 2.1, see (2.40) – (2.42).
(Q.e.d.(case 1)).

Case 2:

$$1/2 < \beta < 1. \tag{4.68}$$

Now, (4.47) is equivalent to

$$\omega_j\Big(1+\frac{a}{b^2\lambda_j^{2\beta-1}}+\frac{\omega_j^2}{b^2\lambda_j^{2\beta}}+\frac{d\omega_j}{b^2\lambda_j^{2\beta-\alpha}}\;e^{-\tau\omega_j}\Big)=-\frac{ad}{b^2}\lambda^{1+\alpha-2\beta}\;e^{-\tau\omega_j} \tag{4.69}$$

We observe that for $\beta<1$ we have $1+\alpha-2\beta>0$. The ansatz

$$\omega=\mu(1+\zeta),\quad (\zeta)<1/2, \tag{4.70}$$

where μ solves

$$\mu=-\frac{ad}{b^2}\lambda^{1+\alpha-2\beta}\;e^{-\tau\mu}, \tag{4.71}$$

plugged into (4.69) yields to solve

$$\underbrace{(1-e^{-\tau\mu\zeta})}_{=:f(\zeta)}\underbrace{(q(\zeta)+\zeta+\zeta q(\zeta))}_{=:g(\zeta)}=0. \tag{4.72}$$

Estimating

$$|q(\zeta)|\le\frac{C}{|\mu|}\Big(\underbrace{|\frac{\mu}{\lambda^{2\beta-1}}+|\frac{\mu^3}{\lambda^{2\beta}}|+|\lambda^{\alpha-2\beta}\mu^2\;e^{-\tau\mu}|}_{=:p_3(\mu)}\Big) \tag{4.73}$$

using — by (4.71) — for $\varrho>0$,

$$\frac{1}{\lambda^\varrho}=\big(\frac{ad}{b^2}\big)^{\delta_\varrho}\frac{e^{-\tau\delta_\varrho\Re\mu}}{|\mu|^{\delta_\varrho}} \tag{4.74}$$

where

$$\delta_\varrho:=\frac{\varrho}{1+\alpha-2\beta}, \tag{4.75}$$

and

$$e^{-\tau\mu}=-\frac{b^2}{ad}\;\frac{\mu}{\lambda^{1+a-2b}}, \tag{4.76}$$

we obtain

$$|p_3(\mu)|\le C\Big(|\frac{\mu}{\lambda^{2\beta-1}}|+|\frac{\mu^3}{\lambda^{2\beta}}|+|\frac{\mu^3}{\lambda}|\Big)\le C\sum_{m=1}^{3}|\mu|^{3-\varrho_m}\;e^{-\tau\delta_{\varrho m}\Re\mu}, \tag{4.77}$$

where

$$\varrho_1:=2\beta-1,\;\varrho_2:=2\beta,\;\varrho_3:=1 \tag{4.78}$$

satisfy $\varrho_m>0,\;m=1,2,3.$
As before we conclude

$$|p_3(\mu)|\le C,\quad |q(\zeta)|\le\frac{C}{|\mu|}$$

implying the existence of exactly one solution ζ_{j_k} in Ω to (4.55), and $\omega_{j_k}=\mu_{j_k}(1+\zeta_{j_k})$ solves (4.47). The relation (4.48) is proved as in (2.39). Regarding (4.49) we estimate, using (4.54),

$$|\lambda_{j_k}^{\alpha-\beta}\;e^{-\tau\omega_{j_k}}|\le C|\frac{\mu_{j_k}}{\lambda_{j_k}^{1+\alpha-\beta}}|\le C|\mu_{j_k}|^{-\frac{b}{1+\alpha-2\beta}}\;e^{-\tau\frac{1+\alpha-\beta}{1+\alpha-2\beta}}\le C. \tag{4.79}$$

Finally, (4.50) follows as in case 1. This finishes the proof of Theorem 4.2.
Q.e.d.

We notice the interesting difference $\mathcal{A}^1_m \neq \mathcal{A}^2_{in}$. For (4.1), (4.2) the damping through θ has to be weak enough ("$\alpha \geq 1/2$") to still guarantee the ill-posedness suggested by the equation (4.1) for u.
One should also notice that the conditions $(\beta,\alpha) \in \mathcal{A}^j_{in}, j = 1,2$, are *sufficient* for the instability, the behavior in the region outside $\mathcal{A}^j_{in}$ is an open question.
It would be possible to study with the methods above even more general systems like

$$u_{tt}(t) + Au(t - g_1\tau) + bA^\beta\theta(t - g_2\tau) = 0, \tag{4.80}$$

$$\theta_t(t) + dA^\alpha\theta(t - g_3\tau) + bA^\beta u_t(t - g_4\tau) = 0, \tag{4.81}$$

where $g_j \in \{0,1\}, j = 1,2,3,4$. Here the cases $g_1 - 1 = g_2 = g_3 = g_4 = 0$ and $g_1 = g_2 = g_3 - 1 = g_4 = 0$ where treated, for simplicity.

5 Appendix

For the reader's convenience, we recall the essential parts of the proof of Theorem 1.1 in [9] which we used in the previous sections. We look for solutions ω_l to the equation

$$\omega_l^n = e^{-\omega_l\tau}\xi_l, \tag{5.1}$$

where $\xi_l \to -\infty$ as $l \to \infty$, and $n \in \mathbb{N}$. Dropping the index l for simplicity and writing $\omega = re^{i\varphi}$ with $0 \leq r < \infty$ and $0 \leq \varphi < 2\pi$, we get from (5.1)

$$\xi = r^n e^{in\varphi} e^{\omega\tau} = r^n e^{r\tau\cos\varphi} e^{i(n\varphi + r\tau\sin\varphi)}. \tag{5.2}$$

Since $\xi < 0$, we wish to solve

$$r^n e^{r\tau\cos\varphi} = |\xi|, \quad n\varphi + r\tau\sin\varphi = \pi, \tag{5.3}$$

which implies

$$r = \frac{\pi - n\varphi}{\tau\sin\varphi}. \tag{5.4}$$

It is advantageous to place an additional restriction on φ:

$$0 < \varphi \leq \frac{\pi}{4n}. \tag{5.5}$$

Substituting (5.4) into (5.2) we obtain

$$\psi(\varphi) := (\pi - n\varphi)^n \, e^{(\pi - n\varphi)\cot\varphi} - |\xi|\tau^n \sin^n\varphi = 0. \tag{5.6}$$

Our aim is to show that (5.6) always has a zero in $(0, \pi/(4n))$ whenever $|\xi|$ is large enough.
We have $\lim_{\varphi\downarrow 0}\psi(\varphi) = \infty$ and

$$\psi\Big(\frac{\pi}{4n}\Big) = \Big(\frac{3\pi}{4}\Big)^n e^{\frac{3\pi}{4}\cot\left(\frac{\pi}{4n}\right)} - |\xi|\tau^n\sin^n\Big(\frac{\pi}{4n}\Big) \to -\infty \text{ as } |\xi| \to \infty. \tag{5.7}$$

which gives us a number $\varphi \in (0, \frac{\pi}{4n})$ solving (5.6). Hence, using $\cos\varphi_l \geq \cos\frac{\pi}{4n} > 0$ due to (5.5), there is an $m \in \mathbb{N}$ such that for all $l \geq m$ there is a solution $\omega_l = r_l e^{i\varphi_l}$ to (5.1) such that

$$\Re\omega_l = r_l\cos\varphi_l \to \infty \text{ as } l \to \infty, \quad r_l \to \infty \text{ as } l \to \infty,$$

where we used $\infty \leftarrow |\xi_l| = r_l^n \, e^{r_l\tau\cos\varphi_l}$.

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