Spatial behavior in phase-lag heat conduction

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Abstract: In this paper we study the spatial behavior of solutions to the equations obtained by taking formal Taylor approximations to the heat conduction dual-phase-lag and three-phase-lag theories, reflecting Saint-Venant's principle. Depending on the relative order of derivation with respect to the time we propose different arguments. One is inspired by the arguments for parabolic problems and the other one is inspired by the arguments for hyperbolic problems. In the first case we obtain a Phragmén-Lindelöf alternative for the solutions, and in the second case we obtain an estimate for the decay as well as a domain of influence result. The main tool to manage these problems is the use of an exponentially weighted Poincaré inequality

1 Introduction

Fourier's heat conduction theory implies that thermal perturbations at some point can be observed in a solid instantly anywhere, however distant. This is a drawback of the model because it implies that heat waves (seem to) propagate with infinite speed. To overcome this difficulty and to satisfy the principle of causality, several alternative heat conduction theories have been suggested in the second part of the last century (see [2, 9, 10]). In the books [14, 34, 36], several mathematical studies concerning the applicability of different alternative thermoelastic theories are presented.

Tzou [35] suggested a modification of the Fourier law in 1995. He proposed a theory where the thermal flux and the gradient of temperature have a delay. The basic constitutive equation is:

$$q(x, t + \tau_q) = -k\nabla u(x, t + \tau_u), \qquad k > 0. \tag{1.1}$$

Here q is the heat flux vector, u is the temperature and τ_q, τ_u are the delay parameters which are assumed positive. This equation says that the temperature gradient established across a material volume at the position x at time $t + \tau_u$ results in a heat flux to flow at a different instant of time $t + \tau_q$. The delays are understood in terms of the microstructure

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of the material. More recently Choudhuri [31] proposed an extension of Tzou's theory. For this new proposition the constitutive equation of the heat flux vector is

$$q(x, t+\tau_q) = -\left(k\nabla u(x, t+\tau_u) + k^*\nabla \nu(x, t+\tau_\nu)\right). \tag{1.2}$$

Here ν is the thermal displacement that satisfies $\nu_t = u$, k^* is a parameter which is typical in the type II and III thermoelastic theories and τ_ν is another delay parameter which is also assumed positive. It seems that Choudhuri wanted to establish a mathematical model based on delays in such a way that the Taylor approximations recover the models proposed by Green and Naghdi [7, 8].

The theories of Tzou and Choudhuri are strongly based on an intuitive point of view, but there is no *a priori* thermomechanical foundation. In fact, it has been shown that, when we combine these constitutive equations with the classical energy equation

$$-\operatorname{div} q(x,t) = cu_t(x,t), \qquad c > 0, \tag{1.3}$$

there exists a sequence of solutions of the form

$$u_n(x,t) = \exp(\omega_n t)\Phi_n(x)$$

such that the real part ω_n tends to infinity [4]. This implies that we cannot obtain continuous dependence on initial data, and the associated mathematical problem is *ill posed* in the sense of Hadamard. This disagrees with the *a priori* expectation. For this reason a big interest has been developed to understand different formal Taylor approximations to these equations [1, 20, 21, 24, 27, 28, 29, 30]. These alternative theories allow to obtain stability of solutions and the well-posedness of the problems, provided certain conditions on the parameters hold.

The study of the *spatial* behavior for partial differential equations is related to Saint-Venant's principle. This aspect has been extensively investigated from the mathematical and also from thermomechanical viewpoints. Spatial decay estimates have been obtained for elliptic [5], parabolic [11, 12], hyperbolic [6] equations and/or combinations of these [23] in the last years. They describe how the influence of the perturbations on a part of the boundary is damped far away from the place where the perturbations were applied. From a mathematical viewpoint, it is usual to consider a semi-infinite cylinder whose finite end is perturbed and to see how the solutions decay when the spatial variable tends to infinity. Spatial behavior of solutions is a topic under deep investigation since a mathematical perspective [16, 17, 19, 22, 32, 33].

It is worth recalling that several cases where the spatial behavior for dual-phase-lag or three-phase lag models have been studied, see [13, 25, 26]. However, in these contributions only equations of third or fourth order with respect to time were analyzed. Arguments for a general type of higher-order equations have not yet been considered in the literature. Here, we propose a contribution of this type, i.e. we will obtain spatial estimates for solutions to equations of higher order.

We study the spatial behavior of solutions to the equations obtained by taking formal Taylor approximations for the dual-phase-lag (1.1) or three-phase-lag theories (1.2) of heat conduction. Plugging these into the energy equation (1.3), we obtain the equation[1]

$$a_0 u + a_1 u^{(1)} + a_2 u^{(2)} + \cdots + a_n u^{(n)} = b_0 \Delta u + b_1 \Delta u^{(1)} + \cdots + b_m \Delta u^{(m)}. \tag{1.4}$$

for $n > m \in \mathbb{N}_0$, where $a_0, \ldots, a_n, b_0, \ldots, b_m$ are constants. We will only assume, that the leading coefficients a_n, b_m are positive.

Typical examples are

$$u^{(1)} + \tau_q u^{(2)} + \frac{\tau_q^2}{2} u^{(3)} = k\Delta u + k\tau_u \Delta u^{(1)} + k\frac{\tau_u^2}{2}\Delta u^{(2)}, \tag{1.5}$$

or,

$$u^{(2)} + \tau_q u^{(3)} + \frac{\tau_q^2}{2} u^{(4)} = k^* \Delta u + \tau_\nu^* \Delta u^{(1)} + \tau_u^* \Delta u^{(2)} + k\frac{\tau_u^2}{2}\Delta u^{(3)}. \tag{1.6}$$

Here $\tau_\nu^* = k^*\tau_\nu + k$ and $\tau_u^* = (k^*\tau_\nu^2 + 2k\tau_u)/2$.

It is known that, in general, these approximations, together with initial and boundary conditions, do not always define well-posed problems; they are ill-posed if $n - m > 2$, (see [4]). So, we will restrict our attention to the cases

$$0 < n - m \leq 2,$$

where we know that they define a well-posed problem, see [1, 37].

In the next section we propose the basic problems and recall the exponentially weighted Poincaré inequality. It will be a fundamental tool in our approach.

As the analysis is different for the cases $n - m = 2$ and $n - m = 1$, respectively, reflecting the different character of the equations (typically: hyperbolic resp. parabolic), we devote a section to each case. The case $n - m = 2$ is studied in section 3, where, in particular, we obtain a *domain of influence* result. When $n - m = 1$ we obtain a Phragmén-Lindelöf alternative in section 4. We then also describe how to obtain an upper bound for the amplitude term when the solution decays in the spatial variable.

2 Preliminaries

In this section we define the basic problem we will study in the paper, and we will recall a fundamental tool that we use in our approach: it is the *exponentially weighted Poincaré*

[1] Here and from now on, $g^{(k)}$ denotes the k-th derivative of the function g with respect to time.

inequality. As spatial domain, we will consider the semi-infinite cylinder $R = [0,\infty) \times D$, where D is a bounded domain in the two-dimensional Euclidean space, being smooth enough to guarantee the use of the divergence theorem.

We consider general Taylor approximations to the dual-phase-lag or three-phase-lag theories of heat conduction of the form:

$$a_0 u + a_1 u^{(1)} + a_2 u^{(2)} + \cdots + a_n u^{(n)} = b_0 \Delta u + b_1 \Delta u^{(1)} + \cdots + b_m \Delta u^{(m)}. \tag{2.1}$$

where $a_0, \ldots, a_n, b_0, \ldots, b_m$ are constants such that $a_n > 0$, $b_m > 0$. As explained above, we here will restrict our attention to the case when $n - m = 1, 2$. We point out that the existence of solutions can be obtained using semigroup theory, see [1, 37], and, hence, we will assume the existence of solutions as well as the necessary regularity required in our calculations.

In addition to the differential equation (2.1), we have the initial conditions

$$u(x,0) = u^{(1)}(x,0) = \cdots = u^{(n-1)}(x,0) = 0, \quad x \in R, \tag{2.2}$$

and the boundary conditions

$$u(x_1,x_2,x_3,t) = 0, \quad (x_2,x_3) \in \partial D, \quad t \geq 0, \tag{2.3}$$

$$u(0,x_2,x_3,t) = f(x_2,x_3,t), \quad (x_2,x_3) \in D, \quad t \geq 0. \tag{2.4}$$

To assure the compatibility, we naturally assume

$$f(x_2,x_3,t) = 0, \quad (x_2,x_3) \in \partial D, \quad t \geq 0.$$

In view of the usual axioms in thermomechanics, it is natural to assume that the thermal conductivity k is positive and that the delay parameters are always positive. Thus, for the Taylor approximations to the model proposed by Tzou, all parameters arising in the differential equation (2.1) are positive. However we do not know any thermomechanical reason to guarantee *a priori* that the parameter k^* is positive[2]. Thus, it is worth covering the case where this parameter can be negative and in this sense we allow that the sign of the parameters a_i $(i = 1 \ldots n-1)$ and b_j $(j = 1 \ldots m-1)$ may be zero or negative, which has not been considered before. But our results are new even in the case that all the coefficients are positive.

An important tool in this paper will be the following result (see the appendix of [18] for a proof), the *exponentially weighted Poincaré inequality*:

[2] We should recall that the stability of solutions in type II and III thermoelastodynamics is related to the positivity of this parameter.

Assume that $f:[0,t]\to\mathbb{R}$ is differentiable and satisfies $f(0)=0$. Then the following inequality

$$\int_0^t \exp(-2\omega s) f^2(s)\, ds \le \frac{4t^2}{\pi^2+4t^2\omega^2}\int_0^t \exp(-2\omega s)\left(f^{(1)}(s)\right)^2 ds, \tag{2.5}$$

holds, for every $\omega>0$. We note that $\varphi(t)=\dfrac{4t^2}{\pi^2+4t^2\omega^2}$ is a growing function, hence

$$\int_0^t \exp(-2\omega s) f^2(s)\, ds \le \omega^{-2}\int_0^t \exp(-2\omega s)\left(f^{(1)}(s)\right)^2 ds. \tag{2.6}$$

As a consequence, we obtain for $n>k+1$ and for f satisfying $f^{(k)}(0)=\cdots=f^{(n-1)}(0)=0$, that the estimate

$$\int_0^t \exp(-2\omega s)|f^{(k)}(s)|^2\, ds \le \omega^{-2(n-k-1)}\int_0^t \exp(-2\omega s)|f^{(n-1)}(s)|^2 ds, \tag{2.7}$$

holds. These inequalities will allow us to deal with lower-order time derivatives in a comparison with higher-order terms.

3 Case $n-m=2$

In this section we analyze the case when $n=m+2$ with $b_m>0$ and $a_{m+2}>0$. However, we do not impose conditions on the sign of the other parameters.

The function, the properties of which will describe the spatial behavior, is given, for $z\ge 0$, $t\ge 0$, by

$$F_\omega(z,t):=$$
$$-\int_0^t\int_0^\tau\int_{D(z)} \exp(-2\omega s)\left(b_0u_{,1}+b_1u_{,1}^{(1)}+\cdots+b_mu_{,1}^{(m)}\right)u^{(m+1)}(a,s)\,da\,ds\,d\tau, \tag{3.1}$$

where ω is a positive constant to be selected later. By $u_{,1}$ the derivative with respect to the first variable (x_1) is denoted, and $D(z):=\{z\}\times D$. It is worth noting that the function F_ω is inspired by the functions usually considered for hyperbolic problems [6, 13]. However, in order to control the low time derivatives by means of the high time derivatives we need to consider an extra integration with respect to the time.

As we shall see later, F_ω is non-negative, decreases with respect to z and converges to zero, describing the asymptotical behavior of u and its time derivatives as $z\to\infty$.

Using the divergence theorem we see that

$$
\begin{aligned}
&F_\omega(z+h,t) - F_\omega(z,t) \\
&= -\frac{1}{2}\int_0^t \int_z^{z+h}\int_D \exp(-2\omega\tau)\left[a_{m+2}|u^{(m+1)}|^2 + b_m|\nabla u^{(m)}|^2\right](v,\tau)dvd\tau \\
&\quad - \int_0^t \int_0^\tau \int_z^{z+h}\int_D \exp(-2\omega s)\left[P + \omega(a_{m+2}|u^{(m+1)}|^2 + b_m|\nabla u^{(m)}|^2)\right](v,s)dv\, dsd\tau,
\end{aligned}
\tag{3.2}
$$

where

$$
P := (b_0\nabla u + \dots + b_{m-1}\nabla u^{(m-1)})\nabla u^{(m+1)} + (a_0 u + \dots + a_{m+1}u^{(m+1)})u^{(m+1)}. \tag{3.3}
$$

For $k = 0, \dots, m-1$, the following relation

$$
\begin{aligned}
\exp(-2\omega s)b_k\nabla u^{(k)}\nabla u^{(m+1)} &= \frac{d}{ds}\left(\exp(-2\omega s)b_k\nabla u^{(k)}\nabla u^{(m)}\right) \\
&\quad - \exp(-2\omega s)b_k\nabla u^{(k+1)}\nabla u^{(m)} + 2\omega\exp(-2\omega s)b_k\nabla u^{(k)}\nabla u^{(m)},
\end{aligned}
\tag{3.4}
$$

is satisfied. So we can write

$$
\begin{aligned}
&F_\omega(z+h,t) - F_\omega(z,t) \\
&= -\frac{1}{2}\int_0^t \int_z^{z+h}\int_D \exp(-2\omega\tau)\left[a_{m+2}|u^{(m+1)}|^2 + b_m|\nabla u^{(m)}|^2\right](x,\tau)dadx_1d\tau \\
&- \int_0^t \int_z^{z+h}\int_D \exp(-2\omega s)\left[b_0\nabla u\nabla u^{(m)} + \dots + b_{m-1}\nabla u^{(m-1)}\nabla u^{(m)}\right](x,\tau)dadx_1d\tau \\
&- \int_0^t \int_0^\tau \int_z^{z+h}\int_D \exp(-2\omega s)\omega\left[a_{m+2}|u^{(m+1)}|^2 + b_m|\nabla u^{(m)}|^2\right](x,\tau)dadx_1dsd\tau \\
&- 2\omega\int_0^t \int_0^\tau \int_z^{z+h}\int_D \exp(-2\omega s)[b_0\nabla u\nabla u^{(m)} + \dots \\
&\qquad\qquad + b_{m-1}\nabla u^{(m-1)}\nabla u^{(m)}](x,\tau)dadx_1dsd\tau \\
&+ \int_0^t \int_0^\tau \int_z^{z+h}\int_D \exp(-2\omega s)[b_0\nabla u^{(1)}\nabla u^{(m)} + \dots + b_{m-1}\nabla u^{(m)}\nabla u^{(m)}]dadx_1dsd\tau \\
&- \int_0^t \int_0^\tau \int_z^{z+h}\int_D \exp(-2\omega s)[a_0uu^{(m+1)} + \dots + a_{m+1}u^{(m+1)}u^{(m+1)}](x,\tau)dadx_1dsd\tau.
\end{aligned}
\tag{3.5}
$$

Thus, we also have

$$
\begin{aligned}
&\frac{\partial F_\omega}{\partial z}(z,t) = -\frac{1}{2}\int_0^t\int_D \exp(-2\omega\tau)\left[a_{m+2}|u^{(m+1)}|^2 + b_m|\nabla u^{(m)}|^2\right](z,a,\tau)dad\tau \\
&-\int_0^t\int_D \exp(-2\omega\tau)\left[b_0\nabla u\nabla u^{(m)} + \cdots + b_{m-1}\nabla u^{(m-1)}\nabla u^{(m)}\right](z,a,\tau)dad\tau \\
&-\int_0^t\int_0^\tau\int_D \exp(-2\omega s)\omega\left[a_{m+2}|u^{(m+1)}|^2 + b_m|\nabla u^{(m)}|^2\right](z,a,s)da\,dsd\tau \\
&-2\omega\int_0^t\int_0^\tau\int_D \exp(-2\omega s)[b_0\nabla u\nabla u^{(m)} + \cdots + b_{m-1}\nabla u^{(m-1)}\nabla u^{(m)}](z,a,s)dadsd\tau \\
&+\int_0^t\int_0^\tau\int_D \exp(-2\omega s)[b_0\nabla u^{(1)}\nabla u^{(m)} + \cdots + b_{m-1}\nabla u^{(m)}\nabla u^{(m)}](z,a,s)dadsd\tau \\
&-\int_0^t\int_0^\tau\int_D \exp(-2\omega s)[a_0uu^{(m+1)} + \cdots + a_{m+1}u^{(m+1)}u^{(m+1)}](z,a,s)dadsd\tau.
\end{aligned}
\tag{3.6}
$$

Now we shall demonstrate that for sufficiently large ω the function $\partial F_\omega/\partial z$ is non-negative. We note that

$$
|\int_0^t\int_D \exp(-2\omega\tau)b_k\nabla u^{(k)}\nabla u^{(m)}(z,a,\tau)dad\tau| \leq \tag{3.7}
$$

$$
|b_k|\left(\int_0^t\int_D \exp(-2\omega\tau)\nabla u^{(k)}\nabla u^{(k)}dad\tau\right)^{1/2}\left(\int_0^t\int_D \exp(-2\omega\tau)\nabla u^{(m)}\nabla u^{(m)}(z,a,\tau)dad\tau\right)^{1/2}
$$

$$
\leq |b_k|b_m^{-1}\omega^{k-m}\int_0^t\int_D \exp(-2\omega s)b_m\nabla u^{(m)}\nabla u^{(m)}(z,a,\tau)dad\tau,
$$

for $k = 0\ldots m-1$. In a similar way

$$
|\int_0^t\int_0^\tau\int_D \exp(-2\omega s)b_k\nabla u^{(k)}\nabla u^{(m)}(z,a,s)dadsd\tau| \tag{3.8}
$$

$$
\leq |b_k|b_m^{-1}\omega^{k-m}\int_0^t\int_0^\tau\int_D \exp(-2\omega s)b_m\nabla u^{(m)}\nabla u^{(m)}(z,a,s)dadsd\tau,
$$

for $k = 0\cdots -1$. We also have

$$
|\int_0^t\int_0^\tau\int_D \exp(-2\omega s)b_k\nabla u^{(k+1)}\nabla u^{(m)}(z,a,s)dadsd\tau| \tag{3.9}
$$

$$
\leq |b_k|b_m^{-1}\omega^{k-m+1}\int_0^t\int_0^\tau\int_D \exp(-2\omega s)b_m\nabla u^{(m)}\nabla u^{(m)}(z,a,s)dadsd\tau,
$$

for $k = 0\ldots m-1$, and

$$
|\int_0^t\int_0^\tau\int_D \exp(-2\omega s)a_k u^{(k)}u^{(m+1)}dadsd\tau| \tag{3.10}
$$

$$\leq |a_k| a_{m+2}^{-1} \omega^{k-m+1} \int_0^t \int_0^\tau \int_D \exp(-2\omega s) a_{m+2} u^{(m+1)} u^{(m+1)}(z,a,s) da ds d\tau,$$

for $k = 0 \ldots m+1$.

We obtain that there exist three polynomials P_1, P_2, P_3 depending on one real variable with $P_i(0) = 0,\ i = 1,2,3$, such that

$$\begin{aligned}\frac{\partial F_\omega}{\partial z}(z,t) \leq -\frac{1}{2} \int_0^t \int_D \exp(-2\omega\tau) \left[a_{m+2} |u^{(m+1)}|^2 + b_m (1 - P_1(\omega^{-1})) |\nabla u^{(m)}|^2\right](z,a,\tau) da d\tau \\ - \int_0^t \int_0^\tau \int_D \exp(-2\omega s) \omega \left[a_{m+2}(1 - P_2(\omega^{-1})) |u^{(m+1)}|^2 \right. \\ \left. + b_m (1 - P_3(\omega^{-1})) |\nabla u^{(m)}|^2\right](z,a,s) da ds d\tau.\end{aligned} \tag{3.11}$$

Since $P_i(0) = 0$, we can select ω large enough to guarantee that the inequality

$$\begin{aligned}\frac{\partial F_\omega}{\partial z}(z,t) \leq -\frac{1}{2} \int_0^t \int_D \exp(-2\omega\tau) \left[a_{m+2} |u^{(m+1)}|^2 + (b_m - \epsilon) |\nabla u^{(m)}|^2\right](z,a,\tau) da d\tau \\ - \int_0^t \int_0^\tau \int_D \exp(-2\omega s) \omega \left[a_{m+2}(1-\epsilon) |u^{(m+1)}|^2 + b_m (1-\epsilon) |\nabla u^{(m)}|^2\right](z,a,\tau) da\, ds d\tau\end{aligned} \tag{3.12}$$

holds, where we can choose a positive ϵ as small as we want, provided ω is chosen large enough. Thus

$$\frac{\partial F_\omega}{\partial z}(z,t) \leq 0, \tag{3.13}$$

i.e, $F_\omega(z,t)$ is decreasing in z.

Our next step is to evaluate the time derivative of F_ω in terms of the spatial derivative. We have that

$$\frac{\partial F_\omega}{\partial t}(z,t) = -\int_0^t \int_{D(z)} \exp(-2\omega s) \left(b_0 u_{,1} + b_1 u_{,1}^{(1)} + \cdots + b_m u_{,1}^{(m)}\right) u^{(m+1)}(a,s) da ds. \tag{3.14}$$

Noting that

$$|\int_0^t \int_D b_k \exp(-2\omega s) u_{,i}^{(k)} u^{(m+1)}(a,s) da ds| \leq \tag{3.15}$$

$$\leq C_k \omega^{k-m} \int_0^t \int_D \exp(-2\omega s)((b_m - \epsilon) \nabla u^{(m)} \nabla u^{(m)} + a_{m+2} u^{(m+1)} u^{(m+1)})(a,s) da ds,$$

for $k = 0 \ldots m$, where

$$C_k := \frac{1}{2} |b_k| (a_{m+2}(b_m - \epsilon))^{-1/2}.$$

It follows

$$|\frac{\partial F_\omega}{\partial t}| \leq -\Omega_\omega \frac{\partial F_\omega}{\partial z}, \tag{3.16}$$

where

$$\Omega_\omega := 2\sum_{k=0}^{m} C_k \omega^{(k-m)/2}. \tag{3.17}$$

This inequality implies that

$$\frac{\partial F_\omega}{\partial t} + \Omega_\omega \frac{\partial F_\omega}{\partial z} \leq 0, \tag{3.18}$$

and

$$\frac{\partial F_\omega}{\partial t} - \Omega_\omega \frac{\partial F_\omega}{\partial z} \geq 0. \tag{3.19}$$

For arbitrary $0 \leq z^* \leq z,$ and $h : [0, \Omega_\omega^{-1}(z - z^*)] \to \mathbb{R}^2,\ r \mapsto (\Omega_\omega r + z^*, r)$ we conclude from (3.18) that

$$\frac{d}{dr} F_\omega(h(r)) \leq 0,$$

hence

$$F_\omega(z, \Omega_\omega^{-1}(z - z^*)) \leq 0 \tag{3.20}$$

Similarly, we obtain from (3.19)

$$F_\omega(z, \Omega_\omega^{-1}(z^{**} - z)) \geq 0, \tag{3.21}$$

for arbitrary $z^{**} \geq z$.

Fixing t, we conclude that when z increases we can find $z^* = z - \Omega_\omega t \geq 0$ and $z^* = z + \Omega_\omega t$ such that f

$$0 = F_\omega(z^*, 0) \leq F(z, t) \leq F_\omega(z^{**}, 0) = 0.$$

These inequalities imply that, for each finite time t,

$$\lim_{z \to \infty} F_\omega(z, t) = 0, \tag{3.22}$$

expressing the next description of the asymptotic behavior in the spirit of Saint-Venant's principle.

As a corollary, taking the limit $h \to \infty$ in (3.5), we obtain

$$
\begin{aligned}
&F_\omega(z,t) = \frac{1}{2}\int_0^t \int_{R(z)} \exp(-2\omega\tau)\left[a_{m+2}|u^{(m+1)}|^2 + b_m|\nabla u^{(m)}|^2\right](v,\tau)dvd\tau \\
&+ \int_0^t \int_{R(z)} \exp(-2\omega\tau)\left[b_0\nabla u^{(0)}\nabla u^{(m)} + \cdots + b_{m-1}\nabla u^{(m-1)}\nabla u^{(m)}\right](v,\tau)dvd\tau \\
&+ \int_0^t \int_0^\tau \int_{R(z)} \exp(-2\omega s)\omega\left[a_{m+2}|u^{(m+1)}|^2 + b_m|\nabla u^{(m)}|^2\right](v,s)dv\,dsd\tau \\
&+ 2\omega \int_0^t \int_0^\tau \int_{R(z)} \exp(-2\omega s)[b_0\nabla u\nabla u^{(m)} + \cdots + b_{m-1}\nabla u^{(m-1)}\nabla u^{(m)}](v,s)dvdsd\tau \\
&- \int_0^t \int_0^\tau \int_{R(z)} \exp(-2\omega s)[b_0\nabla u^{(1)}\nabla u^{(m)} + \cdots + b_{m-1}\nabla u^{(m)}\nabla u^{(m)}](v,s)dvdsd\tau \\
&+ \int_0^t \int_0^\tau \int_{R(z)} \exp(-2\omega s)[a_0 u u^{(m+1)} + \cdots + a_{m+1}u^{(m+1)}u^{(m+1)}](v,s)dvdsd\tau,
\end{aligned} \tag{3.23}
$$

where $R(z) := [z,\infty) \times D$ $(R = R(0))$.

Now the inequality (3.18) implies that solutions are decreasing along the lines of slope Ω_ω^{-1}. Thus we have that

$$F_\omega(z,t) \le F_\omega(z^*,t^*), \tag{3.24}$$

where and $z \ge z^*$ and t are related by $t - t^* = \Omega_\omega^{-1}(z - z^*)$. In a similar way, from (3.19) we get

$$F_\omega(z,t) \ge F_\omega(z^{**},t^{**}), \tag{3.25}$$

for $t - t^{**} = \Omega_\omega^{-1}(z^{**} - z)$, where $z \le z^{**}$. If we consider two points (z,t) and (z^*,t^*) with $z \ge z^*$ such that $|t - t^*| \le \Omega_\omega^{-1}(z - z^*)$, we conclude

$$F_\omega(z,t) \le F_\omega(z^*,t^*), \tag{3.26}$$

for $|t - t^*| \le \Omega_\omega^{-1}(z - z^*)$. Thus, we have proved

Theorem 3.1. *Let u be a solution of the initial-boundary-value problem (2.1)-(2.3) when $n = m+2$. Then the function $F_\omega(z,t)$ defined in (3.23) (which is a measure on the solutions for ω sufficiently large) satisfies the inequality (3.26) whenever $|t-t^*| \le \Omega_\omega^{-1}(z - z^*)$ and $z \ge z^*$.*

Moreover, we can take $z^{**} \ge z$ arbitrary in (3.21), we conclude the non-negativity of F_ω,

$$F_\omega(z,t) \ge 0$$

for any $z \ge 0$, $t \ge 0$. On the other hand, we have from (3.20) for $0 \le t \le \frac{z}{\Omega_\omega}$

$$F_\omega(z,t) \le 0.$$

This implies that for $0 \leq t \leq \frac{z}{\Omega_\omega}$
$$F_\omega(z,t) = 0. \tag{3.27}$$
Hence we obtain for all $z \geq 0$ and $0 \leq t \leq \frac{z}{\Omega_\omega}$,
$$0 = \frac{\partial F_\omega}{\partial z}(z,t) = 0$$
which implies, using (3.12), (3.13) and the fact that the initial values for u vanish,
$$u(x_1, x_2, x_3, t) = 0 \qquad\qquad \text{if } \ 0 \leq t \leq \frac{x_1}{\Omega_\omega}. \tag{3.28}$$
This is a *domain of influence* result. Defining
$$\Omega_\infty := \lim_{\omega\to\infty} \Omega_\omega,$$
we observe that Ω_ω converges to Ω_∞ from above, hence we conclude
$$u(x_1, x_2, x_3, t) = 0 \qquad\qquad \text{if } \ 0 \leq t < \frac{x_1}{\Omega_\infty}. \tag{3.29}$$
If one defines the measure
$$\mathcal{F}_\omega(z,t) = \int_0^t F_\omega(z,r)dr, \tag{3.30}$$
we get from (3.27) for $\Omega_\omega t \geq z$
$$\mathcal{F}_\omega(z,t) = \int_{\Omega_\omega^{-1} z}^t F_\omega(z,r)dr.$$
Now, we set (see [3]) $r := (1 - z/(\Omega_\omega t))s + z/\Omega_\omega$ and obtain
$$\mathcal{F}_\omega(z,t) = \left(1 - \frac{z}{\Omega_\omega t}\right) \int_0^t F_\omega\left(z, \left(1 - \frac{z}{\Omega_\omega t}\right) s + \frac{z}{\Omega_\omega}\right) ds.$$
Since we know from Theorem 3.1 that
$$F_\omega\left(z, \left(1 - \frac{z}{\Omega_\omega t}\right) s + \frac{z}{\Omega_\omega}\right) \leq F_\omega(0,s),$$
we conclude
$$\mathcal{F}_\omega(z,t) \leq \left(1 - \frac{z}{\Omega_\omega t}\right) \int_0^t F_\omega(0,r)dr = \left(1 - \frac{z}{\Omega_\omega t}\right) \mathcal{F}_\omega(0,t).$$
Theorem 3.2. *Let u be a solution of the initial-boundary value problem (2.1)-(2.3) with $n = m + 2$ and ω sufficiently large to guarantee that the function defined in (3.22) is a measure (non-negative) on the solutions. Then*
$$u(x,t) = 0 \qquad \text{if } \ \Omega_\infty t \leq x_1, \tag{3.31}$$
$$\mathcal{F}_\omega(z,t) \leq \left(1 - \frac{z}{\Omega_\omega t}\right)\mathcal{F}_\omega(0,t) \qquad \text{if } \ \Omega_\omega t \geq z. \tag{3.32}$$

An upper bound for the amplitude term $\mathcal{F}_\omega(0,t)$ can be obtained in terms of the boundary conditions, but we do not consider this question here.

4 Case $n - m = 1$

In this section we analyze the case when $n = m+1$, with $b_m > 0$ and $a_{m+1} > 0$. However, as in the previous section, we do not impose conditions on the sign of the other parameters.

In this case the analysis starts by considering the function

$$G_\omega(z,t) = -\int_0^t \int_{D(z)} \exp(-2\omega s) \left(b_0 u_{,1} + b_1 u^{(1)}_{,1} + \cdots + b_m u^{(m)}_{,1} \right) u^{(m)}(a,s)dads, \quad (4.1)$$

where ω is a positive constant to be chosen later. This function is the natural counterpart to the ones used for parabolic linear equations [15].

Using the divergence theorem we see that

$$\begin{aligned} G_\omega(z+h,t) - G_\omega(z,t) &= -\frac{1}{2}\exp(-2\omega t)\int_z^{z+h}\int_D a_{m+1}|u^{(m)}|^2(v,t)dv \\ &- \int_0^t \int_z^{z+h}\int_D \exp(-2\omega s)\left[Q + \omega a_{m+1}|u^{(m)}|^2 + b_m|\nabla u^{(m)}|^2)\right](v,s)dvds, \end{aligned} \quad (4.2)$$

where

$$Q := (b_0\nabla u + b_1\nabla u^{(1)} + \cdots + b_{m-1}\nabla u^{(m-1)})\nabla u^{(m)} + (a_0 u + a_1 u^{(1)} + \cdots + a_m u^{(m)})u^{(m)}. \quad (4.3)$$

Thus, we get

$$\begin{aligned} \frac{\partial G_\omega}{\partial z}(z,t) &= -\frac{1}{2}\int_D \exp(-2\omega t)\left[a_{m+1}|u^{(m)}|^2\right](z,a,t)da \\ &- \int_0^t\int_D \exp(-2\omega s)\left[Q + \omega a_{m+1}|u^{(m)}|^2 + b_m|\nabla u^{(m)}|^2)\right](z,a,s)da\,ds. \end{aligned} \quad (4.4)$$

To control the function Q we can use a similar argument to the one proposed in the previous section. If we consider the exponentially weighted Poincaré inequality, we can obtain the existence of two polynomials Q_1 and Q_2 satisfying $Q_i(0) = 0$, $i = 1, 2$ such that

$$\begin{aligned} \frac{\partial G_\omega}{\partial z}(z,t) &\leq -\frac{1}{2}\int_D \exp(-2\omega t)\left[a_{m+1}|u^{(m)}|^2\right](z,a,t)da \\ &- \int_0^t\int_D \exp(-2\omega s)\left[\omega a_{m+1}(1 - Q_1(\omega^{-1}))|u^{(m)}|^2 + (b_m - Q_2(\omega^{-1}))|\nabla u^{(m)}|^2)\right](z,a,s)da\,ds. \end{aligned} \quad (4.5)$$

In particular, we note that, for ω large enough, the following inequality

$$\begin{aligned} \frac{\partial G_\omega}{\partial z} &\leq -\frac{1}{2}\int_D \exp(-2\omega t)\left[a_{m+1}|u^{(m)}|^2\right](z,a,t)da \\ &\quad -\frac{1}{2}\int_0^t\int_D \exp(-2\omega s)\left[\omega a_{m+1}|u^{(m)}|^2 + b_m|\nabla u^{(m)}|^2)\right](z,a,s)da\,ds \\ &\leq 0 \end{aligned} \quad (4.6)$$

holds.

Our next step is to evaluate the absolute value of G_ω in terms of the spatial derivative. With a sufficiently small ϵ (obtained for sufficiently large ω as in the previous section), we can obtain positive constants

$$D_k := |b_k|(2(b_m - \epsilon)\lambda_1)^{-1/2},$$
$$\Omega^*_\omega := 2\sum_{k=0}^{m} D_k \omega^{(k-m)/2}, \tag{4.7}$$

such that

$$|G_\omega| \le -\Omega^*_\omega \frac{\partial G_\omega}{\partial z}. \tag{4.8}$$

Here, λ_1 is the first eigenvalue of the negative Laplace operator $-\Delta$ with Dirichlet boundary conditions (clamped membrane) in the domain D. It arises in estimating $u^{(m)}$ by $\nabla u^{(m)}$. This inequality is well known in the study of spatial estimates. It implies that

$$G_\omega \le -\Omega^*_\omega \frac{\partial G_\omega}{\partial z} \tag{4.9}$$

and

$$-G_\omega \le -\Omega^*_\omega \frac{\partial G_\omega}{\partial z}. \tag{4.10}$$

For fixed t, we distinguish two cases:

(I) If there exists $z_0 \ge 0$ such that $G_\omega(z_0, t) < 0$, it follows that $G_\omega(z, t) < 0$ for every $z \ge z_0$. We conclude that

$$-G_\omega(z,t) \ge -G_\omega(z_0,t)\exp\left(\frac{z - z_0}{\Omega^*_\omega}\right), \quad z \ge z_0. \tag{4.11}$$

(II) Otherwise we see that $G_\omega(z,t) \ge 0$ for every $z \ge 0$. It then follows the spatial decay estimate

$$G_\omega(z,t) \le G_\omega(0,t)\exp\left(-\frac{z}{\Omega^*_\omega}\right), \quad z \ge 0. \tag{4.12}$$

We can summarize this result in the following way:

Theorem 4.1. *Let u be a solution of the initial-boundary-value problem (2.1)-(2.3) with $n = m+1$. Then either the function $-G_\omega(z,t)$ satisfies the asymptotic condition (4.11), or the function*

$$\begin{aligned} 0 \le G_\omega(z,t) &= \frac{1}{2}\int_{R(z)} \exp(-2\omega t)\left[a_{m+1}|u^{(m)}|^2\right](v,t)dv \\ &+ \int_0^t \int_{R(z)} \exp(-2\omega s)\left[Q + \omega a_{m+1}|u^{(m)}|^2 + b_m|\nabla u^{(m)}|^2)\right](v,s)dv\,ds, \end{aligned} \tag{4.13}$$

satisfies the decay estimate (4.12).

It is worth noting that when ω increases the parameter $(\Omega^*_\omega)^{-1}$ tends to $(2b_m/\lambda_1)^{1/2}$.

In the remaining of this section, we will describe how to obtain an upper bound for the amplitude term $G_\omega(0,t)$ in terms of the boundary data.

From now on, we restrict our attention to solutions satisfying the decay estimate (4.12) where ω is large enough to guarantee that

$$\begin{aligned} G_\omega(z,t) \geq \frac{1}{2}\int_{R(z)} \exp(-2\omega t) a_{m+1}|u^{(m)}|^2(v,t)dv \\ + \frac{1}{2}\int_0^t\int_{R(z)} \exp(-2\omega s)\left[\omega a_{m+1}|u^{(m)}|^2 + b_m|\nabla u^{(m)}|^2)\right](v,s)dv\,ds. \end{aligned} \tag{4.14}$$

We shall denote by $\xi = \xi(x,t)$ a function which tends uniformly to zero, rapidly, as $x_1 \to \infty$, and satisfies the same boundary conditions as u. Below, we shall, typically, choose

$$\xi(x,t) := \exp(-dx_1)f(x_2,x_3,t),$$

with a positive constant d.

Then we have

$$G_\omega(0,t) = -\int_0^t\int_{D(0)} \exp(-2\omega s)(b_0u_{,1} + \cdots + b_m u^{(m)}_{,1})\xi^{(m)}(a,s)da ds. \tag{4.15}$$

After the use of the boundary, asymptotic and the initial conditions, we see that

$$\begin{aligned} G_\omega(0,t) =& \int_0^t\int_R \exp(-2\omega s)\left[b_0\nabla u + \cdots + b_m\nabla u^{(m)}\right]\nabla\xi^{(m)}(v,s)dvds \\ &+ \int_0^t\int_R \exp(-2\omega s)\left[a_0 u + \cdots + a_{m+1}u^{(m+1)}\right]\xi^{(m)}(v,s)dv\,ds. \end{aligned} \tag{4.16}$$

As

$$\begin{aligned} \exp(-2\omega s)u^{(m+1)}\xi^{(m)} =& \frac{d}{ds}(\exp(-2\omega s)u^{(m)}\xi^{(m)}) \\ &+2\omega\exp(-2\omega s)u^{(m)}\xi^{(m)} - \exp(-2\omega s)u^{(m+1)}\xi^{(m+1)}, \end{aligned} \tag{4.17}$$

we obtain

$$G_\omega(0,t) = I_1 + I_2 + I_3 + I_4 + I_5, \tag{4.18}$$

where

$$I_1 := \int_0^t\int_R \exp(-2\omega s)\left[b_0\nabla u + \cdots + b_m\nabla u^{(m)}\right]\nabla\xi^{(m)}(v,s)dvds, \tag{4.19}$$

$$I_2 := \int_0^t\int_R \exp(-2\omega s)\left[a_0u + \cdots + a_m u^{(m)}\right]\xi^{(m)}(v,s)dv\,ds, \tag{4.20}$$

$$I_3 := \exp(-2\omega t)\int_R a_{m+1}u^{(m)}\xi^{(m)}(v,t)dv, \tag{4.21}$$

$$I_4 := 2\omega \int_0^t \int_R \exp(-2\omega s) a_{m+1} u^{(m)} \xi^{(m)}(v,s) dv\, ds, \tag{4.22}$$

$$I_5 := -\int_0^t \int_R \exp(-2\omega s) a_{m+1} u^{(m)} \xi^{(m+1)}(v,s) dv\, ds. \tag{4.23}$$

By choosing ω large enough, we can take $\epsilon_i = 1 \ldots 5$ as small as we want and such that

$$I_1 \le \epsilon_1 \int_0^t \int_R \exp(-2\omega s) b_m |\nabla u^{(m)}|^2(v,s) dv ds + C_1^* \int_0^t \int_R \exp(-2\omega s) |\nabla \xi^{(m)}|^2(v,s) dv ds, \tag{4.24}$$

$$I_2 \le \epsilon_2 \int_0^t \int_R \exp(-2\omega s) a_{m+1} |u^{(m)}|^2(v,s) dv ds + C_2^* \int_0^t \int_R \exp(-2\omega s) |\xi^{(m)}|^2(v,s) dv ds, \tag{4.25}$$

$$I_3 \le \epsilon_3 \exp(-2\omega t) \int_R a_{m+1} |u^{(m)}|^2 dv + C_3^* \exp(-2\omega t) \int_R |\xi^{(m)}|^2(v,t) dv, \tag{4.26}$$

$$I_4 \le \epsilon_4 \int_0^t \int_R \exp(-2\omega s) a_{m+1} |u^{(m)}|^2(v,s) dv ds + C_4^* \int_0^t \int_R \exp(-2\omega s) |\xi^{(m)}|^2(v,s) dv ds, \tag{4.27}$$

$$I_5 \le \epsilon_5 \int_0^t \int_R \exp(-2\omega s) a_{m+1} |u^{(m)}|^2(v,s) dv ds + C_5^* \int_0^t \int_R \exp(-2\omega s) |\xi^{(m+1)}|^2(v,s) dv ds. \tag{4.28}$$

Here $C_i^*, i = 1 \ldots 5$ are constants which can be computed in terms of the data of the problem, ω and ϵ_i. It then follows that

$$G_\omega(0,t) \le 2(\epsilon_1 + \cdots + \epsilon_5) G_\omega(0,t) + C_1^* J_1 + (C_2^* + C_4^*) J_2 + C_3^* J_3 + C_5^* J_5, \tag{4.29}$$

where

$$J_1 := \int_0^t \int_R \exp(-2\omega s) |\nabla \xi^{(m)}|^2(v,s) dv ds \tag{4.30}$$

$$J_2 := \int_0^t \int_R \exp(-2\omega s) |\xi^{(m)}|^2(v,s) dv ds, \tag{4.31}$$

$$J_3 := \exp(-2\omega t) \int_R |\xi^{(m)}|^2(v,t) dv, \tag{4.32}$$

$$J_5 := \int_0^t \int_R \exp(-2\omega s) |\xi^{(m+1)}|^2(v,s) dv ds. \tag{4.33}$$

If we select ϵ_i such that $\epsilon_1 + \cdots + \epsilon_5 < 1/4$, we obtain that

$$G_\omega(0,t) \le 2(C_1^* J_1 + (C_2^* + C_4^*) J_2 + C_3^* J_3 + C_5^* J_5). \tag{4.34}$$

To obtain a precise upper bound for the J_i, we recall the choice of the function ξ, and we note that

$$\xi^{(m)} = \exp(-dx_1) f^{(m)}(x_2, x_3, t), \quad \xi^{(m+1)} = \exp(-dx_1) f^{(m+1)}(x_2, x_3, t), \tag{4.35}$$

and

$$\nabla\xi^{(m)} = \exp(-dx_1)(-df^{(m)}(x_2,x_3,t), f^{(m)}_{,2}(x_2,x_3,t), f^{(m)}_{,3}(x_2,x_3,t)). \tag{4.36}$$

We conclude

$$J_1 \leq \int_0^t \int_{D(0)} \left(\frac{d}{2}|f^{(m)}|^2 + \frac{1}{2d}(|f^{(m)}_{,3}|^2 + |f^{(m)}_{,3}|^2)\right)(a,s)dads, \tag{4.37}$$

$$J_2 \leq \frac{1}{2d}\int_0^t \int_{D(0)} |f^{(m)}|^2(a,s)dads, \quad J_3 \leq \frac{1}{2d}\int_{D(0)} |f^{(m)}|^2(a,t)da, \tag{4.38}$$

$$J_5 \leq \frac{1}{2d}\int_0^t \int_{D(0)} |f^{(m+1)}|^2(a,s)dads. \tag{4.39}$$

From the previous inequalities we finally obtain

$$\begin{aligned} G_\omega(0,t) \quad \leq \quad & (dC_1^* + \frac{C_2^* + C_4^*}{d})\int_0^t \int_{D(0)} |f^{(m)}|^2(a,s)dads \\ & + \frac{C_1^*}{d}\int_0^t \int_{D(0)} \left(|f^{(m)}_{,3}|^2 + |f^{(m)}_{,3}|^2\right)(a,s)dads \\ & + \frac{C_3^*}{d}\int_{D(0)} |f^{(m)}|^2(a,t)da + \frac{C_5^*}{d}\int_0^t \int_{D(0)} |f^{(m+1)}|^2(a,s)dads. \end{aligned} \tag{4.40}$$

We remark that one could optimize the right-hand side by taking a suitable value of the parameter d, but it does not seem to be an easy task.

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