# Global existence and decay properties for solutions of the Cauchy problem in one-dimensional thermoelasticity with second sound

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**Abstract**

We consider the one-dimensional Cauchy problem in nonlinear thermoelasticity with second sound, where the heat conduction is modeled by Cattaneo's law. After presenting decay estimates for solutions to the linearized problem, including refined estimates for data in weighted Lebesgue-spaces, we prove a global existence theorem for small data together with improved decay estimates, in particular for derivatives of the solutions.

**Keywords:** Thermoelasticity; decay rate; Lyapunov functional; second sound.
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## 1 Introduction

We consider the following nonlinear initial-value problem in thermoelasticity:

$$\begin{cases} u_{tt} - (\sigma(u_x))_x + \beta\theta_x = 0, & \\ \theta_t + \eta q_x + \delta u_{tx} = 0, & \\ \tau q_t + q + \kappa\theta_x = 0, & x \in \mathbb{R},\ t \geq 0 \\ u(x,0) = u_0, \qquad u_t(x,0) = u_1, & \\ \theta(x,0) = \theta_0, \qquad q(x,0) = q_0, & \end{cases} \tag{1.1}$$

where $\sigma(\eta)$ is a smooth function of $\eta$ such that $\sigma'(\eta) > 0$ and $\sigma'(0) = a > 0$. The parameters $\beta$, $\eta$, $\delta$, $\tau$, and $\kappa$ are positive constants. The system describes the propagation of nonlinear elastic waves in the presence of thermoelastic effects. The evolution of the deformation, $u$, is coupled to the temperature field, $\theta$, through the dependence of the stress on the temperature, as seen in the first equation in (1.1): the stress is $\sigma - \beta\theta$, where $\beta$ is the coefficient of thermal expansion. The second equation in (1.1) is the equation of heat conduction, where $\eta$ is the heat conduction coefficient, $q$ is the heat flux, and the last term represents the thermal effects of deformation. The third equation in (1.1) is a constitutive law for heat conduction. It is simply the Fourier law, if the relaxation time vanishes: i.e., if $\tau = 0$. If $\tau > 0$, the equation (also called Cattaneo's law [3]) is

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a model of hyperbolic heat conduction. Since Cattaneo's law introduces the extra speed of propagation in addition to the elastic wave speed, the model is called "thermoelasticity with second sound". For reviews of related work, see [4, 5, 13, 25].

In this work, we establish the global existence of small and smooth solutions of (1.1), i.e. solutions in $H^s(\mathbb{R})$ with $s \geq 2$, by using weighted energy estimates. In addition, we obtain the following decay rates of the solution:

$$\left\|\partial_x^k U(t)\right\|_2 \leq C\left(\|U_0\|_1 + \|U_0\|_{H^s}\right)(1+t)^{-1/4-k/2}, \tag{1.2}$$

where $C$ is a positive constant, $0 \leq k \leq s-1$, and $s \geq 3$; see Theorems 4.5 and 4.8.

For the linearized version of (1.1), we first establish the following decay rates in Theorem 3.3:

$$\begin{aligned} \left\|\partial_x^k U(t)\right\|_2 &\leq C(1+t)^{-1/4-k/2}\|U_0\|_1 + Ce^{-ct}\left\|\partial_x^k U_0\right\|_2, && \text{for } 0 \leq k \leq s, \\ \left\|\partial_x^k U(t)\right\|_\infty &\leq Ct^{-1/2-k/2}\left(\|U_0\|_1 + \left\|\partial_x^{k+1} U_0\right\|_2\right), && \text{for } 0 \leq k \leq s-1, \end{aligned} \tag{1.3}$$

where $U_0 = (u_1, u_x(0), \theta_0, q_0) \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$.

The linear decay estimates above can be improved as follows (see Theorems 3.4 and 3.7):

$$\begin{aligned} \left\|\partial_x^k U(t)\right\|_2 &\leq C(1+t)^{-1/4-k/2-\gamma/2}\|U_0\|_{1,\gamma} + Ce^{-ct}\left\|\partial_x^k U_0\right\|_2, && \text{for } 0 \leq k \leq s, \\ \left\|\partial_x^k U(t)\right\|_\infty &\leq Ct^{-1/2-k/2-\gamma/2}\left(\|U_0\|_{1,\gamma} + \left\|\partial_x^{k+1} U_0\right\|_2\right), && \text{for } 0 \leq k \leq s-1, \end{aligned} \tag{1.4}$$

if $U_0 \in H^s(\mathbb{R}) \cap L^{1,\gamma}(\mathbb{R})$ with $\int_{\mathbb{R}} U_0(x)\,dx = 0$. These improvements can be extended further with additional assumptions about the spatial decay of $U_0$ (see Remark 3.6 below).

The one-dimensional nonlinear Cauchy problem of thermoelasticity with second sound, with more general nonlinearities, has been analyzed by Racke and Wang [20]. They derived the following decay rate of the solution:

$$\|U(t)\|_2 + \|\partial_x U(t)\|_2 \leq C\left(\|U_0\|_1 + \|U_0\|_{H^3}\right)(1+t)^{-1/4}. \tag{1.5}$$

With the method used in our paper, which is different from the one used in [20], it is possible to improve the decay rate (1.5) as seen in (1.2); that is, we additionally obtain the decay rates of higher-order derivatives.

We should also note that a global existence result for small initial data for a system similar to that in [20] has been established by Tarabek [26]. He considered the following one-dimensional system:

$$\begin{aligned} u_{tt} - a(u_x, \theta, q)\,u_{xx} + b(u_x, \theta, q)\,\theta_x &= \alpha_1(u_x, \theta)\,qq_x, \\ \theta_t + g(u_x, \theta, q)\,q_x + d(u_x, \theta, q)\,u_{tx} &= \alpha_2(u_x, \theta)\,qq_t, \\ \tau(u_x, \theta)\,q_t + q + k(u_x, \theta)\,\theta_x &= 0, \end{aligned} \tag{1.6}$$

in both bounded and unbounded domains. Although he showed that the solution tends to equilibrium as $t$ tends to infinity, he derived no decay rate.

Concerning the linearized system corresponding to (1.1), the Cauchy problem has been considered in [27] where the authors derived the same decay rates as (1.3). However, our method here differs from the one used in [27]. In addition, we obtain improved decay rates as seen in (1.4).

It is well known that smooth solutions of quasilinear hyperbolic systems may develop singularities in finite time, even for very regular initial data (see for example [14]). System (4.1) includes a damping mechanism that prevents the formation of singularities of solutions for initial data that are close to the equilibrium state in appropriate Sobolev norms. For the Cauchy problem in classical thermoelasticity, Hrusa and Messaoudi [8] have shown that if the initial data are large enough, then the solution will develop singularities in finite time. (See also [6] for a similar result). It has been recently shown [9] that for the nonlinear systems discussed here, a blow-up for large data is expected.

The plan of the remainder of this paper is as follows. In Section 2, we introduce the notation and, for the convenience of the reader, we recall some useful results without proofs. In Section 3, we investigate the linearized model and derive appropriate decay estimates. Section 4 is devoted to demonstrating the existence of global solutions for small and smooth initial data to the nonlinear model. In addition, we provide proofs of new decay estimates.

## 2 Notation and some useful lemmas

Before proceeding, we introduce the notation used in this paper. Throughout, $\|.\|_q$ and $\|.\|_{H^l}$ stand for the $L^q(\mathbb{R}^N)$-norm $(2 \le q \le \infty)$ and the $H^l(\mathbb{R}^N)$-norm. We define the weighted function space, $L^{1,\gamma}(\mathbb{R}^N)$, $N \ge 1$, $\gamma \in [0, +\infty)$, as follows: $u \in L^{1,\gamma}(\mathbb{R}^N)$ iff $u \in L^1\left(\mathbb{R}^N\right)$ and

$$\|u\|_{1,\gamma} = \int_{\mathbb{R}^N} (1+|x|)^\gamma |u(x)| dx < +\infty.$$

By $\hat{f}$, we denote the Fourier transform of $f$:

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} f(x)\, e^{-i\xi x} dx, \quad f(x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \hat{f}(\xi)\, e^{i\xi x} d\xi.$$

The symbol $[A, B] = AB - BA$ denotes the commutator. The constant $C$ denotes a generic positive constant that appears in various inequalities and may change its value in different occurrences.

We next recall several useful inequalities.

**Lemma 2.1** *([16]) Let $N \ge 1$, $1 \le p, q, r \le \infty$, and let $k$ be a positive integer. Then, for any integer $j$ with $0 \le j \le k$, we have*

$$\|\partial_x^j u\|_p \le C \left\|\partial_x^k u\right\|_q^a \|u\|_r^{1-a} \tag{2.1}$$

*where*

$$\frac{1}{p} = \frac{j}{N} + a\left(\frac{1}{q} - \frac{k}{N}\right) + (1-a)\frac{1}{r}$$

*for $a$ satisfying $j/k \le a \le 1$ and $C$ is a positive constant. The following exceptional cases exist:*

1. *If $j = 0$, $qk < N$ and $r = \infty$, then we make the additional assumption that either $u(x) \to 0$ as $|x| \to \infty$ or $u \in L^{q'}$ for some $0 < q' < \infty$.*
2. *If $1 < r < \infty$ and $k - j - N/r$ is a nonnegative integer, then (2.1) holds only for $j/k \le a < 1$.*

**Lemma 2.2** *[7, Lemma 4.1]. Let $1 \le p, q, r \le \infty$ and $1/p = 1/q + 1/r$. Then, we have*

$$\|\partial_x^k (uv)\|_p \le C(\|u\|_q \|\partial_x^k v\|_r + \|v\|_q \|\partial_x^k u\|_r), \quad k \ge 0 \tag{2.2}$$

*and*

$$\|[\partial_x^k, u] v_x\|_p \le C(\|u_x\|_q \|\partial_x^k v\|_r + \|v_x\|_q \|\partial_x^k u\|_r), \quad k \ge 1. \tag{2.3}$$

**Lemma 2.3** *[15, 22]. Let $a > 0$ and $b > 0$ be constants. Then,*

$$\int_0^t (1+t-s)^{-a}(1+s)^{-b}\,ds \leq C(1+t)^{-\min(a,b)}, \quad \textit{if } \max(a,b) > 1, \tag{2.4}$$

$$\int_0^t (1+t-s)^{-a}(1+s)^{-b}\,ds \leq C(1+t)^{-\min(a,b)}\ln(2+t), \quad \textit{if } \max(a,b) = 1, \tag{2.5}$$

$$\int_0^t (1+t-s)^{-a}(1+s)^{-b}\,ds \leq C(1+t)^{1-a-b}, \quad \textit{if } \max(a,b) < 1. \tag{2.6}$$

# 3 The linearized model

This section is devoted to establishing various decay rates for the linear problem in thermoelasticity with second sound. Let us consider the following Cauchy problem:

$$\begin{cases} u_{tt} - au_{xx} + \beta\theta_x = 0, \\ \theta_t + \eta q_x + \delta u_{tx} = 0, \\ \tau q_t + q + \kappa\theta_x = 0, & x \in \mathbb{R},\ t \geq 0, \\ u(x,0) = u_0, \qquad u_t(x,0) = u_1, \\ \theta(x,0) = \theta_0, \qquad q(x,0) = q_0, \end{cases} \tag{3.1}$$

with positive constants $a, \beta, \eta, \delta, \tau, \kappa$.
Setting $v = u_t$ and $z = u_x$, we get from (3.1) the following first-order system:

$$\begin{cases} v_t - az_x + \beta\theta_x = 0, \\ z_t - v_x = 0, \\ \theta_t + \eta q_x + \delta v_x = 0, \\ q_t + \dfrac{1}{\tau}q + \dfrac{\kappa}{\tau}\theta_x = 0, \end{cases} \tag{3.2}$$

which, equivalently, takes the matrix form

$$\begin{cases} U_t + AU_x + LU = 0, \\ U(x,0) = U_0, \end{cases} \tag{3.3}$$

where $U := (v, z, \theta, q)^T$ and the matrices $A$ and $L$ are

$$A = \begin{pmatrix} 0 & -a & \beta & 0 \\ -1 & 0 & 0 & 0 \\ \delta & 0 & 0 & \eta \\ 0 & 0 & \kappa/\tau & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\tau \end{pmatrix}.$$

**Remark 3.1** *By the change of variables, $W = \left(v,\ \sqrt{a}z,\ \sqrt{\beta/\delta}\theta,\ \sqrt{\tau\eta\beta/(\kappa\delta)}q\right)$, system (3.3) can be symmetrized to take the form*

$$W_t + A_1 W_x + BW = 0, \tag{3.4}$$

*where $A_1$ is symmetric and $B = \sqrt{\tau\eta\beta/(\kappa\delta)}L$. For this new system, the Shizuta-Kawashima condition [23] is satisfied. The result in Theorem 3.3 below can therefore be obtained directly from the general result of [1]. We present a direct method of using estimates in Fourier space and obtain more general decay rates in weighted spaces. More importantly, we also employ the technique used to derive the linear results in obtaining the decay rates for the nonlinear system.*

To proceed, we take the Fourier transform of (3.2) and obtain

$$\begin{cases} \hat{v}_t - ai\xi\hat{z} + \beta i\xi\hat{\theta} = 0, \\ \hat{z}_t - i\xi\hat{v} = 0, \\ \hat{\theta}_t + i\xi\eta\hat{q} + \delta i\xi\hat{v} = 0, \\ \hat{q}_t + \dfrac{1}{\tau}\hat{q} + \dfrac{\kappa}{\tau}i\xi\hat{\theta} = 0. \end{cases} \tag{3.5}$$

Let us now define the energy functional

$$\mathscr{E}(t) = \frac{1}{2}\Big(\kappa\delta\,|\hat{v}|^2 + \kappa\delta a\,|\hat{z}|^2 + \kappa\beta|\hat{\theta}|^2 + \eta\beta\tau\,|\hat{q}|^2\Big). \tag{3.6}$$

Multiplying the first equation in (3.5) by $\kappa\delta\bar{\hat{v}}$, the second equation by $\kappa\delta a\bar{\hat{z}}$, the third equation by $\kappa\beta\bar{\hat{\theta}}$ and the fourth equation by $\eta\beta\tau\,\bar{\hat{q}}$, adding these equalities and taking the real part, it easily follows that

$$\frac{d\mathscr{E}(t)}{dt} = -\beta\eta\,|\hat{q}|^2. \tag{3.7}$$

The following crucial lemma will be proved directly with the help of an appropriate Lyapunov functional in the Fourier space.

**Lemma 3.2** *Let $\hat{U}(\xi,t) = (\hat{v},\hat{z},\hat{\theta},\hat{q})(\xi,t)$ be the solution of (3.5). Then, for any $t \geq 0$ and $\xi \in \mathbb{R}$, we have the following pointwise estimate:*

$$\left|\hat{U}(\xi,t)\right|^2 \leq Ce^{-c\rho(\xi)t}\left|\hat{U}(\xi,0)\right|^2, \tag{3.8}$$

*where $\rho(\xi) = \xi^2/(1+\xi^2)$ and $C$ and $c$ are positive constants.*

**Proof.** Our main tool to prove Lemma 3.2 is the energy method in the Fourier space. In the sequel, we make repeated use of Young's inequality:

$$|ab| \leq \epsilon a^2 + C(\epsilon)b^2.$$

Constants $C(\epsilon)$ here and in the sequel denote possibly different values in different places, but they are in principle easy to determine.

First, by multiplying the second equation in (3.5) by $-i\xi\bar{\hat{v}}$, the first equation by $i\xi\bar{\hat{z}}$, adding the resulting equation and taking the real part, we get

$$\left\{\mathrm{Re}\left(i\xi\bar{\hat{z}}\hat{v}\right)\right\}_t + \xi^2\left(a\,|\hat{z}|^2 - |\hat{v}|^2\right) = \mathrm{Re}\left(\beta\xi^2\hat{\theta}\bar{\hat{z}}\right). \tag{3.9}$$

Using Young's inequality, the term on the right-hand side can be estimated as

$$\left|\mathrm{Re}\left(\beta\xi^2\hat{\theta}\bar{\hat{z}}\right)\right| \leq \epsilon\xi^2\,|\hat{z}|^2 + C(\epsilon)\,\xi^2|\hat{\theta}|^2. \tag{3.10}$$

Plugging (3.10) into (3.9), we obtain

$$\left\{\mathrm{Re}\left(i\xi\bar{\hat{z}}\hat{v}\right)\right\}_t+\xi^2\left((a-\epsilon)|\hat{z}|^2-|\hat{v}|^2\right)\leq C(\epsilon)\,\xi^2|\hat{\theta}|^2. \tag{3.11}$$

Similarly, multiplying the third equation in (3.5) by $-i\xi\bar{\hat{v}}$ and the first equation by $i\xi\bar{\hat{\theta}}$, adding the results and taking the real part, we find that

$$\left\{\mathrm{Re}(i\xi\bar{\hat{\theta}}\hat{v})\right\}_t+\xi^2(\delta|\hat{v}|^2-\beta|\hat{\theta}|^2)=-\mathrm{Re}\left(\xi^2\eta\hat{q}\bar{\hat{v}}\right)-\mathrm{Re}\left(a\xi^2\hat{z}\bar{\hat{\theta}}\right). \tag{3.12}$$

Then, Young's inequality gives

$$\left|-\mathrm{Re}\left(\xi^2\eta\hat{q}\bar{\hat{v}}\right)-\mathrm{Re}\left(a\xi^2\hat{z}\bar{\hat{\theta}}\right)\right|\leq\epsilon_1\xi^2\,|\hat{z}|^2+C(\epsilon_1)\,\xi^2|\hat{\theta}|^2+\epsilon\xi^2|\hat{v}|^2+C(\epsilon)\,\xi^2\,|\hat{q}|^2\,.$$

Thus, (3.12) takes the form

$$\left\{\mathrm{Re}(i\xi\bar{\hat{\theta}}\hat{v})\right\}_t+\xi^2((\delta-\epsilon)\,|\hat{v}|^2\leq\epsilon_1\xi^2\,|\hat{z}|^2+C(\epsilon_1)\,\xi^2|\hat{\theta}|^2+C(\epsilon)\,\xi^2\,|\hat{q}|^2\,. \tag{3.13}$$

Next, multiplying the fourth equation in (3.5) by $-i\xi\bar{\hat{\theta}}$, the third equation by $i\xi\bar{\hat{q}}$, adding the results, and taking the real part, we find that

$$\left\{\mathrm{Re}(i\xi\bar{\hat{q}}\hat{\theta})\right\}_t+\xi^2\left(\frac{\kappa}{\tau}|\hat{\theta}|^2-\eta\,|\hat{q}|^2\right)=\mathrm{Re}\left(\frac{1}{\tau}i\xi\bar{\hat{\theta}}\hat{q}\right)+\mathrm{Re}\left(\delta\xi^2\hat{v}\bar{\hat{q}}\right). \tag{3.14}$$

As above, Young's inequality implies

$$\left|\mathrm{Re}\left(\frac{1}{\tau}i\xi\bar{\hat{\theta}}\hat{q}\right)+\mathrm{Re}\left(\delta\xi^2\hat{v}\bar{\hat{q}}\right)\right|\leq\epsilon\xi^2|\hat{\theta}|^2+C(\epsilon,\epsilon_2)\left(1+\xi^2\right)|\hat{q}|^2+\epsilon_5\xi^2|\hat{v}|^2.$$

Consequently, taking into account this last estimate, we obtain from (3.14) that

$$\left\{\mathrm{Re}(i\xi\bar{\hat{q}}\hat{\theta})\right\}_t+\xi^2\left(\frac{\kappa}{\tau}-\epsilon\right)|\hat{\theta}|^2\leq\epsilon_2\xi^2|\hat{v}|^2+C(\epsilon,\epsilon_2)\left(1+\xi^2\right)|\hat{q}|^2\,. \tag{3.15}$$

Now, let $\alpha_1$ and $\alpha_2$ be two positive constants (to be chosen later). Then, (3.11) + $\alpha_1$(3.13) + $\alpha_2$(3.15) gives

$$\begin{aligned}&\xi\frac{d\mathscr{F}(t)}{dt}+\xi^2\left((a-\epsilon)-\epsilon_1\alpha_1\right)|\hat{z}|^2+\left((\alpha_1(\delta-\epsilon)-1-\alpha_2\epsilon_2\right)\xi^2|\hat{v}|^2\\&+\left\{\alpha_2\left(\frac{\kappa}{\tau}-\epsilon\right)-C(\epsilon)-\alpha_1C(\epsilon_1)\right\}\xi^2|\hat{\theta}|^2\\\leq\;&C(\epsilon,\epsilon_1,\epsilon_2,\alpha_1,\alpha_2)\left(1+\xi^2\right)|\hat{q}|^2\,,\end{aligned} \tag{3.16}$$

where

$$\mathscr{F}(t)=\mathrm{Re}\left(i\bar{\hat{z}}\hat{v}\right)+\alpha_1\,\mathrm{Re}(i\bar{\hat{\theta}}\hat{v})+\alpha_2\,\mathrm{Re}(i\bar{\hat{q}}\hat{\theta}). \tag{3.17}$$

At this point, we fix our constants in (3.16) very carefully. First, we choose $\epsilon$ small enough such that $\epsilon<\min(a,\kappa/\tau,\delta)$. Once $\epsilon$ is fixed, we may pick $\alpha_1$ large enough such that $\alpha_1(\delta-\epsilon)/2\geq 1$. After that, we select $\epsilon_1$ small enough such that $\epsilon_1<(a-\epsilon)/\alpha_1$. Then, we may take $\alpha_2$ large enough such that

$$\alpha_2\left(\frac{\kappa}{\tau}-\epsilon\right)-C(\epsilon_1)-\alpha_1C(\epsilon_1)>0.$$

Finally, we fix $\epsilon_2$ small enough such that $\epsilon_2 < \alpha_1 (\delta - \epsilon) / (2\alpha_2)$. Consequently, we deduce from (3.16) that there exists a positive constant, $\lambda > 0$, such that

$$\xi \frac{d\mathscr{F}(t)}{dt} + \lambda\xi^2 \left(|\hat{z}|^2 + |\hat{v}|^2 + |\hat{\theta}|^2\right) \leq C(\epsilon, \epsilon_1, \epsilon_2, \alpha_1, \alpha_2)\left(1 + \xi^2\right)|\hat{q}|^2 . \tag{3.18}$$

Now, let us define the functional, $\mathscr{L}$, as

$$\mathscr{L}(t) := M\mathscr{E}(t) + \frac{\xi}{1+\xi^2}\mathscr{F}(t), \tag{3.19}$$

where $M$ is a large positive constant to be fixed later. Consequently, exploiting (3.7) and (3.18), we may write

$$\frac{d\mathscr{L}(t)}{dt} + \lambda \frac{\xi^2}{1+\xi^2}\left(|\hat{z}|^2 + |\hat{v}|^2 + |\hat{\theta}|^2\right) + \left(M - C(\epsilon, \epsilon_1, \epsilon_2, \alpha_1, \alpha_2)\right)|\hat{q}|^2 \leq 0. \tag{3.20}$$

By choosing $M$ large enough such that $M > C(\epsilon, \epsilon_1, \epsilon_2, \alpha_1, \alpha_2)$, (3.20) takes the form

$$\frac{d\mathscr{L}(t)}{dt} + c\mathscr{W}(t) \leq 0, \tag{3.21}$$

where

$$\mathscr{W}(t) = \frac{\xi^2}{1+\xi^2}\left(|\hat{z}|^2 + |\hat{v}|^2 + |\hat{\theta}|^2\right) + |\hat{q}|^2$$

and $c$ is a positive constant.

On the other hand, and for large enough $M$, there exist three positive constants, $\beta_1$, $\beta_2$, and $\beta_3$, such that for all $t \geq 0$, we have

$$\beta_1\mathscr{E}(t) \leq \mathscr{L}(t) \leq \beta_2\mathscr{E}(t) \quad \text{and} \quad \mathscr{W}(t) \geq \beta_3\rho(\xi)\mathscr{E}(t), \tag{3.22}$$

where $\rho(\xi) = \xi^2/\left(1+\xi^2\right)$. (See [21] for the proof of similar inequalities.)

Consequently, from (3.21) and (3.22) we can find $\eta > 0$ such that

$$\mathscr{E}(t) = \left|\hat{U}(\xi, t)\right|^2 \leq e^{-\eta\rho(\xi)}\mathscr{E}(0).$$

Thus, inequality (3.8) is proved. ■

Dividing the domain of integration into the low-frequency region ($|\xi| < 1$) and into the high-frequency region ($|\xi| \geq 1$), we obtain the following theorem.

**Theorem 3.3** *Let $s$ be a nonnegative integer and assume that $U_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$. Then, the solution $U$ of problem (3.3) satisfies the following decay estimates:*

$$\left\|\partial_x^k U(t)\right\|_2 \leq C(1+t)^{-1/4-k/2}\|U_0\|_1 + Ce^{-ct}\left\|\partial_x^k U_0\right\|_2, \tag{3.23}$$

*for $0 \leq k \leq s$.*

Moreover, using the Fourier representation, we can improve the decay result obtained in Theorem 3.3 by restricting the initial data to $U_0 \in H^s(\mathbb{R}) \cap L^{1,\gamma}(\mathbb{R})$ with $\gamma \in [0, 1]$. For these data, we get faster decay estimates than those given in Theorem 3.3. As in [18, 19], we obtain the following theorem.

**Theorem 3.4** *Let $s$ be a nonnegative integer and assume that $U_0 \in H^s(\mathbb{R}) \cap L^{1,\gamma}(\mathbb{R})$, where $\gamma \in [0, 1]$. Then the solution $U$ of problem (3.3) satisfies the following decay estimates:*

$$\left\|\partial_x^k U(t)\right\|_2 \leq C(1+t)^{-1/4-k/2-\gamma/2}\|U_0\|_{1,\gamma} + Ce^{-ct}\left\|\partial_x^k U_0\right\|_2 + C(1+t)^{-1/4-k/2}\left|\int_{\mathbb{R}} U_0(x)\,dx\right|, \tag{3.24}$$

*for $0 \leq k \leq s$.*

**Remark 3.5** *The estimate (3.24) shows that by taking the initial data* $U_0 \in H^l(\mathbb{R}) \cap L^{1,\gamma}(\mathbb{R})$, $\gamma \in [0,1]$, *such that* $\int_{\mathbb{R}} U_0(x)\,dx = 0$, *the decay rates given in Theorem 3.3 can be improved by* $t^{-\gamma/2}$.

**Remark 3.6** *For* $\gamma \in \mathbb{N}$ *and* $U_0 \in H^s(\mathbb{R}) \cap L^{1,2(\gamma+1)}(\mathbb{R})$ *satisfying*

$$\int_{\mathbb{R}} x^m U_0(x)\,dx = 0, \quad m = 0, ..., 2\gamma,$$

*the following stronger decay estimates are obtained:*

$$\left\|\partial_x^k U(t)\right\|_2 \le C(1+t)^{-1/4-k/2-(2\gamma+1)/2}\left(\|U_0\|_{1,2(\gamma+1)} + \|U_0\|_{1,2\gamma+1} + \left\|\partial_x^k U_0\right\|_2\right), \tag{3.25}$$

*where k is a non-negative integer satisfying* $k \le s$ *and C is a positive constant. This can be proved using [11, Lemma 2.3] and the estimate (3.8).*

Now, we are going to prove the decay rate of the $L^\infty$ norm of the solution of (3.3).

**Theorem 3.7** *Let s be a nonnegative integer and* $\gamma \in [0,1]$. *Then, we have the following decay estimates:*

**(i)** *If* $U_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$, *then the solution U of problem (3.3) satisfies*

$$\left\|\partial_x^k U\right\|_\infty \le Ct^{-\frac{1+k}{2}}\left(\|U_0\|_1 + \left\|\partial_x^{k+1} U_0\right\|_2\right), \qquad k = 0,1,2,..,s-1. \tag{3.26}$$

**(ii)** *If* $U_0 \in H^s(\mathbb{R}) \cap L^{1,\gamma}(\mathbb{R})$, *then the solution U of problem (3.3) satisfies*

$$\left\|\partial_x^k U\right\|_\infty \le Ct^{-\frac{1}{2}-\frac{k+\gamma}{2}}\left(\|U_0\|_{1,\gamma} + \left\|\partial_x^{k+1} U_0\right\|_2\right) + C\left|\int_{\mathbb{R}} U_0(x)\,dx\right| t^{-\frac{1}{2}-\frac{k}{2}}, \;\; k = 0,1,2,..,s-1. \tag{3.27}$$

**Proof.** By using the Fourier transform, we have from (3.8) that

$$\left|\hat{U}(\xi,t)\right| \le Ce^{-\frac{c}{2}\rho(\xi)t}\left|\hat{U}(\xi,0)\right|.$$

Consequently,

$$\begin{aligned}
\left\|\widehat{\partial_x^k U}\right\|_1 = \left\|(i\xi)^k \hat{U}\right\|_1 &\le C\left\||\xi|^k e^{-\frac{c}{2}\rho(\xi)t}\hat{U}_0\right\|_1 \\
&\le C\int_{\mathbb{R}} |\xi|^k e^{-\frac{c}{2}\rho(\xi)t}\left|\hat{U}_0(\xi)\right| d\xi.
\end{aligned} \tag{3.28}$$

By splitting the integral on the right-hand side of (3.28) into the low-frequency part ($\xi < 1$) and the high-frequency part ($\xi \ge 1$), and by using the same methods as in the proof of Theorem 3.3, we get the following estimate:

$$\begin{aligned}
\left\|\widehat{\partial_x^k U}\right\|_1 &\le C\|U_0\|_1 \int_{\xi\le 1} |\xi|^k e^{-c\xi^2 t} d\xi + C\int_{|\xi|\ge 1} |\xi|^k e^{-\frac{c}{2}\rho(\xi)t}\left|\hat{U}(\xi,0)\right| d\xi \\
&\le C\|U_0\|_1 \int_{\xi\le 1} |\xi|^k e^{-c\xi^2 t} d\xi + Ce^{-\frac{c}{4}t}\int_{|\xi|\ge 1} |\xi|^k\left|\hat{U}(\xi,0)\right| d\xi \\
&\le C\|U_0\|_1 t^{-\frac{k+1}{2}} + Ce^{-\frac{c}{4}t}\left(\int_{|\xi|\ge 1} |\xi|^{2(k+1)}\left|\hat{U}(\xi,0)\right|^2 d\xi\right)^{1/2}\left(\int_{|\xi|\ge 1} |\xi|^{-2} d\xi\right)^{1/2} \\
&\le C\|U_0\|_1 t^{-\frac{k+1}{2}} + Ce^{-\frac{c}{4}t}\left\|\partial_x^{k+1} U_0\right\|_2.
\end{aligned} \tag{3.29}$$

By using the inequality

$$\|f\|_p \le \|\hat{f}\|_q, \qquad \frac{1}{p}+\frac{1}{q}=1, \qquad 1\le q\le 2, \tag{3.30}$$

we have

$$\left\|\partial_x^k W\right\|_\infty \le \left\|\widehat{\partial_x^k W}\right\|_1,$$

and therefore, this last inequality together with the estimate (3.29) imply (3.26). Thus, (i) is proved.

To prove (ii), we have, as in (3.28),

$$\left\|\widehat{\partial_x^k U}\right\|_1 \le C\int_{\xi\le 1}|\xi|^k e^{-c\xi^2 t}\left|\hat{U}_0(\xi)\right|d\xi + C\int_{|\xi|\ge 1}|\xi|^k e^{-\frac{c}{2}\rho(\xi)t}\left|\hat{U}(\xi,0)\right|d\xi. \tag{3.31}$$

We can estimate $\left|\hat{U}_0\right|$ as follows [12]:

$$\left|\hat{U}_0(\xi)\right| \le C_\gamma |\xi|^\gamma \|U_0\|_{1,\gamma} + \left|\int_{\mathbb{R}} U_0(x)\,dx\right| \tag{3.32}$$

with $C_\gamma = K_\gamma + M_\gamma$. Consequently, inserting (3.32) in (3.31) yields

$$\begin{aligned}\left\|\widehat{\partial_x^k U}\right\|_1 \quad\le\quad & C\|U_0\|_{1,\gamma}\int_{\xi\le 1}|\xi|^{k+\gamma}e^{-c\xi^2 t}d\xi + C\left|\int_{\mathbb{R}}U_0(x)\,dx\right|\int_{\xi\le 1}|\xi|^k e^{-c\xi^2 t}d\xi \\ & + C\int_{|\xi|\ge 1}|\xi|^k e^{-\frac{c}{2}\rho(\xi)t}\left|\hat{U}(\xi,0)\right|d\xi.\end{aligned} \tag{3.33}$$

This means, by using

$$\int_0^1 |\xi|^\sigma e^{-\frac{c}{2}\xi^2 t}d\xi \le C(1+t)^{-(\sigma+1)/2}, \tag{3.34}$$

that

$$\left\|\widehat{\partial_x^k U}\right\|_1 \le C\|U_0\|_1 t^{-\frac{1}{2}-\frac{k+\gamma}{2}} + Ce^{-\frac{c}{4}t}\left\|\partial_x^{k+1}U_0\right\|_2 + C\left|\int_{\mathbb{R}}U_0(x)\,dx\right| t^{-\frac{k+1}{2}}.$$

Consequently, using the (3.30) once again yields (3.27). Thus, the proof of (ii) is finished as is the proof of Theorem 3.7. ■

**Remark 3.8** *The decay estimates (3.24), (3.26) and (3.27) are new, but the estimate (3.23) can also be deduced from the work of Young and Wang [27]. However, our method of the proof here is different and can be adapted to tackle the nonlinear problem as well.*

## 4 The nonlinear problem

In this section, we extend the decay results obtained in section 3 to the nonlinear problem (1.1). The latter can be written as the first-order system

$$\begin{cases} U_t + F(U)_x + LU = 0, \\ U(x,0) = U_0, \end{cases} \tag{4.1}$$

where $U = (v, z, \theta, q)^T$ and $F(U) = \left(-\sigma(z) + \beta\theta, v, \eta q + \delta v, \frac{\kappa}{\tau}\theta\right)^T$. The linearized problem (3.1) can be obtained by taking the Jacobian of $F$ at $U = 0$ and setting $\sigma'(0) = a$.

**Remark 4.1** *Our system is a symmetrizable hyperbolic system as in [1, 2]. In contrast to [1, 2], however, with our direct approach, we do not need assumptions on the existence of entropy functions satisfying certain inequalities. Moreover, we have additionally obtained results on decay estimates for derivatives of the solution.*

Our goal in this section is to establish the global existence and asymptotic behavior of the solution of (4.1), at least for small initial data.

The local existence result can be obtained by the standard method based on the successive approximation sequence. We omit its details and only derive desired a priori estimates of solutions.

**Remark 4.2** *As we mentioned in the Introduction, Tarabek [26] obtained a global existence result similar to the one in Theorem 4.5 below, by using the energy method from the work of Slemrod [24]. He used a relation from the second law of thermodynamics to overcome the difficulty arising from the lack of the Poincaré inequality (see inequality (2.16) in [26]). However, no decay rate of the solution has been given in [26]. Here, we use a different method based on the weighted energy estimates to prove Theorem 4.5. This method also allows us to find the decay rates of the solution in certain Sobolev spaces, as shown in Theorem 4.8. These decay rates extend the ones given in [20] to estimates for the derivatives.*

Our main goal now is to prove the global existence Theorem 4.5 and Theorem 4.8, which establishes the decay rates. We first need several lemmas and propositions. The aim of the computations that follow is to establish the inequality (4.67) below. To this end, the following basic types of estimates will be used:

- estimates obtained directly from the derivative of the total energy of the system (4.9);
- inequalities obtained by using equations in (4.13) to express estimates of energy of higher order;
- interpolation inequalities.

This approach has successfully been applied to other problems, e.g. Timoshenko systems [18, 19].

In order to state our main result, we introduce the time-weighted energy norm, $E(t)$, and the corresponding dissipation norm, $D(t)$, as follows:

$$E^2(t) = \sum_{j=0}^{s} \sup_{0\le\zeta\le t} (1+\zeta)^j \left\|\partial_x^j U(\zeta)\right\|_{H^{s-j}}^2, \tag{4.2}$$

$$D^2(t) = \sum_{j=0}^{s-1} \int_0^t (1+\zeta)^j \left\|\partial_x^{j+1} U(\zeta)\right\|_{H^{s-j-1}}^2 d\zeta + \sum_{j=0}^{s} \int_0^t (1+\zeta)^j \left\|\partial_x^j q(\tau)\right\|_{H^{s-j}}^2 d\zeta. \tag{4.3}$$

The following quantities are convenient in the computations below:

$$\begin{aligned} M_0(t) &= \sup_{0\le\zeta\le t} \|U(\zeta)\|_\infty, \\ M_1(t) &= \sup_{0\le\zeta\le t} (1+\zeta)\|\partial_x U(\zeta)\|_\infty, \\ M(t) &= \sum_{j=0}^{s} \sup_{0\le\zeta\le t} (1+\zeta)^{1/4+j/2} \left\|\partial_x^j U(\zeta)\right\|_2. \end{aligned}$$

Now, let $T > 0$ and consider solutions of (4.1) that are defined on the time interval $[0, T]$ and satisfy $U \in C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$. Thanks to the assumption $s > N/2 + 1 = 3/2$ (where $N = 1$ is the space dimension), it follows from the Sobolev embedding theorem that

$$\sup_{0\le\zeta\le t} \|U(\zeta)\|_\infty + \sup_{0\le\zeta\le t} \|\partial_x U(\zeta)\|_\infty \le C \|U(t)\|_{H^s}. \tag{4.4}$$

Thus, we derive the energy estimates under the a priori assumption

$$\sup_{0\le t\le T} \|U(t)\|_\infty \le \bar{\alpha}, \tag{4.5}$$

where $\bar{\alpha}$ is a fixed small number independent of $T$.

**Proposition 4.3** *Let $U_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ with $s \ge 3$. Assume that $T > 0$ and let $U$ be the local solution of problem (4.1), such that*

$$U \in C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$$

*and*

$$\sup_{0\le t\le T} \|U(t)\|_\infty \le \bar{\alpha}.$$

*Then, the estimate*

$$E(t)^2 + D(t)^2 \le C \|U_0\|_{H^s}^2 + C(M_0(t) + M_1(t)) D^2(t) \tag{4.6}$$

*holds true for all $t \in [0, T]$, where $C$ is a positive constant independent of $T$.*

**Proof.** In order to prove Proposition 4.3, it suffices to show that, for any $t \in [0, T]$ and for any $0 \le j \le s-1$, the estimates

$$\begin{aligned}(1+\zeta)^j \left\|\partial_x^j U(\zeta)\right\|_{H^{s-j}}^2 + \int_0^t (1+\zeta)^j \left\|\partial_x^{j+1} U(\zeta)\right\|_{H^{s-j-1}}^2 d\zeta + \int_0^t (1+\zeta)^j \left\|\partial_x^j q(\zeta)\right\|_{H^{s-j}}^2 d\zeta \\ \le C \|U_0\|_{H^s}^2 + C(M_0(t) + M_1(t)) D^2(t)\end{aligned} \tag{4.7}$$

and

$$(1+\zeta)^s \left\|\partial_x^s U(\zeta)\right\|_2^2 + \int_0^t (1+\zeta)^s \left\|\partial_x^s q(\zeta)\right\|_2^2 d\zeta \le C \|U_0\|_{H^s}^2 + C(M_0(t) + M_1(t)) D^2(t) \tag{4.8}$$

hold true.

First, let us rewrite (4.1) as:

$$\begin{cases} v_t - \sigma(z)_x + \beta\theta_x = 0, \\ z_t - v_x = 0, \\ \theta_t + \eta q_x + \delta v_x = 0, \\ \tau q_t + q + \kappa\theta_x = 0. \end{cases} \tag{4.9}$$

We proceed with the basic energy estimate by multiplying the first equation in (4.9) by $\kappa\delta v$, the second equation by $\kappa\delta(\sigma(z) - \sigma(0))$, the third equation by $\kappa\beta\theta$, and the fourth equation by $\eta\beta q$. Adding the resulting equations, we obtain:

$$\begin{aligned}&\frac{1}{2}\left(\kappa\delta v^2 + \kappa\beta\theta^2 + \eta\beta\tau q^2 + \kappa\delta F(z)\right)_t \\ &+ (\kappa\beta\delta v\theta + \kappa\beta\eta q\theta - (\sigma(z) - \sigma(0)) v)_x + \eta\beta q^2 = 0,\end{aligned} \tag{4.10}$$

where

$$F(z) = 2\int_0^z (\sigma(s) - \sigma(0))\, ds.$$

Integrating equation (4.10) with respect to $x$ over $\mathbb{R}$, we find that

$$\frac{d}{dt}E^{(0)}(t) + \eta\beta \|q\|_2^2 = 0, \tag{4.11}$$

where

$$E^{(0)}(t) = \frac{1}{2}\left(\kappa\delta \|v\|_2^2 + \kappa\beta \|\theta\|_2^2 + \eta\beta\tau \|q\|_2^2\right) + \kappa\delta \int_{\mathbb{R}} F(z)\, dx. \tag{4.12}$$

To obtain the energy estimates on higher-order terms, we apply $\partial_x^k$ to (4.9) and get

$$\begin{cases} \partial_x^k v_t - \sigma'(z)\, \partial_x^{k+1}(z) + \beta\partial_x^{k+1}\theta = \left[\partial_x^k, \sigma'(z)\right] z_x, \\ \partial_x^k z_t - \partial_x^{k+1} v = 0, \\ \partial_x^k \theta_t + \eta\partial^{k+1} q + \delta\partial_x^{k+1} v = 0, \\ \tau\partial_x^k q_t + \partial_x^k q + \kappa\partial_x^{k+1}\theta = 0. \end{cases} \tag{4.13}$$

where we used $[\partial_x^k, A]B = \partial_x^k(AB) - A\partial_x^k B$.

Now, we define the energy associated with system (4.13) as

$$E^k(t) = \frac{1}{2}\left(\kappa\delta \left\|\partial_x^k v\right\|_2^2 + \kappa\beta \left\|\partial_x^k \theta\right\|_2^2 + \eta\beta\tau \left\|\partial_x^k q\right\|_2^2\right) + \kappa\delta \int_{\mathbb{R}} F^k(z)\, dx, \tag{4.14}$$

where

$$F^k(z) = \frac{1}{2}\sigma'(z)(\partial_x^k z)^2.$$

Next, we construct a Lyapunov functional with appropriate multipliers. Multiplying the first equation in (4.13) by $\kappa\delta\partial_x^k v$, the second equation by $\kappa\delta(\sigma'(z)\partial_x^k z)$, the third equation by $\kappa\beta\partial_x^k\theta$, the fourth equation by $\eta\beta\partial_x^k q$, and adding the resulting equations, we get

$$\begin{aligned} &\frac{1}{2}\left(\kappa\delta(\partial_x^k v)^2 + \kappa\beta(\partial_x^k\theta)^2 + \eta\beta\tau(\partial_x^k q)^2 + \kappa\delta\sigma'(z)(\partial_x^k z)^2\right)_t \\ &+ \left(\kappa\beta\delta\partial_x^k v\partial_x^k\theta + \kappa\beta\eta\partial_x^k q\partial_x^k\theta - \sigma'(z)\,\partial_x^k(z)\,\partial_x^k v\right)_x + \eta\beta(\partial_x^k q)^2 \\ &= \frac{\kappa\delta}{2}\sigma'(z)_t\left(\partial_x^k z\right)^2 - \kappa\delta\sigma'(z)_x\left(\partial_x^k z\right)\partial_x^k v + \kappa\delta\partial_x^k v\left[\partial_x^k, \sigma'(z)\right] z_x. \end{aligned} \tag{4.15}$$

Integrating (4.15) with respect to $x$, we obtain

$$\frac{d}{dt}E^{(k)}(t) + \eta\beta\left\|\partial_x^k q\right\|_2^2 = R_0^{(k)}, \tag{4.16}$$

where

$$R_0^{(k)} = \int_{\mathbb{R}}\left\{\frac{\kappa\delta}{2}\sigma'(z)_t\left(\partial_x^k z\right)^2 - \kappa\delta\sigma'(z)_x\left(\partial_x^k z\right)\partial_x^k v + \kappa\delta\partial_x^k v\left[\partial_x^k, \sigma'(z)\right] z_x\right\} dx.$$

Using the assumption (4.5) and keeping in mind that $z_t = v_x$, we get

$$\left|R_0^{(k)}\right| \leq C\int_{\mathbb{R}} |v_x|\left|\partial_x^k z\right|^2 + |z_x|\left|\partial_x^k z\right|\left|\partial_x^k v\right| + \left|\partial_x^k v\right|\left|\left[\partial_x^k, \sigma'(z)\right]\right| |z_x|,$$

which implies, by using Lemma 2.2 (see [10], where a similar estimate has been proved for the Timoshenko system), that

$$\left|R_0^{(k)}\right| \leq C \|\partial_x U\|_\infty \left\|\partial_x^k U\right\|_2^2. \tag{4.17}$$

On the other hand, recalling (4.5), we deduce that there exist two positive constants, $\beta_1$ and $\beta_2$, such that

$$\beta_1 \left\|\partial_x^k U\right\|_2^2 \leq E^k(t) \leq \beta_2 \left\|\partial_x^k U\right\|_2^2, \qquad k \geq 0. \tag{4.18}$$

Consequently, multiplying (4.11) by $(1+t)^\mu$ and integrating with respect to $t$ and using (4.18) we get

$$\begin{aligned}
&(1+t)^\mu \|U(t)\|_2^2 + \eta\beta \int_0^t (1+\zeta)^\mu \|q(\zeta)\|_2^2 \, d\zeta \\
\leq\quad & \|U_0\|_2^2 + \mu \int_0^t (1+\zeta)^{\mu-1} \|U(\zeta)\|_2^2 \, d\zeta.
\end{aligned} \tag{4.19}$$

Similarly, for $k \geq 1$, the estimates (4.17), (4.18) together with (4.16) yield after multiplication by $(1+t)^\mu$ and integration with respect to $t$ over $(0,t)$ the following:

$$\begin{aligned}
&(1+t)^\mu \left\|\partial_x^k U(t)\right\|_2^2 + \eta\beta \int_0^t (1+\zeta)^\mu \left\|\partial_x^k q(\zeta)\right\|_2^2 d\zeta \\
&\leq C \left\|\partial_x^k U_0\right\|_2^2 + \mu \int_0^t (1+\zeta)^{\mu-1} \|\partial_x^k U(\zeta)\|_2^2 d\zeta \\
&+C \int_0^t (1+\zeta)^\mu \|\partial_x U(\zeta)\|_\infty \|\partial_x^k U(\zeta)\|_2^2 d\zeta.
\end{aligned} \tag{4.20}$$

Adding the estimate (4.19) to (4.20) and taking the summation for $1 \leq k \leq s$, we get the main estimate:

$$\begin{aligned}
&(1+t)^\mu \|U(t)\|_{H^s}^2 + \eta\beta \int_0^t (1+\zeta)^\mu \|q(\zeta)\|_{H^s}^2 \, d\zeta \\
&\leq C \|U_0\|_{H^s}^2 + \mu \int_0^t (1+\zeta)^{\mu-1} \|U(\zeta)\|_{H^s}^2 \, d\zeta \\
&+C \int_0^t (1+\zeta)^\mu \|\partial_x U(\zeta)\|_\infty \|\partial_x U(\zeta)\|_{H^{s-1}}^2 \, d\zeta.
\end{aligned} \tag{4.21}$$

To control the second term on the right-hand side of (4.21), we have to get a dissipative term of the form $\int_0^t \|U(t)\|_{H^s}^2$ on the left-hand side of (4.21). Applying $\partial_x^k$ to (4.9) and putting $\partial_x^k(v,z,\theta,q) = (\tilde{v},\tilde{z},\tilde{\theta},\tilde{q})$, we get

$$\begin{cases}
\tilde{v}_t - a\tilde{z}_x + \beta\tilde{\theta}_x = \partial_x^k g(z)_x, \\
\tilde{z}_t - \tilde{v}_x = 0, \\
\tilde{\theta}_t + \eta\tilde{q}_x + \delta\tilde{v}_x = 0, \\
\tau\tilde{q}_t + \tilde{q} + \kappa\tilde{\theta}_x = 0,
\end{cases} \tag{4.22}$$

where $g(z) = \sigma(z) - \sigma(0) - \sigma'(0)z = O(z^2)$.

Multiplying the second equation in (4.22) by $\tilde{v}_x$, the first equation by $-\tilde{z}_x$ and adding the resulting equations, we get

$$(\tilde{v}\tilde{z}_x)_t - (\tilde{v}\tilde{z}_t)_x + a\tilde{z}_x^2 - \tilde{v}_x^2 - \beta\tilde{\theta}_x\tilde{z}_x = -\tilde{z}_x\partial_x^k g(z)_x.$$

Young's inequality gives

$$\beta\tilde{\theta}_x\tilde{z}_x \leq \varepsilon\tilde{z}_x^2 + C(\varepsilon)\tilde{\theta}_x^2,$$

and, for any $\varepsilon > 0$,

$$(\tilde{v}\tilde{z}_x)_t - (\tilde{v}\tilde{z}_t)_x + (a-\varepsilon)\,\tilde{z}_x^2 - \tilde{v}_x^2 \leq -\tilde{z}_x\partial_x^k g\,(z)_x + C\,(\varepsilon)\,\tilde{\theta}_x^2. \tag{4.23}$$

Similarly, multiplying the third equation in (4.22) by $\tilde{v}_x$, the first equation by $-\tilde{\theta}_x$, adding the two resulting equations, we obtain

$$(\tilde{\theta}\tilde{v}_x)_t - (\tilde{\theta}\tilde{v}_t)_x + \delta\tilde{v}_x^2 - \beta\tilde{\theta}_x^2 = -\eta\tilde{q}_x\tilde{v}_x - a\tilde{z}_x\tilde{\theta}_x - \tilde{\theta}_x\partial_x^k g\,(z)_x\,.$$

Again, Young's inequality implies that, for any $\varepsilon$, $\varepsilon_1 > 0$,

$$(\tilde{\theta}\tilde{v}_x)_t - (\tilde{\theta}\tilde{v}_t)_x + (\delta-\varepsilon)\,\tilde{v}_x^2 \leq\ \varepsilon_1\tilde{z}_x^2 + C\,(\varepsilon_1)\,\tilde{\theta}_x^2 + C\,(\varepsilon)\,\tilde{q}_x^2 - \tilde{\theta}_x\partial_x^k g\,(z)_x\,. \tag{4.24}$$

Next, multiplying the fourth equation in (4.22) by $\frac{1}{\tau}\tilde{\theta}_x$, the third equation by $-\tilde{q}_x$, and adding the results, we get

$$\left(\tilde{q}\tilde{\theta}_x\right)_t - \left(\tilde{q}\tilde{\theta}_t\right)_x + \frac{\kappa}{\tau}\tilde{\theta}_x^2 + \frac{1}{\tau}\tilde{q}\tilde{\theta}_x - \delta\tilde{v}_x\tilde{q}_x = 0,$$

which gives, by Young's inequality,

$$\left(\tilde{q}\tilde{\theta}_x\right)_t - \left(\tilde{q}\tilde{\theta}_t\right)_x + \left(\frac{\kappa}{\tau}-\varepsilon\right)\tilde{\theta}_x^2 \leq \varepsilon_2\tilde{v}_x^2 + C\,(\varepsilon,\varepsilon_2)\left(\tilde{q}_x^2 + q^2\right). \tag{4.25}$$

Now, we let $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ be two positive constants (to be chosen later), take (4.23) + $\tilde{\alpha}_1$(4.24) + $\tilde{\alpha}_2$(4.25), and integrate the result over $\mathbb{R}$ to obtain:

$$\begin{aligned}
&\frac{d}{dt}\tilde{\mathscr{F}}^{(k)}\,(t) + \{(a-\varepsilon) - \varepsilon_1\tilde{\alpha}_1\}\left\|\partial_x^k z_x\right\|_2^2 + \{\tilde{\alpha}_1\,(\delta-\varepsilon) - 1 - \varepsilon_2\tilde{\alpha}_2\}\left\|\partial_x^k v_x\right\|_2^2\\
&+\left\{\tilde{\alpha}_2\left(\frac{\kappa}{\tau}-\varepsilon\right) - C\,(\varepsilon) - \tilde{\alpha}_1 C\,(\varepsilon_1)\right\}\left\|\partial_x^k\theta_x\right\|_2^2\\
&\leq \tilde{\alpha}_2 C\,(\varepsilon,\varepsilon_2,\tilde{\alpha}_1)\left(\left\|\partial_x^k q_x\right\|_2^2 + \left\|\partial_x^k q\right\|_2^2\right) + R^{(k)},
\end{aligned} \tag{4.26}$$

where

$$\tilde{\mathscr{F}}^{(k)}\,(t) = \int_{\mathbb{R}}\partial_x^k v\partial_x^{k+1} z dx + \tilde{\alpha}_1\int_{\mathbb{R}}\partial_x^k\theta\partial_x^{k+1} v dx + \tilde{\alpha}_2\int_{\mathbb{R}}\partial_x^k q\partial_x^{k+1}\theta dx \tag{4.27}$$

and

$$R^{(k)} = -\int_{\mathbb{R}}\left(\partial_x^k z_x\partial_x^k g\,(z)_x + \tilde{\alpha}_1\partial_x^k\theta_x\partial_x^k g\,(z)_x\right)dx.$$

We choose the constants $\varepsilon$, $\varepsilon_1$, $\varepsilon_2$, $\tilde{\alpha}_1$, and $\tilde{\alpha}_2$ exactly as we have chosen $\epsilon$, $\epsilon_1$, $\epsilon_2$, $\alpha_1$, and $\alpha_2$ in (3.16). Then, we deduce that there exists $\tilde{\lambda} > 0$, such that for any $0 \leq k \leq s-1$, the estimate (4.26) takes the form

$$\frac{d}{dt}\tilde{\mathscr{F}}^{(k)}\,(t) + \tilde{\lambda}\left\{\left\|\partial_x^k z_x\right\|_2^2 + \left\|\partial_x^k v_x\right\|_2^2 + \left\|\partial_x^k\theta_x\right\|_2^2\right\} \leq \tilde{\alpha}_2 C\,(\varepsilon,\varepsilon_2,\tilde{\alpha}_1)\left\|\partial_x^k q\right\|_{H^1}^2 + R^{(k)}.$$

This inequality can be rewritten as

$$\frac{d}{dt}\tilde{\mathscr{F}}^{(k)}\,(t) + \tilde{\lambda}_1\left\|\partial_x^{k+1} U\,(t)\right\|_2^2 \leq \tilde{\alpha}_2 C\,(\varepsilon,\varepsilon_2,\tilde{\alpha}_1)\left\|\partial_x^k q\,(t)\right\|_{H^1}^2 + R^{(k)}, \tag{4.28}$$

for any $0 \leq k \leq s-1$ and for some $\tilde{\lambda}_1 > 0$.

On the other hand, there exists a constant $c_1 > 0$, such that

$$\left|\tilde{\mathscr{F}}^{(k)}\,(t)\right| \leq c_1\|\partial_x^k U\,(t)\,\|_{H^1}^2, \qquad \forall t \geq 0. \tag{4.29}$$

We also have the following estimate, as in [10],

$$R_1^{(k)} \leq C\|z\|_\infty \|\partial_x^{k+1} z\|_2^2 + C\|z\|_\infty \|\partial_x^{k+1}\theta\|_2 \|\partial_x^{k+1} z\|_2. \tag{4.30}$$

Now, multiplying (4.28) by $(1+t)^\mu$, integrating with respect to $t$ and using (4.29) and (4.30), we arrive at

$$\begin{aligned}\int_0^t (1+\zeta)^\mu \left\|\partial_x^{k+1} U(\zeta)\right\|_2^2 d\zeta \quad &\leq \quad C\|\partial_x^k U_0\|_{H^1}^2 + C(1+t)^\mu \|\partial_x^k U(t)\|_{H^1}^2 + C\int_0^t (1+\zeta)^\mu \left\|\partial_x^k q(\zeta)\right\|_{H^1}^2 d\zeta \\ &\quad + C\int_0^t (1+\zeta)^\mu \|z\|_\infty \left(\|\partial_x^{k+1} z\|_2^2 + \|\partial_x^{k+1}\theta\|_2 \|\partial_x^{k+1} z\|_2\right)\end{aligned} \tag{4.31}$$

for all $t \geq 0$ and for $0 \leq k \leq s-1$. Taking the summation in (4.31) over $k$ with $0 \leq k \leq s-1$, we get

$$\begin{aligned}\int_0^t (1+\zeta)^\mu \|\partial_x U(\zeta)\|_{H^{s-1}}^2 d\zeta \quad &\leq \quad C\|U_0\|_{H^s}^2 + C(1+t)^\mu \|U(t)\|_{H^s}^2 + C\int_0^t (1+\zeta)^\mu \|q(\zeta)\|_{H^s}^2 d\zeta \\ &\quad + C\int_0^t (1+\zeta)^\mu \|z\|_\infty \left(\|\partial_x z\|_{H^{s-1}}^2 + \|\partial_x\theta\|_{H^{s-1}} \|\partial_x z\|_{H^{s-1}}\right) d\zeta.\end{aligned} \tag{4.32}$$

Let $\omega$ be a positive constant. Then taking (4.21)+$\omega$(4.32) and choosing $\omega$ small enough, we arrive at the following estimate:

$$\begin{aligned}&(1+t)^\mu \|U(t)\|_{H^s}^2 + \int_0^t (1+\zeta)^\mu \|q(\zeta)\|_{H^s}^2 d\zeta + \omega\int_0^t (1+\zeta)^\mu \|\partial_x U(\zeta)\|_{H^{s-1}}^2 d\zeta \\ &\leq C\|U_0\|_{H^s}^2 + \mu\int_0^t (1+\zeta)^{\mu-1} \|U(\zeta)\|_{H^s}^2 d\zeta \\ &+ C\omega\int_0^t (1+\zeta)^\mu \|z\|_\infty \left(\|\partial_x z\|_{H^{s-1}}^2 + \|\partial_x\theta\|_{H^{s-1}} \|\partial_x z\|_{H^{s-1}}\right) d\zeta \\ &+ C\int_0^t (1+\zeta)^\mu \|\partial_x U(\zeta)\|_\infty \|\partial_x U(\zeta)\|_{H^{s-1}}^2 d\zeta.\end{aligned} \tag{4.33}$$

Our goal now is to prove (4.7), which will be done by induction on $j$. Indeed, we deduce from (4.49) that (4.7) holds for $j = 0$. Now, we let $0 \leq l \leq s-1$ and suppose that (4.7) holds true for $j = l$. Then, we will show the validity of (4.7) for $j = l+1$. Taking $\mu = l+1$ in (4.20) and adding over $k$ with $l+1 \leq k \leq s$, we obtain

$$\begin{aligned}&(1+t)^{l+1} \left\|\partial_x^{l+1} U(t)\right\|_{H^{s-l-1}}^2 + \int_0^t (1+\zeta)^{l+1} \left\|\partial_x^{l+1} q(\zeta)\right\|_{H^{s-l-1}}^2 d\zeta \\ &\leq C\|U_0\|_{H^s}^2 + C\int_0^t (1+\zeta)^l \|\partial_x^{l+1} U(\zeta)\|_{H^{s-l-1}}^2 d\zeta \\ &+ C\int_0^t (1+\zeta)^{l+1} \|\partial_x U(\zeta)\|_\infty \|\partial_x^{l+1} U(\zeta)\|_{H^{s-l-1}}^2 d\zeta,\end{aligned} \tag{4.34}$$

where we have used the fact that $\left\|\partial_x^{l+1} U_0\right\|_{H^{s-l-1}}^2 \leq \|U_0\|_{H^s}^2$.

By the same method, taking $\mu = l+1$ in (4.31) and adding over $k$ with $l+1 \leq k \leq s-1$, we have

$$\begin{aligned}\int_0^t (1+\zeta)^{l+1} \left\|\partial_x^{l+2} U(\zeta)\right\|_{H^{s-l-2}}^2 d\zeta \quad &\leq \quad C\|U_0\|_{H^s}^2 + C(1+t)^{l+1} \|\partial_x^{l+1} U(t)\|_{H^{s-l-1}} \\ &\quad + C\int_0^t (1+\zeta)^{l+1} \left\|\partial_x^{l+1} q(\zeta)\right\|_{H^{s-l-1}}^2 d\zeta \\ &\quad + C\int_0^t (1+\zeta)^{l+1} \|z\|_\infty \left(\|\partial_x^{l+2} z\|_{H^{s-l-2}}^2 + \|\partial_x^{l+2}\theta\|_{H^{s-l-2}} \|\partial_x^{l+2} z\|_{H^{s-l-2}}\right) d\zeta.\end{aligned} \tag{4.35}$$

As above, for $\hat{\omega}$ small enough, we have by taking (4.34) + $\hat{\omega}$(4.35),

$$\begin{aligned}(1+t)^{l+1}\left\|\partial_x^{l+1}U(t)\right\|_{H^{s-l-1}}^2 &+ \int_0^t (1+\zeta)^{l+1}\left\|\partial_x^{l+1}q(\zeta)\right\|_{H^{s-l-1}}^2 d\zeta + \int_0^t (1+\zeta)^{l+1}\left\|\partial_x^{l+2}U(\zeta)\right\|_{H^{s-l-2}}^2 d\zeta \\ &\leq C\|U_0\|_{H^s}^2 + C\int_0^t (1+\zeta)^l \|\partial_x^{l+1}U(\zeta)\|_{H^{s-l-1}}^2 d\zeta \\ &\quad + C\int_0^t (1+\zeta)^{l+1}\|\partial_x U(\zeta)\|_\infty \|\partial_x^{l+1}U(\zeta)\|_{H^{s-l-1}}^2 d\zeta, \\ &\quad + C\int_0^t (1+\zeta)^{l+1}\|z\|_\infty \left(\|\partial_x^{l+2}z\|_{H^{s-l-2}}^2 + \|\partial_x^{l+2}\theta\|_{H^{s-l-2}}\|\partial_x^{l+2}z\|_{H^{s-l-2}}\right) d\zeta,\end{aligned} \tag{4.36}$$

where $C$ is a positive constant depending on $\hat{\omega}$.

The second term on the right-hand side of (4.36) is estimated by the induction hypothesis (4.7) with $j = l$ as

$$\int_0^t (1+\zeta)^l \|\partial_x^{l+1}U(\zeta)\|_{H^{s-l-1}}^2 d\zeta \leq C\|U_0\|_{H^s}^2 + C(M_0(t) + M_1(t))D^2(t). \tag{4.37}$$

On the other hand, we have

$$\begin{aligned}\int_0^t (1+\zeta)^{l+1}\|\partial_x U(\zeta)\|_{L^\infty}\|\partial_x^{l+1}U(\zeta)\|_{H^{s-l-1}}^2 d\zeta &\leq CM_1(t)\int_0^t (1+\zeta)^l\|\partial_x^{l+1}U(\zeta)\|_{H^{s-l-1}}^2 d\zeta \\ &\leq CM_1(t)D^2(t).\end{aligned} \tag{4.38}$$

Also,

$$\begin{aligned}\int_0^t (1+\zeta)^{l+1}\|z\|_\infty\|\partial_x^{l+2}z\|_{H^{s-l-2}}^2 d\zeta &\leq CM_0(t)\int_0^t (1+\zeta)^{l+1}\|\partial_x^{l+2}z\|_{H^{s-l-2}}^2 d\zeta \\ &\leq CM_0(t)D^2(t).\end{aligned} \tag{4.39}$$

The last term on the right-hand side of (4.36) can be estimated as follows:

$$\begin{aligned}&\int_0^t (1+\zeta)^{l+1}\|z\|_\infty\|\partial_x^{l+2}\theta\|_{H^{s-l-2}}\|\partial_x^{l+2}z\|_{H^{s-l-2}} d\zeta \\ \leq\ & CM_0(t)\left(\int_0^t (1+\zeta)^{l+1}\|\partial_x^{l+2}\theta\|_{H^{s-l-2}}^2 d\zeta\right)^{1/2}\left(\int_0^t (1+\zeta)^{l+1}\|\partial_x^{l+2}z\|_{H^{s-l-2}}^2 d\zeta\right)^{1/2} \\ \leq\ & CM_0(t)D^2(t).\end{aligned} \tag{4.40}$$

Inserting the estimates (4.37), (4.38), (4.39), and (4.40) into (4.36), we get

$$\begin{aligned}&(1+t)^{l+1}\left\|\partial_x^{l+1}U(t)\right\|_{H^{s-l-1}}^2 + \int_0^t (1+\zeta)^{l+1}\left\|\partial_x^{l+1}q(\zeta)\right\|_{H^{s-l-1}}^2 d\zeta \\ &+ \int_0^t (1+\zeta)^{l+1}\left\|\partial_x^{l+2}U(\zeta)\right\|_{H^{s-l-2}}^2 d\zeta \\ \leq\ & C\|U_0\|_{H^s}^2 + C(M_0(t) + M_1(t))D^2(t),\end{aligned} \tag{4.41}$$

which shows that (4.7) holds true for $j = l+1$. Thus, by induction, we have proved (4.7) for all $0 \leq j \leq s-1$. Now, we are going to prove the estimate (4.8). Indeed, taking $\mu = k = s$ in (4.20), we arrive at

$$\begin{aligned}(1+t)^s\left\|\partial_x^s U(t)\right\|_2^2 + \eta\beta\int_0^t (1+\zeta)^s\left\|\partial_x^s q(\zeta)\right\|_2^2 d\zeta &\leq C\left\|\partial_x^s U_0\right\|_2^2 + s\int_0^t (1+\zeta)^{s-1}\|\partial_x^s U(\zeta)\|_2^2 d\zeta \\ &\quad + C\int_0^t (1+\zeta)^s\|\partial_x U(\zeta)\|_\infty\|\partial_x^s U(\zeta)\|_2^2 d\zeta.\end{aligned} \tag{4.42}$$

From (4.7), we have for $j = s - 1$

$$\int_0^t (1+\zeta)^{s-1} \|\partial_x^s U(\zeta)\|_2^2 d\zeta \le C\|U_0\|_{H^s}^2 + C(M_0(t) + M_1(t)) D^2(t). \tag{4.43}$$

On the other hand, the last term on the right hand side of (4.42) is estimated as follows:

$$\int_0^t (1+\zeta)^s \|\partial_x U(\zeta)\|_\infty \|\partial_x^s U(\zeta)\|_2^2 d\zeta \quad \le \quad CM_1(t) D^2(t). \tag{4.44}$$

Inserting the estimates (4.43) and (4.44) into (4.42) yields (4.8). Consequently, the proof of Proposition 4.3 is finished. ■

**Proposition 4.4** *Assume that $U_0 \in H^s(\mathbb{R})$ where $s \ge 2$ is an integer. Let $U(x,t)$ be the local solution of problem (4.1) on the interval $[0,T]$. Then, there exists a small positive constant $\bar{\alpha}$, independent of $T$, such that, if $\sup_{0\le t\le T} \|U(t,x)\|_{H^s} \le \bar{\alpha}$, then the solution satisfies the following uniform energy estimate for $t \in [0,T]$ :*

$$\|U(t)\|_{H^s}^2 + \int_0^t \|q(\zeta)\|_{H^s}^2 d\zeta + \int_0^t \|\partial_x U(\zeta)\|_{H^{s-1}}^2 d\zeta \le C\|U_0\|_{H^s}^2. \tag{4.45}$$

**Proof.** Let us take $\mu = 0$ in (4.33). Then, we obtain

$$\begin{aligned} &\|U(t)\|_{H^s}^2 + \int_0^t \|q(\zeta)\|_{H^s}^2 d\zeta + \omega \int_0^t \|\partial_x U(\zeta)\|_{H^{s-1}}^2 d\zeta \\ &\le C\|U_0\|_{H^s}^2 + C\omega \int_0^t \|z\|_\infty \left( \|\partial_x z(\zeta)\|_{H^{s-1}}^2 + \|\partial_x \theta(\zeta)\|_{H^{s-1}} \|\partial_x z(\zeta)\|_{H^{s-1}} \right) d\zeta \\ &+ C\int_0^t \|\partial_x U(\zeta)\|_\infty \|\partial_x U(\zeta)\|_{H^{s-1}}^2 d\zeta. \end{aligned} \tag{4.46}$$

Let $\bar{\alpha}$ be a small positive constant independent of $T$ such that $\sup_{0\le t\le T} \|U(t,x)\|_{H^s} \le \bar{\alpha}$, which implies, by (4.4) and $s \ge 2$, that

$$\sup_{0\le t\le T} \|\partial_x U(t)\|_\infty \le \bar{\alpha}.$$

The last term on the right-hand side of (4.46) is estimated as

$$\int_0^t \|\partial_x U(\zeta)\|_\infty \|\partial_x U(\zeta)\|_{H^{s-1}}^2 d\zeta \le \bar{\alpha} \int_0^t \|\partial_x U(\zeta)\|_{H^{s-1}}^2 d\zeta. \tag{4.47}$$

Similarly, we have

$$\begin{aligned} &\int_0^t \|z\|_\infty \left( \|\partial_x z(\zeta)\|_{H^{s-1}}^2 + \|\partial_x \theta(\zeta)\|_{H^{s-1}} \|\partial_x z(\zeta)\|_{H^{s-1}} \right) d\zeta \\ \le \quad & \bar{\alpha} \int_0^t \left( \|\partial_x z(\zeta)\|_{H^{s-1}}^2 + \|\partial_x \theta(\zeta)\|_{H^{s-1}} \|\partial_x z(\zeta)\|_{H^{s-1}} \right) d\zeta \\ \le \quad & 2\bar{\alpha} \int_0^t \|\partial_x U(\zeta)\|_{H^{s-1}}^2 d\zeta. \end{aligned} \tag{4.48}$$

Plugging (4.47) and (4.48) into (4.46), we get

$$\begin{aligned} &\|U(t)\|_{H^s}^2 + \int_0^t \|q(\zeta)\|_{H^s}^2 d\zeta + \int_0^t \|\partial_x U(\zeta)\|_{H^{s-1}}^2 d\zeta \\ &\le C\|U_0\|_{H^s}^2 + C\bar{\alpha} \int_0^t \|\partial_x U(\zeta)\|_{H^{s-1}}^2 d\zeta. \end{aligned}$$

This last inequality implies (4.45) provided that $\bar{\alpha}$ is sufficiently small. This completes the proof of Proposition 4.4. ∎

**Theorem 4.5** *Assume that $U_0 \in H^s(\mathbb{R})$ where $s \geq 2$ is an integer. Then, there exists a small positive constant $\delta_0$ such that, if $\|U_0\|_{H^s} \leq \delta_0$, then (4.1) has a unique global solution $U(x,t)$ such that*

$$U \in C\left([0,\infty); H^s\right) \cap C^1([0,\infty); H^{s-1}).$$

*Moreover, the solution satisfies the uniform energy estimate:*

$$\|U(t)\|_{H^s}^2 + \int_0^t \|q(\zeta)\|_{H^s}^2 \, d\zeta + \int_0^t \|\partial_x U(\zeta)\|_{H^{s-1}}^2 \, d\zeta \leq C \|U_0\|_{H^s}^2 . \tag{4.49}$$

**Proof.** By using Proposition 4.4, the proof of Theorem 4.5 is straightforward. Indeed, using the a priori estimate (4.45), we can apply the continuity argument and get a global solution to problem (4.1) as long as $\|U_0\|_{H^s}^2$ is suitably small, i.e., $\|U_0\|_{H^s}^2 \leq \delta_0$. The solution thus obtained satisfies the estimate (4.45) for all $t \geq 0$. The proof of Theorem 4.5 is therefore finished. ∎

**Lemma 4.6** *Suppose that $U_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ with $s \geq 3$. Then we have*

$$M(t) \leq CE_s + CM(t)^2 + CM_0(t)E(t), \tag{4.50}$$

*for all $t \in [0,T]$, where $C$ is a positive constant independent of $T$ and $E_s = \|U_0\|_{H^s} + \|U_0\|_{L^1}$.*

**Proof.** In order to prove (4.50), it suffices to establish the estimate

$$\left\|\partial_x^j U(t)\right\|_2 \leq CE_s(1+t)^{-1/4-j/2} + C\left(M(t)^2 + M_0(t)E(t)\right)(1+t)^{-1/4-j/2}, \tag{4.51}$$

for all $t \in [0,T]$ and $0 \leq j \leq s-1$.

By the Duhamel principle, the solution of (4.1) can be written as

$$U(t) = e^{t\Phi}U_0 + \int_0^t e^{(t-\zeta)\Phi} G(U)_x(\zeta)\, d\zeta, \tag{4.52}$$

where

$$\left(e^{t\Phi}\omega\right)(x) = \mathcal{F}^{-1}\left[e^{t\hat{\Phi}(i\xi)}\hat{\omega}(\xi)\right](x)$$

with $\hat{\Phi}(i\xi) = -(i\xi A + L)$ and $G(U) = (g(z), 0, 0, 0)$. Therefore, estimate (3.23) can be rewritten as

$$\left\|\partial_x^k e^{t\Phi} U_0\right\|_2 \leq C(1+t)^{-1/4-k/2}\|U_0\|_1 + Ce^{-ct}\left\|\partial_x^k U_0\right\|_2 . \tag{4.53}$$

Let $j$ be a nonnegative integer and apply $\partial_x^j$ to (4.52) to obtain

$$\begin{aligned} \left\|\partial_x^j U(t)\right\|_2 &\leq \left\|\partial_x^j e^{t\Phi} U_0\right\|_2 + \int_0^t \left\|\partial_x^{j+1} e^{(t-\zeta)\Phi} G(U)\right\|_2 d\zeta \\ &= I_1 + I_2. \end{aligned} \tag{4.54}$$

Since $e^{t\Phi}U_0$ is the solution of the linear problem, then from (3.23), we get

$$I_1 \leq CE_s(1+t)^{-1/4-j/2}. \tag{4.55}$$

To estimate $I_2$, split it into two parts:

$$\begin{aligned} I_2 &= \int_0^{t/2} \left\| \partial_x^{j+1} e^{(t-\zeta)\Phi} G\left(U(\zeta)\right) \right\|_2 d\zeta + \int_{t/2}^{t} \left\| \partial_x^{j+1} e^{(t-\zeta)\Phi} G\left(U(\zeta)\right) \right\|_2 d\zeta \\ &= J_1 + J_2, \end{aligned}$$

Using Lemma 2.2, recalling that $g(z) = O(z^2)$, and applying (3.23), we obtain

$$\begin{aligned} J_1 &\le C \int_0^{t/2} (1+t-\zeta)^{-3/4-j/2} \left\| G\left(U(\zeta)\right) \right\|_1 d\zeta + C \int_0^{t/2} e^{-c(t-\zeta)} \left\| \partial_x^{j+1} G\left(U(\zeta)\right) \right\|_2 d\zeta \\ &\le C \int_0^{t/2} (1+t-\zeta)^{-3/4-j/2} \left\| U(\zeta) \right\|_2^2 d\zeta + C \int_0^{t/2} e^{-c(t-\zeta)} \left\| \partial_x^{j+1} G\left(U(\zeta)\right) \right\|_2 d\zeta \\ &\le CM(t)^2 \int_0^{t/2} (1+t-\zeta)^{-3/4-j/2} (1+\zeta)^{-1/2} d\zeta + C \int_0^{t/2} e^{-c(t-\zeta)} \left\| \partial_x^{j+1} G\left(U(\zeta)\right) \right\|_2 d\zeta. \end{aligned} \tag{4.56}$$

The first term on the right-hand side of (4.56) can be estimated as

$$CM(t)^2 \int_0^{t/2} (1+t-\zeta)^{-3/4-j/2} (1+\zeta)^{-1/2} d\zeta \le CM(t)^2 (1+t)^{-1/4-j/2}, \tag{4.57}$$

where we used (2.6).

On the other hand,

$$\left\| \partial_x^{j+1} G(U) \right\|_2 \le \|U\|_\infty \left\| \partial_x^{j+1} U \right\|_2,$$

and, for $j+1 \le s$, we obtain

$$\|U\|_\infty \left\| \partial_x^{j+1} U \right\|_2 \le M_0(t) \left\| \partial_x^{j+1} U \right\|_{H^{s-j-1}} \le M_0(t) E(t) (1+t)^{-1/2-j/2}.$$

Consequently, the last term on the right-hand side of (4.56) can be estimated as

$$\begin{aligned} C \int_0^{t/2} e^{-c(t-\zeta)} \left\| \partial_x^{j+1} G\left(U(\zeta)\right) \right\|_2 d\zeta &\le CM_0(t) E(t) \int_0^{t/2} e^{-c(t-\zeta)} (1+\zeta)^{-1/2-j/2} d\zeta \\ &\le CM_0(t) E(t) (1+t)^{-1/2-j/2}. \end{aligned} \tag{4.58}$$

Using (4.57) and (4.58), we deduce that

$$J_1 \le C\left(M(t)^2 + M_0(t) E(t)\right) (1+t)^{-1/4-j/2}. \tag{4.59}$$

Next, $J_2$ is estimated by applying (4.53) with $j = 1$ and using $\partial_x^j G(U)$ instead of $U_0$, to obtain

$$\begin{aligned} J_2 &= \int_{t/2}^{t} \left\| \partial_x e^{(t-\zeta)\Phi} \partial_x^j G\left(U(\zeta)\right) \right\|_2 d\zeta \\ &\le C \int_{t/2}^{t} (1+t-\zeta)^{-\frac{5}{4}} \left\| \partial_x^j G\left(U(\zeta)\right) \right\|_1 d\zeta + C \int_{t/2}^{t} e^{-c(t-\zeta)} \left\| \partial_x^{j+1} G\left(U(\zeta)\right) \right\|_2 d\zeta \\ &= J_{21} + J_{22}. \end{aligned}$$

On the other hand, we have (see [10, page 1021]):

$$\begin{aligned} \left\| \partial_x^j G(U) \right\|_1 &\le C \|U\|_2 \left\| \partial_x^j U \right\|_2 \\ &\le CM^2(t) (1+t)^{-1/2-j/2}, \end{aligned}$$

for $j \leq s-1$. Thus,

$$\begin{aligned} J_{21} &\leq CM^2(t)\int_{t/2}^{t}(1+t-\zeta)^{-5/4}(1+\zeta)^{-1/2-j/2}\,d\zeta \\ &\leq CM^2(t)(1+t)^{-1/2-j/2}. \end{aligned} \tag{4.60}$$

Since

$$\begin{aligned} \left\|\partial_x^{j+1}G(U)\right\|_2 &\leq C\|U\|_\infty\left\|\partial_x^{j+1}U\right\|_2 \\ &\leq CM_0(t)\left\|\partial_x^{j+1}U\right\|_{H^{s-j-1}} \\ &\leq CM_0(t)E(t)(1+t)^{-1/2-j/2}, \end{aligned}$$

we get

$$\begin{aligned} J_{22} &\leq CM_0(t)E(t)\int_{t/2}^{t}e^{-c(t-\zeta)}(1+\zeta)^{-1/2-j/2}\,d\zeta \\ &\leq CM_0(t)E(t)(1+t)^{-1/2-j/2}. \end{aligned} \tag{4.61}$$

From (4.60) and (4.61), it follows that

$$J_2 \leq CM_0(t)E(t)(1+t)^{-1/2-j/2}. \tag{4.62}$$

Therefore, (4.55), (4.59), and (4.62) lead to

$$\left\|\partial_x^j U(t)\right\|_2 \leq CE_s(1+t)^{-1/4-j/2} + C\left(M(t)^2 + M_0(t)E(t)\right)(1+t)^{-1/4-j/2} \tag{4.63}$$

for all $0 \leq j \leq s-1$. Estimate (4.51) is now proved, which concludes the proof of Lemma 4.6. ■

**Lemma 4.7** *Let $U_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ with $s \geq 3$ and put $E_s = \|U_0\|_{H^s} + \|U_0\|_1$. Let $T > 0$ and let $U(x,t)$ be the solution of (3.3), satisfying*

$$U \in C([0,T];H^s) \cap C^1(0,T];H^{s-1}).$$

*Then, we have the a priori estimates:*

$$\begin{aligned} E^2(T) + D^2(T) &\leq CE_s^2, \qquad (4.64)\\ M(T) &\leq CE_s, \qquad (4.65) \end{aligned}$$

*where $C$ is a positive constant independent of $T$ and $E_s$.*

**Proof.** By using the following interpolation inequality:

$$\|U\|_\infty \leq \sqrt{2}\,\|U\|_2^{1/2}\,\|\partial_x U\|_2^{1/2}, \tag{4.66}$$

we can see that

$$M_0(t) \leq CM(t),$$

provided that $s \geq 2$. Similarly, applying (4.66), with $U_x$ instead of $U$, we get for $s-1 \geq 2$ (i.e., for $s \geq 3$)

$$M_1(t) \leq CM(t).$$

For $s \geq 3$, by using (4.6) and (4.50), we therefore get,

$$(E(t) + D(t) + M(t))^2 \leq CE_s^2 + C(E(t) + D(t) + M(t))^3 . \tag{4.67}$$

By standard arguments (cf. [17]), we conclude that for sufficiently small $E_s$,

$$E(t) + D(t) + M(t) \leq \hat{C}. \tag{4.68}$$

Indeed, let $x = (E(t) + D(t) + M(t))^2$ and $h(x) = C\left(E_s^2 + x^{3/2}\right) - x$. Then, (4.67) implies $h(0) = CE_s^2$ and $f(x) \geq 0$. On the other hand, we have

$$h'(x) = \frac{3}{2}Cx^{1/2} - 1 \leq -\frac{1}{2},$$

for small enough $x$, say $0 \leq x \leq 1/\left(9C^2\right)$. From the identity $f(x) = f(0) + \int_0^t f'(x)\,dx$, we deduce that $f(x)$ changes its sign in $0 \leq x \leq 2CE_s^2$. Let $\hat{C}$ be the first zero of the function $h$. Then, we deduce that (4.68) holds. This proves Lemma 4.7. ■

**Theorem 4.8** *Let $U_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ and $E_s = \|U_0\|_{H^s} + \|U_0\|_{L^1}$, where $s \geq 3$ . Then, there exists a positive constant, $\delta_1 > 0$, such that, if $E_s \leq \delta_1$, then the global solution obtained in Theorem 4.5 satisfies the weighted energy estimate:*

$$E^2(t) + D^2(t) \leq CE_s^2, \tag{4.69}$$

*and the decay estimate:*

$$\left\|\partial_x^k U(t)\right\|_2 \leq CE_s(1+t)^{-1/4-k/2}, \tag{4.70}$$

*where $C$ is a positive constant and $0 \leq k \leq s-1$.*

**Proof.** The proof of Theorem 4.8 is a direct consequence of Lemma 4.7 and can be given from a chain of estimates of the energy type. ■

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